Chapter 25
Capacitance and physics of dielectric
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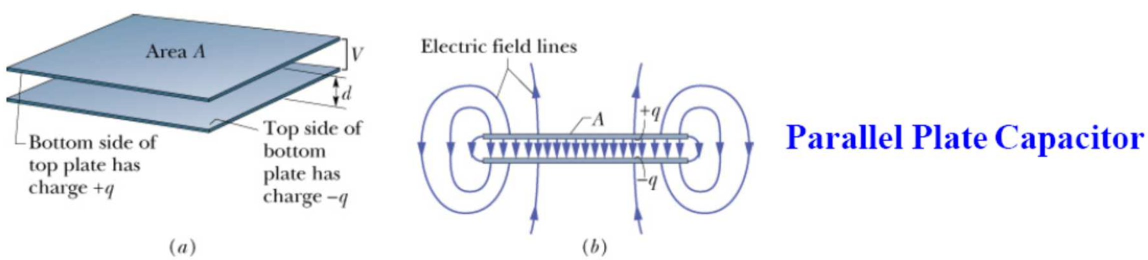
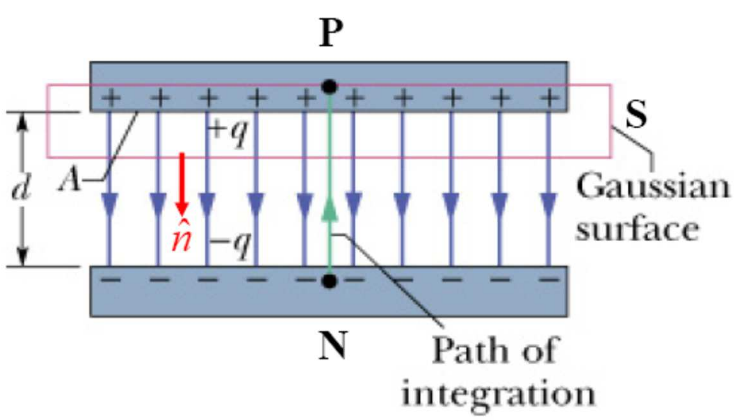
Capacitance (F)

1 F = 1 C/V	
1 μF = 10^{-6} F	(μ: micro)
1 nF = 10^{-9} F	(n: nano)
1 pF = 10^{-12} F	(p: pico)
1 fF = 10^{-15} F	(f: femto)
1 aF = 10^{-18} F	(a: atto)

1. Parallel plate capacitance





$$V = Ed$$

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A}$$

The capacitance C is defined by

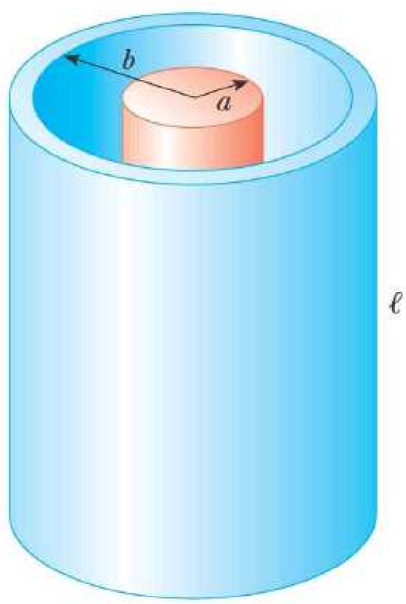
$$C = \frac{Q}{V} = \frac{\varepsilon_0 AE}{Ed} = \frac{\varepsilon_0 A}{d}$$ (parallel-plate capacitor)

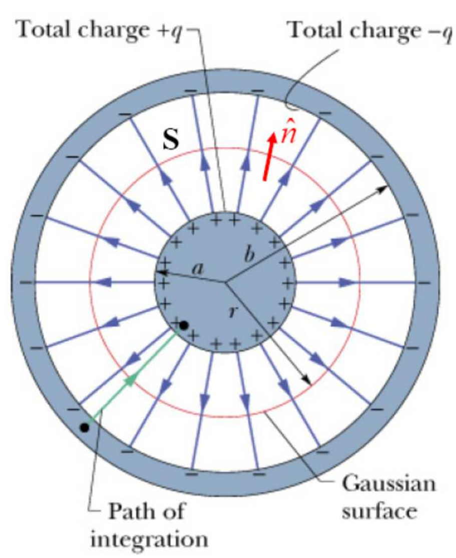
((**Note**)) **Example**

$A = 25m \times 5cm = 25 \times 0.05m^2$, $d = 0.01mm = 10^{-5}m$

$$C = 1.11\mu F$$

2. Cylindrical capacitor





$$\boldsymbol{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$$

$$\begin{aligned} V_{ba} &= -\int_a^b \boldsymbol{E}\cdot d\boldsymbol{r} = -\int_a^b \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}\cdot\hat{r}dr \\ &= -\int_a^b \frac{\lambda}{2\pi\varepsilon_0 r}dr = -\frac{\lambda}{2\pi\varepsilon_0}\ln(\frac{b}{a}) \\ &= -\frac{1}{2\pi\varepsilon_0}\frac{q}{L}\ln(\frac{b}{a}) \end{aligned}$$

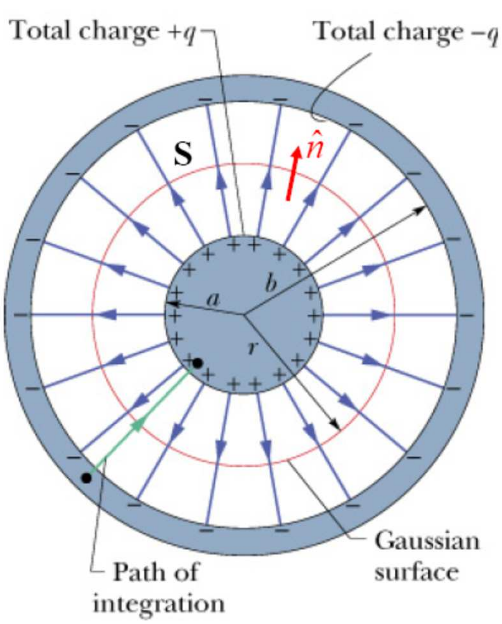
Since $V_{\mathrm{ba}}<0$ (the higher potential at $r = a$ and the lower potential ar $r = b$), we put

$V_{\mathrm{ba}} = -V$ ($V>0$).

The capacitance C is given by

$$C = \frac{q}{V} = \frac{2\pi\varepsilon_0 V}{V}\frac{L}{\ln(\frac{b}{a})} = 2\pi\varepsilon_0\frac{L}{\ln(\frac{b}{a})},$$ (cylindrical capacitor).

3. Spherical capacitance



$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r}$$

$$V_{ba} = -\int_a^b \boldsymbol{E}\cdot d\boldsymbol{r} = -\int_a^b \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r}\cdot\hat{r}dr$$

$$= -\frac{q}{4\pi\varepsilon_0}\int_a^b \frac{1}{r^2}dr = \frac{q}{4\pi\varepsilon_0}[\frac{1}{r}]_a^b = -\frac{q}{4\pi\varepsilon_0}(\frac{1}{a}-\frac{1}{b}) = -\frac{q}{4\pi\varepsilon_0}\left(\frac{b-a}{ab}\right)$$

Since V_{ba}<0 (the higher potential at $r = a$ and the lower potential ar $r = b$), we put

$V_{ba} = -V$ (V>0).

The capacitance C is given by

$$C = \frac{q}{V} = \frac{4\pi\varepsilon_0 V}{V}\left(\frac{ab}{b-a}\right) = 4\pi\varepsilon_0\left(\frac{ab}{b-a}\right).$$ (spherical capacitance)

4. Isolated capacitance

What is the capacitance when $a = R$ and $b \to\infty$, we have

$$C = 4\pi\varepsilon_0 R$$

where

$\varepsilon_0 = 8.854187817 \times 10^{-12}$ C^2/Nm^2

((Units))

[F] = C/V = C^2/(CV$^)$ = C^2/J = C^2/(Nm)

or

[F] = C^2/(Nm)

What is the capacitance of the Earth?

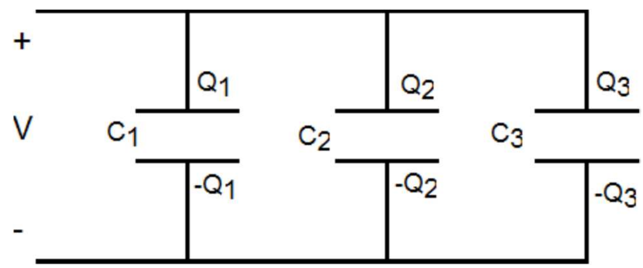
C=708.981 μF.

where the radius of the Earth (R) is

$R = 6.372 \times 10^6$ m

5. Capacitors in parallel and in series

5.1 Parallel connection



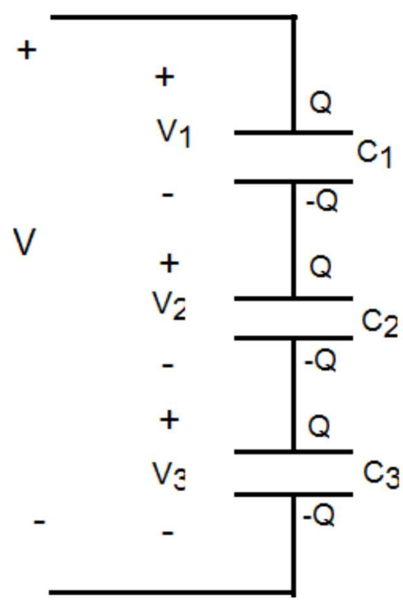
$$Q_1 = C_1 V$$
$$Q_2 = C_2 V$$
$$Q_3 = C_3 V$$

$$Q = Q_1 + Q_2 + Q_3 = (C_1 + C_2 + C_3)V$$

or

$$C = \frac{Q}{V} = C_1 + C_2 + C_3$$

5.2 Series connection



$$Q = C_1 V_1 = C_2 V_2 = C_3 V_3$$
$$V = V_1 + V_2 + V_3$$

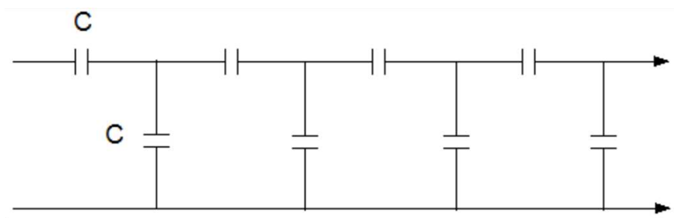
or

$$\frac{V}{Q} = \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

6. Examples

6.1 Example-1

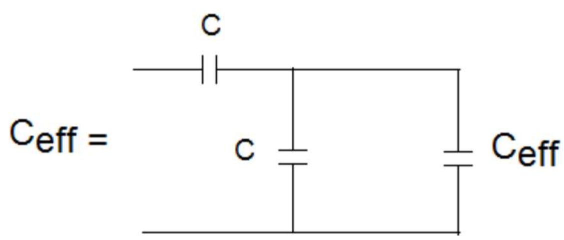
One frequency models real physical system (for example, transmission lines or nerve axons) with an infinitely repeating series of discrete circuit elements such as capacitors. Such an array is shown here. What is the capacitance between terminals X and Y for such a line, assuming it extends indefinitely? All of the capacitors are identical and have capacitance C.



(Schaum's outlines Physics for Engineering and Science, by M.E. Browne) p.281.

((**Solution**))

We assume that the effective capacitance C_{eff} is defined by the equivalent circuit given by



From this equivalent circuit, we have the following relation

$$C_{eff} = \frac{C(C + C_{eff})}{2C + C_{eff}}$$

or

$$C_{eff}{}^2 + CC_{eff} - C^2 = 0$$

or

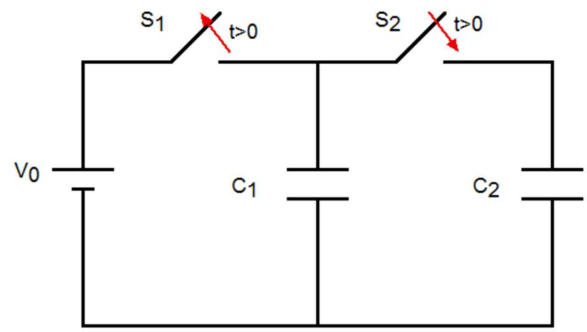
$$C_{eff} = \frac{\sqrt{5} - 1}{2} C = 0.62C$$

6.2 Example-2

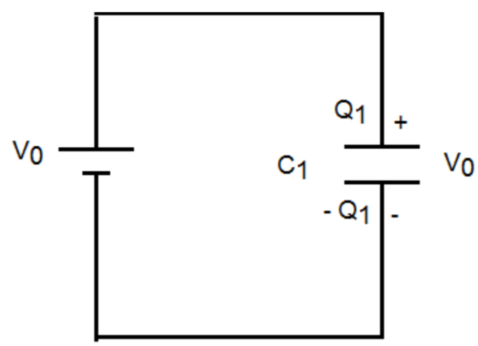
A 6 μF capacitor is charged by a 12 V battery and then disconnected. It is then connected to an uncharged 3 μF capacitor. What is the final potential difference across each capacitor?

((**Solution**))

$V_0 = 12$ V
$C_1 = 6$ μF
$C_2 = 3$ μF

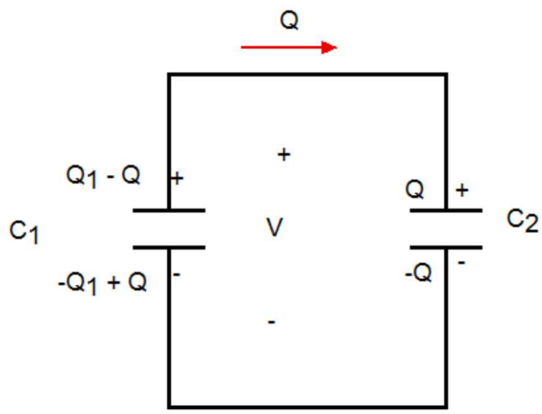


(a) $t<0$



$$Q_1 = C_1 V_0 == 72 \text{ μC}.$$

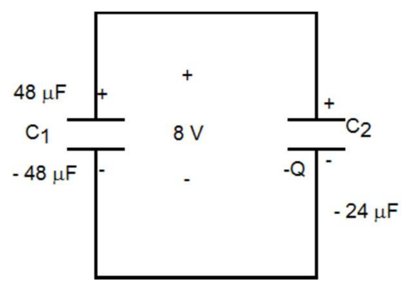
(b) $t>0$



$$Q_1 - Q = C_1 V$$
$$Q = C_2 V$$

or

$$Q_1 = (C_1 + C_2)V$$
$$V = \frac{Q_1}{C_1 + C_2} = \frac{C_1 V_0}{C_1 + C_2} = 8V$$

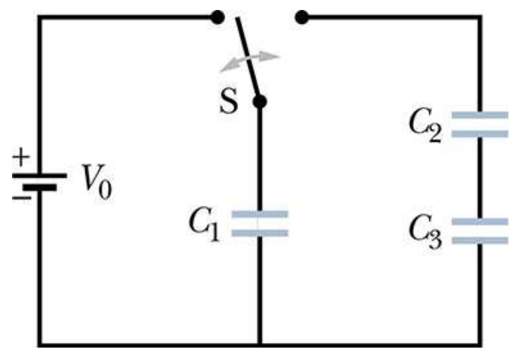


Then we have

$$V = 8V$$
$$Q = C_2 V = 24\mu C$$

7. Typical examples ((25-26))

Figure displays a 12.0 V battery and three uncharged capacitors of capacitances C_1 = 4.00 μF, C_2 = 6.00 μF, and C_3 = 3.00 μF. The switch is thrown to the left side until capacitor 1 is fully charged. Then the switch is thrown to the right. What is the final charge on (a) capacitor 1, (b) capacitor 2, and (c) capacitor 3?



C_1 = 4.00 μF, C_2 = 6.00 μF, C_3 = 3.00 μF. V_0 = 12.0 V

((**Solution**))

$$Q_1 = C_1 V_0 = 4\mu F \times 12V = 48\mu C$$

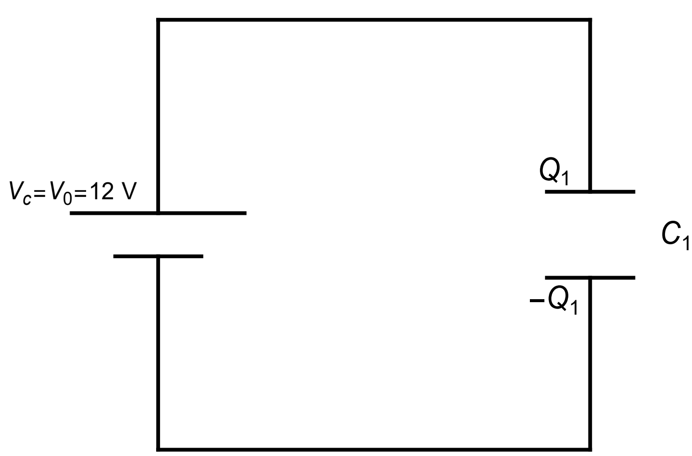
$$Q_1 - Q = C_1 V_1, \qquad Q = C_2 V_2 = C_3 V_3, \qquad V_1 = V_2 + V_3$$

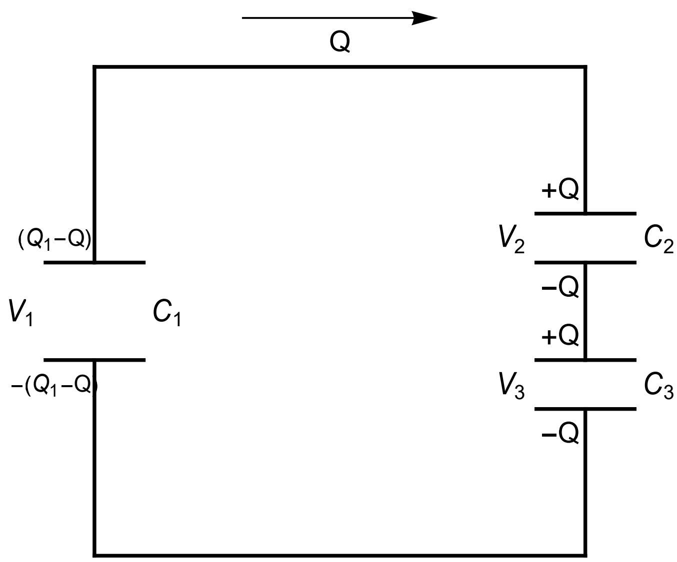
$$V_3 = \frac{C_2}{C_3} V_2 = \frac{6\mu F}{3\mu F} V_2 = 2V_2$$

From these relations we get

$$V_1 = 8\text{V}, \qquad V_2 = \frac{8}{3}\text{ V}, \qquad \text{and} \qquad V_3 = \frac{16}{3}\text{ V}.$$

$$Q_2 = Q_3 = 16\,\mu\text{C}.$$





8. The Energy of capacitance

To "charge up" a capacitor, we have to remove electrons from the positive plate and carry them to the negative plate. In doing so, one fight against the electric field, which is pulling them back toward the positive conductor and pushing them away from the negative one. How much work does it take, then, to charge the capacitor up to a final amount Q? Suppose that at some intermediate stage in the process the charge on the positive plate is q, so that the potential difference is q/C. the work you must do to transport the next piece of charge, dq, is

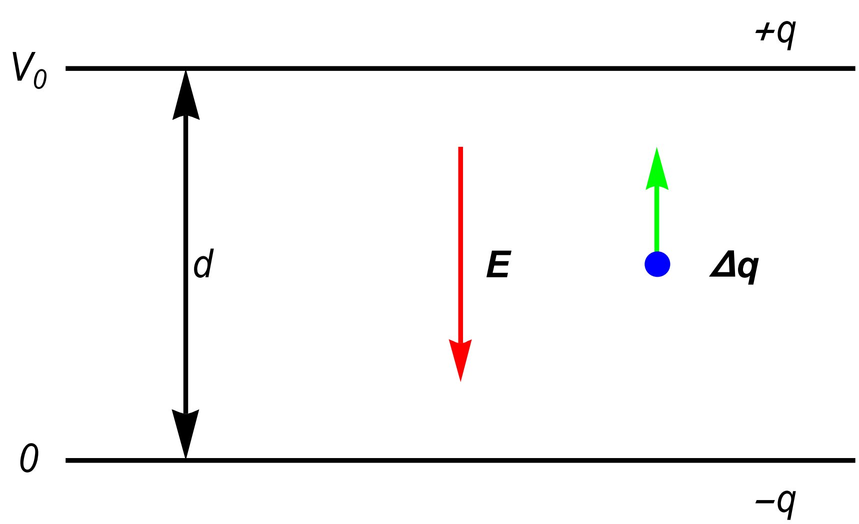


Fig. $V_0 = Ed$. $q = CV_0$

$$\Delta W = -\Delta q(Ed)$$
$$= -V_0 \Delta q$$
$$= -\frac{q}{C}\Delta q$$

where $\boldsymbol{F}$ is the force and $\boldsymbol{F} = \Delta q \boldsymbol{E}$ and $V_0 = Ed = \frac{q}{C}$ The total work necessary, then, to go from $q = 0$ to $q = Q$, is

$$W = -\int_0^Q \frac{q}{C} dq = -\frac{Q^2}{2C} = -\frac{1}{2}CV^2$$

where Q is the total charge, $Q = CV$, V is the final electric potential of the capacitor. Using the work-energy theorem, we have the potential energy U as

$$U = -W = \frac{Q^2}{2C} = \frac{CV^2}{2}$$

((**Note-1**)) **Feynman**

Recalling that the capacity of a conducting sphere (relative to infinity) is

$$C = 4\pi\varepsilon_0 R$$

where R is the radius of sphere. Thus the energy of a charged sphere is

$$U = \frac{Q^2}{8\pi\varepsilon_0 R}.$$

((**Note-2**))

The energy density u is defined by

$$u = \frac{U}{Ad} = \frac{1}{Ad}\frac{1}{2}\frac{\varepsilon_0 A}{d}(Ed)^2 = \frac{1}{2}\varepsilon_0 E^2$$

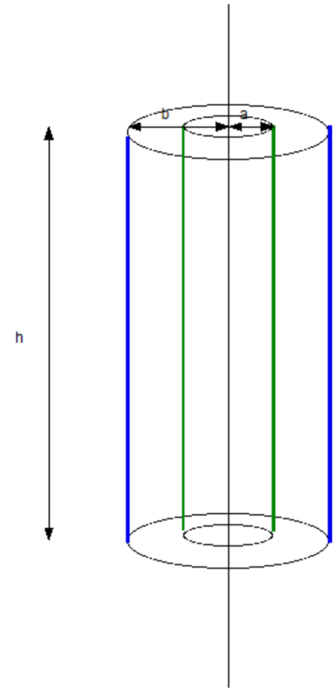
where Ad is the volume, $C = \frac{\varepsilon_0 A}{d}$, and $V = Ed$. The total energy of the capacitance can be rewritten as

$$U = \int \frac{1}{2}\varepsilon_0 \boldsymbol{E}^2 d^3\boldsymbol{r}$$

9. Example Problem ((25-68))

A cylindrical capacitor has radii a and b in Fig. Show that half the stored electric potential energy lies within a cylinder whose radius is $r = \sqrt{ab}$.

((**Solution**))



((**Solution**))

From the Gauss' theorem, we have

$$E = \frac{1}{2\pi rh}\frac{1}{\varepsilon_0}\lambda h = \frac{\lambda}{2\pi\varepsilon_0 r}$$

The energy density is

$$u = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0\left(\frac{\lambda}{2\pi\varepsilon_0 r}\right)^2 = \frac{\lambda^2}{8\pi^2\varepsilon_0 r^2}$$

The total energy U is

$$U = \int_a^b u(2\pi r)hdr = \int_a^b \frac{1}{2}\varepsilon_0\left(\frac{\lambda}{2\pi\varepsilon_0 r}\right)^2 (2\pi r)hdr$$
$$= \frac{\lambda^2 h}{4\pi\varepsilon_0}\int_a^b \frac{1}{r}dr = \frac{\lambda^2 h}{4\pi\varepsilon_0}\ln(\frac{b}{a})$$

U_{half} is defined as

$$U_{half} = \frac{\lambda^2 h}{4\pi\varepsilon_0}\ln(\frac{r}{a})$$

$$\frac{U_{half}}{U} = \frac{1}{2} = \frac{\ln(\frac{r}{a})}{\ln(\frac{b}{a})}$$

or

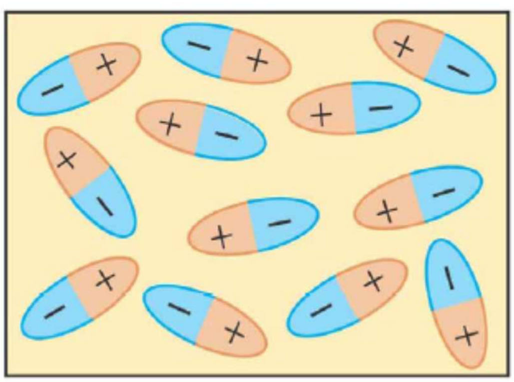
$$\ln(\frac{r^2}{a^2}) = \ln(\frac{b}{a}),$$

or

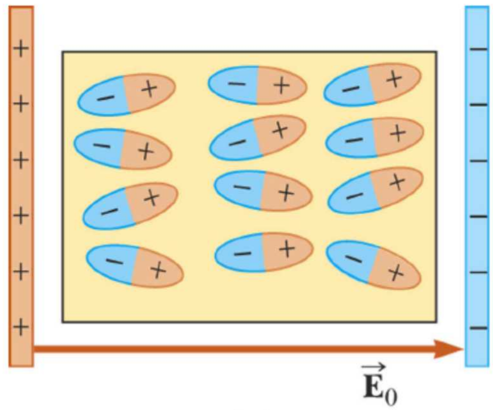
$$r = \sqrt{ab}.$$

10. Dielectrics in the presence of electric field: atomic view

The molecules that make up the dielectric are modeled as dipoles. The molecules are randomly oriented in the absence of an electric field.

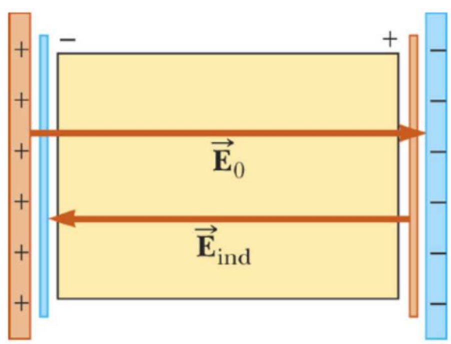


Suppose that an external electric field is applied. This produces a torque on the molecules. The molecules partially align with the electric field.



(b)

An external field can polarize the dielectric whether the molecules are polar or nonpolar. The charged edges of the dielectric act as a second pair of plates producing an induced electric field in the direction opposite the original electric field



(c)

11. Experiment (I) Charge remained constant

Walter Lewin" 8.02X MIT Physics, Electricity and Magnetism
Lecture 8

The capacitance of a set of charged parallel plates is increased by the insertion of a dielectric material.

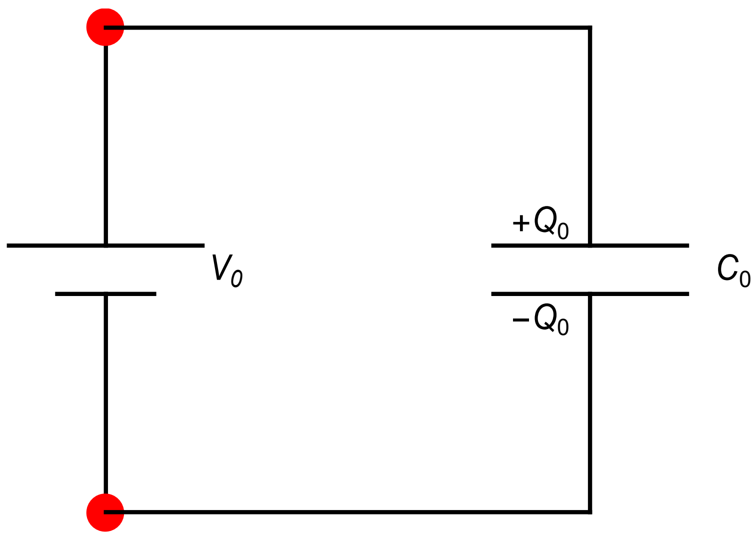
$$C_0 = \frac{\varepsilon_0 A}{d}, \qquad C = \frac{\varepsilon_0 \kappa A}{d}, \qquad \frac{C}{C_0} = \kappa \qquad \text{(dielectric constant)}$$

We discuss the physical meaning of κ using the following experiments.

(a) Step-1 (closed circuit)

The capacitance (C_0) is charged to the charge Q_0 by connecting a voltage source V_0.

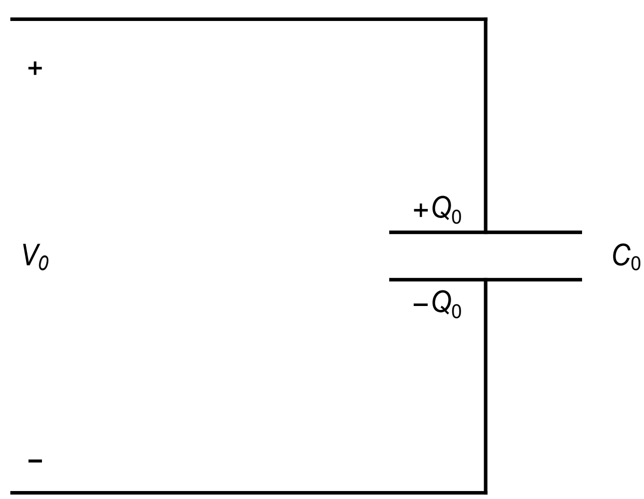
$$Q_0 = C_0 V_0$$



(b) Step-2 (open circuit)

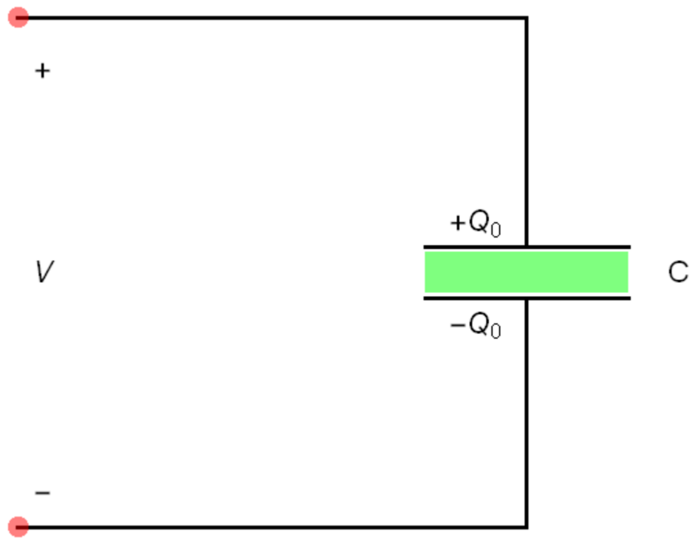
The voltage source is disconnected from the circuit. The charge remains unchanged during this process.

$$Q_0 = C_0 V_0$$



(c) Step-3 (open circuit)

A dielectric material is inserted into the space between two electrodes of the capacitance. The capacitance changes from $C_0 \rightarrow C$. The voltage across the capacitance changes.



The free charge Q_0 remains unchanged, while the voltage across the capacitance changes from V_0 to V,

$$Q_0 = CV$$

or

$$Q_0 = C_0 V_0 = CV$$

Suppose that

$$\frac{C}{C_0} = \kappa$$

(The dielectric medium is inserted into the interlamellar space of the capacitance)

Then we have

$$\frac{V}{V_0} = \frac{C_0}{C} = \frac{1}{\kappa} \qquad\qquad V = \frac{1}{\kappa} V_0$$

Since $V_0 = E_0 d$ and $V = Ed$

$$E = \frac{1}{\kappa} E_0$$

where *d* is the separation distance between two electrodes of the capacitance. We note that

$$C = \frac{C_0 V_0}{V} = \frac{Q_0}{V}$$

((**Note**))

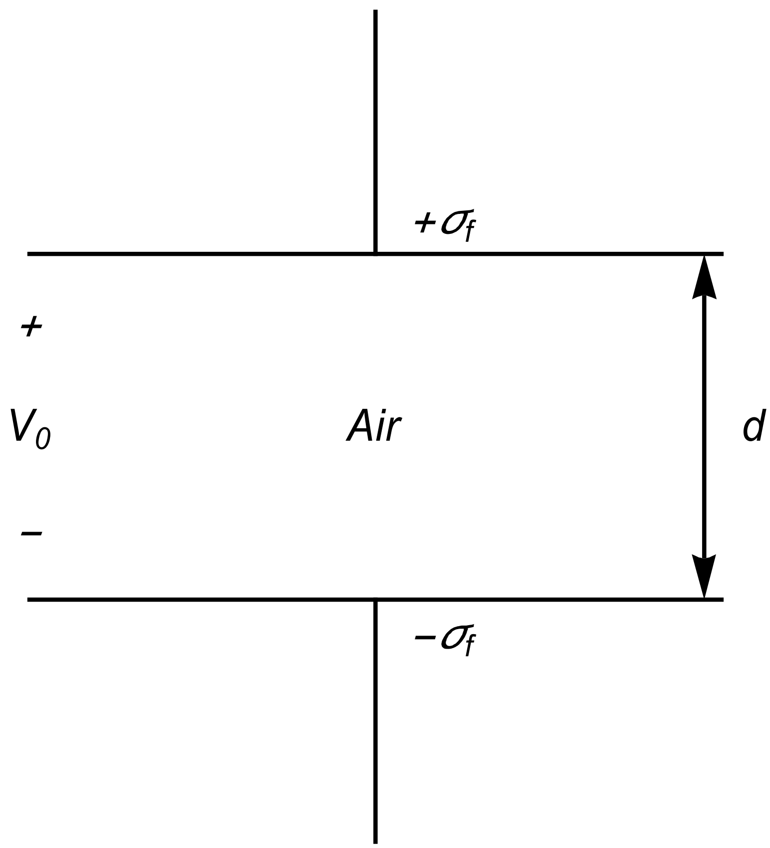
$$E = \frac{1}{\kappa} E_0 \qquad V = \frac{1}{\kappa} V_0$$

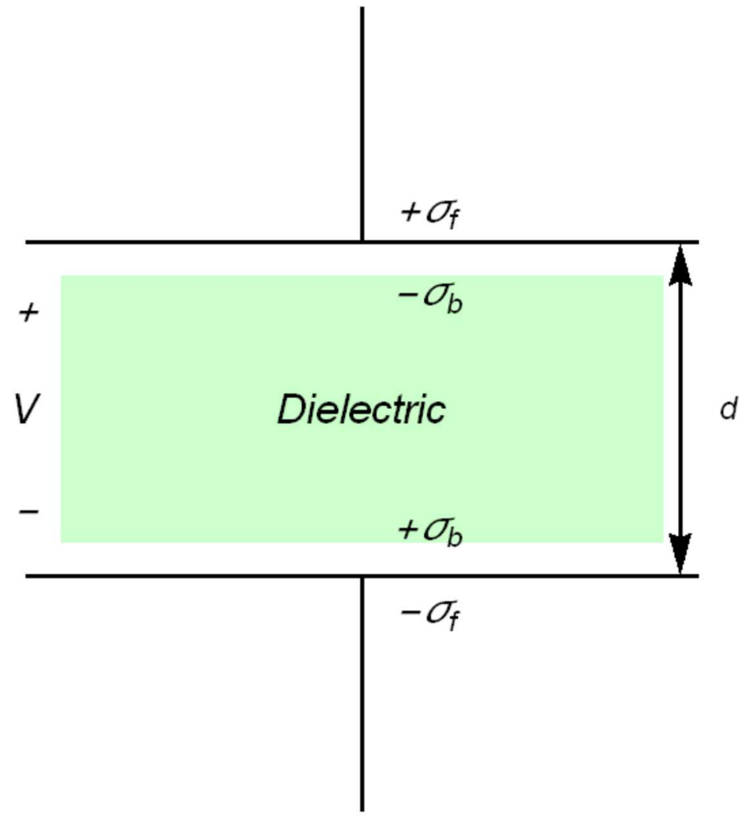
Since

$$E = \frac{\sigma_f - \sigma_b}{\varepsilon_0} = \frac{V}{d}, \qquad E_0 = \frac{\sigma_f}{\varepsilon_0} = \frac{V_0}{d}$$

we get

$$\frac{\sigma_f - \sigma_b}{\varepsilon_0} = \frac{1}{\kappa} \frac{\sigma_f}{\varepsilon_0}, \qquad \text{or} \qquad \sigma_b = (1 - \frac{1}{\kappa}) \sigma_f$$



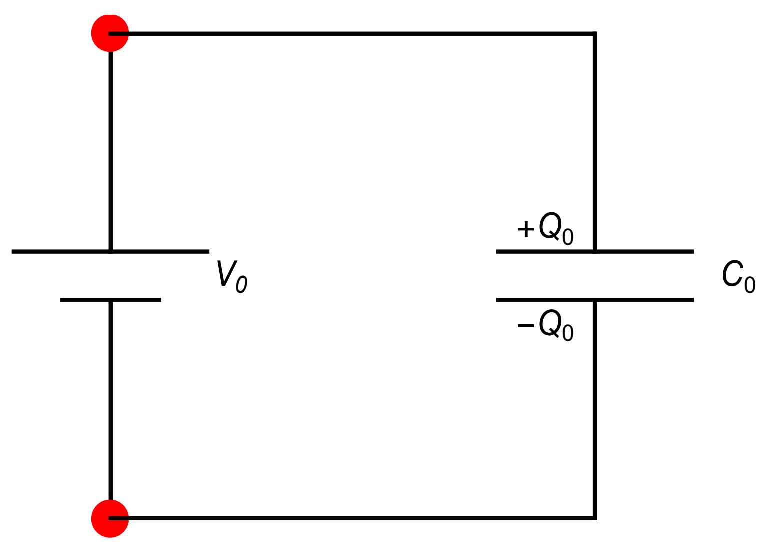


12. Experiment II Constant voltage source

(a) Step-I

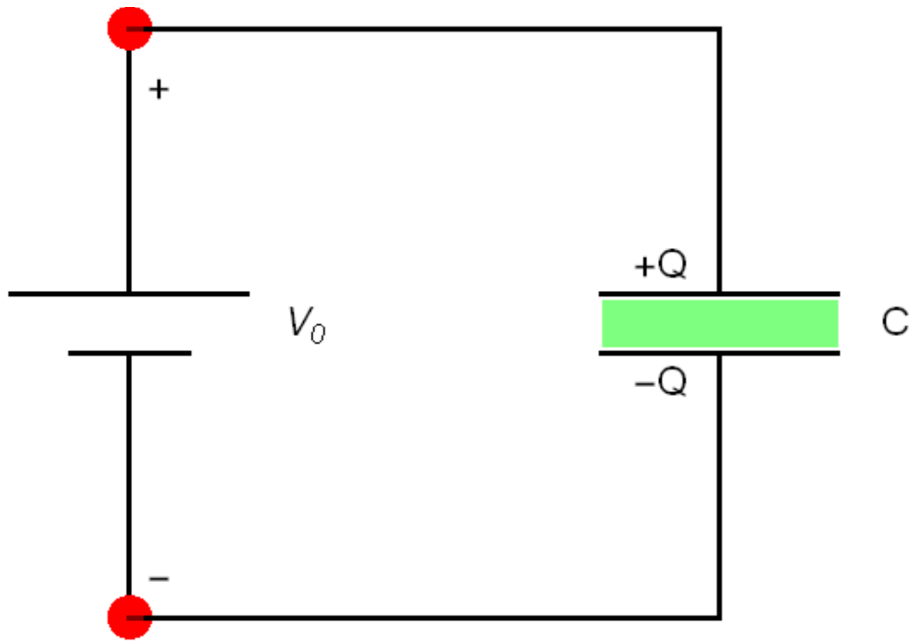
The capacitance (C_0) is charged to the charge Q_0 by connecting a voltage source V_0.

$$Q_0 = C_0 V_0$$



(b) Step-II

While the battery continues to be connected, the dielectric is inserted into a gap between two electrodes of the capacitance. While the voltage remains unchanged as V_0, the charge changes from Q_0 to Q.



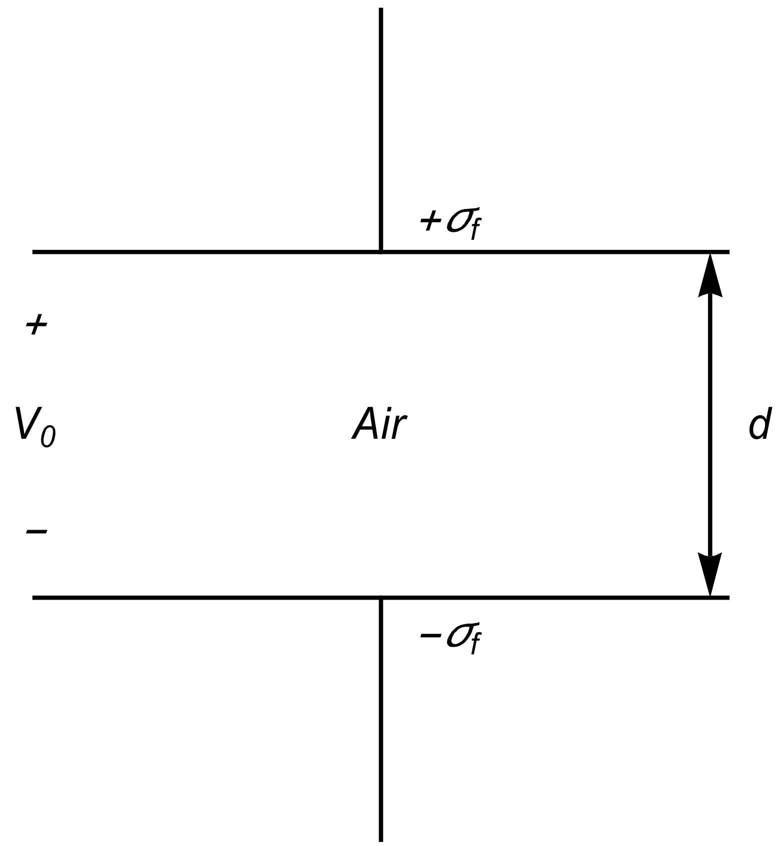
$$Q = CV_0$$

Since $C = \kappa C_0$, we have

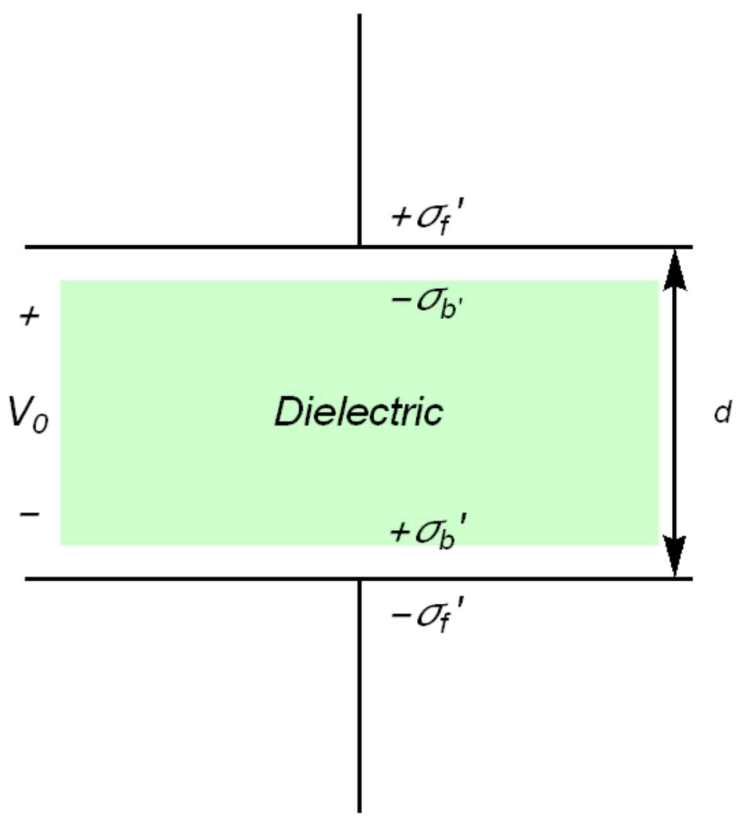
$$\frac{Q}{Q_0} = \frac{CV_0}{C_0 V_0} = \frac{C}{C_0} = \kappa, \qquad Q = \kappa Q_0$$

The electric field E remains unchanged during this process, since the applied voltage is kept constant.

((**Note**))



$C_0 V_0 = A\sigma_f$



$$CV_0 = \kappa C_0 V_0 = \kappa A\sigma_f = A(\sigma_f' - \sigma_P'),$$

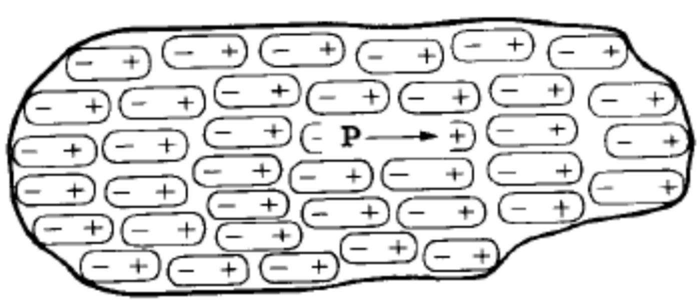
or

$$\sigma_f' - \sigma_P' = \kappa\sigma_f$$

or

$$\sigma_f' = \kappa\sigma_f + \sigma_P'$$

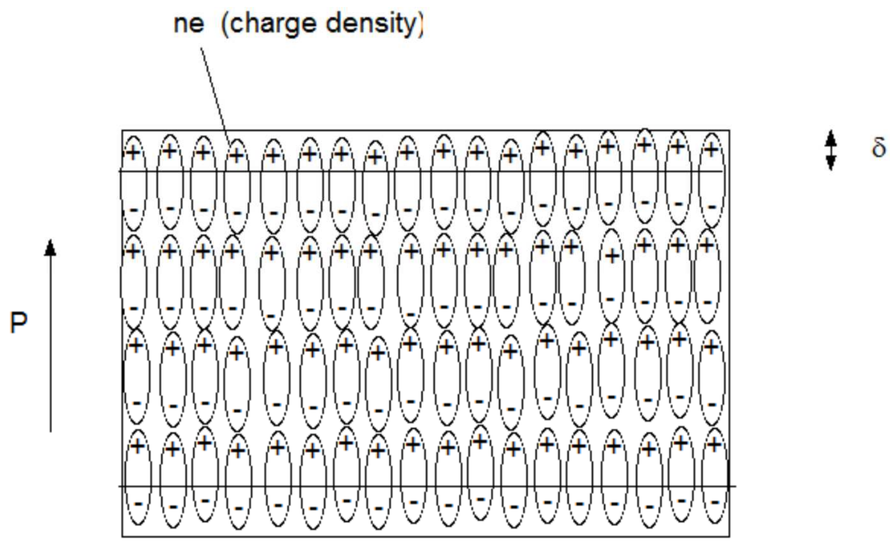
13. Polarization vector *P*



Suppose that the molecules with permanent electric dipole moments are lined neatly, all pointing the same way, and frozen in position. There are N dipoles (with electric dipole moment $\boldsymbol{p}$) per cubic meters. We shall assume that N is so large that any macroscopically small volume $d\tau$ contains quite a large number of dipoles. The total dipole strength in such a volume is $\boldsymbol{p}Nd\tau$. At any point far away from this volume element compared with its size, the electric field from these particular dipoles would be practically the same if they were replaced by a single dipole moment of strength $\boldsymbol{p}Nd\tau$. We shall call $\boldsymbol{p}N$ the density of polarization, and denoted it by $\boldsymbol{P}$. Then $\boldsymbol{P}d\tau$ is the dipole moment to be associated with any small volume element $d\tau$.

14. Feynman's comment on the expression of ρ_b and σ_b

Feynman's lecture on physics



We consider the above situation, where P is uniform in the above figure. We have a positive charge at the one side

$$\Delta Q = enA\delta$$

and a negative charge

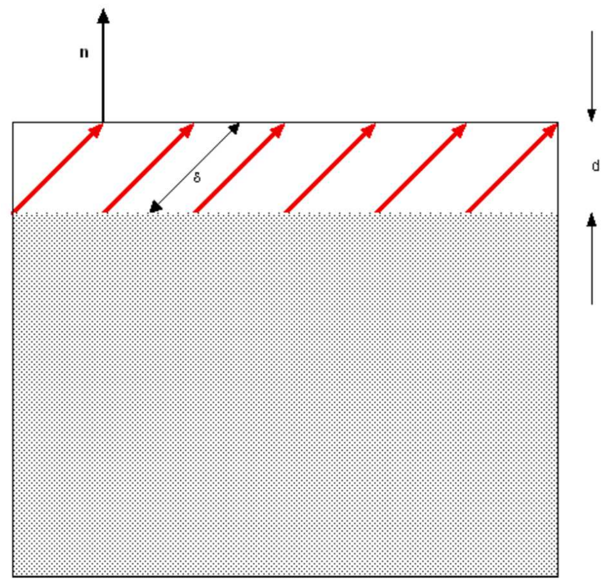
$$-\Delta Q = -enA\delta,$$

where A is the surface area, δ is the displacement, $-e$ is the electron charge, and n is the number of electrons per unit volume. From the definition, the surface charge density is given by

$$\sigma_b = \frac{\Delta Q}{A} = (e\delta)n = pn = P$$

where p (= $e\delta$) is the electric dipole moment. The vector P is the polarization vector. The magnitude P is the electric dipole moment per unit volume.

What happens to σ_b when $\boldsymbol{P}$ does not point to the direction perpendicular to the surface?

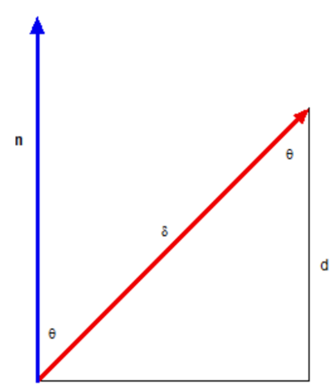


The total charge in the surface region (d) is equal to

$$\Delta Q' = enAd$$

When the angle between $\boldsymbol{P}$ and the normal unit vector $\boldsymbol{n}$ (perpendicular to the surface) is θ, the relation between d and δ is given by

$$d = \delta \cos\theta$$



Then the surface charge density is

$$\sigma_b = \frac{\Delta Q'}{A} = (ed)n = (e\delta)n\cos\theta = pn\cos\theta = P\cos\theta$$

or

$\boxed{\sigma_b = \boldsymbol{P} \cdot \boldsymbol{n}}$

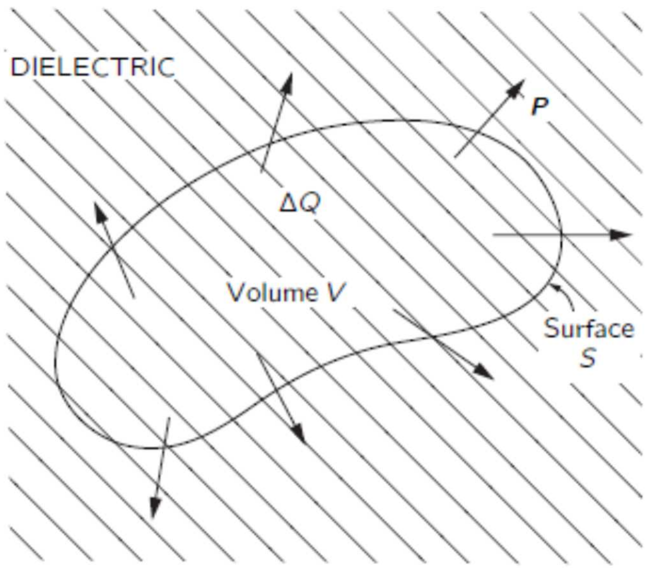
From the Gauss' law,

$$\int \nabla \cdot \boldsymbol{P} d\tau = \int \boldsymbol{P} \cdot \boldsymbol{n} da = \int \sigma_b da \,. \tag{1}$$

Since the total charge is equal to zero, we have

$$\int \rho_b d\tau + \int \sigma_b da = 0 \,, \tag{2}$$

where ρ_b is the volume charge density.



From Eqs.(1) and (2), we get

$$\int \nabla \cdot \boldsymbol{P} d\tau = -\int \rho_b d\tau$$

or

$\boxed{\rho_b = -\nabla \cdot \boldsymbol{P}}$

((**Note**)) We define the current density due to the polarization vector $\boldsymbol{P}$ as

$\boxed{\boldsymbol{J}_b = \dfrac{\partial \boldsymbol{P}}{\partial t}}$

$$\nabla \cdot \boldsymbol{J}_b = \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{P}) = -\frac{\partial}{\partial t}\rho_b, \quad \text{or} \qquad \nabla \cdot \mathbf{J}_b + \frac{\partial}{\partial t}\rho_b = 0$$

which corresponds to the continuity of the polarization current.

15. Displacement vector: Derivation of the σ_b and ρ_b from the electric potential

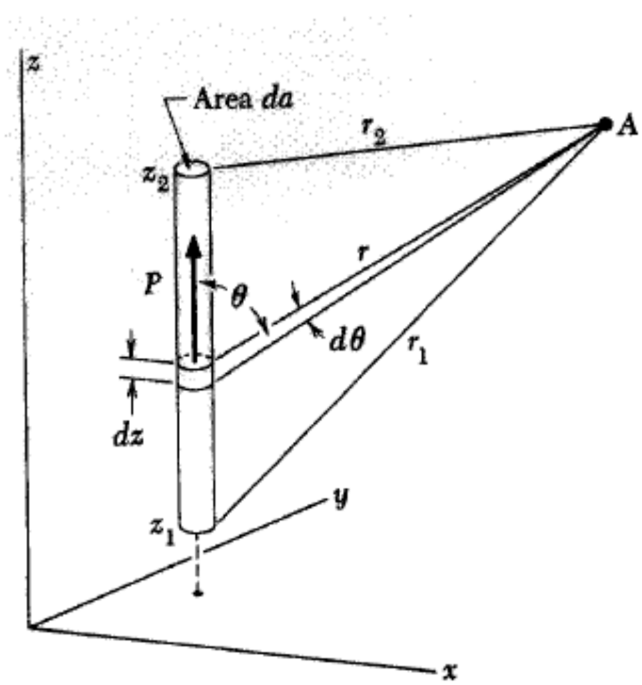
(a) 1D case

Here we also assume that there is no net charge in the system. So we have only the dipole moments to consider as sources of a distant field. The figure shows a slender column, or cylinder, of this polarized material. Its cross section is da, and it extends vertically from z_1 to z_2. The polarization density $\boldsymbol{P}$ within the column is uniform over the length and points in the positive z direction. Now we calculate the electrical potential, at some external point, of this column polarization. An element of the cylinder, of height dz, has a dipole moment $\boldsymbol{P}$dadz. It contribution to the potential at the point A can be described by

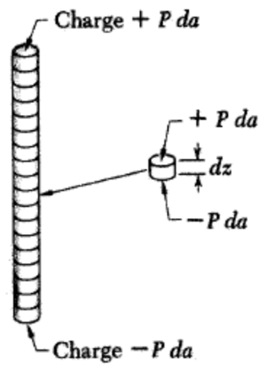
$$dV_A = \frac{1}{4\pi\varepsilon_0}\frac{(\boldsymbol{P}dadz)\cdot\boldsymbol{r}}{r^3} = \frac{1}{4\pi\varepsilon_0}\frac{Pdadz\cos\theta}{r^2}$$

The potential due to the entire column is

$$V_A = \frac{1}{4\pi\varepsilon_0}\int_{z_1}^{z_2}\frac{Pdadz\cos\theta}{r^2}$$



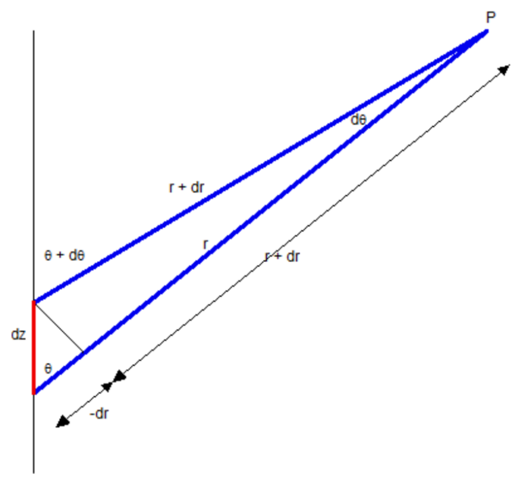
(a)



(b)

Since $dz\cos\theta = -dr$,

$$V_A = \frac{1}{4\pi\varepsilon_0}\int_{r_1}^{r_2}\frac{Pda(-dr)}{r^2} = \frac{Pda}{4\pi\varepsilon_0}(\frac{1}{r_2}-\frac{1}{r_1})$$



This is precisely the same as the expression for the potential at A that would be produced by two point charges, a positive charge of magnitude Pda sitting on the top of the column at a distant r_2 from A, and a negative charge of the same magnitude at the bottom of the column. *The source consisting of a column of uniformly polarized matter is equivalent to two concentrated charges.*

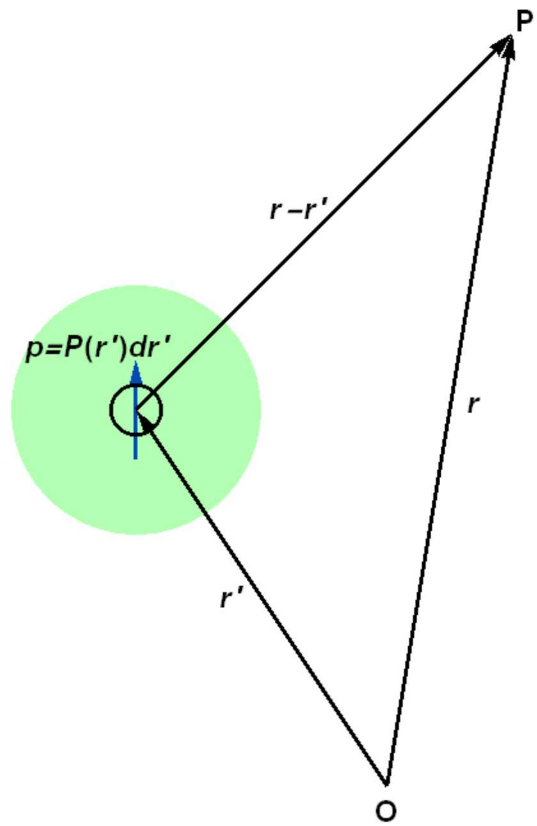
(b) General case

We consider a finite piece of dielectric material which is polarized. We define a polarization $\boldsymbol{P}(\boldsymbol{r}')$ at each point $\boldsymbol{r}'$ in the system. Each volume $d\tau'$ is characterized by an electric dipole moment $\boldsymbol{P}(\boldsymbol{r}')d\tau'$. The contribution of the electric potential at the point $\boldsymbol{r}$ from the moment $\boldsymbol{P}(\boldsymbol{r}')d\tau'$ is given by

$$dV(\boldsymbol{r}) = \frac{\boldsymbol{P}(\boldsymbol{r}')d\tau' \cdot (\boldsymbol{r} - \boldsymbol{r}')}{4\pi\varepsilon_0 \left|\boldsymbol{r} - \boldsymbol{r}'\right|^3}$$

Then the entire potential at point *r* is obtained as

$$V(\boldsymbol{r}) = \int \frac{\boldsymbol{P}(\boldsymbol{r}') \cdot (\boldsymbol{r} - \boldsymbol{r}')}{4\pi\varepsilon_0 \left|\boldsymbol{r} - \boldsymbol{r}'\right|^3} d\tau'$$



We use the formula of the vector analysis,

$$\nabla' \frac{1}{\left|\boldsymbol{r} - \boldsymbol{r}'\right|} = \frac{\boldsymbol{r} - \boldsymbol{r}'}{\left|\boldsymbol{r} - \boldsymbol{r}'\right|^3},$$

and

$$\nabla' \cdot (f\boldsymbol{A}) = f\nabla' \cdot \boldsymbol{A} + \boldsymbol{A} \cdot \nabla' f,$$

where *f* is any scalar point function and $\boldsymbol{A}$ is an arbitrary vector point function. The prime indicates differentiation with respect to the prime coordinates. Letting $\boldsymbol{A} = \boldsymbol{P}$ and

$$f = \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}.$$

Using the relation

$$\frac{\boldsymbol{P}(\boldsymbol{r}') \cdot (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} = \boldsymbol{P}(\boldsymbol{r}') \cdot \nabla' \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = \nabla' \cdot \left(\frac{\boldsymbol{P}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}\right) - \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \nabla' \cdot \boldsymbol{P}(\boldsymbol{r}'),$$

we have

$$\begin{aligned} V(\boldsymbol{r}) &= \frac{1}{4\pi\varepsilon_0} \int \left[\nabla' \cdot \left(\frac{\boldsymbol{P}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}\right) - \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \nabla' \cdot \boldsymbol{P}(\boldsymbol{r}')\right] d\tau' \\ &= \frac{1}{4\pi\varepsilon_0} \left[\int \frac{\boldsymbol{P}(\boldsymbol{r}') \cdot \boldsymbol{n}'}{|\boldsymbol{r} - \boldsymbol{r}'|} da' + \int \frac{-\nabla' \cdot \boldsymbol{P}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} d\tau'\right] \end{aligned},$$

where the volume integral of $\nabla' \cdot \left(\frac{\boldsymbol{P}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}\right)$ is replaced by a surface integral through the application of the Gauss' theorem and ***n'*** is the outward normal to the surface element d*a'*.

Here we define

$$\sigma_b = \boldsymbol{P} \cdot \boldsymbol{n} = P_n,$$

and

$$\rho_b = -\nabla \cdot \boldsymbol{P}.$$

The surface charge density σ_b is given by the component of the polarization $\boldsymbol{P}$ normal to the surface and the volume charge density ρ_b is a measure of the nonuniformity of the polarization $\boldsymbol{P}$ in side the system. So we have the final form of $V(\boldsymbol{r})$ as

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \left[\int \frac{\sigma_P}{|\boldsymbol{r} - \boldsymbol{r}'|} da' + \int \frac{\rho_P}{|\boldsymbol{r} - \boldsymbol{r}'|} d\tau'\right]$$

(c) Electric displacement D

$$\varepsilon_0 \nabla \cdot \boldsymbol{E} = \rho_f + \rho_P$$

with

$$\rho_P = -\nabla \cdot \boldsymbol{P}$$

where P is the polarization vector (charge per unit area). Thus we have

$$\nabla \cdot (\varepsilon_0 \boldsymbol{E} + \boldsymbol{P}) = \rho_f$$

or

$$\nabla \cdot \boldsymbol{D} = \rho_f$$

It is customary to give the combination $\varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$ a special name, the electric displacement vector and its own symbol ***D***,

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$$

Using the Gauss's law, we have

$$\int \nabla \cdot \boldsymbol{D} \, d^3\boldsymbol{r} = \int \rho_f \, d^3\boldsymbol{r} = Q_f$$

or

$$\int \boldsymbol{D} \cdot d\boldsymbol{a} = \int \rho_f \, d^3\boldsymbol{r} = Q_f$$

We have

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}$$

$$\boldsymbol{P} = \varepsilon_0 \chi_e \boldsymbol{E}$$

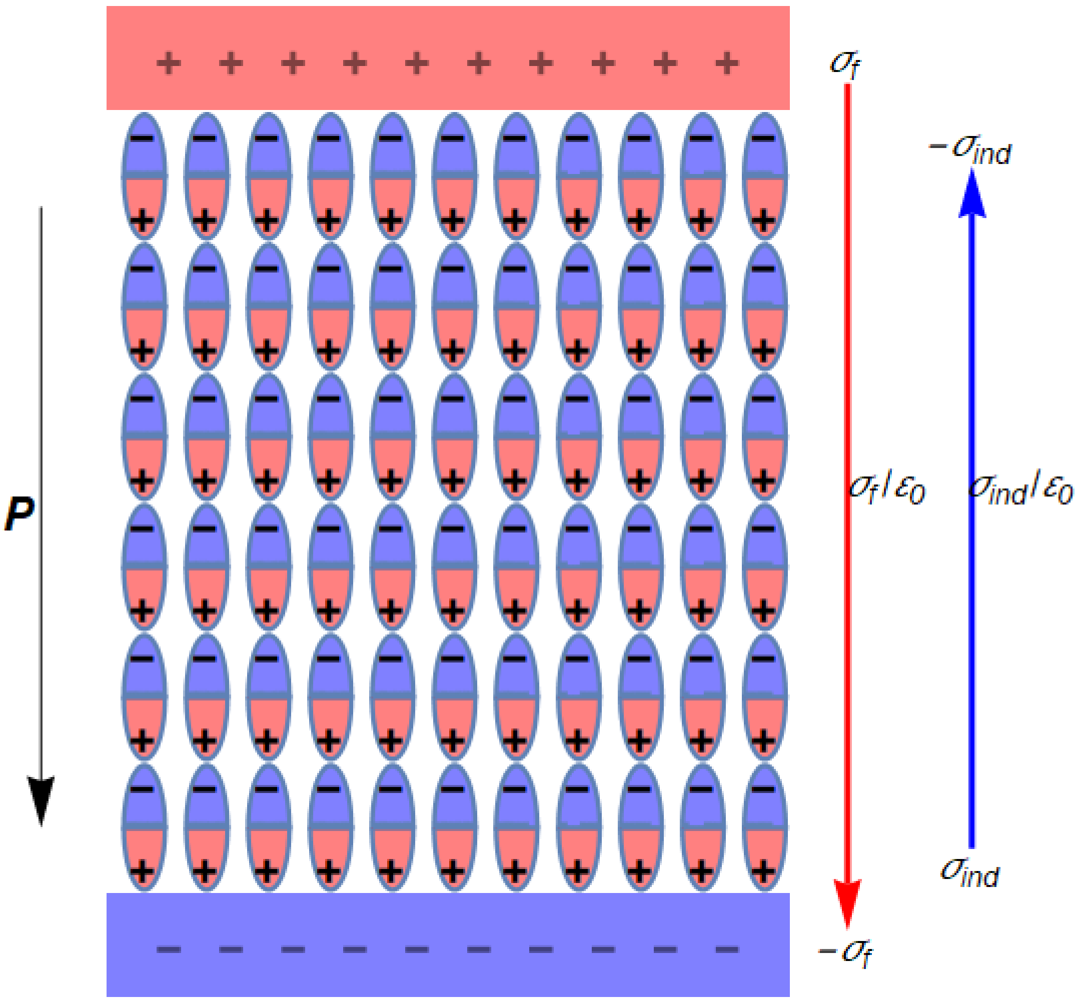
$$\varepsilon \boldsymbol{E} = \varepsilon_0 \boldsymbol{E} + \varepsilon_0 \chi_e \boldsymbol{E}$$

where

$$\frac{\varepsilon}{\varepsilon_0} = \varepsilon_r = 1 + \chi_e = \kappa$$

16 Capacitance with dielectric (I)

Here we discuss the capacitance of the dielectric.



In this figure, σ_f is the free charge. $\boldsymbol{P}$ is the polarization vector. The inductive charge $\sigma_{ind} = \sigma_b$ is given by

$$\sigma_{ind} = \sigma_b = \boldsymbol{P} \cdot \boldsymbol{n} = P$$

where $\boldsymbol{n}$ is the vector normal to the boundary. The total electric field E is obtained as

$$E = \frac{\sigma_f}{\varepsilon_0} - \frac{\sigma_{ind}}{\varepsilon_0} = \frac{1}{\kappa} E_f = \frac{\sigma_f}{\kappa \varepsilon_0}$$

using the Gauss's theorem. Note that

$$\frac{E}{E_f} = \frac{1}{\kappa}$$

where

$$E_f = \frac{\sigma_f}{\varepsilon_0}$$

Then we have

$$\sigma_{ind} = \sigma_f (1 - \frac{1}{\kappa})$$

or

$$E_f (1 - \frac{1}{\kappa}) = \frac{\sigma_{ind}}{\varepsilon_0}$$

where κ is dielectric constant of dielectric.

Gauss' law with dielectrics

$$\varepsilon_0 \int \kappa \boldsymbol{E} \cdot d\boldsymbol{a} = q_f$$

Table: Dielectric constants

Dielectric constants of various substances

Substance	Conditions	Dielectric constant (κ)
Air	gas, 0 °C, 1 atm	1.00059
Methane, CH_4	gas, 0 °C, 1 atm	1.00088
Hydrogen chloride, HCl	gas, 0 °C, 1 atm	1.0046
Water, H_2O	gas, 110 °C, 1 atm	1.0126
	liquid, 20 °C	80.4
Benzene, C_6H_6	liquid, 20 °C	2.28
Methanol, CH_3OH	liquid, 20 °C	33.6
Ammonia, NH_3	liquid, −34 °C	22.6
Mineral oil	liquid, 20 °C	2.24
Sodium chloride, NaCl	solid, 20 °C	6.12
Sulfur, S	solid, 20 °C	4.0
Silicon, Si	solid, 20 °C	11.7
Polyethylene	solid, 20 °C	2.25–2.3
Porcelain	solid, 20 °C	6.0-8.0
Paraffin wax	solid, 20 °C	2.1–2.5
Pyrex glass 7070	solid, 20 °C	4.00

κ(vacuum) = 1.000000
κ(paper) = 3.5
κ(transformer oil) = 4.5

$\kappa(SrTiO_3) = 310$
κ(liquid water at 25°C) = 78.5

The polarization vector is defined as

$$\boldsymbol{P}\cdot\boldsymbol{n} = P = \sigma_{ind} = \varepsilon_0 \chi E$$

or

$$\frac{\sigma_{ind}}{\varepsilon_0} = \frac{P}{\varepsilon_0} = \chi E$$

or

$$P = \varepsilon_0 \chi E$$

Thus we have

$$E = \frac{\sigma_f}{\varepsilon_0} - \chi E$$

or

$$(1+\chi)E = \frac{\sigma_f}{\varepsilon_0}$$

or

$$E = \frac{1}{1+\chi}\frac{\sigma_f}{\varepsilon_0} = \frac{1}{1+\chi}E_f = \frac{1}{\kappa}E_f$$

leading to the relation

$$\kappa = 1+\chi$$

where χ is called the *electric susceptibility*.

17. Capacitance of dielectric (II)

The capacitance C of the dielectric is defined by

$$C = \frac{Q_f}{V}.$$

where C_0 is the capacitance of the vacuum. The validity of this definition is explained in Sec.

$$C_0 = \frac{Q_f}{V_f} = \frac{Q_f}{E_f d} = \frac{A\sigma_f}{\frac{\sigma_f}{\varepsilon_0} d} = \varepsilon_0 \frac{A}{d}$$

and

$$V = Ed = \frac{1}{\kappa} E_f d = \frac{1}{\kappa} V_f$$

Thus we have the capacitance,

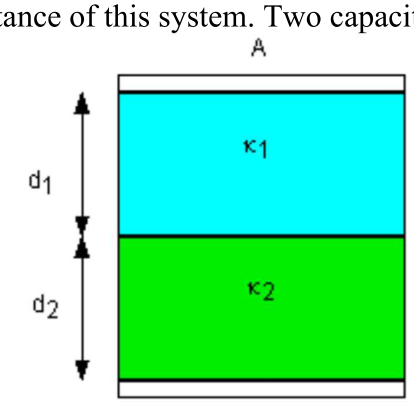
$$C = \frac{Q_f}{V} = \frac{Q_f}{V_f}\frac{V_f}{V} = \kappa C_0 = \kappa\varepsilon_0 \frac{A}{d}$$

or

$$\frac{C}{C_0} = \kappa$$

18. Capacitors with dielectrics in series and in parallel connections

We calculate the capacitance of this system. Two capacitors are connected in series.

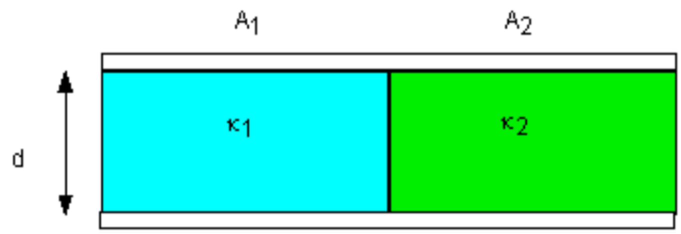


$$C_1 = \varepsilon_0\kappa_1 \frac{A}{d_1}$$

$$C_2 = \varepsilon_0\kappa_2 \frac{A}{d_2}$$

$$C = \frac{C_1C_2}{C_1 + C_2} = \frac{\varepsilon_0\kappa_1 \frac{A}{d_1}\varepsilon_0\kappa_2 \frac{A}{d_2}}{\varepsilon_0\kappa_1 \frac{A}{d_1} + \varepsilon_0\kappa_2 \frac{A}{d_2}} = \frac{\varepsilon_0 A \frac{\kappa_1}{d_1}\frac{\kappa_2}{d_2}}{\frac{\kappa_1}{d_1} + \frac{\kappa_2}{d_2}}$$

Next we calculate the capacitance of the system where two capacitors are connected in parallel



$$C_1 = \varepsilon_0\kappa_1 \frac{A_1}{d}$$

$$C_2 = \varepsilon_0\kappa_2 \frac{A_2}{d}$$

$$C = C_1 + C_2 = \varepsilon_0\kappa_1 \frac{A_1}{d} + \varepsilon_0\kappa_2 \frac{A_2}{d}$$

$$= \frac{\varepsilon_0}{d}(\kappa_1 A_1 + \kappa_2 A_2)$$

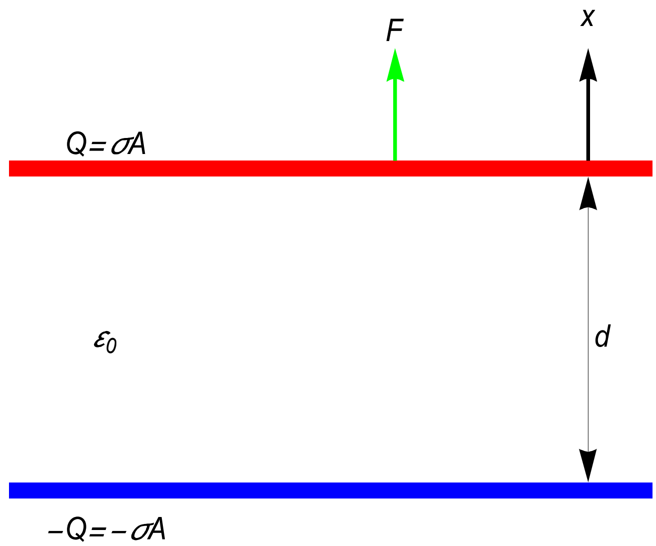
19. Work-energy theorem for capacitance (I)
Walter Lewin: 8.02X Electricity and Magnetism

We consider the capacitance consisting of two conducting plates which are parallel to each other. The separation distance between two plates id *d*. The upper plate is positively charged; $Q = \sigma A$, while the lower late is negatively charged as $-Q = -\sigma A$. The electric field is constant is given by

$$E = \frac{\sigma}{\varepsilon_0}$$

We now consider a case when the upper plate is moved upward by a force *F* (along the *x* direction). Note that the weight of the upper plate is negligibly small. We use the work-energy theorem,

$$\Delta K = W = -\Delta U$$



The work is given by

$$W = Fdx = -\Delta U$$

where F is the conservative force, and U is the potential energy

$$U = \frac{1}{2}QV = \frac{1}{2}(\sigma A)\frac{\sigma}{\varepsilon_0}d = \frac{\sigma^2 Ad}{2\varepsilon_0}$$

$$Q = \sigma A, \qquad V = Ed = \frac{\sigma}{\varepsilon_0}d$$

Since

$$\Delta U = \frac{\sigma^2 A}{2\varepsilon_0}\Delta x$$

the force F is

$$F_x = -\frac{dU}{dx} = -\frac{\sigma^2 A}{2\varepsilon_0} \ (<0)$$

which is an attractive force. If you want to move the plate to the upward, you need to apply an external force

$$F_{ext} = \frac{\sigma^2 A}{2\varepsilon_0}$$

So we have the work

$$W = F_{ext} x = \frac{\sigma^2 Ax}{2\varepsilon_0} = \frac{\sigma^2 Ax}{2\varepsilon_0}$$

where Ax is the volume. The electric field energy density is obtained as

$$\frac{W}{Ax} = \frac{\sigma^2}{2\varepsilon_0} = \frac{\sigma^2}{2\varepsilon_0} = \frac{1}{2}\varepsilon_0 E^2$$

20. Work energy theorem for capacitance

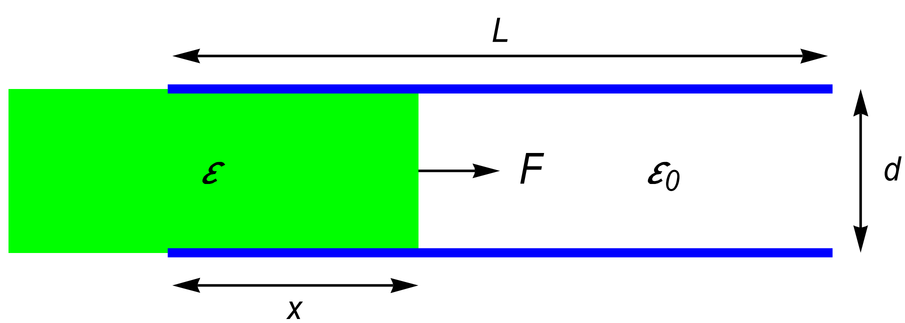
(1) Force on a capacitance plate: (Problem 3-26) Purcell and Morin

A parallel-plate capacitor consists of a fixed plate and a moval plate that is allowed to slide in the direction parallel to the plates. Let x be the distance of overlap as shown in Fig. The separation between the plates is fixed.

(a) Assume that the plates are electrically isolated, so that their charges $\pm Q$ are constant. In terms of Q and the (variable) capacitance C, derive an expression for the leftward force on the movable plate.

(b) Now assume that the plates are connected to a battery, so that the potential difference V is held constant. In terms of V and the capacitance C, derive an expression for the force.

(c) If the movable plate is held in place by an opponent force, then either of the above two setups could be the relevant one, because nothing is moving. So the forces in (a) and (b) should be equal. Verify that this is the case.



(a) Q = constant

Work-energy theorem

$$\Delta K = \Delta W = -\Delta U$$

$$\Delta W = \boldsymbol{F} \cdot d\boldsymbol{r} = -\Delta U$$

where

$$U = \frac{1}{2}QV = \frac{Q^2}{2C}$$

with $Q = CV$. The force $\boldsymbol{F}$ is given by

$$\boldsymbol{F} = -\nabla U$$

or

$$F_x = -\frac{dU}{dx} = -\frac{Q^2}{2}\frac{d}{dx}\frac{1}{C} = \frac{Q^2}{2C^2}\frac{dC}{dx} \qquad (F_x > 0).$$

for the 1D system. Note that

$$C = \varepsilon\frac{Lx}{d} + \varepsilon_0\frac{L(L-x)}{d} = \frac{L}{d}[\varepsilon x + \varepsilon_0(L-x)]$$

The capacitance C increases with increasing x.

$$\frac{dC}{dx} = \frac{L}{d}(\varepsilon - \varepsilon_0) \qquad (>0)$$

(b) V = constant

Work-energy theorem:

$$\Delta K = \Delta W = \Delta W_b - \Delta U = \boldsymbol{F}\cdot\Delta\boldsymbol{r}$$

where W_b is the work required to move each of the charge increment.

$$\Delta W_b = V\Delta Q, \qquad \Delta U = \frac{1}{2}V\Delta Q$$

or

$$\Delta W_b = 2\Delta U$$

Then we have

$$\Delta W = \Delta W_b - \Delta U = 2\Delta U - \Delta U = \Delta U = \boldsymbol{F}\cdot\Delta\boldsymbol{r}$$

or

$$F_x = \frac{dU}{dx} = \frac{1}{2}V^2\frac{dC}{dx} \qquad (F_x > 0).$$

for the 1D system, where

$$\frac{dC}{dx} = \frac{L}{d}(\varepsilon - \varepsilon_0) \qquad (>0)$$

21. Displacement vector *D*

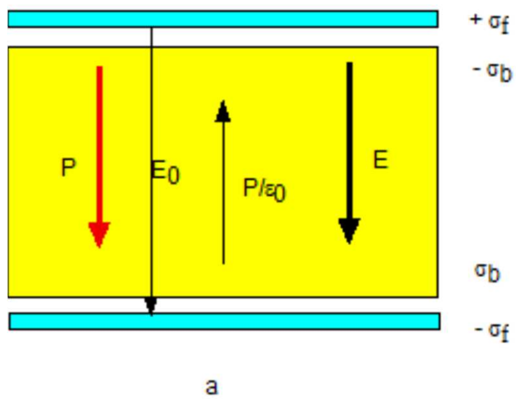


Fig. *P* is the polarization of the dielectric. σ_f is the free surface charge density due to the free charges located on the two parallel plates. $\sigma_b = \sigma_{ind}$, σ_b is the bound surface charge density due to the polarization of the dielectric. E_0 is an external electric field. *E* is an electric field inside the dielectrics. σ_b is equal to *P*. $E = E_0 - \frac{P}{\varepsilon_0}$. *P* is related to *E* through $P = \varepsilon_0 \chi E$.

The external field $\boldsymbol{E}_0$ inside the air (the space between two parallel metal plates is air) is given by

$$E_0 = E_f = \frac{\sigma_f}{\varepsilon_0}$$

or

$$\varepsilon_0 \oint \boldsymbol{E}_f \cdot d\boldsymbol{a} = q_f$$

The electric field inside the dielectric (the space between two parallel metal plates is filled with dielectric) is given by

$$E = \frac{\sigma_f - \sigma_b}{\varepsilon_0}$$

or

$$\varepsilon_0 \oint \boldsymbol{E} \cdot d\boldsymbol{a} = q_{eff} = q_f - q_b$$

where q_f is the free charge density and q_b is the bound charge density.

Here we define the electric displacement $\boldsymbol{D}$ by

$$D = \sigma_f = \varepsilon_0 E_f \qquad \text{(electric displacement)}$$

or

$$\oint \boldsymbol{D} \cdot d\boldsymbol{a} = q_f$$

This equation states Gauss' law in its general form. It is applicable to any dielectric medium as well as to a vacuum. This is a useful way to express Gauss' law, in the context of dielectrics, because it makes reference only to free charges, and free charge is the stuff we control (Griffiths, Introduction to electrodynamics).

Since $\boldsymbol{E}_f = \kappa \boldsymbol{E}$, $\boldsymbol{D}$ is described as

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E}_f = \varepsilon_0 \kappa \boldsymbol{E}$$

Then we get

$$q_f = \varepsilon_0 \oint \boldsymbol{E}_f \cdot d\boldsymbol{a} = \varepsilon_0 \kappa \oint \boldsymbol{E} \cdot d\boldsymbol{a} = \oint \boldsymbol{D} \cdot d\boldsymbol{a}$$

or

$$\varepsilon_0 \kappa \oint \boldsymbol{E} \cdot d\boldsymbol{a} = q_f \qquad \text{(Gauss' law with dielectric)}$$

or

$$\int (\nabla \cdot \kappa \boldsymbol{E}) dV = \oint \boldsymbol{E}_f \cdot d\boldsymbol{a} = \frac{1}{\varepsilon_0} Q_f = \frac{1}{\varepsilon_0} \int \rho_f dV$$

leading to the formula

$$\boxed{\nabla \cdot (\kappa \boldsymbol{E}) = \frac{1}{\varepsilon_0}\rho_f}$$

22. Application of the Gauss' law

We apply the Gauss theorem on the Gaussian surface (cylindrical surface)

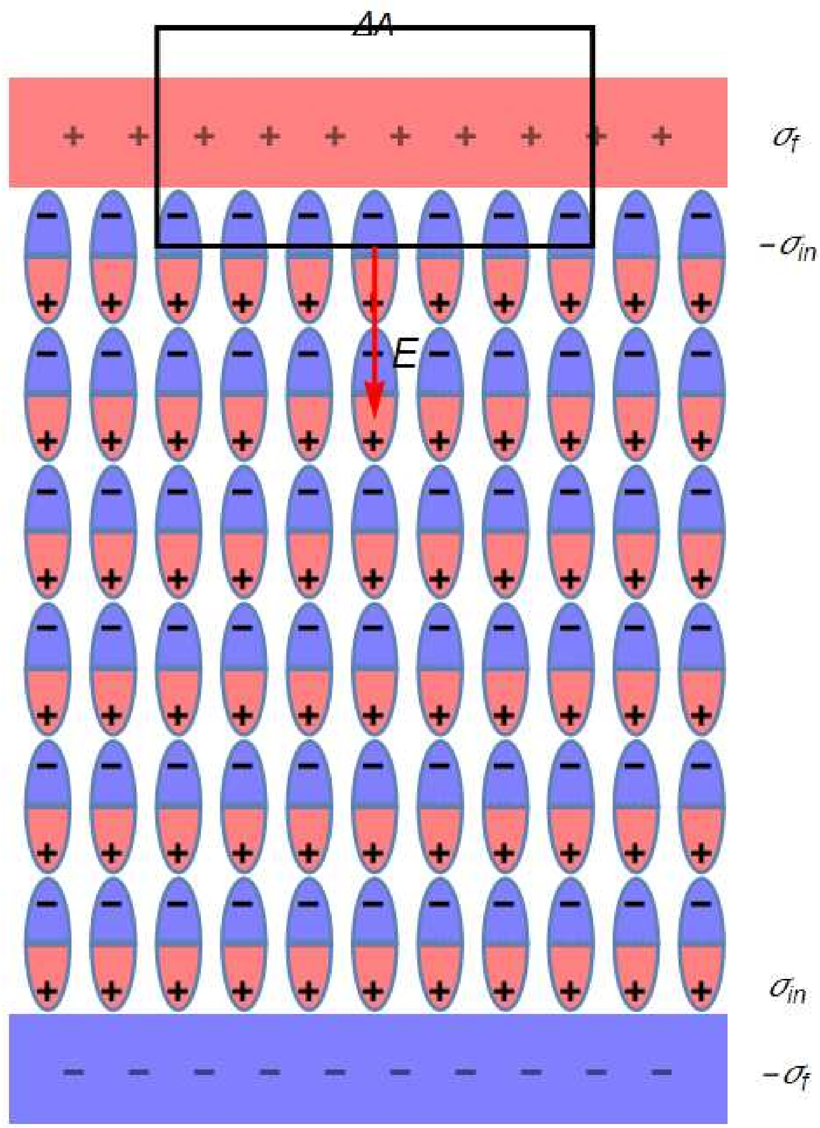
$$E\Delta A = \frac{1}{\varepsilon_0}(\sigma_f - \sigma_{in})\Delta A$$

or

$$E = \frac{1}{\varepsilon_0}(\sigma_f - \sigma_{in})$$

Since $\sigma_f = D$ and $\sigma_{in} = P$, we have

$$\varepsilon_0 E = D - P, \qquad D = \varepsilon_0 E + P$$



23. Example: *D* and *E* for the capacitor

We consider the simple case of the capacitor where the dielectric with *k* between two parallel plates.

The displacement vector *D* is given by

$$D = \sigma_f$$

D is related to the electric field *E* by

$$D = \varepsilon_0 \kappa E$$

or

$$E = \frac{D}{\varepsilon_0 \kappa} = \frac{\sigma_f}{\varepsilon_0 \kappa}$$

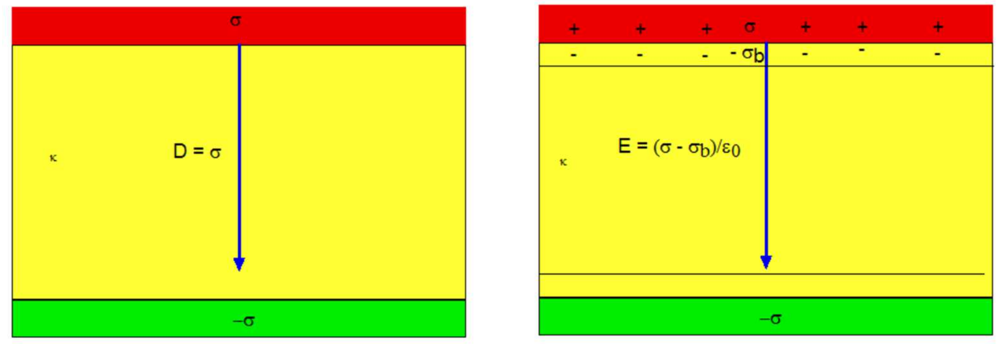


Fig. $\sigma = \sigma_f$ and $\sigma_{ind} = \sigma_b$ in this Fig.

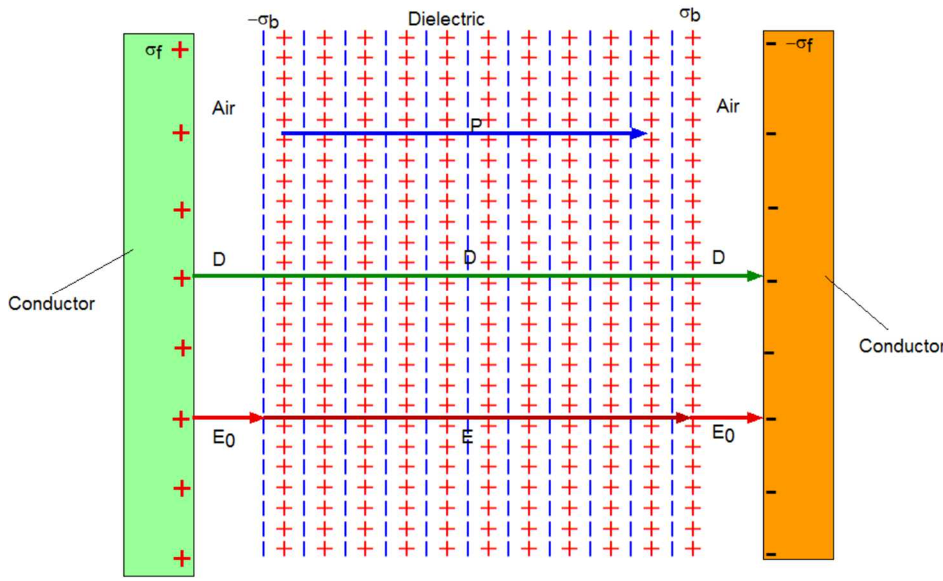
The electric field E is also derived as

$$E = \frac{\sigma_f - \sigma_b}{\varepsilon_0}$$

The bound surface charge σ_b is obtained as

$$\sigma_b = \sigma_f - \varepsilon_0 E = \sigma_f - \varepsilon_0 \frac{\sigma_f}{\varepsilon_0 \kappa} = \sigma_f (1 - \frac{1}{\kappa})$$

In summary, we show the schematic diagram for the fields ***D*** and ***E*** in the dielectric in the parallel-plate capacitor; the displacement vector ***D*** depends only on the free charge and is the same inside and outside (air gaps).



$$D=\sigma_f \qquad E_0=\frac{\sigma_f}{\varepsilon_0}$$

$$D=\sigma_f \qquad P=\sigma_b \qquad E=\frac{\sigma_f-\sigma_b}{\varepsilon_0}=\frac{\sigma_f}{\varepsilon_0\kappa}$$

$$D=\sigma_f \qquad E_0=\frac{\sigma_f}{\varepsilon_0}$$

24. Maxwell/s equation with *E*, *D*, and *P*

The effect of the polarization is equivalent to a charge density ρ_b given by

$$\rho_b=-\nabla\cdot\boldsymbol{P}$$

The divergence of $\boldsymbol{E}$ is related to the effective charge density ρ_{eff} by

$$\nabla\cdot\boldsymbol{E}=\frac{\rho_{eff}}{\varepsilon_0}=\frac{\rho_f+\rho_b}{\varepsilon_0}=\frac{\rho_f}{\varepsilon_0}-\frac{1}{\varepsilon_0}\nabla\cdot\boldsymbol{P}$$

where ρ_f is a free charge density. This equation is rewritten as

$$\nabla\cdot(\varepsilon_0\boldsymbol{E}+\boldsymbol{P})=\rho_f$$

We define $\boldsymbol{D}$ as

$$\boldsymbol{D}=\varepsilon_0\boldsymbol{E}+\boldsymbol{P}$$

with

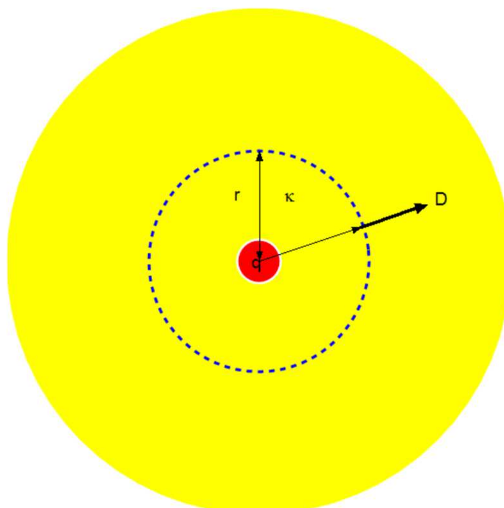
$$\nabla \cdot \boldsymbol{D} = \rho_f$$

In summary

$$\boldsymbol{D} = \varepsilon_0 \kappa \boldsymbol{E} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}, \qquad \nabla \cdot \boldsymbol{D} = q_f$$

25. Example: *D* and *E* for the simple case with spherical symmetry

We consider the simple case of dielectric sphere where the point charge is located at the center.



We apply the Gauss' law for the Gaussian surface (dashed line)

$$\int \boldsymbol{D} \cdot d\boldsymbol{a} = q_f = q$$

or

$$D(4\pi r^2) = q_f$$

or

$$D = \frac{q_f}{4\pi r^2}$$

where q_f (= q) is the free charges. The electric field $\boldsymbol{E}$ is related to $\boldsymbol{D}$ by a relation

$$D = \varepsilon_0 \kappa E$$

Then we have

$$E = \frac{D}{\varepsilon_0 \kappa} = \frac{q}{4\pi\varepsilon_0 \kappa r^2}$$

The effective charge inside the dashed line (q_{eff}) is evaluated as

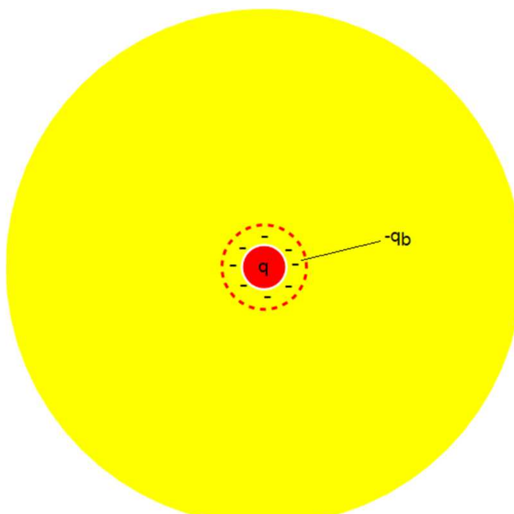
$$q_{eff} = \varepsilon_0 \int \boldsymbol{E} \cdot d\boldsymbol{a} = \varepsilon_0 \frac{q}{4\pi\varepsilon_0 \kappa r^2}(4\pi r^2) = \frac{q}{\kappa}$$

Here q_{eff} consists of free charge (q) and bound charge (q_{b}).

$$q_{eff} = q - q_b = \frac{q}{\kappa}$$

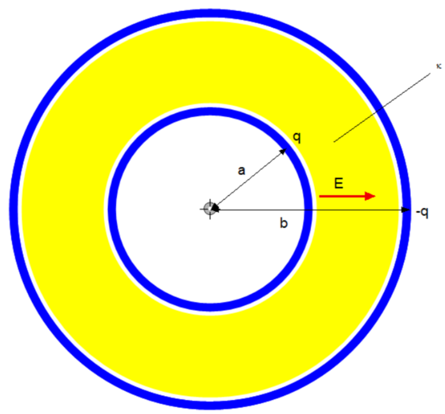
or

$$q_b = q(1 - \frac{1}{\kappa})$$



26. Example Problem 25-53 (SP25-53)

The space between two concentric spherical shells of radii b = 1.70 cm and a = 1.20 cm is filled with a substance of dielectric constant κ = 23.5. A potential difference V = 73.0 V is applied across the inner and outer shells. Determine (a) the capacitance of the device, (b) the free charge q on the inner shell, and (c) the charge q' induced along the surface of the inner shell.



((**Solution**))
a = 1.20 cm
b = 1.70 cm
κ = 23.5
V_{ba} = 73.0 V

We apply the Gauss' law for the Gaussian surface (dashed line)

$$\int \boldsymbol{D} \cdot d\boldsymbol{a} = q_f = q \qquad \text{(true charge)}$$

Then we have

$$D(4\pi r^2) = q$$

or

$$D = \frac{q}{4\pi r^2}$$

Using the relation give by

$$D = \varepsilon_0 \kappa E$$

The electric field E is derived as

$$E = -\frac{dV}{dr} = \frac{D}{\varepsilon_0 \kappa} = \frac{q}{4\pi\varepsilon_0 \kappa r^2}$$

Then we have

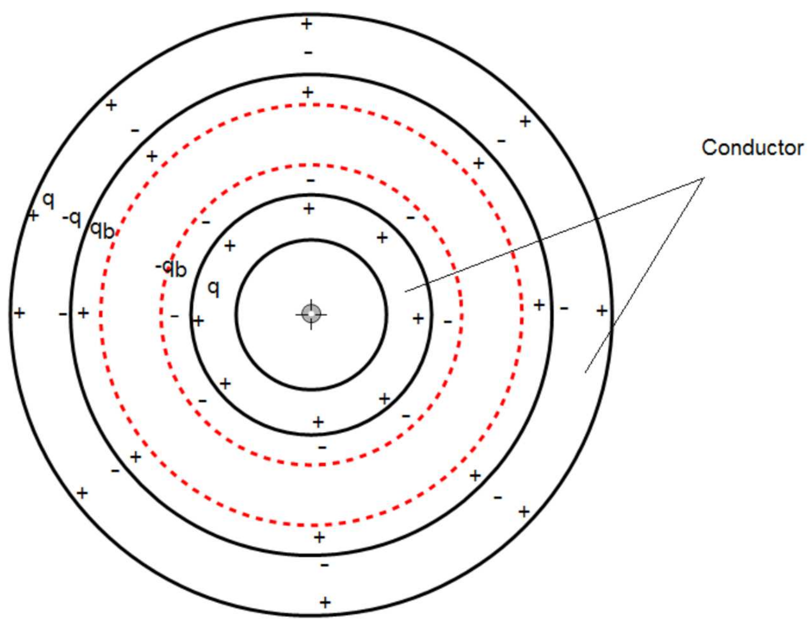
$$V_{ab} = -\int_a^b \frac{q}{4\pi\varepsilon_0 \kappa r^2} dr = \frac{q}{4\pi\varepsilon_0 \kappa}\left(\frac{1}{b} - \frac{1}{a}\right) = -V_{ba}$$

(a)

$$Q_a = q = CV_{ba}$$
$$C = \frac{q}{V_{ba}} = \frac{4\pi\varepsilon_0\kappa}{\frac{1}{a}-\frac{1}{b}} = 0.1067nF$$

(b)

$$q = CV_{ba} = 0.1067nF \times 73V = 7.79nC$$



(c)
Gaussian surface (dotted line in the vicinity of $r = a$)

$$\varepsilon_0 \oint \boldsymbol{E} \cdot d\boldsymbol{a} = q_{eff} = q - q_b$$

where q_b is the bound charge (induced charge)

For the Gaussian surface just outside $r = a$,

$$E(4\pi a^2) = \frac{q - q_b}{\varepsilon_0} \qquad \text{or} \qquad E = \frac{q - q_b}{4\pi\varepsilon_0 a^2}$$

The electric field E is also given by

$$E = \frac{q}{4\pi\varepsilon_0 \kappa a^2}$$

Using the dielectric constant κ, we have

$$q - q_b = \varepsilon_0 E(4\pi a^2) = \varepsilon_0 \frac{q}{4\pi\varepsilon_0 \kappa a^2}(4\pi a^2) = \frac{q}{\kappa}$$

or

$$q_b = q(1 - \frac{1}{\kappa}) = 7.46 \text{ nC}$$

REFERENCES

J. Reitz, F.J. Milford, and R.W. Christy, *Foundations of Electromagnetic Theory*, 3rd edition (Addison-Wesley, 1980).

E.M. Purcell and D.J. Morin, *Electricity and Magnetism*, 3rd edition (Cambridge, 2013).

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APPENDIX

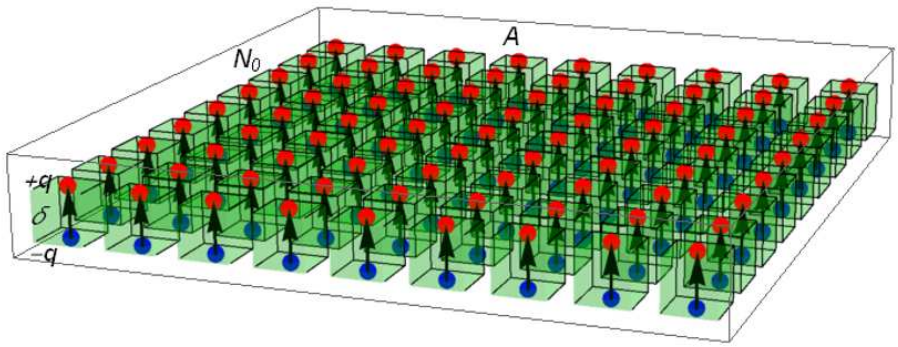
Surface charge density in terms of the polarization vector ((Feynman))

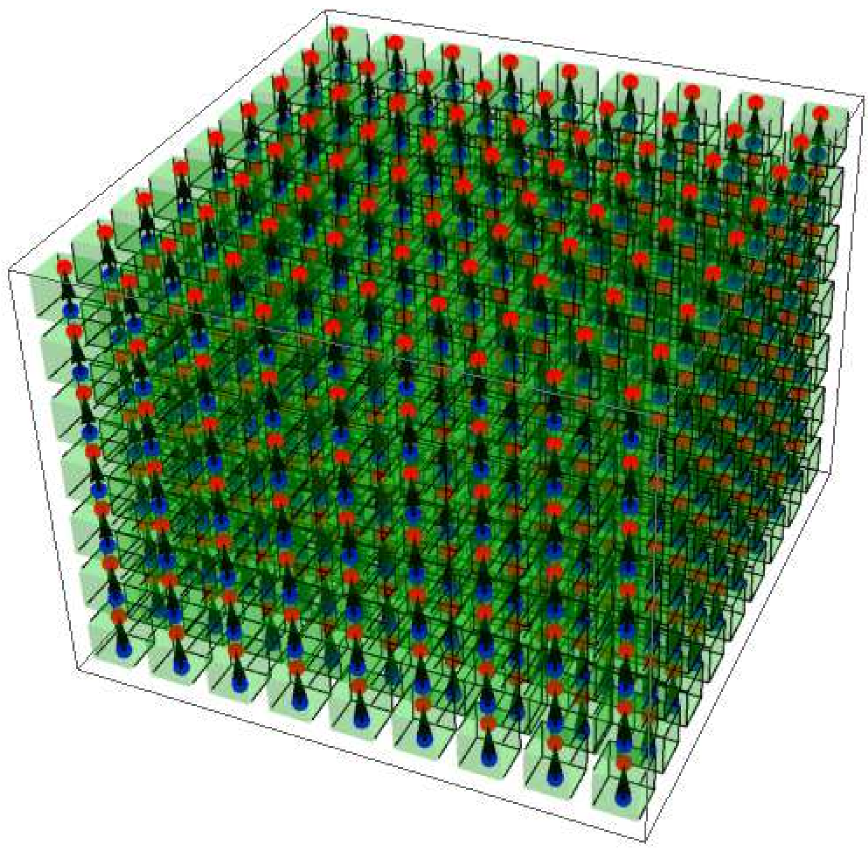
We now consider the situation in which the polarization vector ***P*** is not everywhere the same. If the polarization is not constant, we would expect in general to find a charge density in the volume, because more charge might come into one side of a small volume element than leaves it on the other. How can we find out how much charge is gained or lost from a small volume?

We calculate how much charge moves across any imaginary surface when the material is polarized. The amount of charge that goes across a surface is just P times the surface area if the polarization is *normal* to the surface. Of course, if the polarization is *tangential* to the surface, no charge moves across it. Following the same arguments, it is easy to see that the charge moved across any surface element is proportional to the *component* of ***P*** *perpendicular* to the surface.

(a) The case of polarization vector which is normal to the top of the surface

We assume that the electric dipole moment is normal to the top of the surface.





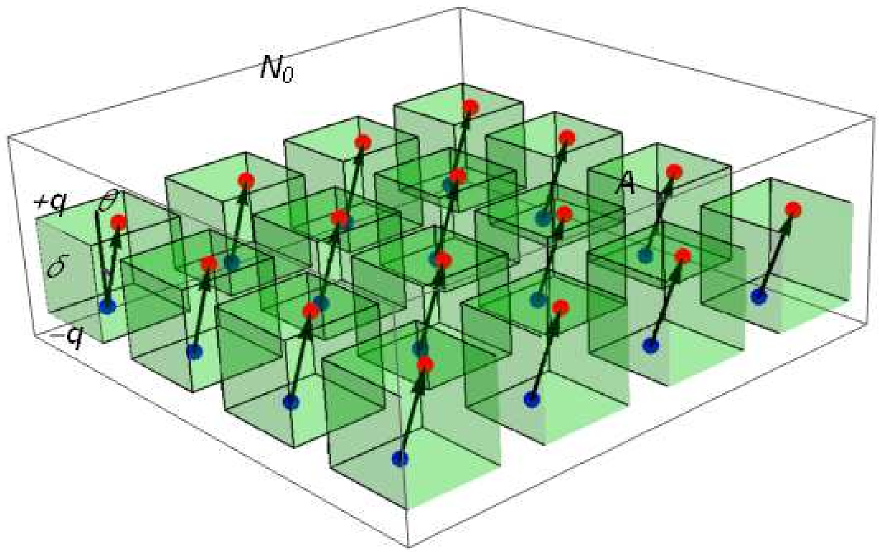
The surface charge density is obtained as

$$\sigma_P = \frac{N_0 q}{A} = \frac{N_0 (q\delta)}{A\delta} = \frac{N_0 p}{V} = P$$

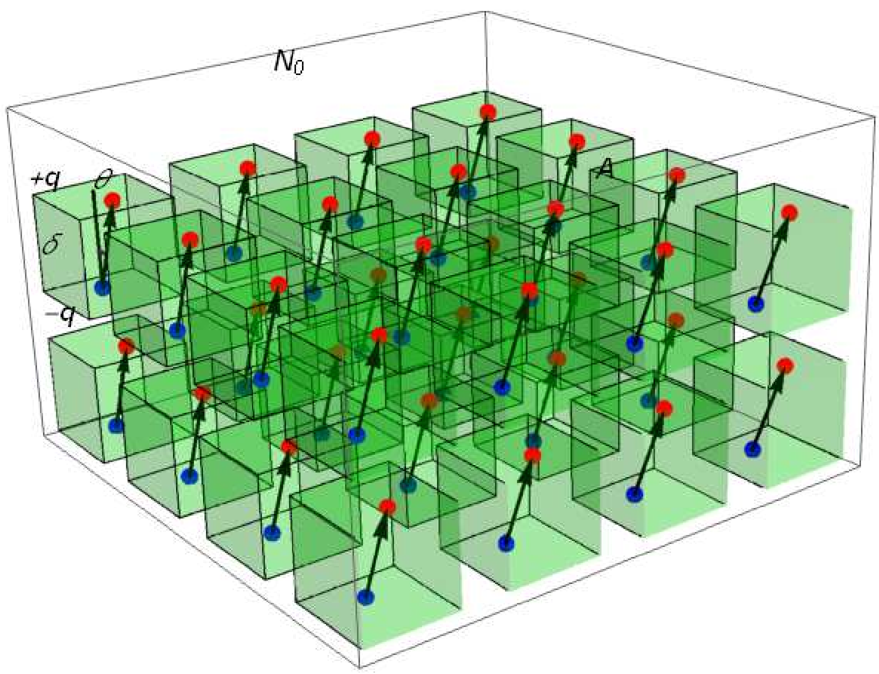
which is equal to the magnitude of the polarization vector, where $V = A\delta$

(b) The case of polarization vector which is not normal to the top of the surface

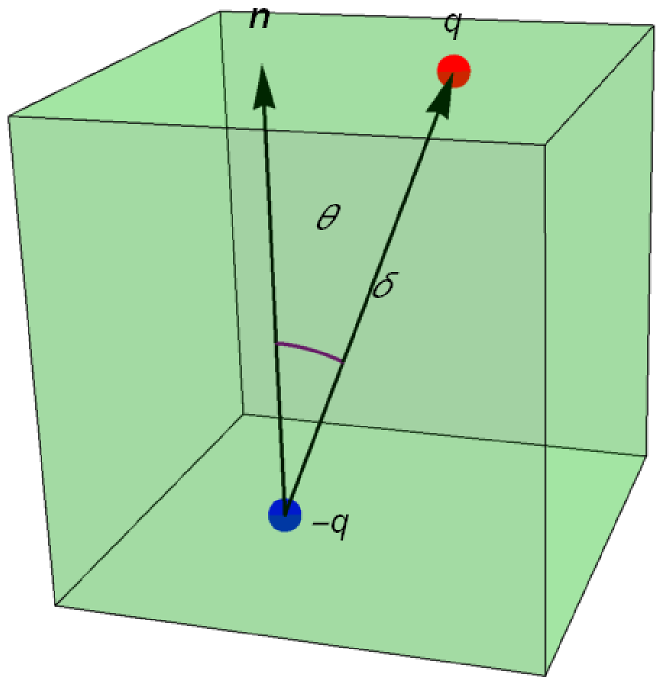
We assume that the electric dipole moment is not normal to the top of the surface.



N_0
$+q$
θ
δ
$-q$
A



N_0
$+q$
θ
δ
$-q$
A



The surface charge density is

$$\sigma_P = \frac{N_0 q}{A} = \frac{N_0 q \delta \cos\theta}{A\delta\cos\theta} = \frac{N_0 p \cos\theta}{V} = P\cos\theta\,,$$

or

$$\sigma_P = \boldsymbol{P}\cdot\boldsymbol{n}\,.$$

where $\qquad V = A\delta\cos\theta\,.$

((**Feynman**))

In his book, Feynman derived the expression $\sigma_P = \boldsymbol{P}\cdot\boldsymbol{n}$ using the following Fig.

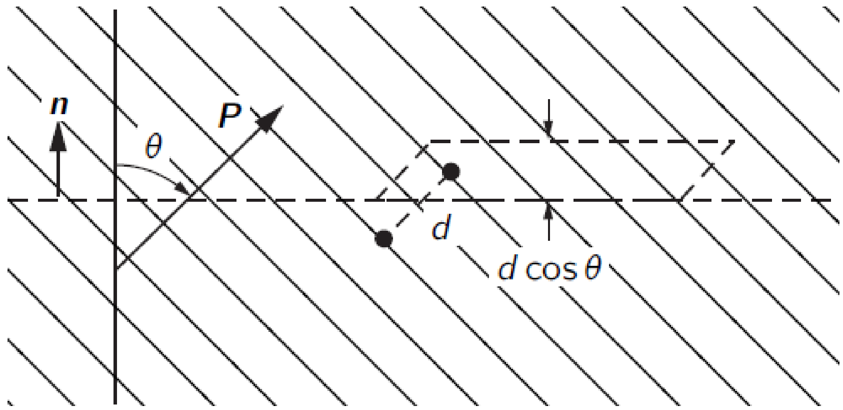


Fig. The charge moved across an element of an imaginary surface in a dielectric is proportional to the component of ***P*** normal to the surface. $d = \delta$.

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R.P. Feynman, R.B. Leighton, and M. Sands, The Feynman Lectures on Physics Vol.2 (Basic Book, 2010).