Chapter 27 Circuits

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1. Basic elements and law

1.1 Lumped circuits

Lumped circuit which is obtained by connecting lumped elements such as resistors, capacitors, inductors, transformers, and so on.

(1) The key property associated with lumped elements is their small size L_0 compared to the wavelength λ corresponding to their normal frequency of operation: $L_0 << \lambda$.

(2) From the more general electromagnetic field point of view, lumped elements are point singularities, that is, they have negligible physical dimensions.

(3) To exhibit the implications of the restriction on size, let us consider the following case. f is the frequency and λ is the wavelength of waves.

$c = 3 \times 10^8$ m/s

For f = 1 GHz = 10^9 Hz, $\lambda = \dfrac{c}{f} = \dfrac{3\times 10^8 m/s}{10^9 1/s} = 0.3m$

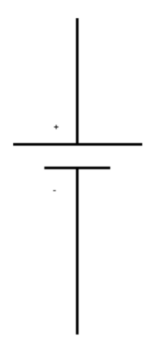
For f = 100 GHz = 10^{11} Hz, $\lambda = \dfrac{c}{f} = \dfrac{3\times 10^8 m/s}{10^{11} 1/s} = 0.3cm$

Consequently, in this case, the lumped approximation may not be good (f = 100 GHz). At low frequency f (= Hz, kHz, MHz), this approximation is very good.

For a two-terminal lumped elements, the current through the element and the voltage across it are well-defined quantities.

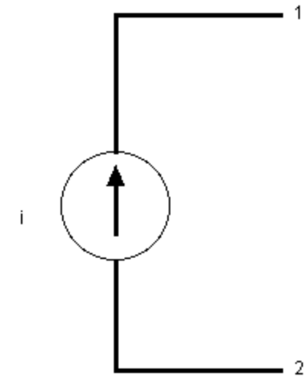
1.2 Ideal battery

If the value of an ideal voltage source is constant, that is, it does not change with time, an ideal voltage source is commonly represented by the equivalent notation.



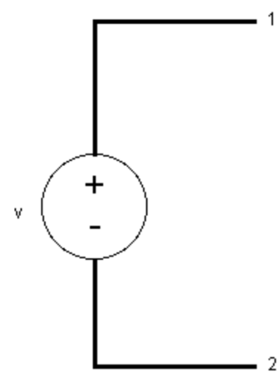
1.3 Ideal current source

An ideal current source, which is a device that, when connected anything, will always pull I (A) into terminal 2 and push I (A) out of terminal 1.



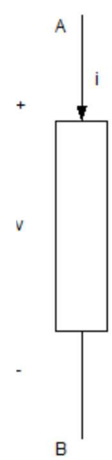
1.4 Ideal voltage source

This is a device that produces a voltage of *v* (V) across its terminals regardless of what is connected to it. The terminal 1 is at an electric potential that is *v* Volts higher than that of terminal 2.



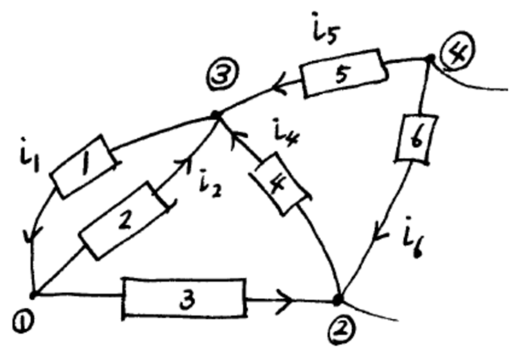
1.5 Reference directions of voltages

The reference direction for the voltage is indicated by the plus and minus symbols located near the terminals A and B.



1.6 Branch and node

In a lumped circuit, the two-terminal elements are called branches, and the terminals of the elements are called nodes.



A lumped circuit with 6 branches and four nodes

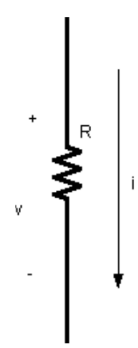
1.7 Open-circuit and short circuit

A two-terminal element is called an open-circuit if it has a branch current identical to zero, whatever the branch voltage may be.

A two-terminal element is called a short-circuit if it has a branch voltage identical to zero, whatever the branch voltage may be.

2. Ohm's law

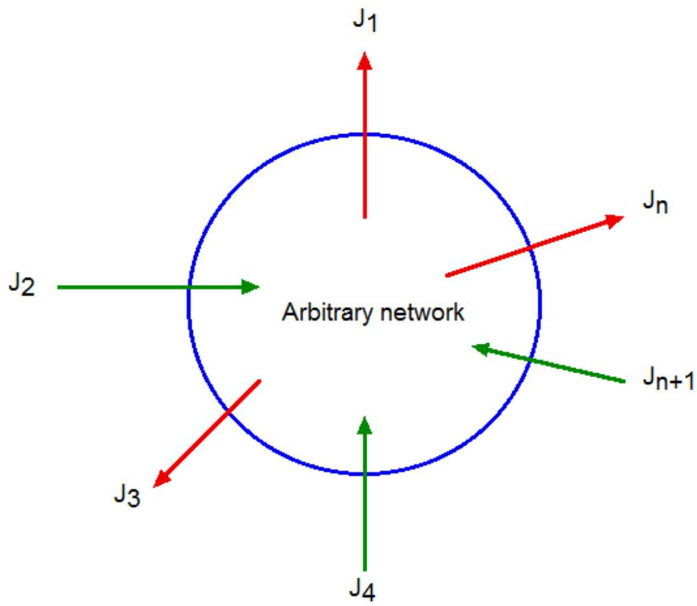
If a resistor R has a voltage v across it and a current i going through it, then it is true that $v = iR$.



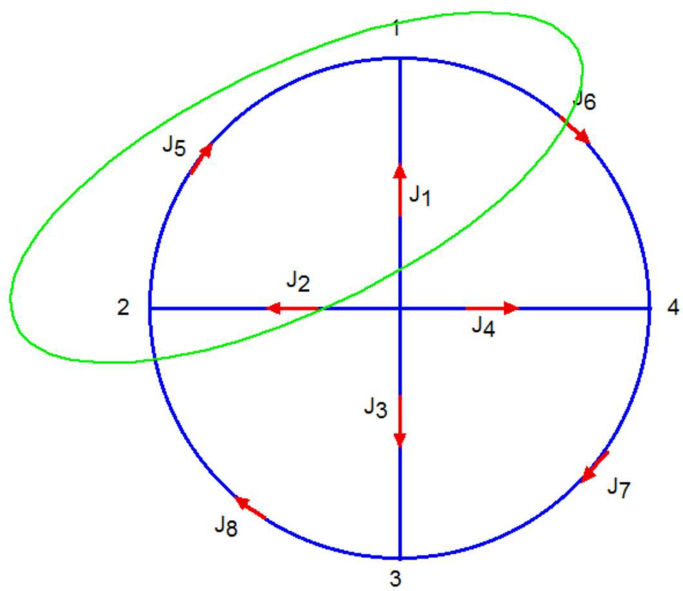
where R is a resistance.

3. KCL (Kirchhoff's current law)

For any lumped electric circuit, for any of its nodes and at any time, the algebraic sum of all branch currents leaving the node is zero.



((**Example**))



KCL at node 1

$$-j_5 - j_1 + j_6 = 0 \tag{1}$$

KCL at node 2

$$-j_8 - j_2 + j_5 = 0 \tag{2}$$

Addition of Eqs.(1) and (2);

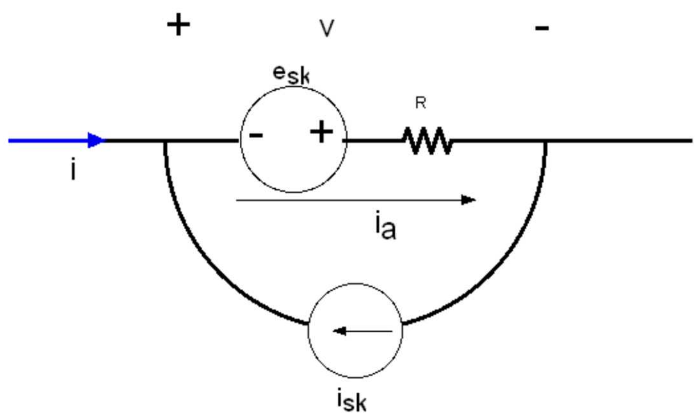
$$-j_8 - j_2 - j_1 + j_6 = 0$$

which corresponds to the KCL at the network denoted by the green line

((Note))
1. KCL is independent of the nature of the elements.
2. KCL expresses the conservation of charge at every node.

$$\sum_n i_n = 0 \qquad \text{where} \qquad i_n = \frac{dq_n}{dt}$$

((**Example**)) general source



By KCL,

$$i_a = i + i_{sk} \tag{1}$$

We also have the relation

$$v = -e_{sk} + Ri_a \tag{2}$$

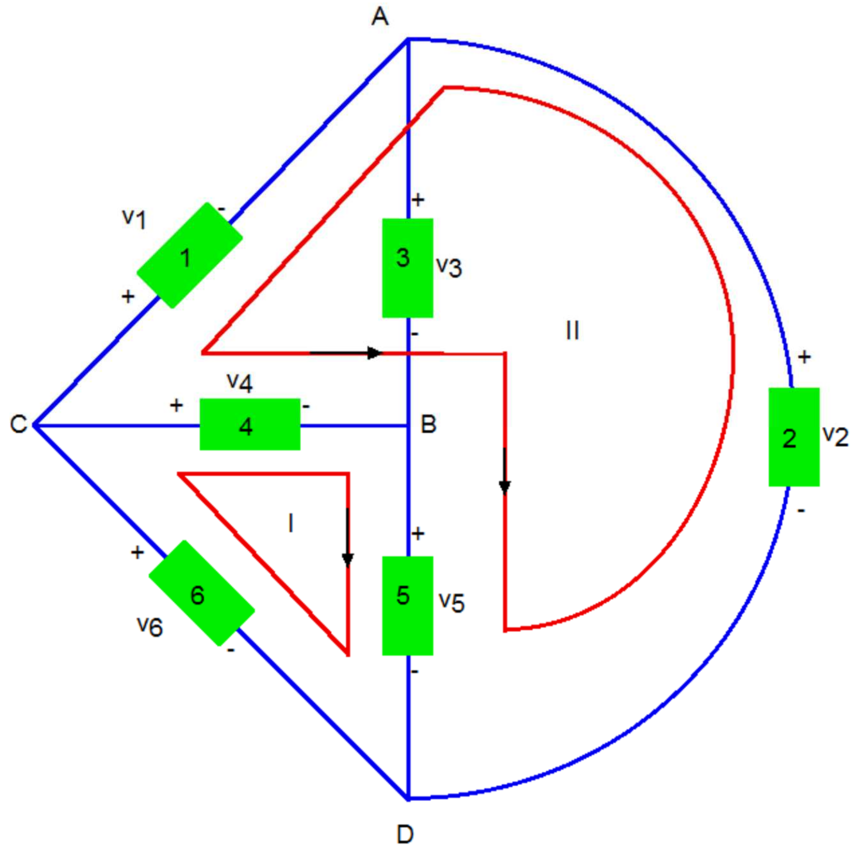
From Eqs.(1) and (2), we have

$$v + e_{sk} = R(i + i_{sk})$$

4. KVL (Kirchhoff's voltage law)

The algebraic sum of the potential difference across all elements in complete around any closed-circuit loop must be zero.

$$\sum_{closedloop} \Delta V = 0$$



(a) KVL applied to the loop I, asserts that

$$v_4 + v_5 - v_6 = 0$$

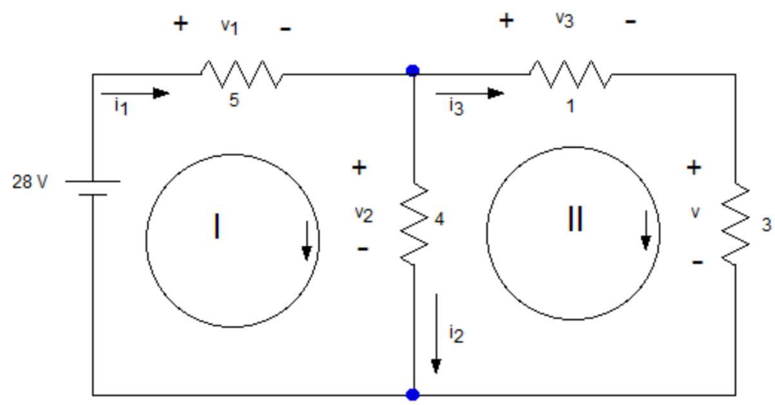
The reference directions of branches 4 and 5 agree with the reference direction of loop I, whereas the reference direction of branch 6 does not agree with that of loop I. We, therefore, assign plus signs to v_4 and v_5, and a minus sign to v_6.

(b) KVL applied to the loop II, asserts that

$$-v_1 + v_4 + v_5 - v_2 = 0$$

((**Note**)) KVL is independent of the nature of the elements.

((**Example**))



By KVL,

$$-28 + v_1 + v_2 = 0 \qquad \text{for loop I} \qquad (1)$$

and

$$v_3 + v - v_2 = 0 \qquad \text{for loop II} \qquad (2)$$

By using Ohm's law, we obtain

$$\begin{aligned} i_2 &= \frac{v_2}{4}, \\ i_3 &= \frac{v}{3} \\ i_1 &= \frac{v_1}{5} \\ v_3 &= i_3 = \frac{v}{3} \end{aligned} \qquad (3)$$

From Eq.(2),

$$v_2 = v_3 + v = \frac{v}{3} + v = \frac{4v}{3}$$

$$i_2 = \frac{1}{4}v_2 = \frac{1}{4}\frac{4v}{3} = \frac{v}{3}$$

By KCL at the node A,

$$-i_1 + i_2 + i_3 = 0$$

$$i_1 = i_2 + i_3 = \frac{v}{3} + \frac{v}{3} = \frac{2v}{3}$$

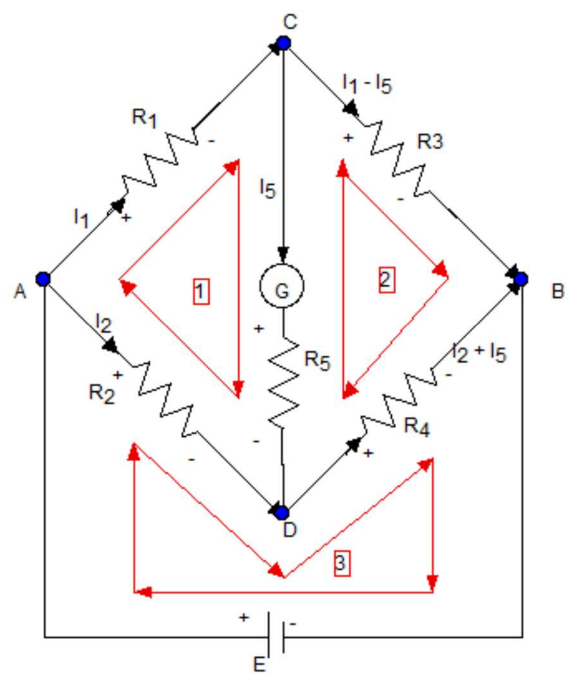
Substituting these into Eq.(1) yields that

$$28 = \frac{10v}{3} + \frac{4v}{3} = \frac{14v}{3}$$
$$v = 6V$$

5. typical circuits

(a) Wheatstone bridge

A Wheatstone bridge is a measuring instrument invented by Samuel Hunter Christie in 1833 and improved and popularized by Sir Charles Wheatstone in 1843. It is used to measure an unknown electrical resistance by balancing two legs of a bridge circuit, one leg of which includes the unknown component. (Wikipedia).



For loop 1

$$R_1 I_1 + R_5 I_5 - R_2 I_2 = 0$$

For loop 2

$$R_3 (I_1 - I_5) - R_4 (I_2 + I_5) - R_5 I_5 = 0$$

For loop 3

$$R_2 I_2 + R_4 (I_2 + I_5) - E = 0$$

We use Mathematica to solve these equations.

$$I_5 = \frac{E(R_2 R_3 - R_1 R_4)}{R_1 R_2 R_3 + R_1 R_2 R_4 + R_1 R_3 R_4 + R_2 R_3 R_4 + (R_1 + R_3)(R_2 + R_4) R_5}$$

$$I_1 = \frac{E\{R_4 R_5 + R_2 (R_3 + R_4 + R_5)\}}{R_1 R_2 R_3 + R_1 R_2 R_4 + R_1 R_3 R_4 + R_2 R_3 R_4 + (R_1 + R_3)(R_2 + R_4) R_5}$$

$$I_2 = \frac{E\{R_3 R_5 + R_1 (R_3 + R_4 + R_5)\}}{R_1 R_2 R_3 + R_1 R_2 R_4 + R_1 R_3 R_4 + R_2 R_3 R_4 + (R_1 + R_3)(R_2 + R_4) R_5}$$

When $I_5 = 0$, we have the condition

$$R_1 R_4 = R_2 R_3$$

Suppose that R_1, R_2, and R_3 are known and R_4 is unknown. Using the above relation, R4 is determined as

$$R_4 = \frac{R_2 R_3}{R_1}$$

((**Mathematica**))

For mesh 1

```
In[1]:= eq1 = R1 I1 + R5 I5 - R2 I2 == 0
Out[1]= I1 R1 - I2 R2 + I5 R5 == 0
```

For mesh 2

```
In[2]:= eq2 = R3 (I1 - I5) - R4 (I2 +  I5) - R5 I5 == 0
Out[2]= (I1 - I5) R3 - (I2 + I5) R4 - I5 R5 == 0
```

For mesh 3

```
In[3]:= eq3 = R2 I2 + R4 (I2 + I5) == E1
Out[3]= I2 R2 + (I2 + I5) R4 == E1
```

```
In[4]:= eq4 = Solve[{eq1, eq2, eq3}, {I1, I2, I5}] //
          FullSimplify
Out[4]= {{I1 → (E1 (R4 R5 + R2 (R3 + R4 + R5))) /
            (R1 R2 R3 + R1 R2 R4 + R1 R3 R4 +
              R2 R3 R4 + (R1 + R3) (R2 + R4) R5),
          I2 → (E1 (R3 R5 + R1 (R3 + R4 + R5))) /
            (R1 R2 R3 + R1 R2 R4 + R1 R3 R4 +
              R2 R3 R4 + (R1 + R3) (R2 + R4) R5),
          I5 → (E1 (R2 R3 - R1 R4)) /
            (R1 R2 R3 + R1 R2 R4 + R1 R3 R4 +
              R2 R3 R4 + (R1 + R3) (R2 + R4) R5)}}
```

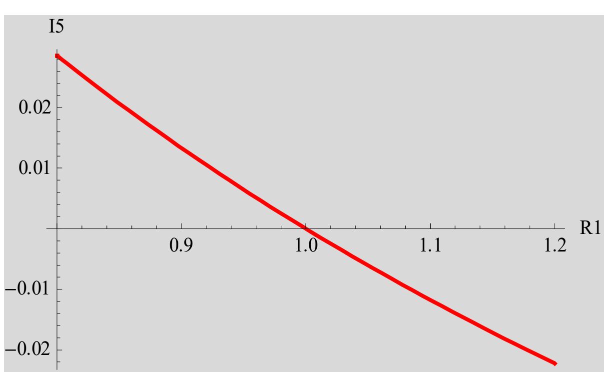
We consider the case R1=x, R2 = R3 = R4= R5=1 Ω, and E=1 V. What is the dependnce of x on the current I5?

```
In[8]:= I51 = I5 /. eq4[[1]] /.
          {R1 → x, R2 → 1, R3 → 1, R4 → 1, R5 → 1,
           E1 → 1}
```

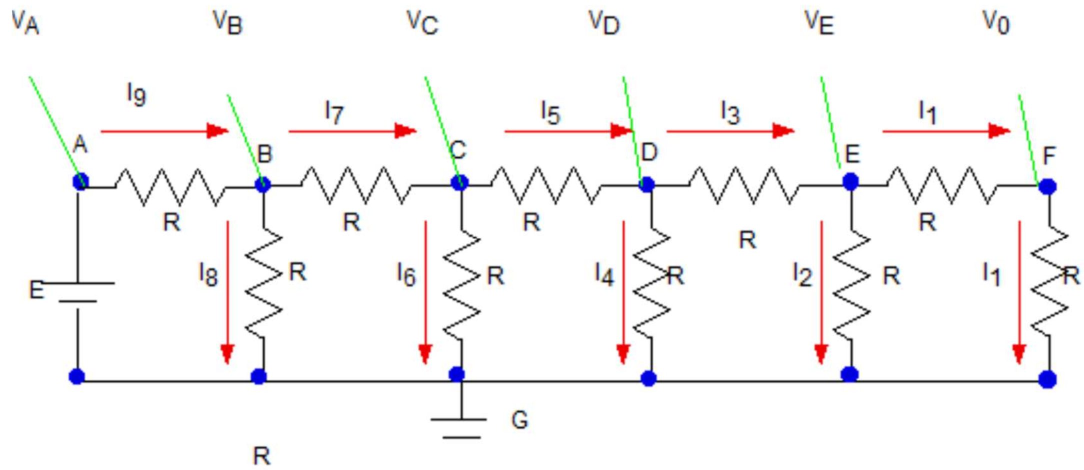
Out[8]= $$\frac{1 - x}{1 + 3\,x + 2\,(1 + x)}$$

```
In[11]:= Plot[I51, {x, 0.8, 1.2},
          AxesLabel → {"R1", "I5"},
          PlotStyle → {Thick, Red},
          Background → LightGray]
```

Out[11]=



(b) Ladder circuit: application of the principle of superposition



We consider a ladder circuit where all the resistors has R (Ω). What are the current of the branches and voltages of nodes. Suppose that $V_F = V_0$.

$$V_F = V_0$$

$$I_1 = \frac{V_0}{R}$$

$$V_E = V_F + I_1 R = 2V_0$$

$$I_2 = \frac{V_E}{R} = \frac{2V_0}{R}$$

$$I_3 = I_1 + I_2 = \frac{3V_0}{R}$$

$$V_D = V_E + RI_3 = 2V_0 + 3V_0 = 5V_0$$

$$I_4 = \frac{V_D}{R} = \frac{5V_0}{R}$$

$$I_5 = I_3 + I_4 = \frac{3V_0}{R} + \frac{5V_0}{R} = \frac{8V_0}{R}$$

$$V_C = V_D + RI_5 = 5V_0 + 8V_0 = 13V_0$$

$$I_6 = \frac{V_C}{R} = \frac{13V_0}{R}$$

$$I_7 = I_5 + I_6 = \frac{21V_0}{R}$$

$$V_B = V_C + RI_7 = 13V_0 + 21V_0 = 34V_0$$

$$I_8 = \frac{V_B}{R} = \frac{34V_0}{R}$$

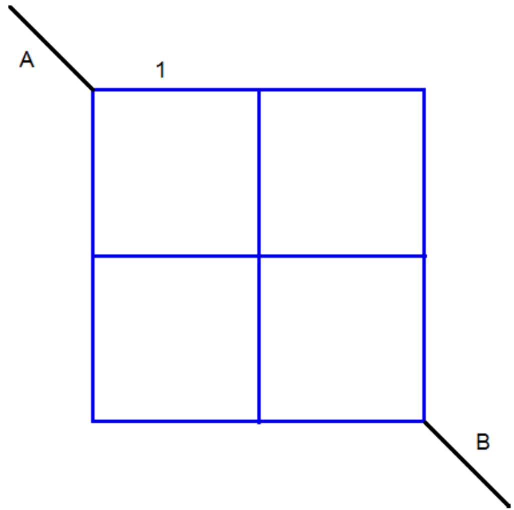
$$I_9 = I_7 + I_8 = \frac{55V_0}{R}$$

$$V_A = V_B + RI_9 = 34V_0 + 55V_0 = 89V_0$$

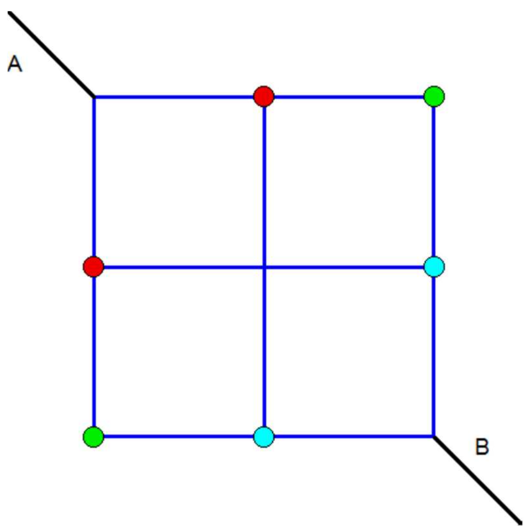
where $E = 89V_0.$, or $V_0 = \frac{E}{89}$. Using this value of V_0, we can determine the current and voltages.

(c) **2x2 square lattice**

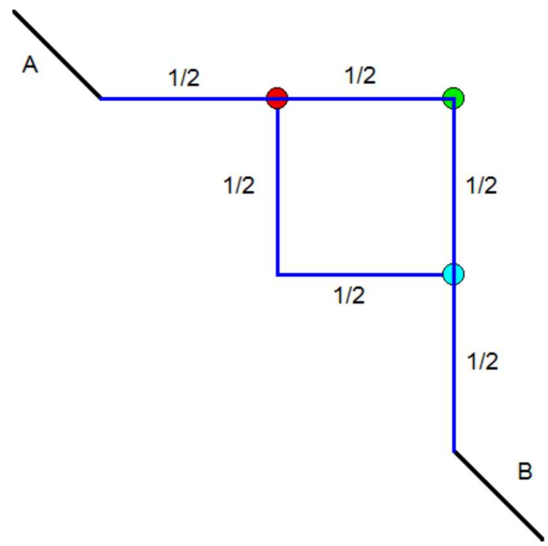
We consider a 2x2 square lattice. Each branch has a 1Ω. What is the total resistance between A and B?



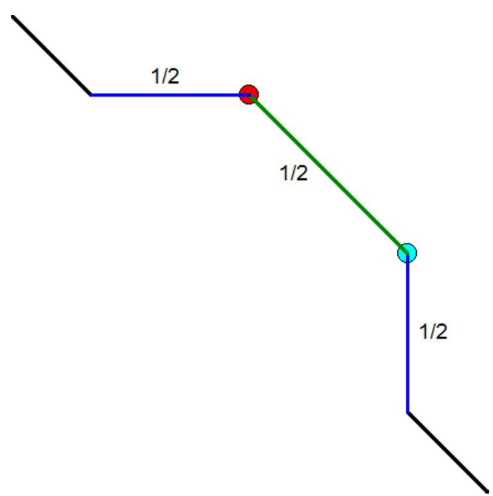
From the symmetry, the points with red circles has the same electric potential. The same things occur for the points with green circles and the points with blue circles.



Then we can fold this square lattice along the AB line (symmetric line).



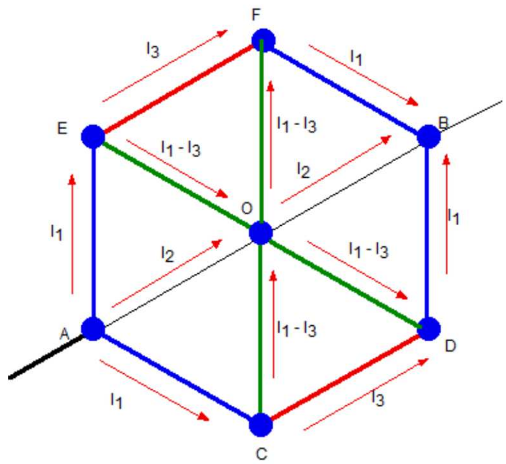
or



Then the total resistance between A and B is 3/2 Ω.

(d) Hexagon

Each branch has a resistance of 1 Ω. From the symmetry, we choose the currents of branches. We calculate the resistance between A and B.



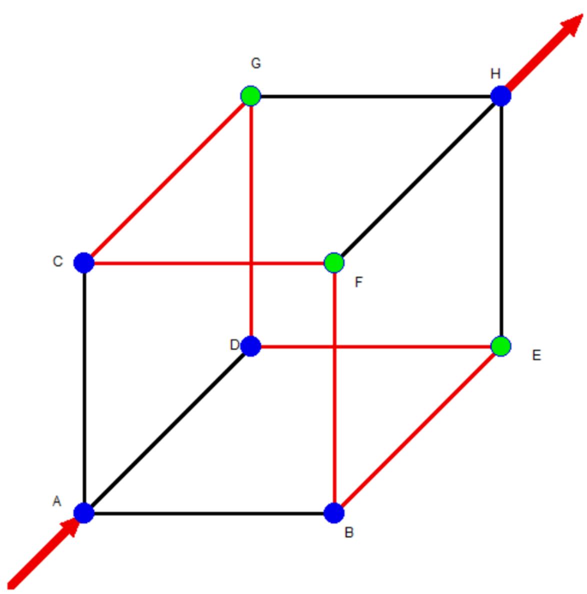
$$V_{tot} = V_{AB} = 2i_2 = 2i_1 + i_3$$
$$V_{AO} = i_2 = 2i_1 - i_3$$
$$I_{tot} = 2i_1 + i_2$$

We get

$$i_2 = 2i_3$$
$$i_1 = \frac{3}{2} i_3$$

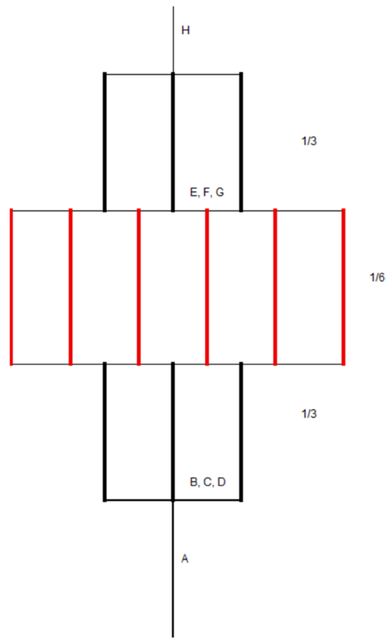
$$\boxed{R = \frac{V_{tot}}{I_{tot}} = \frac{4i_3}{5i_3} = \frac{4}{5}\Omega}$$

(e) Three dimensional circuits with high symmetry



We consider a cubic circuit where each side has a resistance of 1 Ω. What is the total resistance between the points A and B?

We notice that the points B, C, and D are at the same electric potential. The points E, F and G are at the same potential. So we can put together these points. The equivalent circuit is given by



Then the total resistance between A and H is

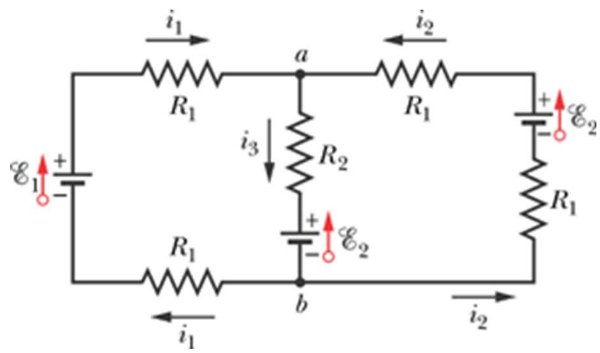
$$R_{tot} = \frac{2}{3} + \frac{1}{6} = \frac{5}{6}\Omega$$

5. Examples of circuits from textbook

((**Example-1**))

Figure 27-12 shows a circuit whose elements have the following values:

$$\mathcal{E}_1 = 3.0\text{V}, \quad \mathcal{E}_2 = 6.0\text{V},$$
$$R_1 = 2.0\ \Omega, \quad R_2 = 4.0\ \Omega\ .$$



The three batteries are ideal batteries. Find the magnitude and direction of the current in each of the three branches.

i_1 = -0.5 A, i_2 = 0.25 A.

((Mathematica))

```
eq1 = 2 i1 - 3 + 2 i1 + 4 (i1 + i2) + 6 == 0 // Simplify
```

```
3 + 8 i1 + 4 i2 == 0
```

```
eq2 = -6 + 2 i2 + 4 (i1 + i2) + 6 + 2 i2 == 0 // Simplify
```

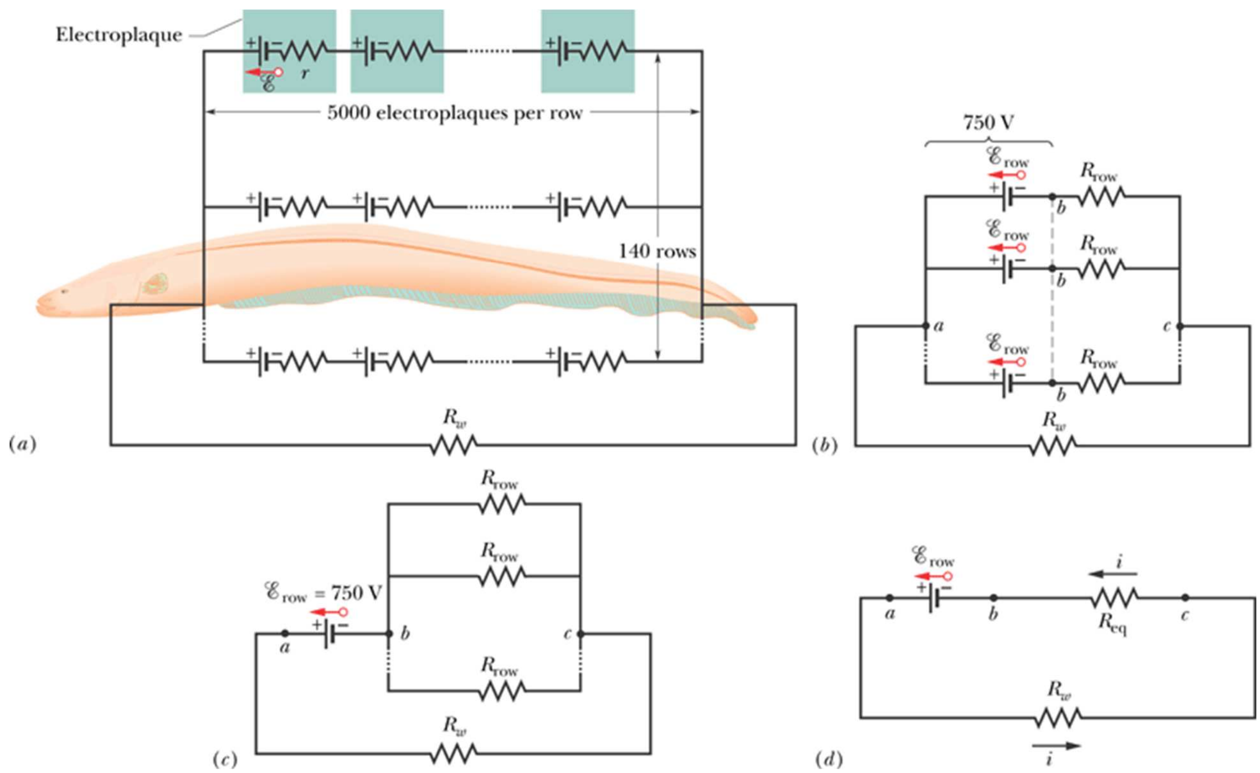
```
i1 + 2 i2 == 0
```

```
Solve[{eq1, eq2}, {i1, i2}]
```

$$\left\{\left\{i1 \to -\frac{1}{2},\ i2 \to \frac{1}{4}\right\}\right\}$$

((**Example-2**))

Electric fish is able to generate current with biological cells called electroplaques, which are physiological emf devices. The electroplaques in the type of electric fish known as a South American eel are arranged in 140 rows, each row stretching horizontally along the body and each containing 5000 electroplaques. The arrangement is suggested in Fig. 27-13 a; each electroplaque has an emf of 0.15 V and an internal resistance *r* of 0.25 Ω.



We now consider the circuit shown in Fig.b. From the symmetry of the circuit, the point b are at the same electric potential. Then we can put together these points b, leading to much more simple circuit shown in Fig.c.

$$-\varepsilon_{row} + R_w i + R_{eq} i = 0$$

$$i = \frac{\varepsilon_{row}}{R_w + R_{eq}} = \frac{750}{808.92} = 0.93A$$

where

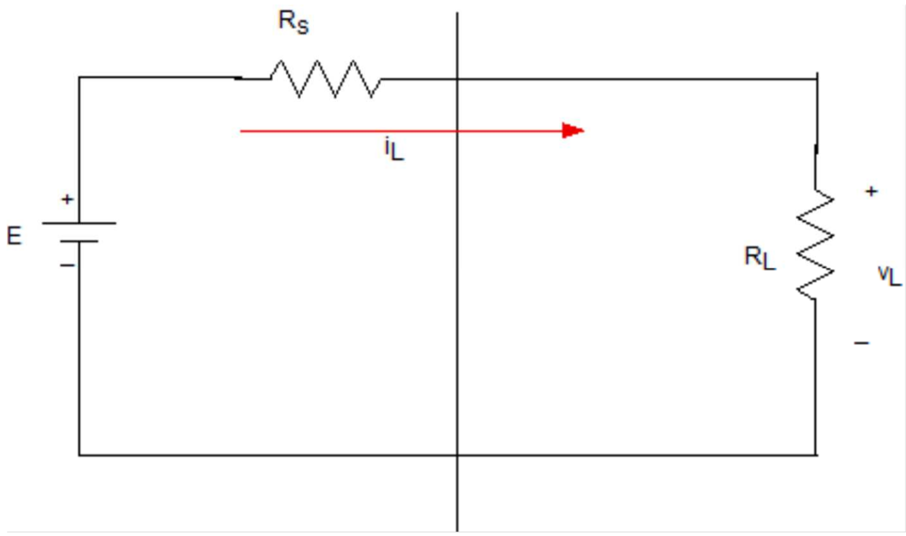
$$\varepsilon_{row} = 750V$$

$$R_{eq} = \frac{5000 \times 0.25}{140} = 8.92\Omega$$

$$R_w = 800\Omega$$

6. Maximum power transfer (impedance matching)

What value of R_L will result in the maximum amount of power absorbed by the load resistance R_L? Here R_s is the internal resistance of the battery.



By Ohm's law,

$$v_s = (R_s + R_L)i_L$$

$$i_L = \frac{v_s}{R_s + R_L}$$

The power dissipated at the load R_L is given by

$$P = R_L i_L{}^2 = \frac{R_L v_s{}^2}{(R_s + R_L)^2} = \frac{v_s{}^2}{R_s}\frac{x}{(x+1)^2}$$

with $x = \frac{R_L}{R_s}$.

P has a maximum at $x = 1$. The maximum value of P is $P_{\max} = \frac{v_s{}^2}{4R_s}$.

((**Impedance matching**))

Given a practical voltage source with an internal resistance R_s, the maximum power that can be delivered to a resistance load R_L is obtained when $R_L = R_s$ and the power is

$$P_{\max} = \frac{v_s{}^2}{4R_s}.$$

((**Mathematica**))

```
f1 = x/(1 + x)^2 ; D[f1, x] // Simplify
```

$$\frac{1-x}{(1+x)^3}$$

```
Plot[f1, {x, 0, 10}, PlotStyle → {Thick, Red}, Background → LightGray,
 AxesLabel → {"x=RL/Rs", "P/(vs^2/Rs)"}]
```

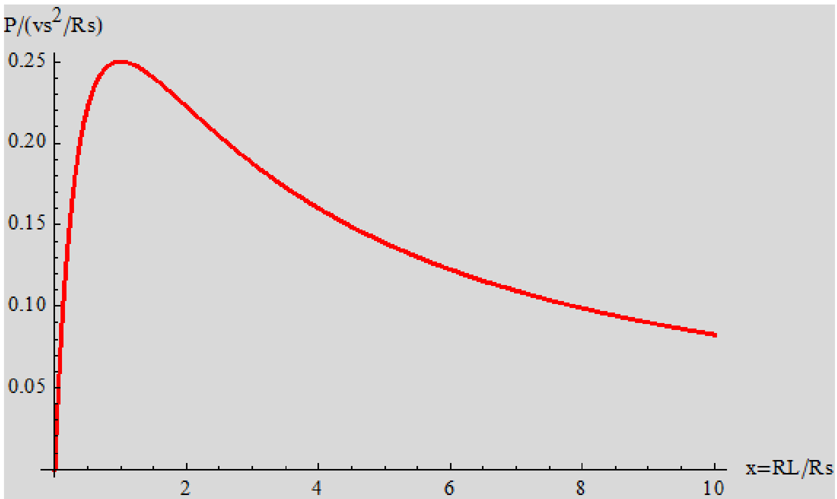
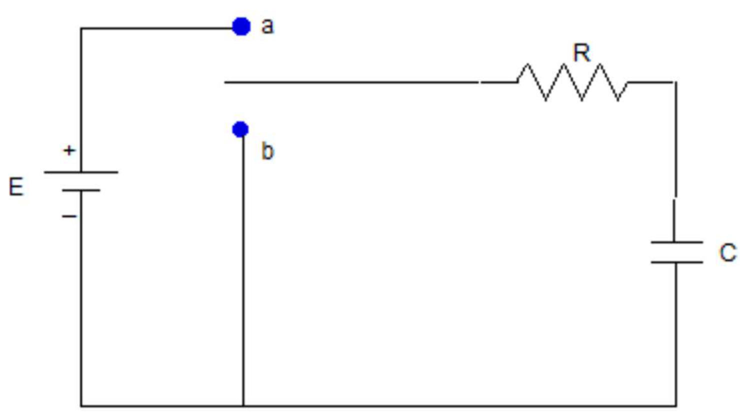


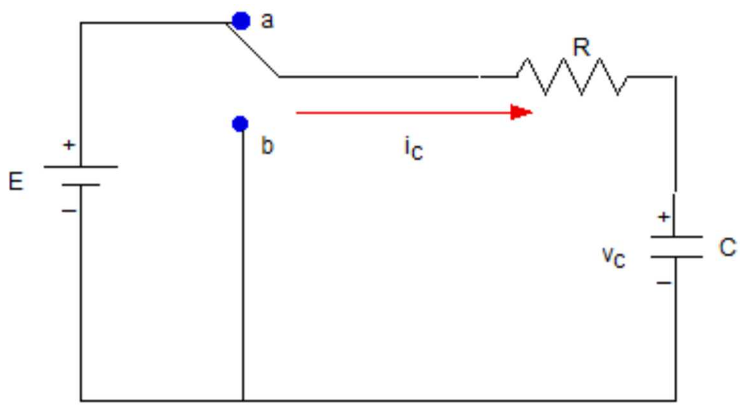
Fig. $P/(v_s^2/R_s)$ vs $x = R_L/R$s. This has a maximum at $x = 1$.

7. RC circuit

(a) Charging



In the above circuit, the switch is connected to the terminal a at $t = 0$,



$$-\varepsilon + Ri_C + v_C = 0$$

$$i_C = C\frac{dv_C}{dt}$$

Then we have a first-order differential equation,

$$RC\frac{dv_C}{dt} + v_C = \varepsilon$$

with an initial condition, $v_C(0) = 0$ and $i_C(0) = 0$.

The solution of this differential equation is

$$v_C(t) = \varepsilon(1 - e^{-\frac{t}{RC}})$$

The current i_C is obtained as

$$i_C(t) = \frac{\varepsilon}{R}e^{-\frac{t}{RC}}$$

Note that $\tau = RC$ is a time constant. $v_C(t)$ is continuous at $t = 0$, while iC(t) is discontinuous at $t = 0$. In this sense, v_C(t) is a variable suitable to the solution of the differential equation.

((**Mathematica**))

```
i1 = If[x < 0, 0, 0] + If[x > 0, Exp[-x], 0];
v1 = If[x < 0, 0, 0] +
   If[x > 0, 1 - Exp[-x], 0];
Plot[{v1, i1}, {x, -1, 5},
 PlotStyle → {{Thick, Red},
    {Thick, Green}},
 Background → LightGray,
 AxesLabel → {"t/RC", "vC/ϵ, iC/(ϵ/R)"}]
```

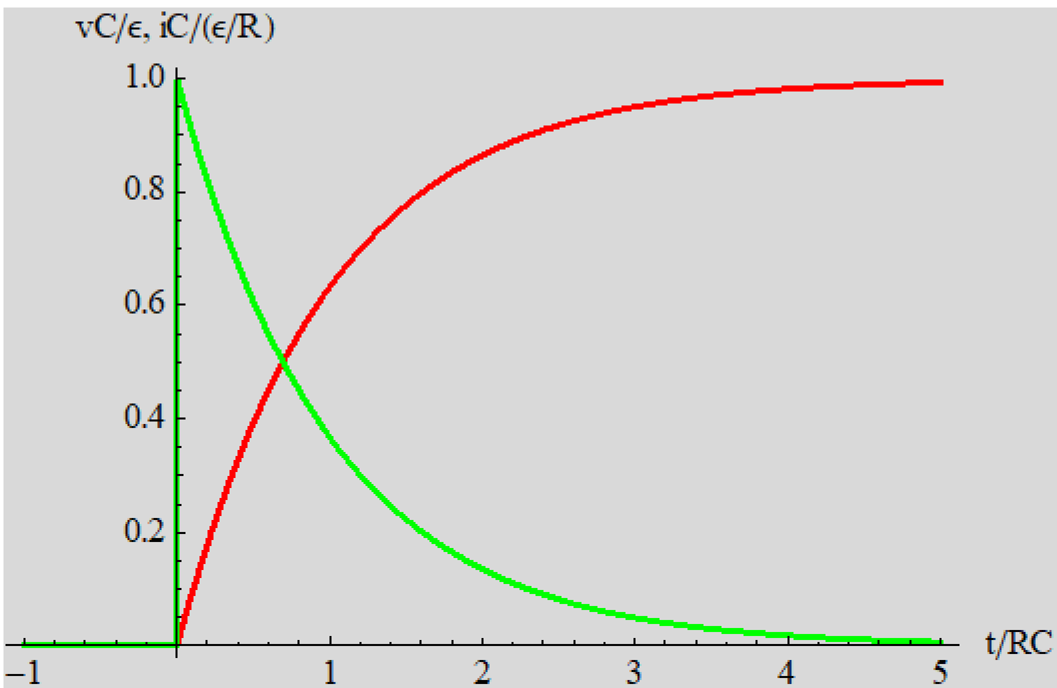
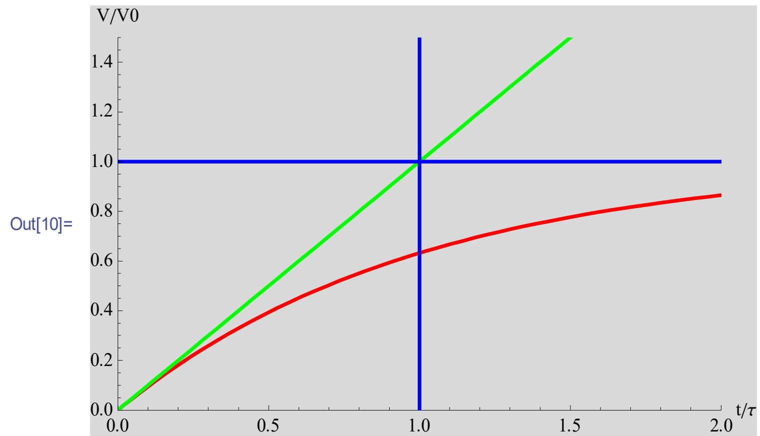
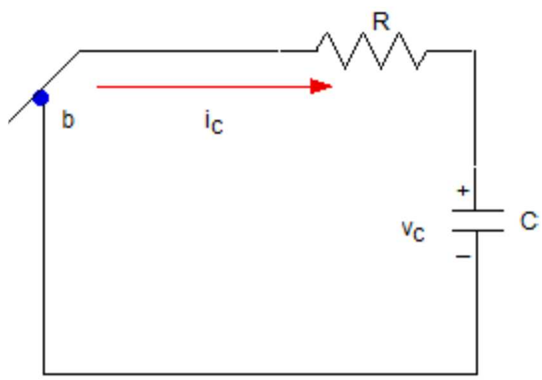


Fig. time dependence of v_C/ε (red, continuous at $t = 0$) and $i_C/(\varepsilon/R)$ (green, discontinuous at $t = 0$).

((**Note**)) How to determine the relaxation time τ. The tangential line at $t = 0$, passes at ($t = \tau$, $V = V_0$).



(b) Discharging



$$Ri_C + v_C = 0$$

$$i_C = C\frac{dv_C}{dt}$$

Then we have a first-order differential equation,

$$RC\frac{dv_C}{dt} + v_C = 0$$

with an initial condition, $v_C(0) = \varepsilon$ and $i_C(0) = 0$. The solution of this differential equation is

$$v_C(t) = \varepsilon e^{-\frac{t}{RC}}$$

The current i_C is obtained as

$$i_C(t) = -\frac{\varepsilon}{R} e^{-\frac{t}{RC}}$$

((**Mathematica**))

```
i1 = If[x < 0, 0, 0] + If[x > 0, -Exp[-x], 0];
v1 = If[x < 0, 1, 0] + If[x > 0, Exp[-x], 0];
Plot[{v1, i1}, {x, -1, 5},
 PlotStyle → {{Thick, Red},
   {Thick, Green}},
 Background → LightGray,
 AxesLabel → {"t/RC", "vC/ϵ, iC/(ϵ/R)"}]
```

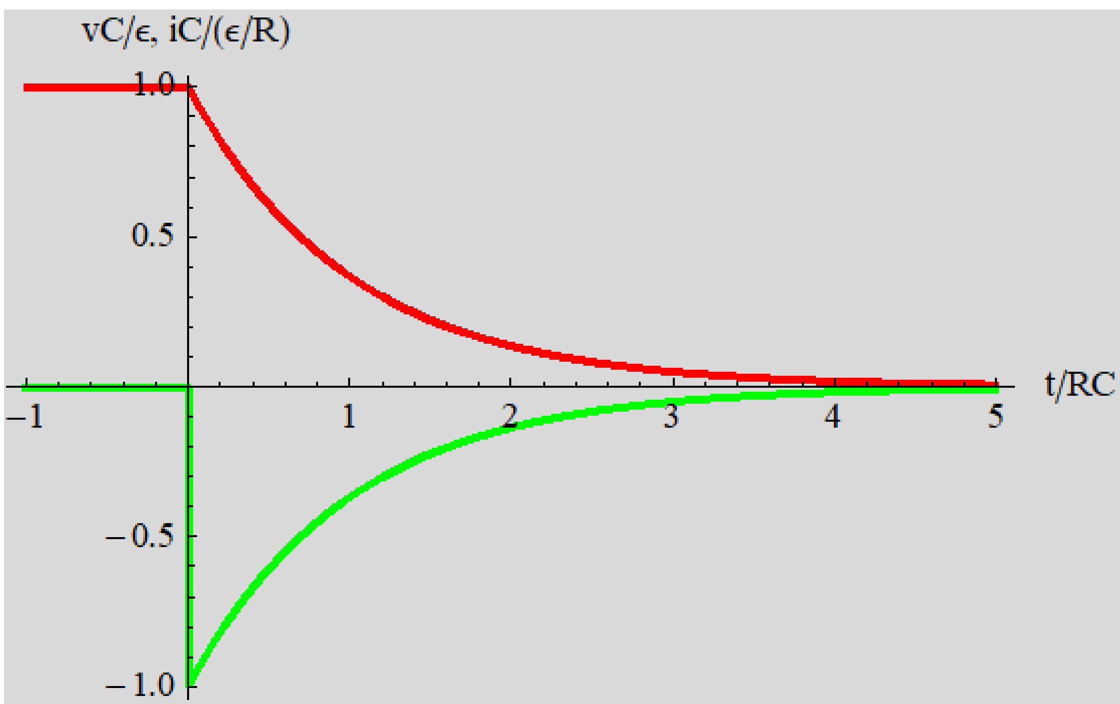


Fig. time dependence of v_C/ε (red, continuous at $t = 0$) and $i_C/(\varepsilon/R)$ (green, discontinuous at $t = 0$).

8. Circuit including the capacitor C: technique how to solve the problem

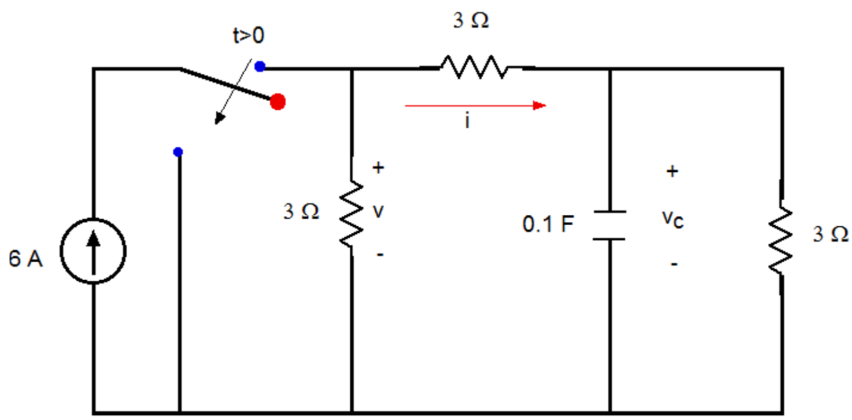
We now consider the difference between the voltage v_C across a capacitor at a time $t = t_0$ and that at a time $t = t_0+\varepsilon$,

$$v_C(t_0+\varepsilon) - v_C(t_0) = \frac{1}{C}\int_{t_0}^{t_0+\varepsilon} i_C(t)dt$$

where $i_C(t)$ is the current flowing through the capacitor. This integral gets arbitrarily small if ε gets arbitrarily small. This means that $v_C(t)$ is continuous at any time t_0.

In conclusion, if we choose i_L and v_C as the variables for the analysis of the circuits, we can determined smoothly the time dependence of i_L and v_C from their initial conditions.

((Example-1)) RC circuits



For t<0, there is no current across the capacitance.

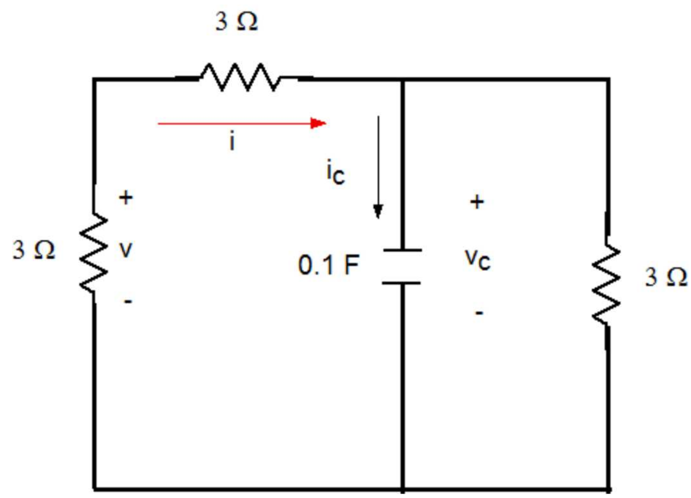
$$6 = \frac{v}{3} + \frac{v}{6} = \frac{v}{2} \qquad \text{or} \qquad v(0) = 12V$$

$$v = 6i \qquad \text{or} \qquad i(0) = 2\text{A}.$$

$$v_C = 3i = 6V \qquad \text{or} \qquad v_C(0) = 6\text{V}$$

$$i_C(0) = 0$$

For $t \geq 0$



$$-i + i_C + \frac{1}{3} v_C = 0$$

$$-i = \frac{v_C}{6}$$

$$i_C = \frac{1}{10} \frac{dv_C}{dt}$$

$$v = -3i = \frac{1}{2} v_C$$

Then we have a first order differential equation,

$$\frac{dv_C}{dt} + 5v_C = 0\,, \qquad \text{or} \qquad v_C(t) = v_C(0)e^{-5t} = 6e^{-5t}$$

$$i_C(t) = \frac{1}{10} \frac{dv_C}{dt} = -3e^{-5t}$$

$$v(t) = 3e^{-5t}$$

((**Mathematica**))

```
VC = If[t < 0, 6, 0] + If[t > 0, 6 e^(-5 t), 0];
IC = If[t < 0, 0, 0] + If[t > 0, -3 e^(-5 t), 0];
V = If[t < 0, 12, 0] + If[t > 0, 3 e^(-5 t), 0];
Plot[{VC, IC, V}, {t, -0.5, 1},
 PlotStyle → {{Thick, Red}, {Thick, Green}, {Thick, Blue}},
 AxesLabel → {"t(s)", "vC,iC,v"}]
```

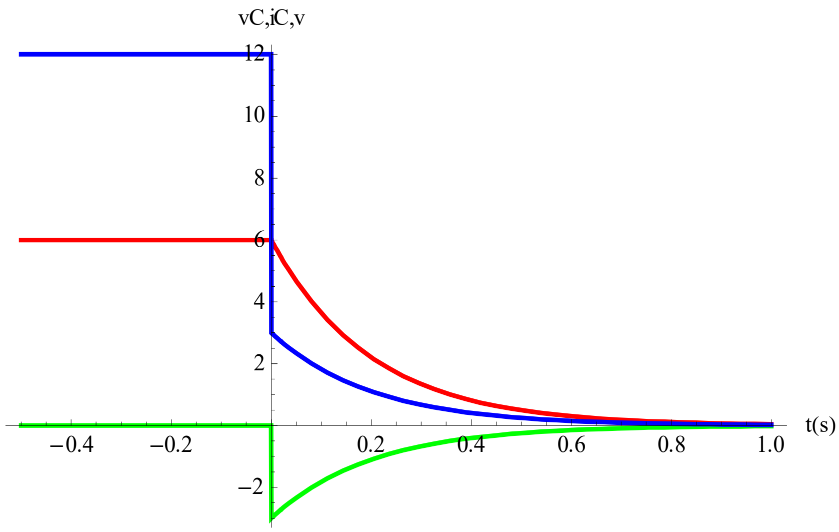
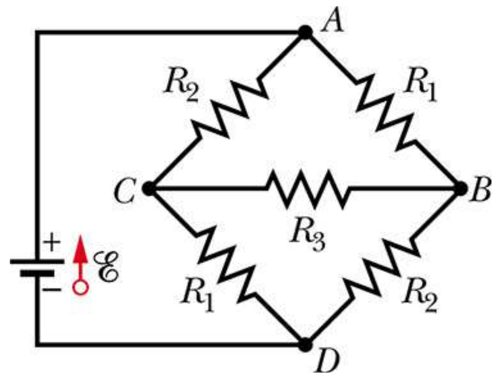


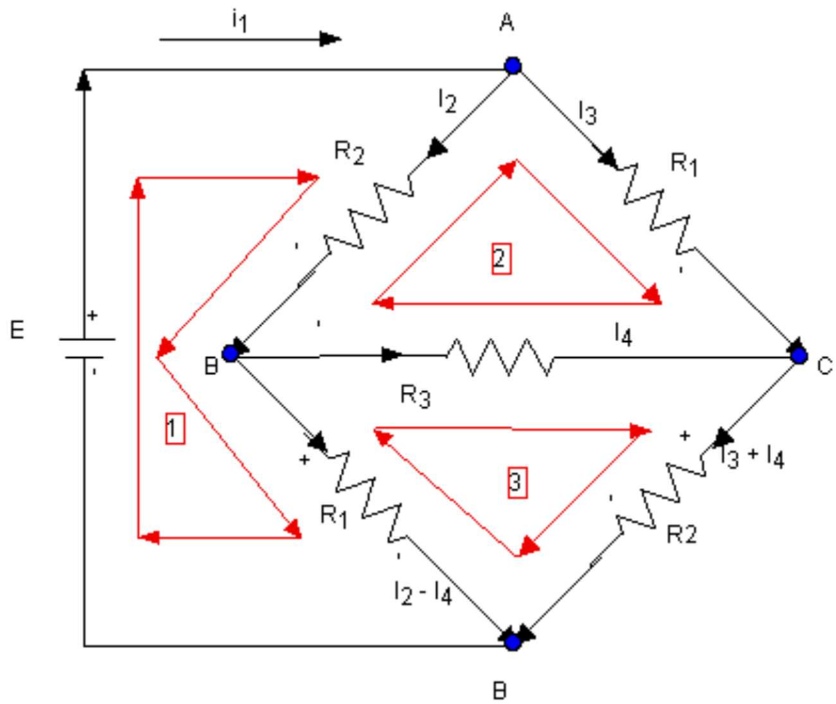
Fig time dependence of v_C (red), i_C (green), and v (blue). Only $v_C(t)$ ic continuous at t = 0.

10 Typical examples
10.1 Problem 27-39 (SP-27)

In Fig., ε = 12.0 V, R_1 = 2000 Ω, R_2 = 3000 Ω, and R_3 = 4000 Ω. What are the potential differences (a) $V_A - V_B$, (b) $V_B - V_C$, (c) $V_C - V_D$, and (d) $V_A - V_C$?



((Solution))



$\varepsilon = 12.0$ V
$R_1 = 2$ kΩ
$R_2 = 3$ kΩ
$R_3 = 4$ kΩ

$$i_1 = i_2 + i_3 .$$

Loop 1:

$$-\varepsilon + R_2 i_2 + R_1(i_2 - i_4) = 0$$

Loop 2:

$$R_1 i_3 - R_3 i_4 - R_2 i_2 = 0$$

Loop 3

$$R_3 i_4 + R_2(i_3 + i_4) - R_1(i_2 - i_4) = 0$$

We have 4 unknown parameters and 4 equations. So we can solve. See the Mathematica calculations for detail.

$i_1 = 4.875$ mA, $i_2 = 2.250$ mA

i_3 = 2.625 mA, i_4 = -0.375 mA.

(a) $v_A - v_B = R_1 i_3 = 5.25V$

(b) $v_B - v_C = -R_3 i_4 = 1.50V$

(c) $v_C - v_D = R_1(i_2 - i_4) = 5.25V$

(d) $v_A - v_C = R_2 i_2 = 6.75V$

((Note))

$$i_1 = \frac{(R_1 + R_2 + R_3)\varepsilon}{2R_1R_2 + R_1R_3 + R_2R_3} = 4.875mA$$
$$i_2 = \frac{(R_1 + R_3)\varepsilon}{2R_1R_2 + R_1R_3 + R_2R_3} = 2.250mA$$
$$i_3 = \frac{(R_2 + R_3)\varepsilon}{2R_1R_2 + R_1R_3 + R_2R_3} = 2.625mA$$
$$i_4 = \frac{(R_1 - R_2)\varepsilon}{2R_1R_2 + R_1R_3 + R_2R_3} = -0.375mA$$

((**Mathematica**))

```
eq1 = i1 == i2 + i3
i1 == i2 + i3

eq2 = -ϵ + R2 i2 + R1 (i2 - i4) == 0
(i2 - i4) R1 + i2 R2 - ϵ == 0

eq3 = R1 i3 - R3 i4 - R2 i2 == 0
i3 R1 - i2 R2 - i4 R3 == 0

eq4 = R3 i4 + R2 (i3 + i4) - R1 (i2 - i4) == 0
- (i2 - i4) R1 + (i3 + i4) R2 + i4 R3 == 0

sq1 = Solve[{eq1, eq2, eq3, eq4}, {i1, i2, i3, i4}] //
  Simplify
```

$$\left\{\left\{i1 \to \frac{(R1 + R2 + 2\,R3)\,\epsilon}{R2\,R3 + R1\,(2\,R2 + R3)},\ i3 \to \frac{(R2 + R3)\,\epsilon}{R2\,R3 + R1\,(2\,R2 + R3)},\ i2 \to \frac{(R1 + R3)\,\epsilon}{R2\,R3 + R1\,(2\,R2 + R3)},\ i4 \to \frac{(R1 - R2)\,\epsilon}{R2\,R3 + R1\,(2\,R2 + R3)}\right\}\right\}$$

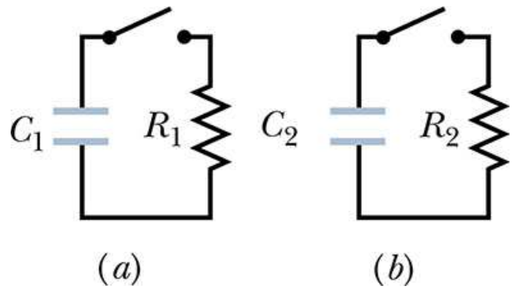
```
rule1 = {R1 → 2000, R2 → 3000, R3 → 4000, ϵ → 12.0}
{R1 → 2000, R2 → 3000, R3 → 4000, ϵ → 12.}

sq2 = sq1 /. rule1
{{i1 → 0.004875, i3 → 0.002625,
  i2 → 0.00225, i4 → -0.000375}}

vAB = R1 i3 /. sq1[[1]] /. rule1
5.25

vBC = -R3 i4 /. sq1[[1]] /. rule1
1.5

vCD = R1 (i2 - i4) /. sq1[[1]] /. rule1
5.25

vAC = R2 i2 /. sq1[[1]] /. rule1
6.75
```

11. Hint of HW-26

Problem 27-68

Figure displays two circuits with a charged capacitor that is to be discharged through a resistor whena switch is closed. In Fig.(a), $R_1 = 20.0\ \Omega$ and $C_1 = 5.00\ \mu F$. In Fig.(b), $R_2 =$

10.0 Ω and C_2 = 8.00 μF. the ratio of the initial charges on the two capacitors is q_{20}/q_{01} = 1.50. At time t = 0, both switches are closed. At what time t do the two capacitors have the same charge?



((**Hint**))
$R_1 = 20.0\ \Omega$
$C_1 = 5.00\ \mu\text{F}$
$R_2 = 10.0\ \Omega$
$C_2 = 8\ \mu\text{F}$
$q_{02}/q_{01} = 1.5$

$$v_1 + R_1 i_1 = 0$$
$$i_1 = C\frac{dv_1}{dt}$$

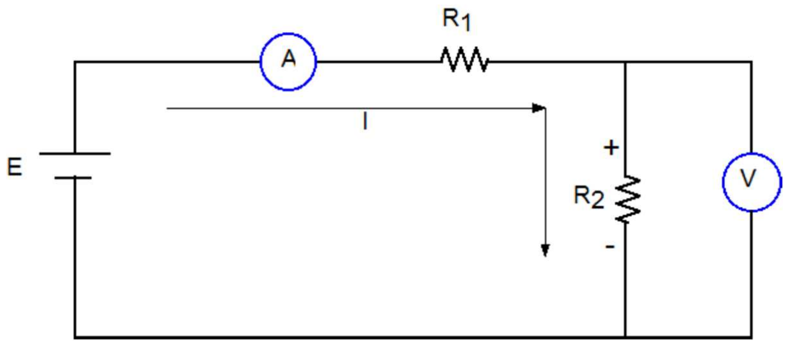
or

$$v_1 + R_1 C_1 \frac{dv_1}{dt} = 0$$
$$v_1(t) = v_1(0)e^{-t/(R_1C_1)}$$
$$Q_1(t) = C_1 v_1(t) = C_1 v_1(0)e^{-t/(R_1C_1)} = q_{01}e^{-t/(R_1C_1)}$$

APPENDIX

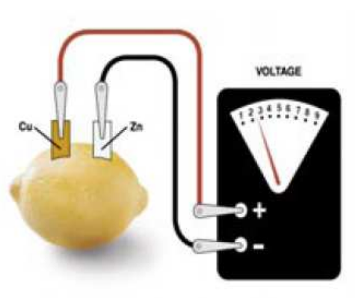
1. Ammeter and voltmeter

In carrying out experiments on electric circuits in the laboratory, it is frequently useful to have available a means for measuring the current flow through – and the potential drop across – a given circuit element. The two instruments that have been developed for these purposes are the ammeter and the voltmeter. An ammeter is a device used to measure current. Normally it is connected in series with other circuit elements. In ideal case, there is no internal resistance. In other words, there is no potential drop across the ammeter. A voltmeter measures the potential difference between any two points in a circuit. In ideal case, the internal resistance is infinity. No current flows in the voltmeter.



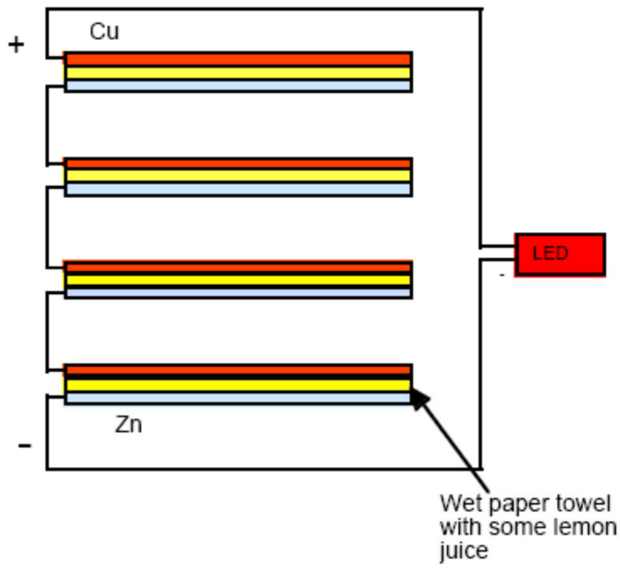
In this circuit, the ammeter (denoted by A in the circle) measures the current flowing in the resistor R_1, while the voltmeter (denoted by V in the circle) measures the voltage across the resistor R_2.

2. Fruit battery; how to get an ideal battery



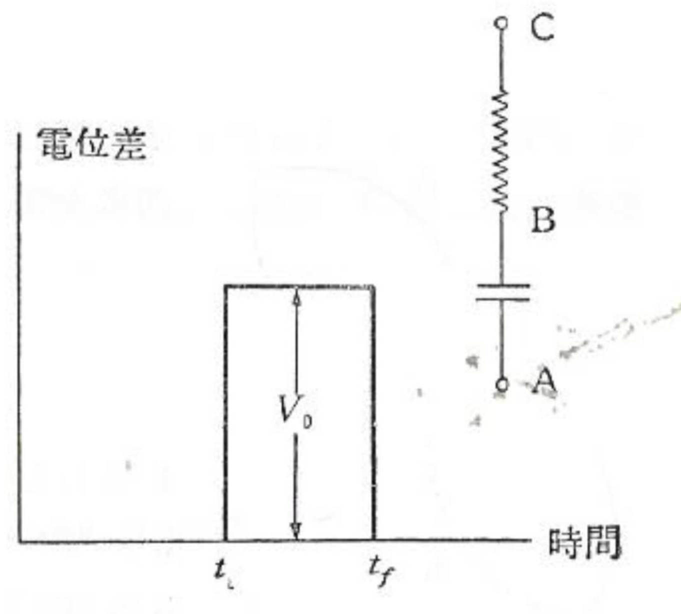
I had a good opportunity to do some experiment of fruit battery at my home. Through the experiments, I learned how to get a high quality fruit battery which has a high voltage and a small internal resistance. Note that an ideal battery has no internal resistance.

When copper and zinc electrodes were installed in fruit (lemon), a voltage is generated between two electrodes. The voltage was about 0.95 V. I found a relatively large internal resistance between two electrodes. When the distance between two electrodes was less than 2 mm, the internal resistance drastically decreased. Based on this, my fruit battery was greatly improved. I found one of the best way to get a battery with relatively high voltage and small internal resistance. The copper and zinc electrodes are separated by a wet paper towel with a small amount of lemon drops and salt (NaCl), so that the distance between two electrodes is less than 2 mm. When four of these batteries are connected in series, I get a battery with the voltage 3.8 V and a small internal resistance. When a LED is connected, a bright light was on.



3. Exercises

I got the following problem from Japanese book on Problems and Solutions in Electriciy and Magnetism, which I used when I was a undergraduate student.



The circuit consists of resistance R and capacitance C in series. The following voltage is applied between A and C. Find the time dependence of the voltage between B and A.

$$v_s = v_R + v_C = Ri + v_C = V_0[u_{-1}(t - t_i) - u_{-1}(t - t_f)]$$

$$v_C = \frac{1}{C}\int i dt, \qquad i = C\frac{dv_C}{dt}$$

Thus we have

$$v_s = CR\frac{dv_C}{dt} + v_C$$

For $0 < t < t_i$, $\qquad v_C = 0$

The initial condition at $t = t_i$: $v_C = 0$. For $t_i < t < t_f$,

$$CR\frac{dv_C}{dt} + v_C = V_0$$

The solution of the first order differential equation

$$v_C(t) = V_0[1 - e^{-(t-t_i)/(RC)}]$$

At $t = t_f$

$$v_C(t_f) = V_0[1 - e^{-(t_f-t_i)/(RC)}]$$

For $t > t_f$

$$CR\frac{dv_C}{dt} + v_C = 0$$

The solution is

$$\begin{aligned} v_c(t) &= v_C(t_f)e^{-(t-t_f)/(RC)} \\ &= V_0[1 - e^{-(t_f-t_i)/(RC)}]e^{-(t-t_f)/(RC)} \\ &= V_0[e^{-(t-t_f)/(RC)} - e^{-(t-t_i)/(RC)}] \\ &= V_0[e^{t_f/(RC)} - e^{t_i/(RC)}]e^{-t/(RC)} \end{aligned}$$

The charge Q_0 is stored in the capacitor *C*, When the resistance *R* is connected to both sides of capacitor. Find the time dependence of voltage and charge of the capacitor.

The charge q

$$v_C + iR = 0. \qquad\qquad v_C + RC\frac{dv_C}{dt} = 0$$

with the initial condition

$$v_C = \frac{Q_0}{C}$$

where

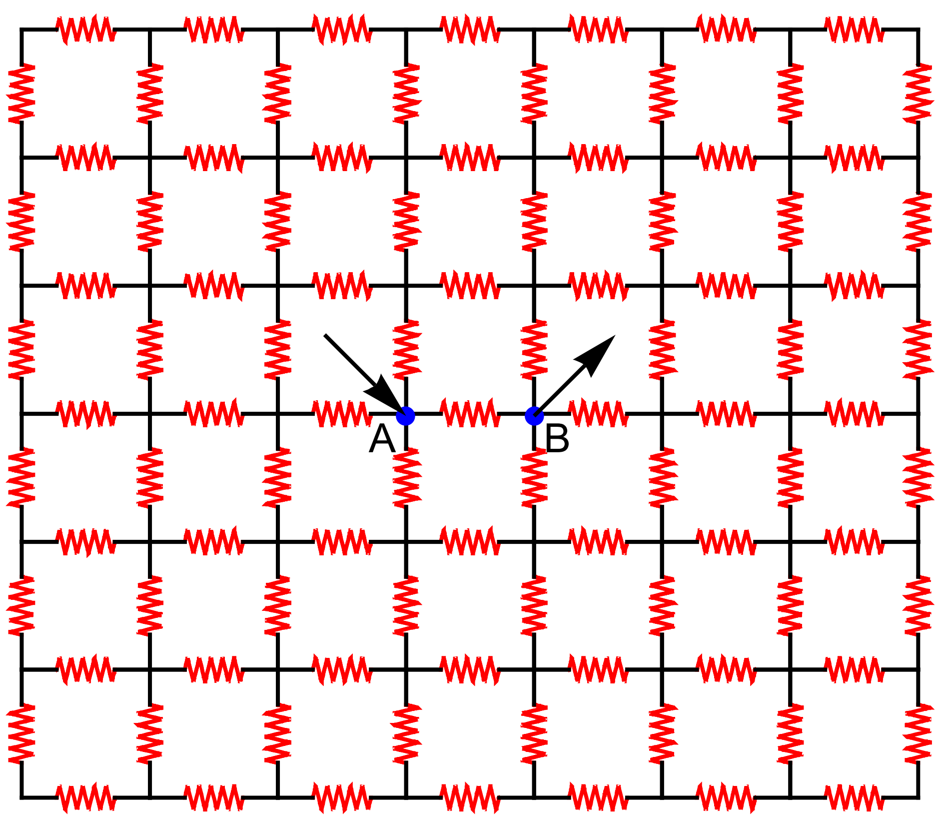
$$i = \frac{dq}{dt} = C\frac{dv}{dt}$$

The solution of the first order differential equation

$$v_C = \frac{Q_0}{C}e^{-t/(RC)}$$

4. Infinite resistor network (square)

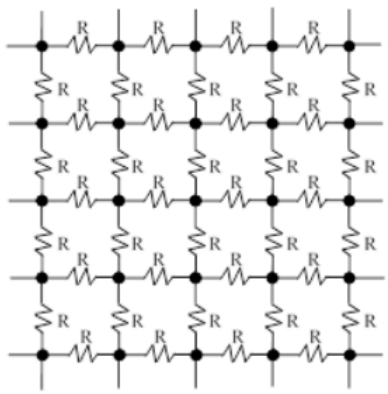
https://www.mathpages.com/home/kmath668/kmath668.htm



Infinite Grid of Resistors

Remain, remain thou here,
While sense can keep it on. And, sweetest, fairest,
As I my poor self did exchange for you,
To your so infinite loss, so in our trifles
I still win of you: for my sake wear this...
Shakespeare

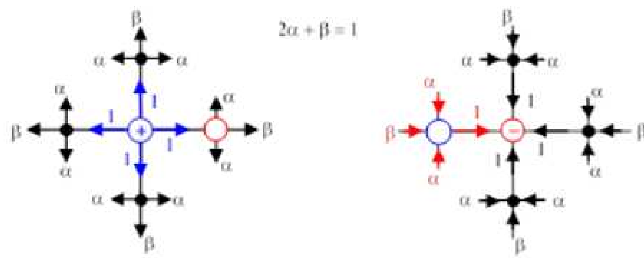
There is a well-known puzzle based on the premise of an "infinite" grid of resistors connecting adjacent nodes of a square lattice. A small portion of such a grid is illustrated below.



Between every pair of adjacent nodes is a resistance R, and we're told that this grid of resistors extends "to infinity" in all direction, and we're asked to determine the effective resistance between two adjacent nodes, or, more generally, between any two specified nodes of the lattice.

For adjacent nodes, the usual solution of this puzzle is to consider the current flow field as the sum of two components, one being the flow field of a grid with current injected into a single node, and the other being the flow field of a grid with current extracted from a single (adjacent) node. The symmetry of the two individual cases then enables us to infer the flow rates through the immediately adjacent resistors, and hence we can conclude (as explained in more detail below) that the effective resistance between two adjacent nodes is R/2. This solution has a certain intuitive plausibility, since it's similar to how the potential field of an electric dipole can be expressed as the sum of the fields of a positive and a negative charge, each of which is spherically symmetrical about its respective charge. Just as the electric potential satisfies the Laplace equation, the voltages of the grid nodes satisfy the discrete from of the Laplace equation, which is to say, the voltage at each node is the average of the voltages of the four surrounding nodes. It's also easy to see that solutions are additive, in the sense that the sum of any two solutions for given boundary conditions is a solution for the sum of the boundary conditions.

If we accept the premise of an infinite grid of resistors, along with some tacit assumption about the behavior of the voltages and currents "at infinity", and if we accept the idea that we can treat the current fields for the positive and negative nodes separately, and that applying a voltage to a single node of the infinite grid will result in some current flow into the grid, the puzzle is easily solved by simple symmetry considerations. We assert (somewhat naively) that if we inject (say) four Amperes of current into a given node, with no removal of current at any finite point of the grid, the current will flow equally out through the four resistors, so one Ampere will flow toward each of the four adjacent nodes. This one Ampere must flow out through the three other lines emanating from that adjacent node, as indicated in the left hand figure below.



The figure on the right shows the four nodes surrounding the "negative" node, assuming we are extracting four Amperes from that node (with no current injected at any finite node of the grid). Again, simple symmetry dictates the distribution of currents indicated in the figure. Adding the two current fields together, we see that the link between the positive and negative nodes carries a total of 2 Amperes away from the positive node, and the other three links emanating from the positive node carry away a combined total of $1 + 1 + 1 - (2\alpha + \beta) = 2$ Amperes. Thus the direct link carries the same current as all the other paths, so the resistance of the direct link equals the effective resistance of the entire grid excluding that link. The direct link is in parallel with the remainder of the grid, so the combined resistance is simply R/2.