Chapter 29
Magnetic fields due to currents

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Here we discuss the Ampere's law and Biot-Savart law for the magnetic field arising from electric currents. These laws can be derived from the vector potential $\boldsymbol{A}$, which is related to the magnetic field $\boldsymbol{B}$ as $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. For the gauge transformation such that $\boldsymbol{A}' = \boldsymbol{A} + \nabla \chi$, one can get the same $\boldsymbol{B}$, where χ is arbitrary scalar function. So A cannot be determined uniquely. In the Aharonov-Bohm effect (quantum mechanics), we realize that the vector potential $\boldsymbol{A}$ is more essential to the magnetism compared with magnetic field $\boldsymbol{B}$.

The Aharonov–Bohm effect is important conceptually because it bears on three issues apparent in the recasting of (Maxwell's) classical electromagnetic theory as a gauge theory, which before the advent of quantum mechanics could be argued to be a mathematical reformulation with no physical consequences. The Aharonov–Bohm thought experiments and their experimental realization imply that the issues were not just philosophical.

1. Biot Savart law

The magnetic field B is given by the integral of the contribution d$\boldsymbol{B}$ to from a length d$\boldsymbol{s}$ of a current I,

$$\boldsymbol{B}(P) = \frac{\mu_0 I}{4\pi} \int \frac{d\boldsymbol{s} \times \hat{r}}{r^2} = \frac{\mu_0 I}{4\pi} \int \frac{d\boldsymbol{s} \times \boldsymbol{r}}{r^3}$$

where $\boldsymbol{r}$ is the displacement vector from the current element to the point P. μ_0 is a permeability of free space.

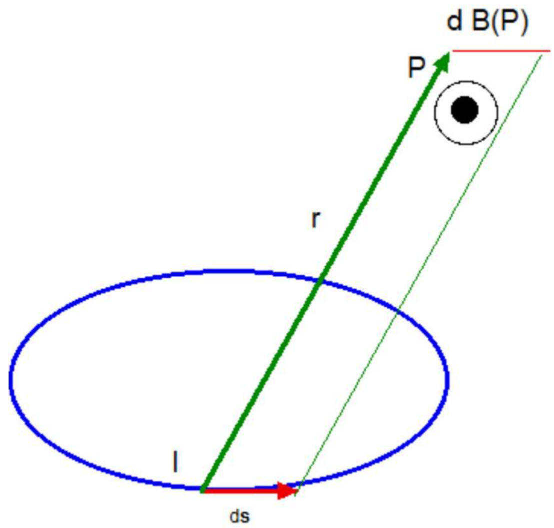
$$\mu_0 = 4\pi \times 10^{-7} N/A^2 = 4\pi \times 10^{-7} Tm/A$$

1T=1N/(Am)=10^4 Gauss

((**Note**))
ε_0 is the permittivity constant.

$$\varepsilon_0 = 8.854187817\text{x}10^{-12}\ \text{C}^2/(\text{N m}^2).$$

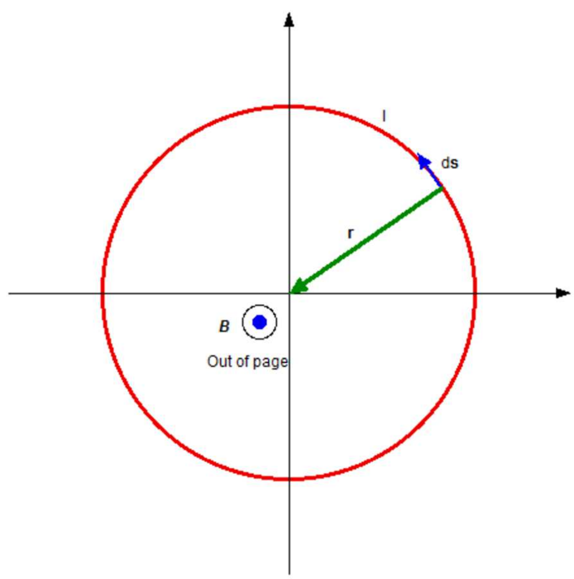
The derivation of the formula of Biot Savart law will be derived later using the vector potential.



The contribution d**B** is in the direction $\boldsymbol{ds} \times \boldsymbol{dr}$, by the right-hand rule, this is perpendicularly out of the plane of paper for the point P.

2. Magnetic field at the center point from a circular loop current

We apply the Biot-Savart law to find the magnetic field at the center point P of a circular loop of radius *r*. The direction of *B* is perpendicular to the plane containing the loop and its center.

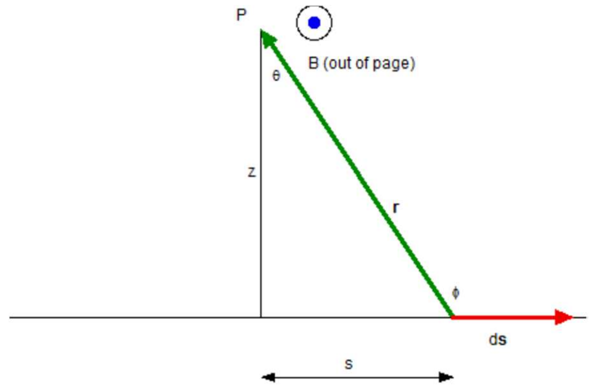


$$\boldsymbol{B}(P) = \frac{\mu_0 I}{4\pi}\int \frac{ds}{r^2} = \frac{\mu_0 I}{4\pi r^2} 2\pi r = \frac{\mu_0 I}{2r}$$

((**Right-hand rule**))

Grasp the wire in the right hand with the thumb pointing along the direction of the current. The fingers will then circle the wire in the sense of the direction of the ***B*** field.

3. Magnetic field from a finite straight current wire



We apply the Biot-Savart's law to a finite length of straight current wire to find the magnitude at the point P (see Fig.). Note that d***s*** x ***r*** points out of the page. The magnitude of d***s*** x ***r*** is given by

$$rds\sin\phi = rds\sin(\theta + \frac{\pi}{2}) = rds\cos\theta$$

There are geometrical relations given by

$$r\cos\theta = z, \quad \text{or} \quad r = \frac{z}{\cos\theta}$$

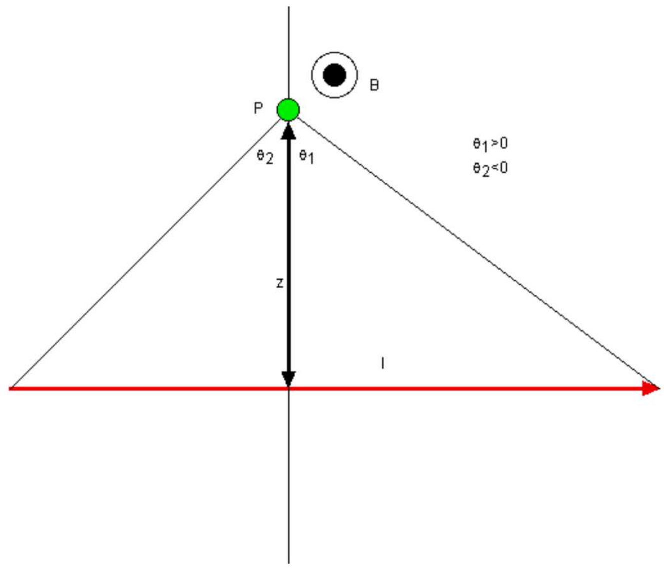
$$s = z\tan\theta$$
$$ds = z\sec^2\theta d\theta$$

The magnetic field at the point P is

$$\begin{aligned} B &= \frac{\mu_0 I}{4\pi}\int_{\theta_2}^{\theta_1}\frac{dsr\cos\theta}{r^3} \\ &= \frac{\mu_0 I}{4\pi}\int_{\theta_2}^{\theta_1} d\theta\frac{z\sec^2\theta\cos^3\theta}{z^2} \\ &= \frac{\mu_0 I}{4\pi z}\int_{\theta_2}^{\theta_1} d\theta\cos\theta \\ &= \frac{\mu_0 I}{4\pi z}[\sin\theta]_{\theta_2}^{\theta_1} \\ &= \frac{\mu_0 I}{4\pi z}(\sin\theta_1 - \sin\theta_2) \end{aligned}$$

or

$$B = \frac{\mu_0 I}{4\pi z}(\sin\theta_1 - \sin\theta_2)$$



Note: in this figure, θ_1> and θ_2<0.

When $\theta_1 = \pi/2$ and $\theta_2 = -\pi/2$ in the above equation, we have

$$B = \frac{\mu_0 I}{2\pi z}$$

This formula can be derived from the Ampere's law (later). In other words, the Ampere's law can be derived from the Biot-Savart law.

4. Magnetic field at the center of the square loop current

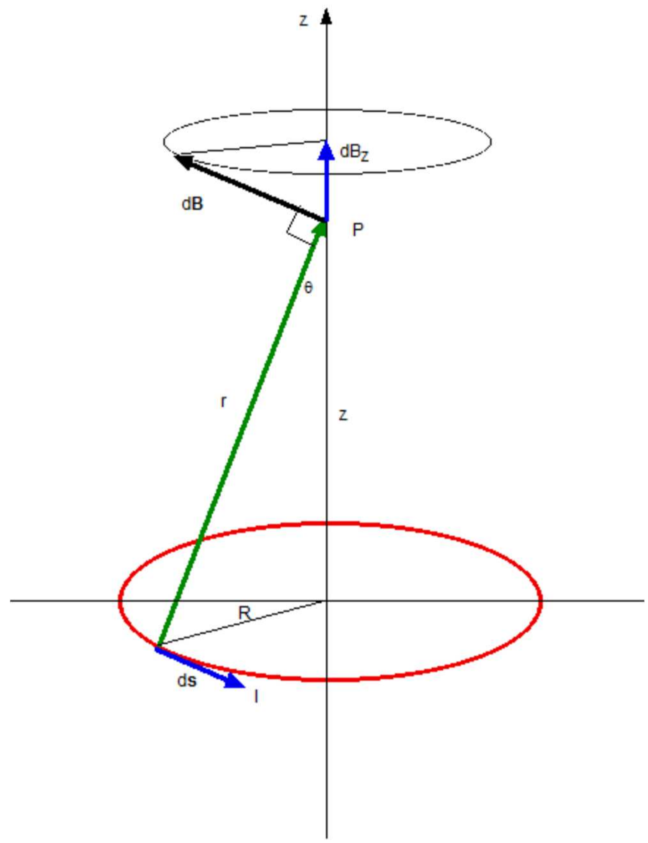
The magnitude and direction of magnetic field due to a square current (side a).

When $\theta_1 = \pi/4$, $\theta_2 = -\pi/4$, and $z = a/2$, we have

$$B = 4\frac{\mu_0 I}{4\pi(a/2)}(\sin\theta_2 - \sin\theta_1) = 4\frac{\mu_0 I}{2\pi a}\sqrt{2} = \frac{\mu_0 2\sqrt{2} I}{\pi a}$$

5. Magnetic field at a point on the axis of a circular current loop of radius *R*.

We consider a point P on the axis of a circular current loop.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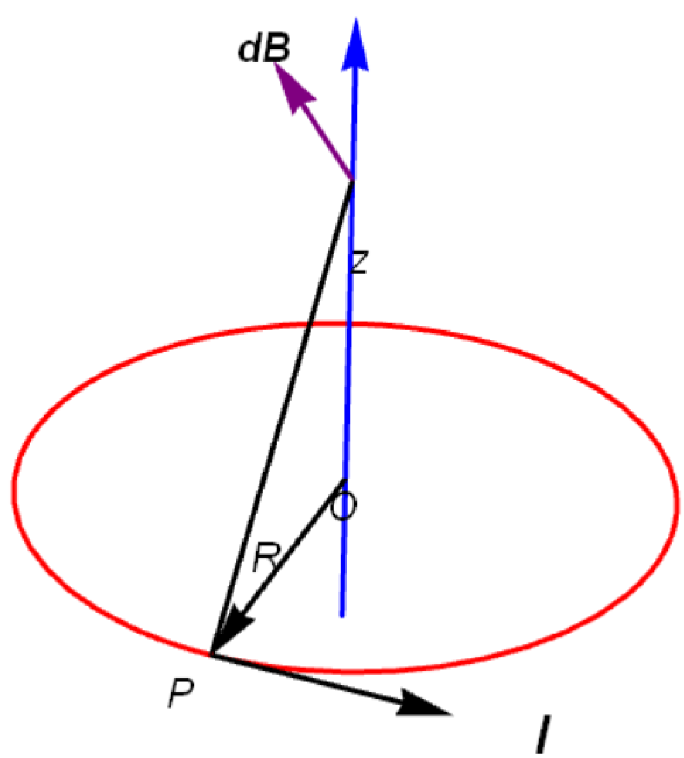
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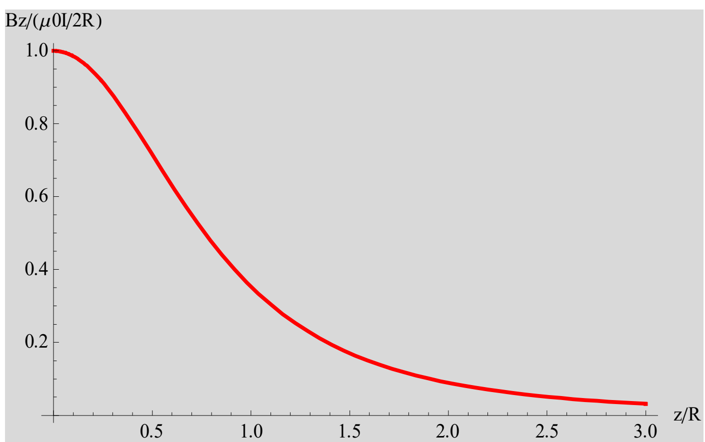
By the symmetry, only the z component of the magnetic field due to a segment d**s** of the loop contributes to the net magnetic field.

$$B_z(P) = \frac{\mu_0 I}{4\pi}\int \frac{rds}{r^3}\cos(\frac{\pi}{2}-\theta) = \frac{\mu_0 I}{4\pi}\frac{2\pi R}{r^2}\sin\theta$$
$$= \frac{\mu_0 I}{4\pi}\frac{2\pi R}{r^2}\frac{R}{r} = \frac{\mu_0 I}{2}\frac{R^2}{r^3}$$
$$= \frac{\mu_0 I}{2}\frac{R^2}{(z^2+R^2)^{3/2}}$$

where

$$\sin\theta = \frac{R}{r} = \frac{R}{\sqrt{z^2+R^2}}$$

Out[5]=



((**Note**))
The point z/R = ½ is the point of inflection at which the curvature changes, $\partial^2 B_z/\partial z^2 = 0$ at z/R = ½.

In the limit of z>>R,

$$B_z \approx \frac{\mu_0 I}{2}\frac{R^2}{z^3}$$

or

$$B_z \approx \frac{\mu_0 I\pi R^2}{2\pi z^3} = \frac{\mu_0 m}{2\pi z^3}$$

where m is the magnetic moment, $m = I(\pi R^2)$.

((**Note**))

From the Bio-Savart law, the magnetic field at the origin ($x = y = z = 0$) arising from the circulating current I around the z axis is

$$B_z = \frac{\mu_0 I}{2R} = \frac{\mu_0 IA}{2RA} = \frac{\mu_0 \mu}{2\pi R^3}$$

where $A = \pi r^2$ and $\mu = IA$ (magnetic moment).

6. The magnetic field at a focal point of the parabola-shaped current path

We consider a magnetic field at the focal point (a, 0), when the current I flows on the parabola ($y^2 = 4ax$). From the Biot-Savart law, the magnetic field at the focal point is along the out-of-page direction.

$$B_z = \frac{\mu_0 I}{4\pi}\int \frac{rds\sin\phi}{r^3} = \frac{\mu_0 I}{4\pi}\int \frac{d\theta}{r} = \frac{\mu_0 I}{4\pi}\frac{1}{2a}\int_0^{2\pi}(1-\cos\theta)d\theta = \frac{\mu_0 I}{4\pi}\frac{1}{2a}\int_0^{2\pi}d\theta = \frac{\mu_0 I}{4a}$$

We note that $rd\theta = ds\sin\phi$ and $r = \dfrac{2a}{1-\cos\theta}$ (from the definition of the parabola, $r\cos\theta + 2a = r$),

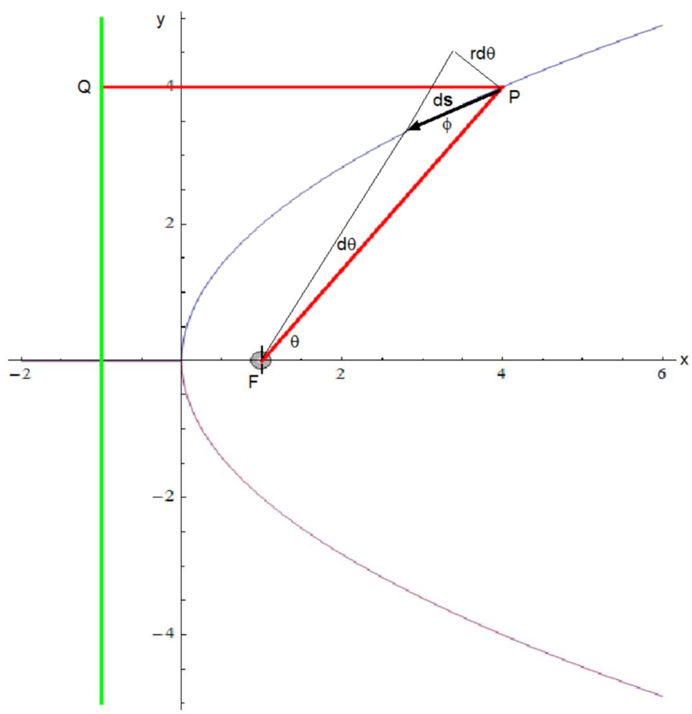


Fig. The parabola-shaped current path with $y^2 = 4ax$ in the x-scale, where $a = 1$. From the definition of the parabola, we have the lengths, PQ = PF. F is the focal point at $(a, 0)$.

7. Force between current wires

7.1 Force between parallel currents

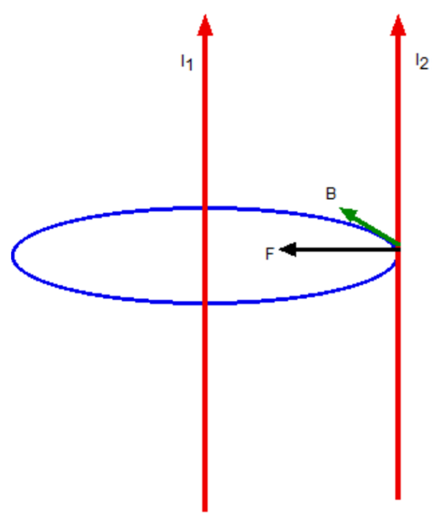
A force between parallel currents is attractive. The proof is as follows. The magnetic field due to the current I_1 is

$$B = \frac{\mu_0 I_1}{2\pi d}$$

Then the force per unit length of the wire flowing the current I_2, is given by

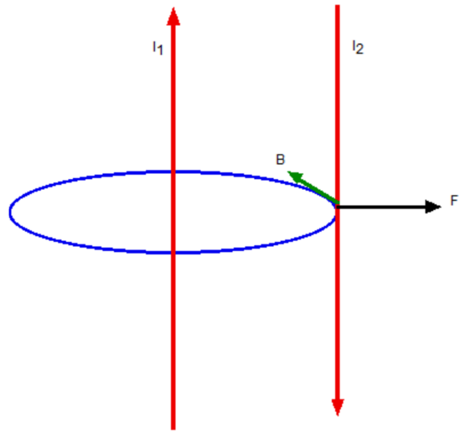
$$f = I_2 B = \frac{\mu_0 I_1 I_2}{2\pi d} \qquad \text{(attractive force)}$$

following the right-hand rule.



7.2. Force between antiparallel current wires

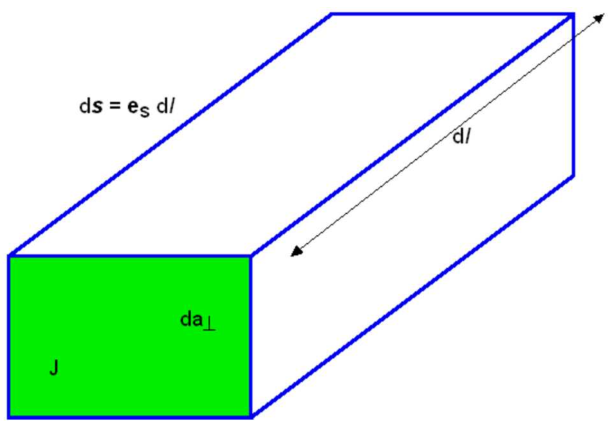
Force between antiparallel currents is the same magnitude but repulsive in comparison with the parallel case.



8. The Biot-Savart law (general expression)

8.1 Expression for the 3D system

The magnetic field due to the current density ***J*** can be expressed as follows.



From the Bio-Savart law, we have

$$\boldsymbol{B}(r) = \frac{\mu_0}{4\pi}\int \frac{Id\boldsymbol{s}\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$$

with

$$Id\boldsymbol{s} = J(\boldsymbol{r}')da_\perp d\boldsymbol{s} = \boldsymbol{e}_s J(\boldsymbol{r}')da_\perp dl = \boldsymbol{J}(\boldsymbol{r}')d\tau'$$

where ***J*** (current per unit area) is the current density, $da_\perp$ is the cross section of the system, and $d\boldsymbol{s} = \boldsymbol{e}_s dl$. Then we have

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{r}')\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}d\tau' \qquad \text{(3D expression)}$$

where $\hat{s}$ is the unit vector along the direction of $\boldsymbol{s}$.

8.2 Expression for the 2D system

The current I flowing the cross section (area = $dl_{\perp}\delta$) is expressed by

$$I = J(dl_{\perp}\delta) = (J\delta)dl_{\perp} = Kdl_{\perp}$$

where δ is the thickness, K (= $J\delta$) is the surface current density (A/m), and J is the current density (A/m^2)

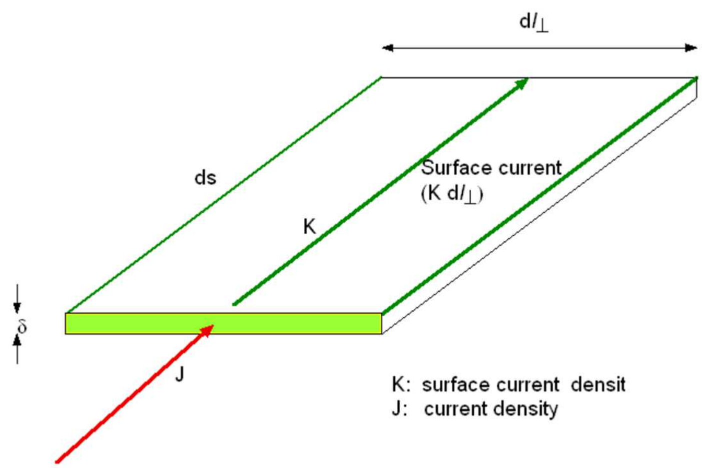


Fig. surface current density $\boldsymbol{K}$. The direction of $\boldsymbol{dl}_{\perp}$ is perpendicular to that of d$\boldsymbol{s}$.

The magnetic field due to the surface current density is expressed by

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{K}(\boldsymbol{r}')\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}da' \qquad \text{(2D expression)}$$

with

$$I d\boldsymbol{s} = (\boldsymbol{K}dl_{\perp})ds = \boldsymbol{K}(\boldsymbol{r}')da'$$

where $da' = dl_{\perp}ds$, $\boldsymbol{K}$ is the surface current density (current per unit length).

9. Ampere's law

9.1 Ampere's law: simple case

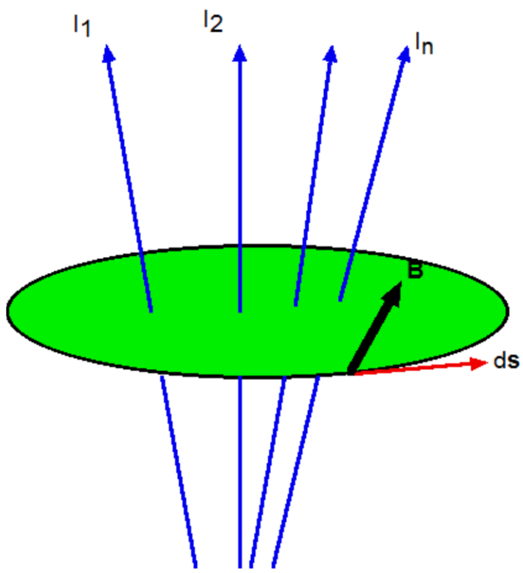
We find a magnetic field outside of a long current wire (current I). The magnetic field lines of $\boldsymbol{B}$ go around the wire in closed circle is

$$\oint \boldsymbol{B}\cdot d\boldsymbol{s} = \frac{\mu_0 I}{2\pi R}(2\pi R) = \mu_0 I$$

using the Bio-Savart law. Since the final result does not depend on the radius of the circle, we surmise that the result is independent of the exact shape of the path used in going around the wire. The expression

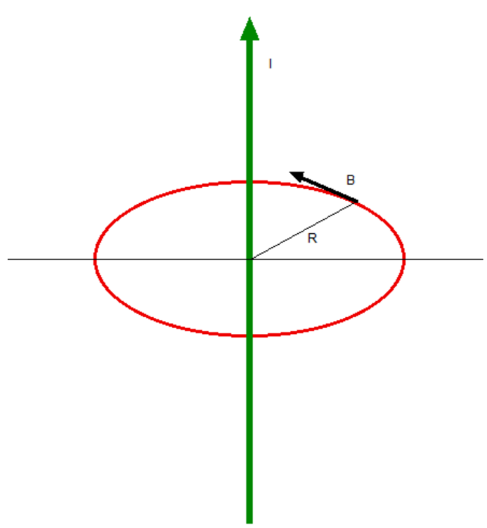
$$\oint \boldsymbol{B} \cdot d\boldsymbol{s} = \mu_0 I \qquad \text{(Ampere's law)}$$

is, in fact, a general theorem that holds true regardless of the exact path of integration. This theorem is called Ampere's law. Note that *I* is the current enclosed by the path.



$$\oint \boldsymbol{B} \cdot d\boldsymbol{s} = \mu_0 I_{enclosed} = \mu_0 \sum_{k=1}^{n} I_k$$

9.2 Example-: map of magnetic field



Using the Ampere's law, we can directly calculate the magnetic field around a circular path (radius r) surrounding an infinitely long wire carrying the current I.

$$\oint \boldsymbol{B}\cdot d\boldsymbol{s} = B(2\pi r) = \mu_0 I$$

or

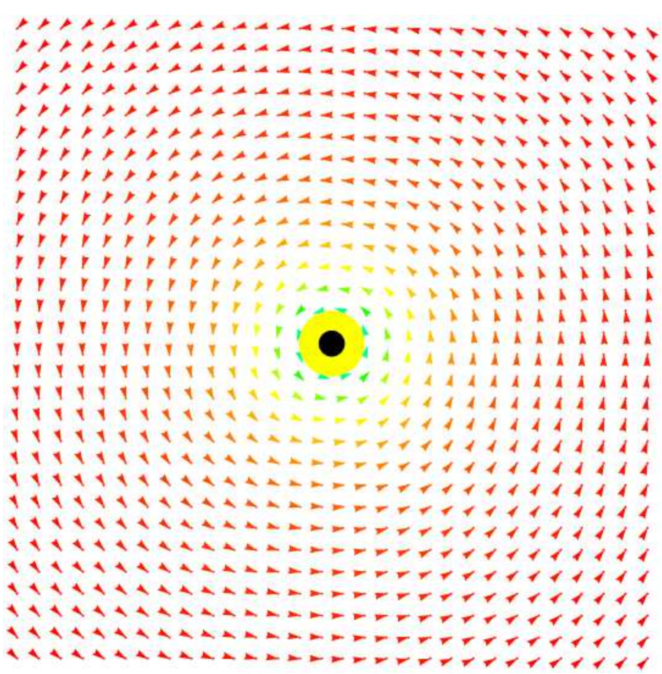
$$B = \frac{\mu_0 I}{2\pi r}$$

We can write this equation in vector form. Remembering that $\boldsymbol{B}$ is at right angles both to $\boldsymbol{I}$ and to $\boldsymbol{r}$, we have

$$\boldsymbol{B} = \frac{\mu_0}{2\pi}\frac{\boldsymbol{I}\times\boldsymbol{e}_r}{r} = \frac{\mu_0}{4\pi}\frac{2\boldsymbol{I}\times\boldsymbol{e}_r}{r} = \frac{1}{4\pi\varepsilon_0 c^2}\frac{2\boldsymbol{I}\times\boldsymbol{e}_r}{r}$$

where $c = \dfrac{1}{\sqrt{\varepsilon_0\mu_0}}$ and $\dfrac{1}{4\pi\varepsilon_0 c^2}=10^{-7}$ (SI units)

((**Mathematica**))
We use the VectorFieldPlot of Mathematica for the map of the magnetic field produced by a long straight current wire (flowing out of page).

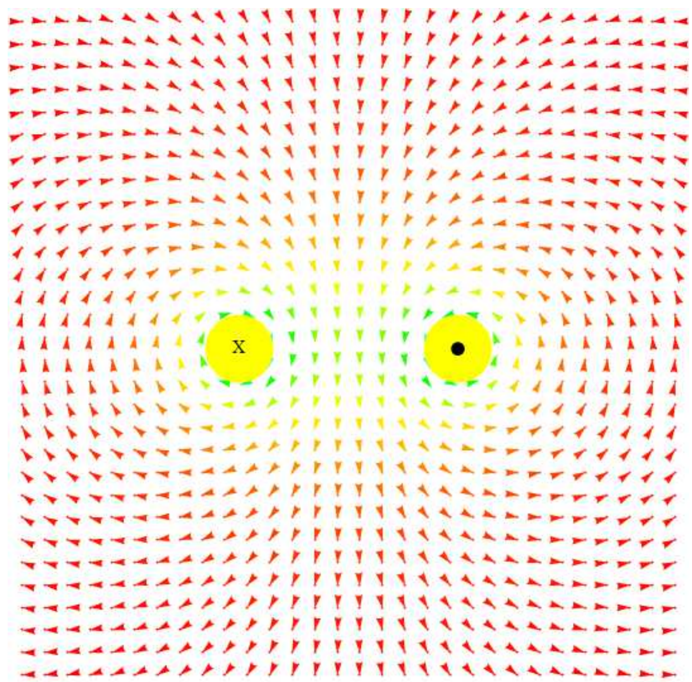


((**Feynman**))

It is worth remembering that it is exactly 10^{-7}, since this equation is used to define the unit of current, the ampere (A). At one meter from a current of 1 A the magnetic field is $2\text{x}10^{-7}$ Wb/m^2 = $2\text{x}10^{-7}$ T = 2.0 mGauss = 2.0 mOe.

9.3 Example: map of magnetic field

This figure shows the VectorFieldPlot of the magnetic field produced by two long wires carrying equal currents in opposite directions. The magnetic field lines always form closed loops. In other words, magnetic field lines never have end points



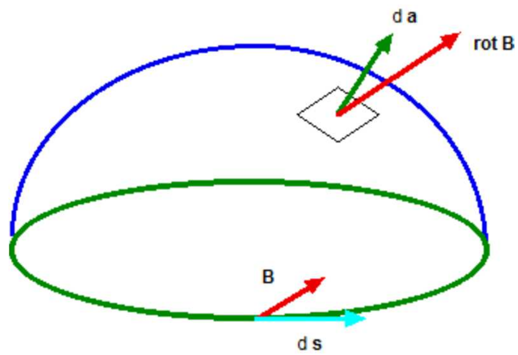
9.4 Ampere's law in differential form

Using the Stoke's theorem, the Ampere's law can be rewritten as

$$\oint \boldsymbol{B}\cdot d\boldsymbol{s} = \oint (\nabla\times\boldsymbol{B})\cdot d\boldsymbol{a} = \mu_0 I = \mu\oint(\boldsymbol{J}\cdot d\boldsymbol{a})$$

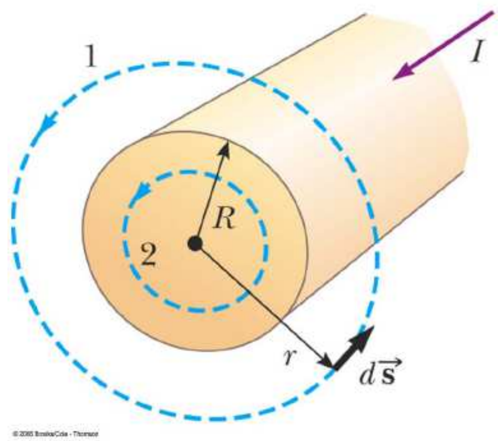
or

$$\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J}$$



This expression is one of the Maxwell's equations.

10. Magnetic field inside a long straight wire with current



A long, straight wire of radius R carries a steady current I that is uniformly distributed through the cross-section of the wire. Find the magnetic field a distance r from the center of the wire in the regions $r \geq R$ and $r \leq R$.

(a) For $r \geq R$,

$$B(2\pi r) = \mu_0 I \text{ , or } \qquad B = \frac{\mu_0 I}{2\pi r}$$

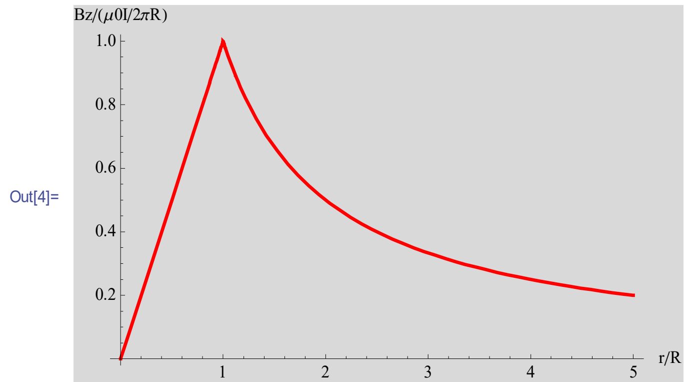
(b) For $r \leq R$,

The current I' passes through the plane of the circle ($r<R$).

$$\frac{I'}{I} = \frac{\pi r^2}{\pi R^2} = \frac{r^2}{R^2}$$

Using the Ampere's law, we have

$$B(2\pi r) = \mu_0 I' = \mu_0 I \frac{r^2}{R^2}, \qquad \text{or} \qquad B = \frac{\mu_0 I}{2\pi R^2} r$$



11. The magnetic field of a solenoid

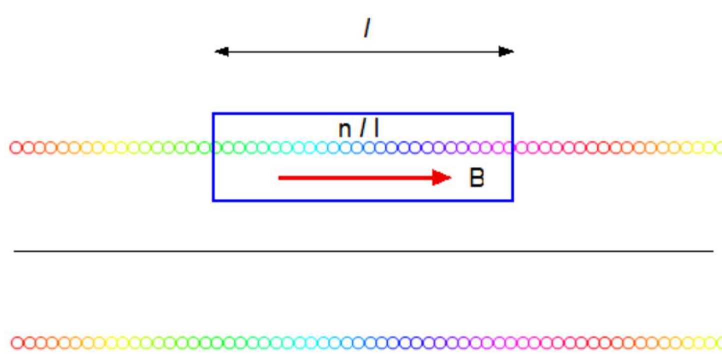
The magnetic field is produced by the current in a infinitely long, tightly wound helical coil of wire (called solenoid) is obtained using the Ampere's law;

$$\oint \boldsymbol{B} \cdot d\boldsymbol{s} = \mu_0 I_{enclosed}$$

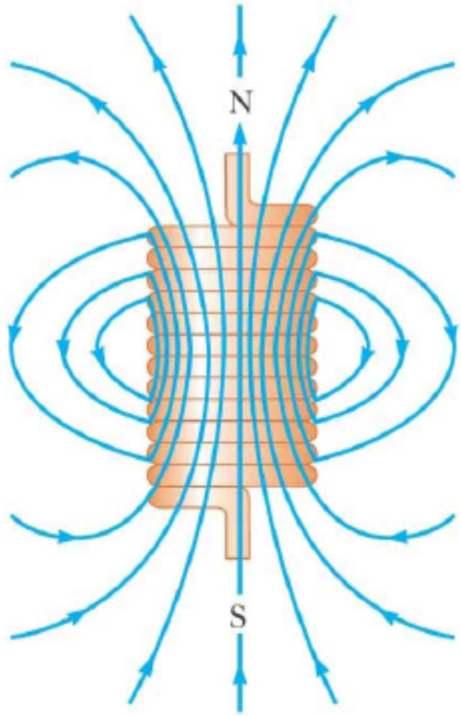
or

$$Bl = \mu_0 (nlI), \text{ or } \qquad B = \mu_0 nI$$

where n is the number of turns per unit length.



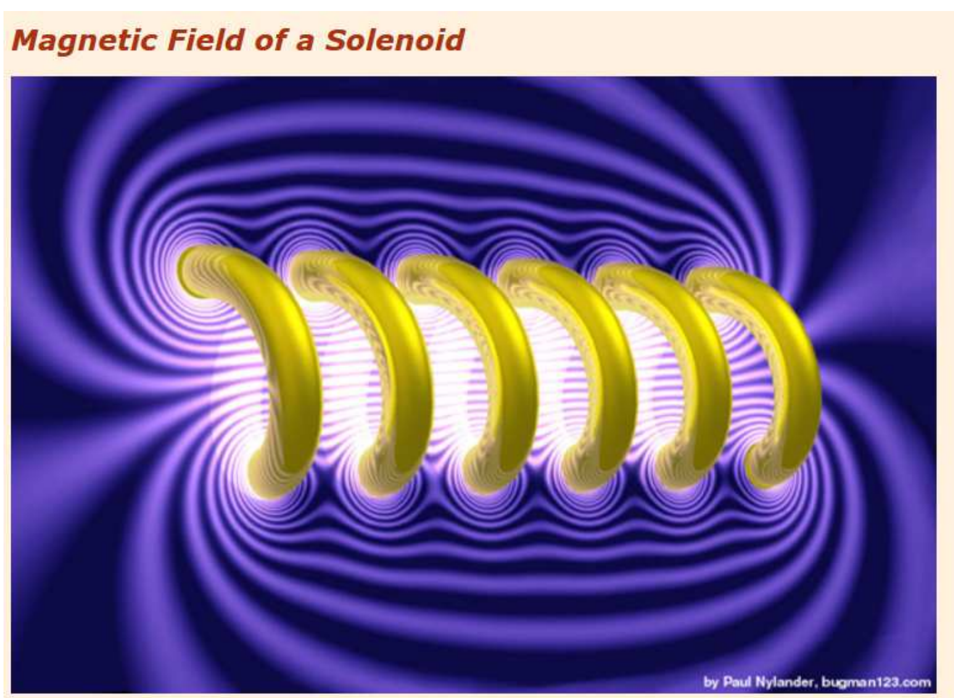
Next figure shows the field lines associated with the current in an ordinary solenoid. Note that there is an outward leakage of field lines between the wires, and that in the region near the center the field lines tends to be parallel to the axis.



((**Link**))

Scientific visualization and graphics with Mathematica. Magnetic field of solenoid calculated using Mathematica

http://members.wri.com/jeffb/visualization/solenoid.shtml



12. Magnetic field of a toroid

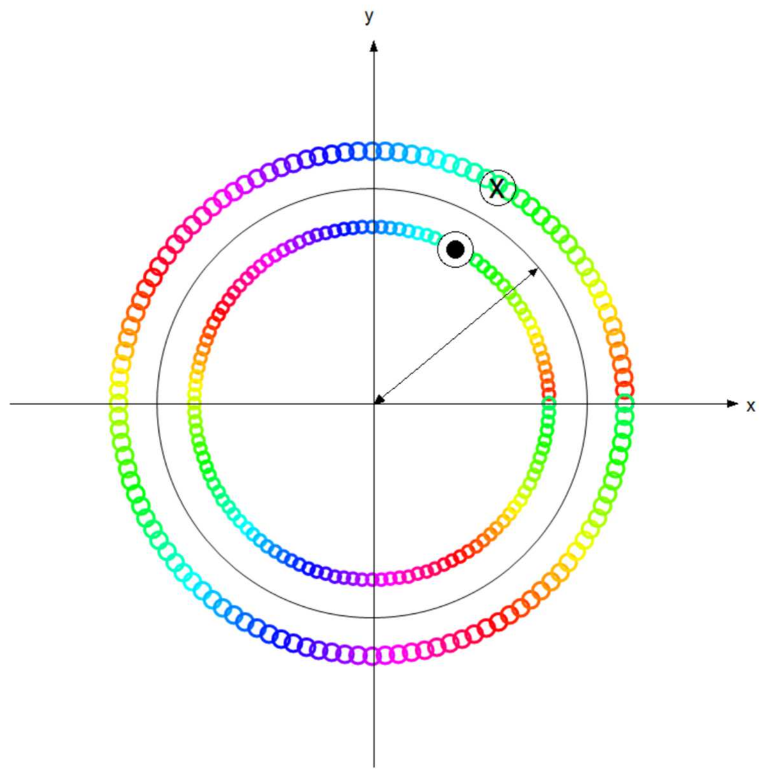
A toroid is a solenoid that is curved until two ends meet, forming a sort of hollow solenoid. Using the Ampere's law, we can calculate the magnetic field inside the toroid,

$$\oint \boldsymbol{B} \cdot d\boldsymbol{s} = 2\pi r B = \mu_0 (NI)$$

or

$$B = \frac{\mu_0 NI}{2\pi r}$$

where N is the total number of turns.



Note that the magnetic field is dependent on r.

13. A current-carrying coil as a magnetic dipole

The magnetic field on the axis of a circular loop current (NI) is approximated by

$$B_z \approx \frac{\mu_0 NI}{2} \frac{R^2}{z^3} = \frac{\mu_0 NI}{2\pi} \frac{\pi R^2}{z^3} = \frac{\mu_0}{2\pi} \frac{NIA}{z^3} = \frac{\mu_0}{2\pi} \frac{\overline{\mu}}{z^3},$$

where $A = \pi R^2$ is the area of the circle and $\overline{\mu} = NIA$ is the magnetic dipole moment.

14. Vector potential *A* and scalar potential ϕ

We introduce the vector potential ***A***:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \ (= \text{rot}\boldsymbol{A} = \text{curl } \boldsymbol{A})$$

since $\nabla \cdot \boldsymbol{B} = 0$ $(= \text{div}\boldsymbol{B})$

$$\nabla \times \boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{B} = -\frac{\partial}{\partial t}\nabla \times \boldsymbol{A} \qquad \text{(Faraday's law, see Chapter 30))}$$

or

$$\nabla \times (\boldsymbol{E} + \frac{\partial}{\partial t}\boldsymbol{A}) = 0$$

or

$$\boldsymbol{E} + \frac{\partial}{\partial t}\boldsymbol{A} = -\nabla \phi$$

where ϕ is a scalar potential.

Then we have

$$\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{A} - \nabla \phi \qquad \text{and} \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A}$$

In the simple case, we have

$$\nabla \times \boldsymbol{B} = \nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{J}$$

$$\nabla \cdot (-\frac{\partial}{\partial t}\boldsymbol{A} - \nabla \phi) = \frac{\rho}{\varepsilon_0}$$

15. Gauge transformation

We have a gauge transformation

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla \chi ,$$

$$\phi' = \phi - \frac{\partial \chi}{\partial t} ,$$

where

$$\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{A} - \nabla\phi \qquad \text{and} \qquad \boldsymbol{B} = \nabla\times\boldsymbol{A}$$

Let us calculate

$$-\frac{\partial}{\partial t}\boldsymbol{A}' - \nabla\phi' = -\frac{\partial}{\partial t}(\boldsymbol{A} + \nabla\chi) - \nabla(\phi - \frac{\partial\chi}{\partial t}) = -\frac{\partial\boldsymbol{A}}{\partial t} - \nabla\phi = \boldsymbol{E}$$

$$\nabla\times\boldsymbol{A}' = \nabla\times(\boldsymbol{A} + \nabla\chi) = \nabla\times\boldsymbol{A}$$

Therefore $(\boldsymbol{A}', \phi')$ and $(\boldsymbol{A}, \phi)$ gives the same expression for $\boldsymbol{E}$ and $\boldsymbol{B}$.

16. Coulomb gauge

Here we use the Coulomb gauge such that

$$\nabla\cdot\boldsymbol{A} = 0$$

Then we have

$$\nabla^2\boldsymbol{A} = -\mu_0\boldsymbol{J}$$

and

$$\nabla^2\phi = -\frac{\rho}{\varepsilon_0}$$

The solutions of $\boldsymbol{A}$ and ϕ are obtained by

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int\frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3\boldsymbol{r}', \qquad \phi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3\boldsymbol{r}'$$

Here we use the Green function such that

$$\boldsymbol{A} = \int\mu_0 J(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')d^3\boldsymbol{r}'$$

Note that

$$\begin{aligned}\nabla^2\boldsymbol{A} &= \int\mu_0 J(\boldsymbol{r}')\nabla^2 G(\boldsymbol{r},\boldsymbol{r}')d\boldsymbol{r}' \\ &= -\int\mu_0 J(\boldsymbol{r}')\delta(\boldsymbol{r}-\boldsymbol{r}')d\boldsymbol{r}' \\ &= -\mu_0 J(\boldsymbol{r})\end{aligned}$$

where

$$\nabla^2 G(\boldsymbol{r},\boldsymbol{r}') = -\delta(\boldsymbol{r}-\boldsymbol{r}')$$

$$G(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{4\pi}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \qquad \text{(Green function)}$$

((**Note-2**)) **Green function method**

The above formula can be derived using the Green function method.

$$\nabla_1{}^2 y(\boldsymbol{r}_1) = -f(\boldsymbol{r}_1)$$

The solution of this equation is given by

$$y(\boldsymbol{r}_1) = \int G(\boldsymbol{r}_1,\boldsymbol{r}_2) f(\boldsymbol{r}_2) d\boldsymbol{r}_2$$

where $G(\mathbf{r}_1,\mathbf{r}_2)$ is the Green function defined by

$$\nabla_1{}^2 G(\boldsymbol{r}_1,\boldsymbol{r}_2) = -\delta(\boldsymbol{r}_1-\boldsymbol{r}_2)$$

or

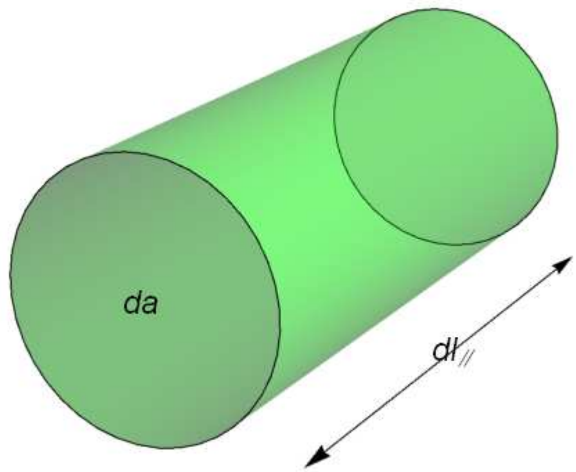
$$G(\boldsymbol{r}_1,\boldsymbol{r}_2) = \frac{1}{4\pi|\boldsymbol{r}_1-\boldsymbol{r}_2|}.$$

17. The expression of the vector potential

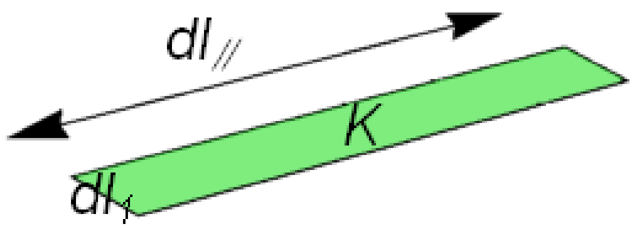
The vector potential is expressed by

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} d^3\boldsymbol{r}'$$

$$\boldsymbol{J}(\boldsymbol{r}')d\boldsymbol{r}' = \boldsymbol{J}(\boldsymbol{r}')da\,dl_{//} = \boldsymbol{I}(\boldsymbol{r}')dl_{//} \qquad \text{(3D case)}$$



$$I(\boldsymbol{r}')dl_{//} = K(\boldsymbol{r}')dl_{\perp}dl_{//} = K(\boldsymbol{r}')da$$



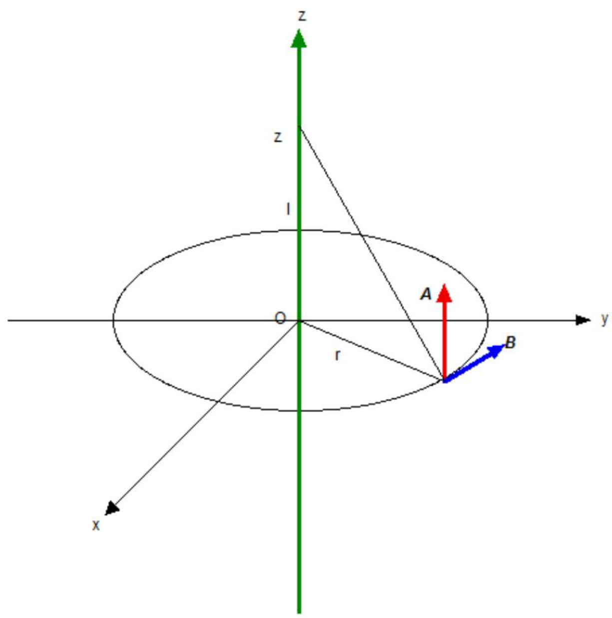
$$A(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{I(\boldsymbol{r}')dl'}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{\mu_0 I}{4\pi}\int \frac{dl'}{|\boldsymbol{r}-\boldsymbol{r}'|} \qquad \text{(1D case)}$$

or

$$A(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{K(\boldsymbol{r}')da'}{|\boldsymbol{r}-\boldsymbol{r}'|} \qquad \text{(2D case)}$$

18. Derivation of Ampere's law from the vector potential

We now consider the derivation of ***A*** for the one-dimensional system (current wire along the *z* axis).



$$A(\boldsymbol{r})=\frac{\mu_0}{4\pi}\int\frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}d\tau'=\frac{\mu_0}{4\pi}\int\frac{I d\boldsymbol{s}}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

$$d\boldsymbol{s}=\boldsymbol{e}_z dz$$

The vector potential $\boldsymbol{A}$ has only the z component.

$$\begin{aligned}A_z(\boldsymbol{r})&=\frac{\mu_0}{4\pi}\int\frac{Idz}{|\boldsymbol{r}-\boldsymbol{r}'|}=\frac{\mu_0 I}{4\pi}\int_{-L}^{L}\frac{dz}{\sqrt{r^2+z^2}}\\&=\frac{\mu_0 I}{4\pi}\ln[\frac{L+\sqrt{L^2+r^2}}{-L+\sqrt{L^2+r^2}}]\\&=\frac{\mu_0 I}{4\pi}\ln[\frac{(L+\sqrt{L^2+r^2})^2}{r^2}]\\&\approx\frac{\mu_0 I}{2\pi}(\ln L-\ln r)\\&=-\frac{\mu_0 I}{2\pi}\ln r+const\end{aligned}$$

Since $A_\theta = 0$ and $A_\phi = 0$, and A_z is only dependent on r,

$$\nabla\times\boldsymbol{A}=(\frac{1}{r}\frac{\partial A_z}{\partial\theta}-\frac{\partial A_\theta}{\partial z})\boldsymbol{e}_r+(\frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r})\boldsymbol{e}_\theta+\frac{1}{r}[\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}]\boldsymbol{e}_z \quad \text{(vector formula)}$$

in the cylindrical coordinates (usually we use $r=\rho$ and $\theta=\phi$). Then we have

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = (-\frac{\partial A_z}{\partial r})\boldsymbol{e}_\theta = \frac{\mu_0 I}{2\pi r}\boldsymbol{e}_\theta$$ (Ampere's law)

((**Mathematica**))

```
∫_{-L}^{L} 1/Sqrt[r^2 + z^2] dz //
  Simplify[#, {r > 0, L > 0}] &

Log[(L + Sqrt[L^2 + r^2])/(-L + Sqrt[L^2 + r^2])]
```

((**Mathematica**))

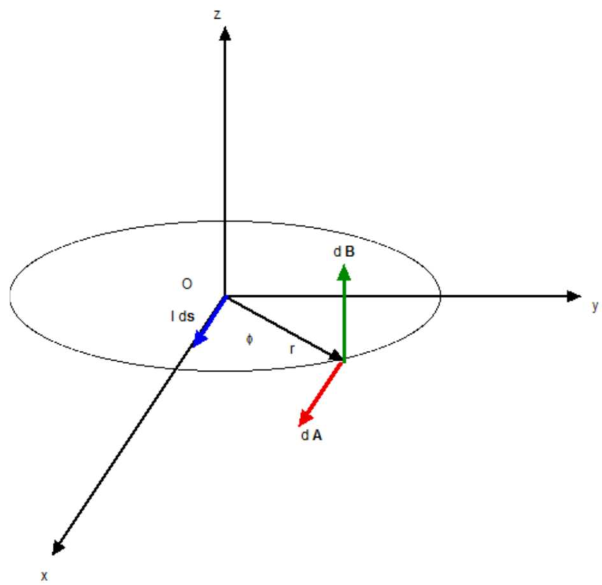
```
<< "VectorAnalysis`"

SetCoordinates[Cylindrical[r, ϕ, z]]; A = {0, 0, -(μ0 I0)/(2 π) Log[r]};

Curl[A]

{0, (I0 μ0)/(2 π r), 0}
```

19. Derivation of the Biot-Savart law from the vector potential

We use the formula

$$A(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{Id\boldsymbol{s}}{|\boldsymbol{r} - \boldsymbol{r}'|}$$

to find the vector potential d$\boldsymbol{A}$ at point (r, ϕ, 0) [cylindrical coordinate] from the small segment of the current ($Id\boldsymbol{s}$) pointing in the positive x axis at the origin (0, 0) [see Fig.]. Note that usually we use the notation $r = \rho$ in the cylindrical coordinate.



$$d\boldsymbol{A} = \frac{\mu_0}{4\pi}\frac{Ids}{r}\boldsymbol{e}_x = \frac{\mu_0 Ids}{4\pi}(\frac{\cos\phi}{r}\boldsymbol{e}_r - \frac{\sin\phi}{r}\boldsymbol{e}_\phi)$$

where

$$\boldsymbol{e}_x = \cos\phi\boldsymbol{e}_r - \sin\phi\boldsymbol{e}_\phi$$

The magnetic field $\boldsymbol{B}$ is obtained as

$$d\boldsymbol{B} = \nabla\times d\boldsymbol{A} = \frac{\mu_0 Ids}{4\pi}\frac{\sin\phi}{r^2}\boldsymbol{e}_z = \frac{\mu_0}{4\pi}\frac{Id\boldsymbol{s}\times\boldsymbol{e}_r}{r^2}$$ (Biot-Savart law)

where

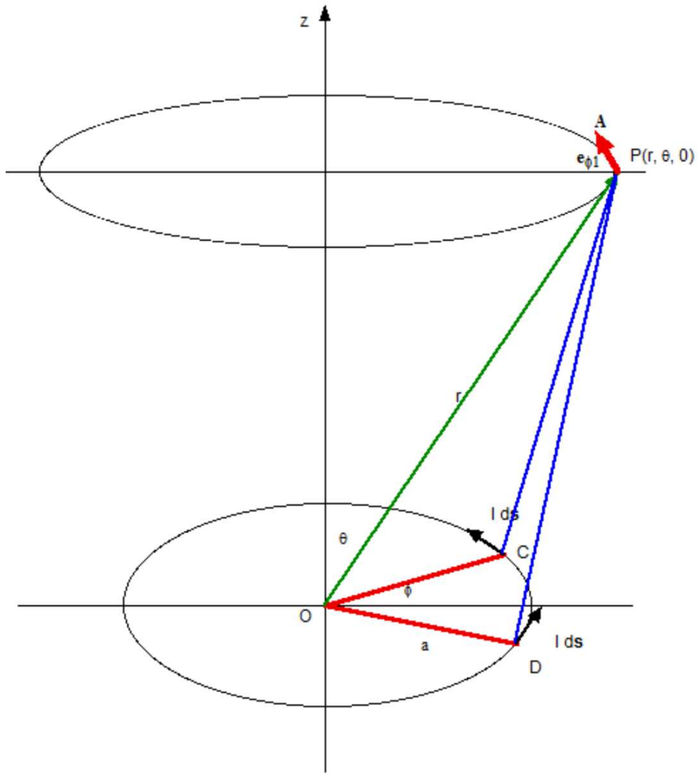
((**Mathematica**))

```
<< "VectorAnalysis`"

SetCoordinates[Cylindrical[r, ϕ, z]]; A = {Cos[ϕ]/r, -Sin[ϕ]/r, 0};
Curl[A]
{0, 0, Sin[ϕ]/r^2}
```

20. Magnetic dipole moment (I)

In this Fig. the polar coordinate of the observation point P is (r, θ, 0). We assume that the circular loop current is formed of the pair of current elements, C (a, $\pi/2$, ϕ) and D (a, $\pi/2$, -ϕ). The sum of vector potentials from these current elements has a direction of $\boldsymbol{e}_{\phi 1}$.

$$\boldsymbol{A} = A_\phi \boldsymbol{e}_{\phi 1}$$



$$A_\phi = \frac{\mu_0}{4\pi}\int_0^\pi \frac{2I(ad\phi)\cos\phi}{\sqrt{r^2 + a^2 - 2ar\cos(\angle COP)}}$$

$$\cos(\angle COP) = \frac{OC}{a}\cdot\frac{\mathbf{r}}{r} = (\cos\phi, \sin\phi, 0)\cdot(\sin\theta, 0, \cos\theta) = \sin\theta\cos\phi$$

Then we have

$$A_\phi = \frac{\mu_0}{4\pi}\int_0^\pi \frac{2I(ad\phi)\cos\phi}{\sqrt{r^2 + a^2 - 2ar\cos(\sin\theta\cos\phi)}}$$

Suppose that $r \gg a$. We use the Legendre generating function

$$\frac{1}{\sqrt{r^2+a^2-2ra\cos(\sin\theta\cos\phi)}}=\frac{1}{r}\frac{1}{\sqrt{1+(\frac{a}{r})^2-2\frac{a}{r}\cos(\sin\theta\cos\phi)}}$$

$$=\frac{1}{r}\sum_{n=0}^{\infty}P_n(\sin\theta\cos\phi)(\frac{a}{r})^n$$

$$=\frac{1}{r}[P_0(\sin\theta\cos\phi)+\frac{a}{r}P_1(\sin\theta\cos\phi)+(\frac{a}{r})^2P_2(\sin\theta\cos\phi)+...]$$

where $P_n(x)$ is the Legendre polynomial,

$$P_0(x)=1$$
$$P_1(x)=x$$
$$P_2(x)=\frac{1}{2}(3x^2-1)$$
$$P_3(x)=\frac{1}{2}x(5x^2-3)$$
$$P_4(x)=\frac{1}{8}x(35x^4-30x^2+3)$$
$$P_5(x)=\frac{1}{8}x(63x^4-70x^2+15)$$

$$A_\phi=\frac{\mu_0 2Ia}{4\pi r}\int_0^\pi d\phi\cos\phi[P_0(\sin\theta\cos\phi)+\frac{a}{r}P_1(\sin\theta\cos\phi)+(\frac{a}{r})^2P_2(\sin\theta\cos\phi)+...]$$

$$=\frac{\mu_0 2Ia}{4\pi r}(\frac{a\pi\sin\theta}{2r}-\frac{3a^3\pi\sin\theta}{4r^3}+\frac{15a^3\pi\sin^3\theta}{16r^3}+...)$$

((**Mathematica**))

```
f1 =
 Cos[ϕ]
  Sum[((a/r)^n LegendreP[n, Sin[θ] Cos[ϕ]]), {n, 0, 3}];

eq1 = Integrate[f1, {ϕ, 0, π}] // Expand
```

$$-\frac{3\,a^3\,\pi\,\mathrm{Sin}[\theta]}{4\,r^3}+\frac{a\,\pi\,\mathrm{Sin}[\theta]}{2\,r}+\frac{15\,a^3\,\pi\,\mathrm{Sin}[\theta]^3}{16\,r^3}$$

In the approximation up to $1/r$, we have

$$A_\phi = \frac{\mu_0 2Ia}{4\pi r}(\frac{a\pi \sin\theta}{2r}) = \frac{\mu_0 \pi a^2 I \sin\theta}{4\pi r^2} = \frac{\mu_0 m \sin\theta}{4\pi r^2}$$

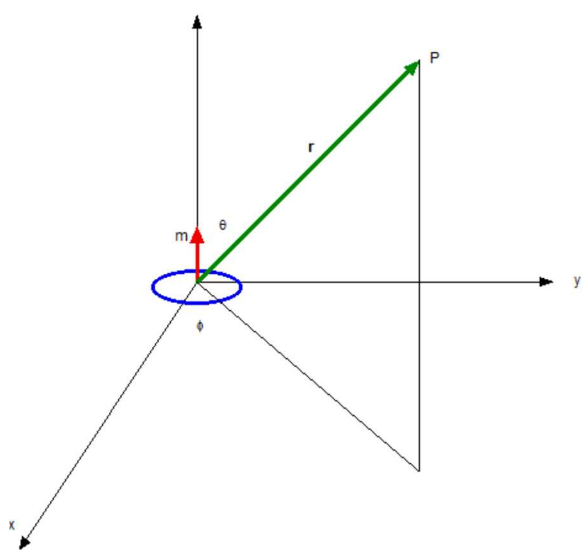
where $\boldsymbol{m} = \pi a^2 I\hat{z}$ is the magnetic dipole moment (or simply magnetic moment) of circular current loop.

21. Magnetic field from the magnetic dipole moment (more simple method)

The magnetic vector potential ***A*** due to a localized current at large distance can be evaluated relatively easily. We show that ***A*** is obtained as

$$\boldsymbol{A} = \mu_0 \frac{\boldsymbol{m} \times \boldsymbol{r}}{4\pi r^3}$$

where ***m*** is the magnetic moment due to the small circular current and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$.



Here we use the formula

$$\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \approx \frac{1}{r} + \frac{\boldsymbol{r} \cdot \boldsymbol{r}'}{r^3}$$

The expression for the vector potential ***A*** is applied to current circuit by making the substitution: $\boldsymbol{J}(\boldsymbol{r}')d\tau' \rightarrow I d\boldsymbol{s}$,

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \oint \frac{I}{|\boldsymbol{r} - \boldsymbol{s}|} d\boldsymbol{s} \approx \frac{\mu_0 I}{4\pi} [\frac{1}{r} \oint d\boldsymbol{s} + \frac{1}{r^3} \oint (\boldsymbol{r} \cdot \boldsymbol{s}) d\boldsymbol{s}$$

The first integral vanishes. The second integrand is one term in the expansion

$$(\boldsymbol{s} \times d\boldsymbol{s}) \times \boldsymbol{r} = -\boldsymbol{s}(\boldsymbol{r} \cdot d\boldsymbol{s}) + d\boldsymbol{s}(\boldsymbol{r} \cdot \boldsymbol{s})$$

Here we use the relation

$$d[\boldsymbol{s}(\boldsymbol{r} \cdot \boldsymbol{s})] = \boldsymbol{s}(\boldsymbol{r} \cdot d\boldsymbol{s}) + d\boldsymbol{s}(\boldsymbol{r} \cdot \boldsymbol{s})$$

From the addition of these equations, we have

$$d\boldsymbol{s}(\boldsymbol{r} \cdot \boldsymbol{s}) = \frac{1}{2}(\boldsymbol{s} \times d\boldsymbol{s}) \times \boldsymbol{r} + \frac{1}{2} d[\boldsymbol{s}(\boldsymbol{r} \cdot \boldsymbol{s})]$$

Since the last term is an exact differential, it contributes nothing to the second integral.

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi r^3}[\frac{I}{2}\oint (\boldsymbol{s} \times d\boldsymbol{s})] \times \boldsymbol{r}$$

or

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m} \times \boldsymbol{r}}{r^3}$$

where

$$\boldsymbol{m} = \frac{I}{2}\oint (\boldsymbol{s} \times d\boldsymbol{s})$$

The magnetic field can be calculated as

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \frac{\mu_0}{4\pi r^5}[3(\boldsymbol{m} \cdot \boldsymbol{r})\boldsymbol{r} - \boldsymbol{m} r^2]$$

((**Mathematica**))

```
Clear["Gobal`"]

Needs["VectorAnalysis`"]

SetCoordinates[Cartesian[x, y, z]]

Cartesian[x, y, z]

m = {m1, m2, m3}

{m1, m2, m3}

r = {x, y, z}

{x, y, z}
```

$$A = \frac{\mu 0\ \mathrm{Cross}[m, r]}{4\pi\ (r.r)^{3/2}}$$

$$\left\{\frac{(-m3\ y + m2\ z)\ \mu 0}{4\pi\left(x^2+y^2+z^2\right)^{3/2}},\ \frac{(m3\ x - m1\ z)\ \mu 0}{4\pi\left(x^2+y^2+z^2\right)^{3/2}},\ \frac{(-m2\ x + m1\ y)\ \mu 0}{4\pi\left(x^2+y^2+z^2\right)^{3/2}}\right\}$$

```
B = Curl[A] // Simplify
```

$$\left\{\frac{\left(3\ x\ (m2\ y + m3\ z) + m1\left(2x^2 - y^2 - z^2\right)\right)\mu 0}{4\pi\left(x^2+y^2+z^2\right)^{5/2}},\ \frac{\left(3\ y\ (m1\ x + m3\ z) - m2\left(x^2 - 2y^2 + z^2\right)\right)\mu 0}{4\pi\left(x^2+y^2+z^2\right)^{5/2}},\ \frac{\left(3\ (m1\ x + m2\ y)\ z - m3\left(x^2 + y^2 - 2z^2\right)\right)\mu 0}{4\pi\left(x^2+y^2+z^2\right)^{5/2}}\right\}$$

```
Div[B] // Simplify

0
```

If $\boldsymbol{m} = m\hat{z}$ (along the z axis), in the Cartesian coordinate we have

$$B_x = \frac{3m\mu_0 zx}{4\pi r^5}$$

$$B_y = \frac{3m\mu_0 yz}{4\pi r^5}$$

$$B_z = \frac{m\mu_0 (3z^2 - r^2)}{4\pi r^5}$$

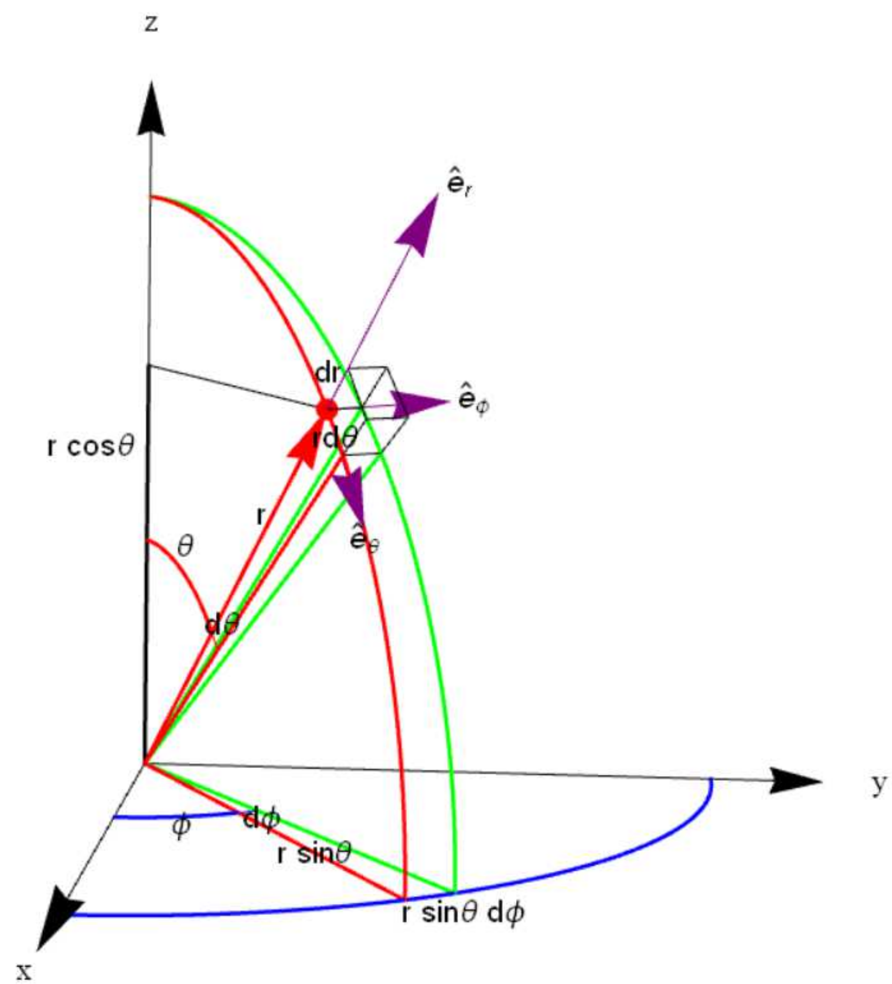
or in spherical coordinates, we have

$$\boldsymbol{B} = B_r \hat{r} + B_\theta \hat{\theta} + B_\phi \hat{\phi}$$

$$B_r = \frac{2\mu_0 m \cos\theta}{4\pi r^3}$$

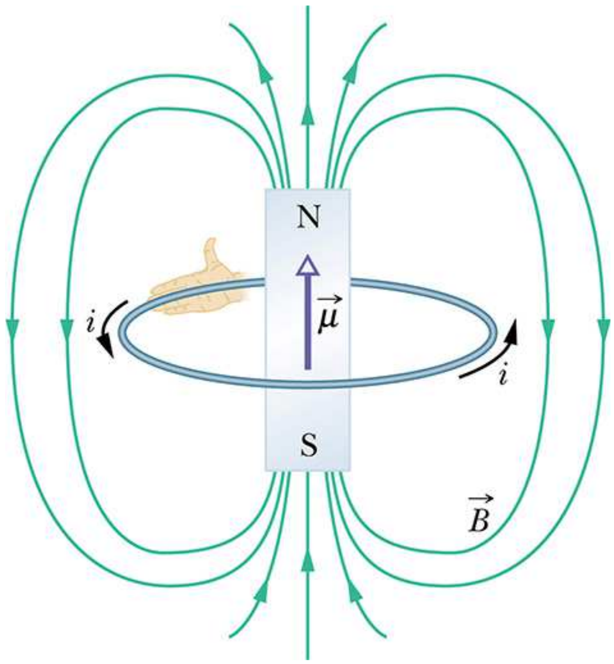
$$B_\theta = \frac{\mu_0 m \sin\theta}{4\pi r^3}$$

$$B_\phi = 0$$

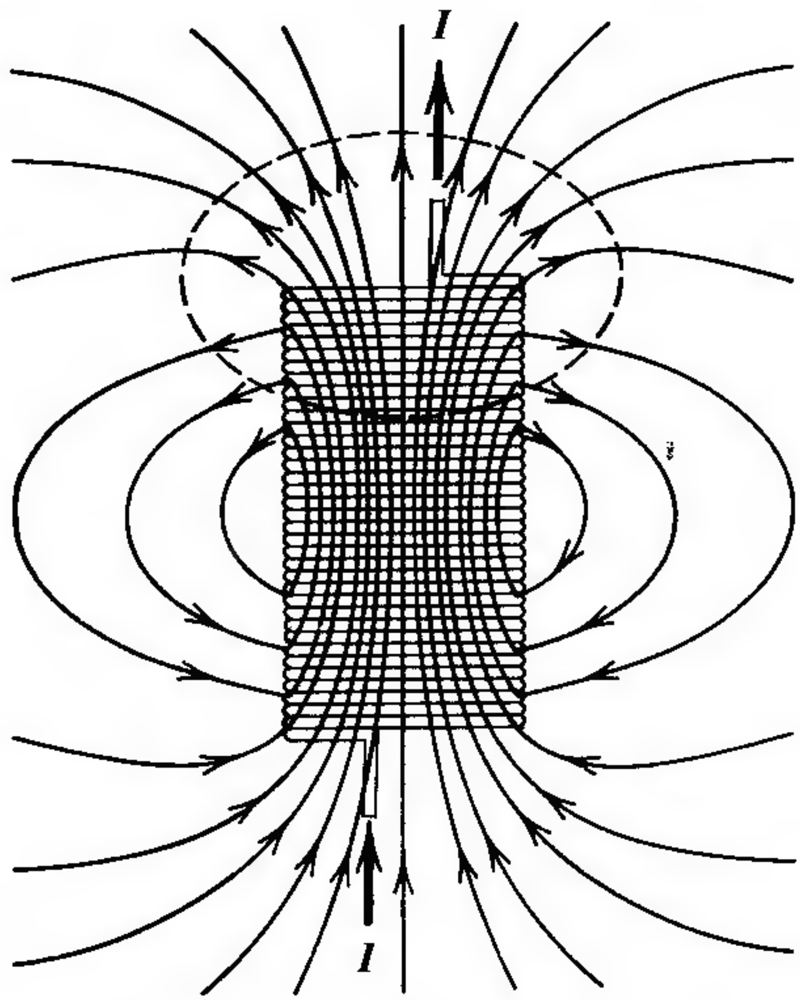


where

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$



N
S
i
i
$\vec{\mu}$
$\vec{B}$



(*b*)

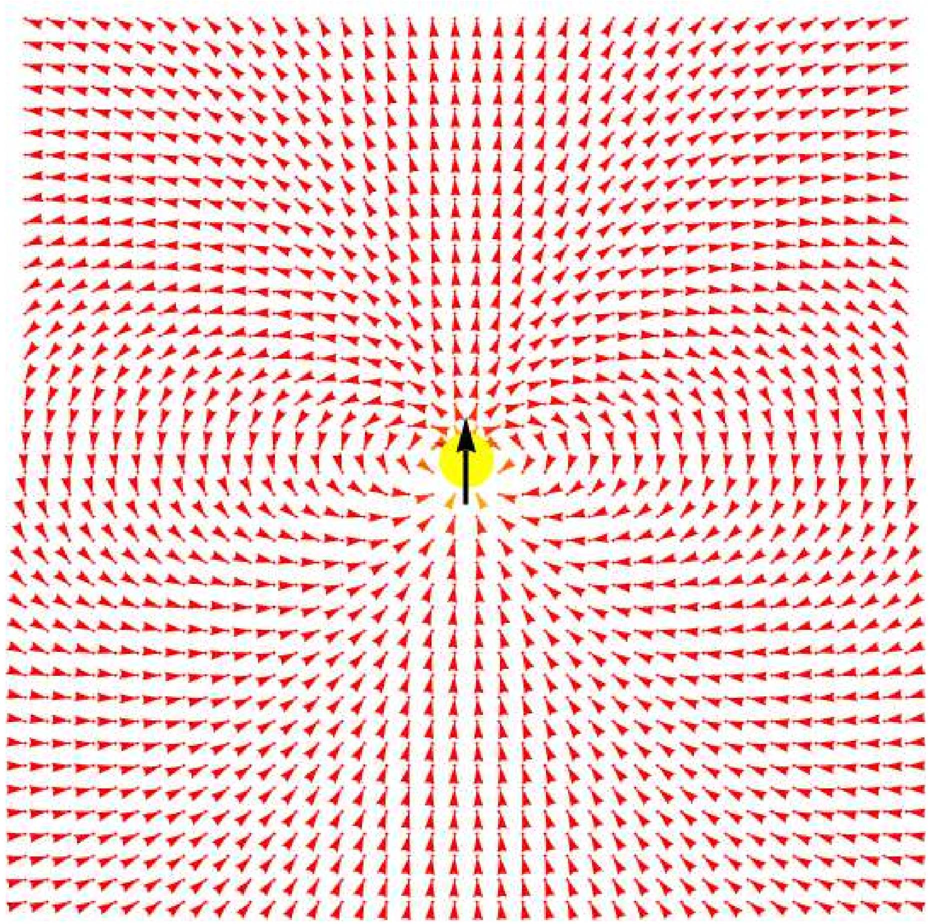
((**Mathematica**))

```
SetCoordinates[Spherical[r, θ, ϕ]]; M = {m Cos[θ], -m Sin[θ], 0};
R = {r, 0, 0}; A = μ0/(4 π r^3) Cross[M, R];
Curl[A]
```

$$\left\{\frac{m\,\mu 0\,\mathrm{Cos}[\theta]}{2\,\pi\,r^3},\ \frac{m\,\mu 0\,\mathrm{Sin}[\theta]}{4\,\pi\,r^3},\ 0\right\}$$

((**Mathematica**)) **VectorFieldPlot (*zx* plane) using the Mathematica**

The map of he magnetic field produced by a magnetic dipole moment directed along the *z* axis (*x*-*z* plane).



22. Vector potential from magnetic moment (Mathematica)

We start with a vector potential given by

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{r}}{r^3}$$

where

$$\boldsymbol{r} = (x, y, z), \qquad \boldsymbol{m} = (0, 0, m)$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

The vector potential:

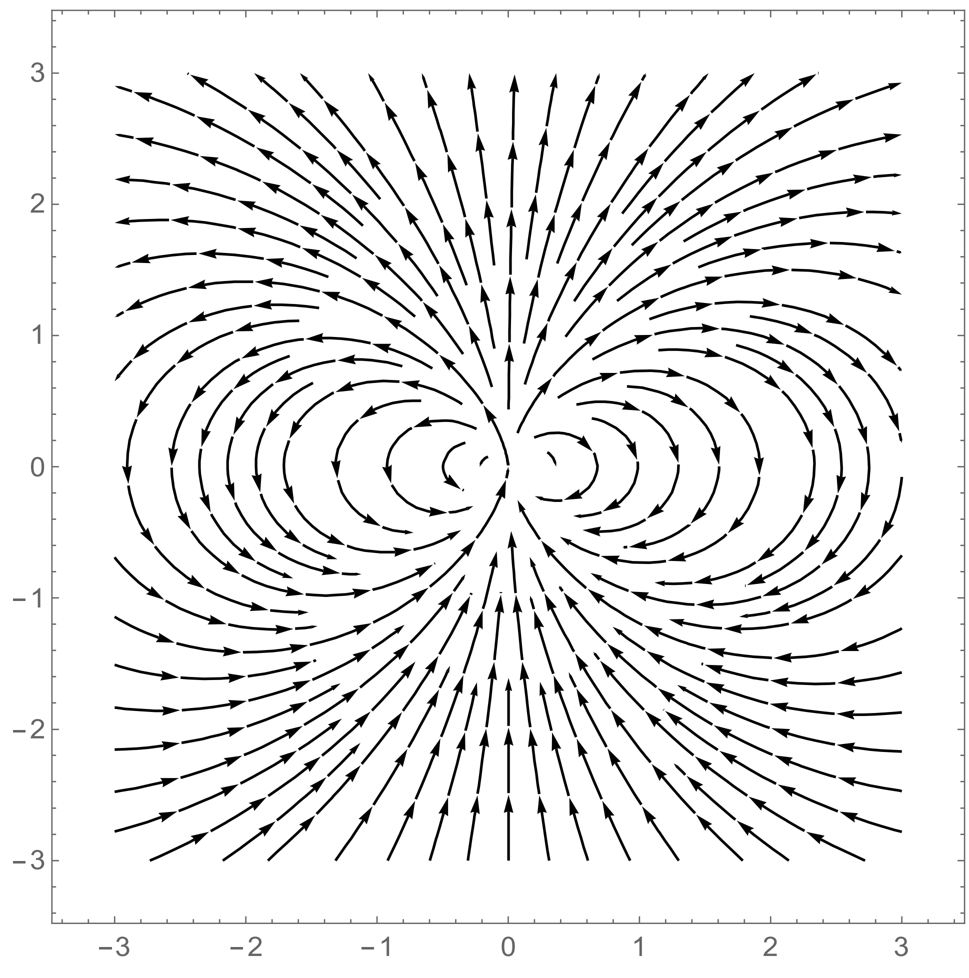
$$\begin{aligned}\boldsymbol{A} &= \frac{m\mu_0}{4\pi}\left(\frac{-y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{x}{(x^2 + y^2 + z^2)^{3/2}}, 0\right) \\ &= \frac{m\mu_0}{4\pi(x^2 + y^2 + z^2)^{3/2}}(-y, x, 0)\end{aligned}$$

The magnetic field:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \frac{m\mu_0}{4\pi(x^2 + y^2 + z^2)^{5/2}}(3zx, 3yz, -(x^2 + y^2 - 2z^2)$$

In the *z*-*x* plane (with $y = 0$),

$$\boldsymbol{B} = \frac{m\mu_0}{4\pi(x^2 + z^2)^{5/2}}(3zx, 0, -x^2 + 2z^2)$$



((**Note**))

Experimental result: magnetic field distribution from a spherical magnet (dipole field)
(from the book of O.D. Jefimenko, Electricity and Magnetism).

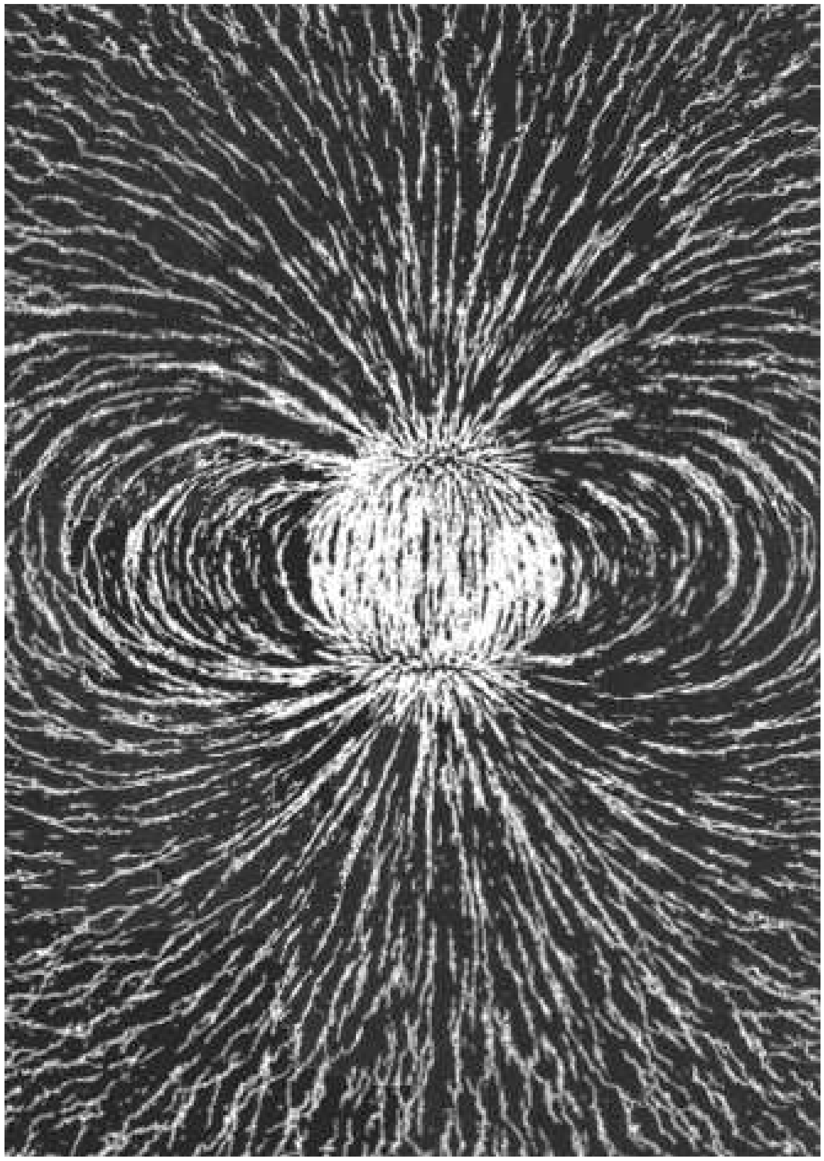


Fig. Magnetic field of a spherical magnet (dipole field).

23. Magnetic scalar potential of magnetic moment

We note that $\nabla\times\boldsymbol{B}=\mu_0\boldsymbol{J}$. When $\boldsymbol{J}=0$, $\nabla\times\boldsymbol{B}=0$. In this case $\boldsymbol{B}$ can be expressed by the gradient of a scalar potential, such that

$$\boldsymbol{B}=-\mu_0\nabla\phi^*.$$

Since $\nabla\cdot\boldsymbol{B}=0$, we have

$$\nabla\cdot\boldsymbol{B}=-\mu_0\nabla\cdot\nabla\phi^*=-\mu_0\nabla^2\phi^*=0$$

Thus ϕ^*, which is called the magnetic scalar potential, satisfies Laplace's equation.

The expression for the scalar potential of a magnetic dipole is particularly useful.

$$\boldsymbol{B}(\boldsymbol{r}) = -\mu_0 \nabla\left(\frac{\boldsymbol{m}\cdot\boldsymbol{r}}{4\pi r^3}\right), \qquad B(\boldsymbol{r}) = -\mu_0 \nabla\phi^*$$

Magnetic scalar potential

$$\phi^* = \frac{\boldsymbol{m}\cdot\boldsymbol{r}}{4\pi r^3}$$

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}$$

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{r}}{r^3}$$ (electric dipole moment)

Thus we have

$$\begin{aligned}\boldsymbol{B} &= \nabla\times\boldsymbol{A}\\ &= \frac{\mu_0}{4\pi}\nabla\times\left(\frac{\boldsymbol{m}\times\boldsymbol{r}}{r^3}\right)\\ &= -\frac{\mu_0}{4\pi}(\boldsymbol{m}\cdot\nabla\frac{\boldsymbol{r}}{r^3})\\ &= -\frac{\mu_0}{4\pi}\nabla(\boldsymbol{m}\cdot\frac{\boldsymbol{r}}{r^3})\\ &= -\mu_0\nabla\phi^*\end{aligned}$$

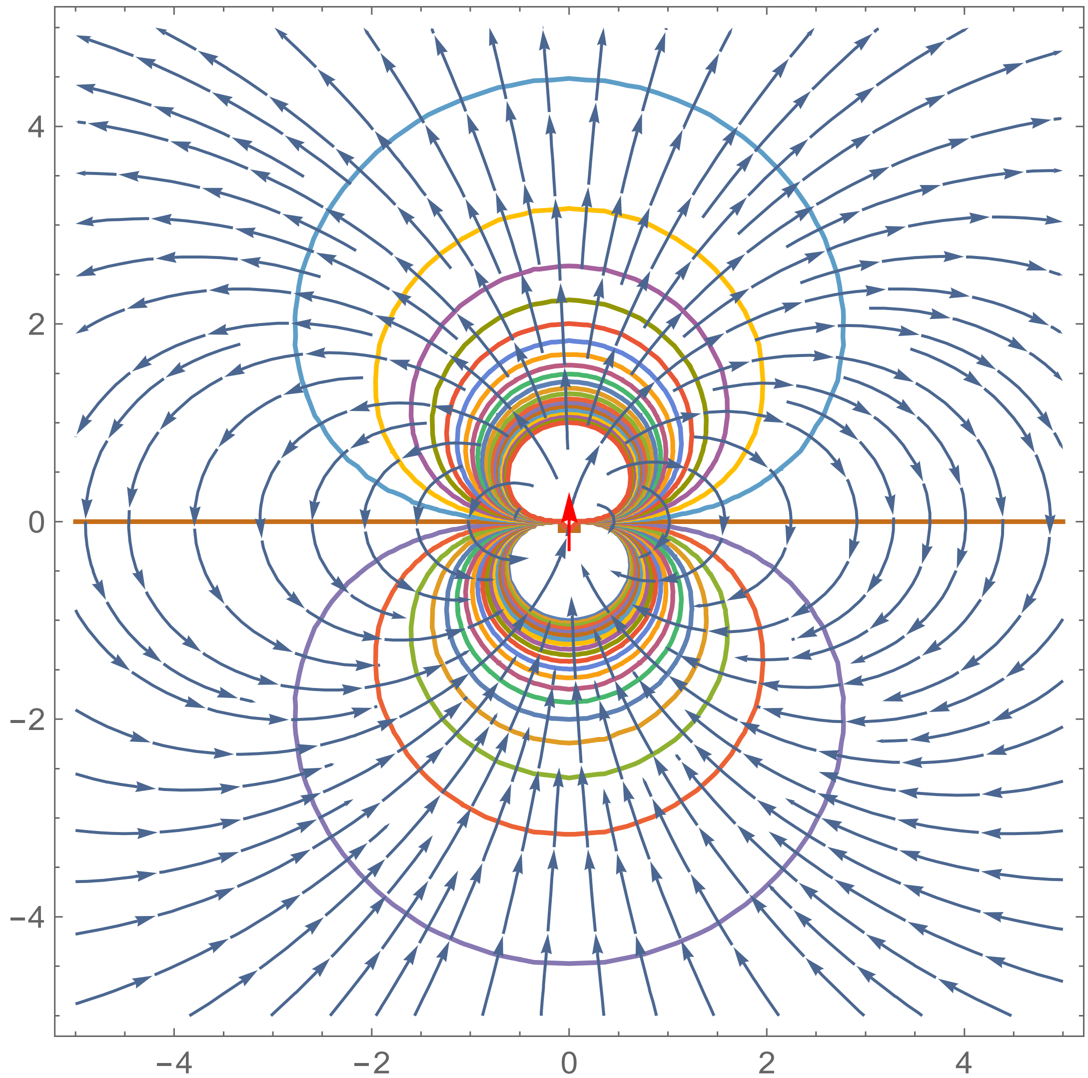
with

$$\phi^* = \frac{\boldsymbol{m}\cdot\boldsymbol{r}}{4\pi r^3}$$

If the magnetic moment $\boldsymbol{m}$ is directed along the z axis,

$$\phi^* = \frac{mz}{4\pi(x^2+z^2)^{3/2}}$$

in the z-x plane ($y = 0$).



24. Vector potential of the magnetic moment

The vector potential $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{r}}{r^3}$$

Where

$$\boldsymbol{m} = m\boldsymbol{e}_z = m(\cos\theta\,\boldsymbol{e}_r - \sin\theta\,\boldsymbol{e}_\theta), \qquad \boldsymbol{r} = r\boldsymbol{e}_r$$

Then we have

$$\boldsymbol{A} = \frac{m\mu_0}{4\pi}\frac{\sin\theta}{r^2}\boldsymbol{e}_\phi$$

in the spherical coordinates. The magnetic field $\boldsymbol{B}$ is evaluated from

$$
\begin{aligned}
\boldsymbol{B} &= \nabla \times \boldsymbol{A} \\
&= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & r\sin\theta \boldsymbol{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & r\sin\theta A_\phi(r,\theta) \end{vmatrix} \\
&= \frac{1}{r^2 \sin\theta} \{\boldsymbol{e}_r \frac{\partial}{\partial \theta}[r\sin\theta A_\phi(r,\theta)] - \boldsymbol{e}_\theta r \frac{\partial}{\partial r}[r\sin\theta A_\phi(r,\theta)]\} \\
&= \frac{m\mu_0}{4\pi} \frac{1}{r^2 \sin\theta} (\boldsymbol{e}_r \frac{1}{r} \frac{\partial}{\partial \theta} \sin^2\theta - \boldsymbol{e}_\theta r \sin^2\theta \frac{\partial}{\partial r} \frac{1}{r}) \\
&= \frac{m\mu_0}{4\pi} (\frac{2\cos\theta}{r^3} \boldsymbol{e}_r + \frac{\sin\theta}{r^3} \boldsymbol{e}_\theta)
\end{aligned}
$$

or

$$
\begin{aligned}
\boldsymbol{B} &= \frac{m\mu_0}{4\pi} [\frac{2\cos\theta}{r^3} (\sin\theta\cos\varphi \boldsymbol{e}_x + \sin\theta \sin\varphi \boldsymbol{e}_y + \cos\theta \boldsymbol{e}_z) \\
&\quad + \frac{\sin\theta}{r^3} (\cos\theta\cos\varphi \boldsymbol{e}_x + \cos\theta\sin\varphi \boldsymbol{e}_y - \sin\theta \boldsymbol{e}_z)] \\
&= \frac{m\mu_0}{4\pi r^3} [2\cos\theta(\sin\theta\cos\varphi \boldsymbol{e}_x + \sin\theta\sin\varphi \boldsymbol{e}_y + \cos\theta \boldsymbol{e}_z) \\
&\quad + \sin\theta(\cos\theta\cos\varphi \boldsymbol{e}_x + \cos\theta\sin\varphi \boldsymbol{e}_y - \sin\theta \boldsymbol{e}_z)]
\end{aligned}
$$

or

$$
\boldsymbol{B} = \frac{m\mu_0}{4\pi r^3} [3\sin\theta\cos\theta\cos\phi \boldsymbol{e}_x + 3\sin\theta\cos\theta\sin\phi \boldsymbol{e}_y + (3\cos^2\theta - 1)\boldsymbol{e}_z]
$$

with the use of the unit vectors of cartesian co-ordinate. When $\phi = 0$ (in the *z*-*x* plane), we have

$$
\boldsymbol{B} = \frac{m\mu_0}{4\pi r^3} [3\sin\theta\cos\theta \boldsymbol{e}_x + (3\cos^2\theta - 1)\boldsymbol{e}_z]
$$

The torque is given by

$$
\boldsymbol{\tau} = \boldsymbol{m} \times \boldsymbol{B} = \boldsymbol{e}_\phi \frac{m^2}{4\pi r^3} 3\sin\theta\cos\theta
$$

where

$$\boldsymbol{m} = m\boldsymbol{e}_z = m(\cos\theta\, \boldsymbol{e}_r - \sin\theta\, \boldsymbol{e}_\theta)$$

$$\boldsymbol{B} = \frac{m\mu_0}{4\pi}\left(\frac{2\cos\theta}{r^3}\boldsymbol{e}_r + \frac{\sin\theta}{r^3}\boldsymbol{e}_\theta\right)$$

((**Mathematica**))

```
Clear["Gobal`"];

Aϕ = (m μ0)/(4 π r^2) Sin[θ];

B1 = Curl[{0, 0, Aϕ}, {r, θ, ϕ}, "Spherical"] // Simplify

{(m μ0 Cos[θ])/(2 π r^3), (m μ0 Sin[θ])/(4 π r^3), 0}

Div[B1, {r, θ, ϕ}, "Spherical"] // Simplify

0
```

25. Magnetic flux arising from the magnetic dipole moment

The magnetic flux is related to the vector potential $\boldsymbol{A}$ by

$$\Phi = \oint \boldsymbol{B}\cdot d\boldsymbol{a} = \oint \boldsymbol{A}\cdot d\boldsymbol{l}$$

$$d\boldsymbol{a} = \boldsymbol{e}_r 2\pi r^2 \sin\theta d\theta$$

$$\begin{aligned}\boldsymbol{B}\cdot d\boldsymbol{a} &= \frac{m\mu_0}{4\pi}\left(\frac{2\cos\theta}{r^3}\boldsymbol{e}_r + \frac{\sin\theta}{r^3}\boldsymbol{e}_\theta\right)\cdot\boldsymbol{e}_r 2\pi r^2\sin\theta d\theta \\ &= \frac{m\mu_0}{4\pi}\frac{2\cos\theta}{r^3}2\pi r^2\sin\theta d\theta \\ &= \frac{m\mu_0}{2r}\sin(2\theta)d\theta\end{aligned}$$

$$\begin{aligned}\Phi &= \oint \boldsymbol{B}\cdot d\boldsymbol{a} \\ &= \frac{m\mu_0}{2r}\int_0^{\theta_0}\sin(2\theta)d\theta \ , \\ &= \frac{m\mu_0}{4r}[1-\cos(2\theta_0)]\end{aligned}$$

How about $\oint \boldsymbol{A} \cdot d\boldsymbol{l}$?

$$\boldsymbol{A} = \boldsymbol{e}_\phi \frac{m\mu_0}{4\pi r^2} \sin\theta_0 . \qquad d\boldsymbol{l} = \boldsymbol{e}_\phi dl = \boldsymbol{e}_\phi (2\pi r \sin\theta_0)$$

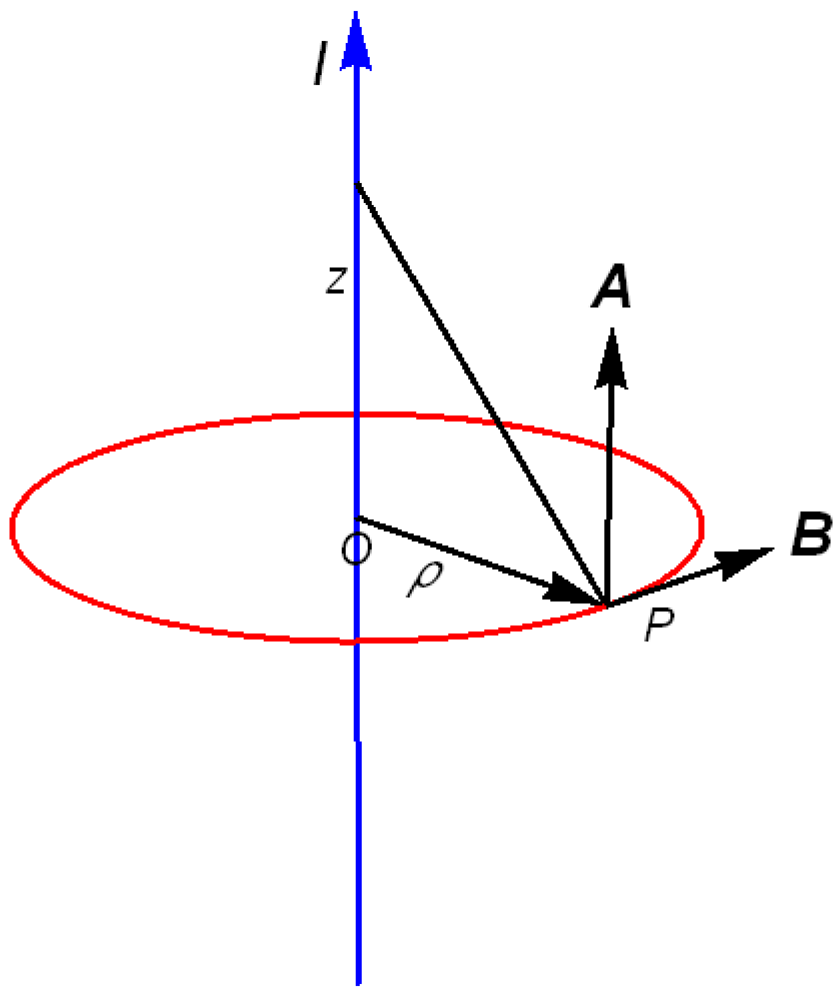
$$\boldsymbol{A} \cdot d\boldsymbol{l} = \boldsymbol{e}_\phi \frac{m\mu_0}{4\pi r^2} \sin\theta_0 \cdot \boldsymbol{e}_\phi dl = \frac{m\mu_0}{4\pi r^2} \sin\theta_0 dl$$

$$\begin{aligned} \oint \boldsymbol{A} \cdot d\boldsymbol{l} &= \int \frac{m\mu_0}{4\pi r^2} \sin\theta_0 dl \\ &= \frac{m\mu_0}{4\pi r^2} \sin\theta_0 (2\pi r \sin\theta_0) \\ &= \frac{m\mu_0}{4r} [1 - \cos(2\theta_0)] \end{aligned}$$

Using the above example, we show that

$$\Phi = \oint \boldsymbol{B} \cdot d\boldsymbol{a} = \oint \boldsymbol{A} \cdot d\boldsymbol{l}$$

26. Vector potential and magnetic field due to a straight infinitely long current wire



We calculate the magnetic field due to an infinitely long electric wire in which the current I flows.

We first calculate the vector potential,

$$\begin{aligned}\boldsymbol{A} &= \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{r}')d^3\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|} \\ &= \boldsymbol{e}_z \frac{\mu_0 I}{4\pi}\int_{-L}^{L}\frac{dz}{\sqrt{\rho^2+z^2}} \\ &= \boldsymbol{e}_z \frac{\mu_0 I}{2\pi}\int_{0}^{L}\frac{dz}{\sqrt{\rho^2+z^2}}\end{aligned}$$

or

$$A_z = \frac{\mu_0 I}{2\pi}\int_0^L \frac{dz}{\sqrt{\rho^2+z^2}} = \frac{\mu_0 I}{2\pi}\ln\left(\frac{L+\sqrt{L^2+\rho^2}}{\rho}\right)$$

which depends only on ρ. The magnetic field $\boldsymbol{B}$ can be calculated using the formula

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}$$

as

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} = \frac{1}{\rho}\begin{vmatrix} \boldsymbol{e}_\rho & \rho\boldsymbol{e}_\varphi & \boldsymbol{e}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\varphi & A_z \end{vmatrix} = \frac{1}{\rho}\begin{vmatrix} \boldsymbol{e}_\rho & \rho\boldsymbol{e}_\varphi & \boldsymbol{e}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ 0 & 0 & A_z(\rho) \end{vmatrix}$$

or

$$\boldsymbol{B} = -\frac{1}{\rho}\rho\boldsymbol{e}_\phi\frac{\partial}{\partial\rho}A_z(\rho) = -\boldsymbol{e}_\phi\frac{\partial}{\partial\rho}A_z(\rho)$$

using the cylindrical coordinate. Noting that

$$\frac{\partial}{\partial\rho}\ln\left(\frac{L+\sqrt{L^2+\rho^2}}{\rho}\right) = -\frac{L}{\rho\sqrt{L^2+\rho^2}} \approx -\frac{1}{\rho}$$

we get the final result

$$\boldsymbol{B} = \boldsymbol{e}_\phi\frac{\mu_0 I}{2\pi\rho}.$$

in the limit of $L\to\infty$.

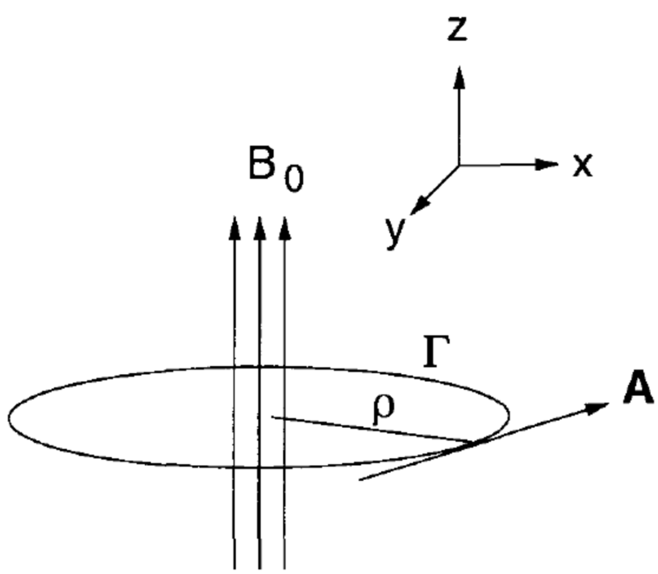
27. Vector potential for a solenoid

A solenoid has radius R, current I, and n turns per unit length. Given that the magnetic field is $B = \mu_0 nI$ inside and $B = 0$ outside, find the vector potential $\boldsymbol{A}$ both inside and outside. Do this in two ways as follows.

(a) Find the magnetic field inside and outside the solenoid using the Ampere's law.

(b) Use the expression for the curl in cylindrical coordinates to find the forms of $\boldsymbol{A}$ that yield the correct values of $\boldsymbol{B} = \nabla\times\boldsymbol{A}$ in the two regions.

[Problem 6-18, **E.M. Purcell and D.J. Morin, Electricity and Magnetism, 3rd edition (Cambridge, 2013))**



There is a magnetic field due to the infinitely long solenoid (the flowing current is *I*). The magnetic field inside the solenoid is

$$B_z = \mu_0 n I$$

Outside the solenoid, there is no magnetic field; $B = 0$.

$$\Phi = \oint \boldsymbol{B} \cdot d\boldsymbol{a} = \oint \boldsymbol{A} \cdot d\boldsymbol{l}$$

(i) $\rho < R$ (inside)

$$B(\pi\rho^2) = 2\pi\rho A_\phi, \qquad A_\phi = \frac{B(\pi\rho^2)}{2\pi\rho} = \frac{B\rho}{2} = \frac{1}{2}\mu_0 n I \rho$$

which is the same form as $\boldsymbol{A} = \frac{1}{2}\boldsymbol{B} \times \boldsymbol{r}$.

(ii) $\rho > R$ (outside)

$$B(\pi R^2) = 2\pi\rho A_\phi, \qquad A_\phi = \frac{B(\pi R^2)}{2\pi\rho} = \frac{BR^2}{2\rho} = \frac{R^2}{2\rho}\mu_0 n I$$

We make a plot of $A_\phi / A_0 = \begin{cases} x & (x<1) \\ \dfrac{1}{x} & (x>1) \end{cases}$, where $A_0 = \dfrac{R}{2}\mu_0 nI$

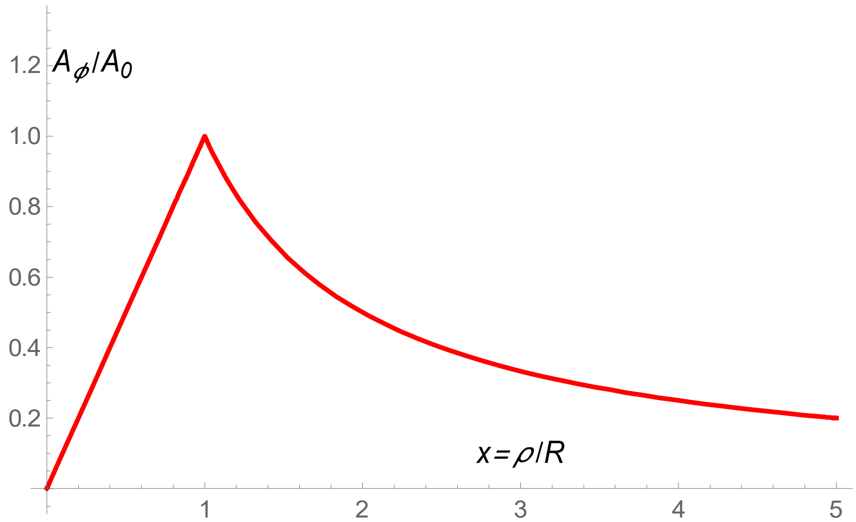


Fig. Plot of $\dfrac{A_\phi}{A_0}$ vs $x = \rho / R$, where $A_0 = \dfrac{R}{2}\mu_0 nI$.

We now calculate the magnetic field using the formula:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \frac{1}{\rho}\begin{vmatrix} \boldsymbol{e}_\rho & \rho\boldsymbol{e}_\varphi & \boldsymbol{e}_z \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\varphi} & \dfrac{\partial}{\partial z} \\ A_\rho & \rho A_\varphi & A_z \end{vmatrix} = \frac{1}{\rho}\begin{vmatrix} \boldsymbol{e}_\rho & \rho\boldsymbol{e}_\varphi & \boldsymbol{e}_z \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\varphi} & \dfrac{\partial}{\partial z} \\ 0 & \rho A_\varphi(\rho) & 0 \end{vmatrix}$$

or

$$\boldsymbol{B} = \frac{1}{\rho}\boldsymbol{e}_z \frac{\partial}{\partial\rho}[\rho A_\varphi(\rho)]$$

The magnetic field ***B*** can be calculated as

$$\boldsymbol{B} = \begin{cases} \boldsymbol{e}_z \mu_0 nI & (\rho < R) \\ 0 & (\rho > R) \end{cases}$$

We note that there is some arbitrariness in choosing the form of A_ϕ

$$A_\phi = \frac{R^2}{2\rho}\mu_0 nI + \frac{C}{\rho}$$

as an addition of the form C/ρ, where C is a constant, since

$$\rho A_\phi = \frac{R^2}{2}\mu_0 nI + C = \text{constant}$$

and

$$\frac{\partial}{\partial\rho}[\rho A_\varphi(\rho)] = 0$$

28. Ampere's law and vector potential

From the Ampere's law, we have

$$B_\phi(2\pi r) = \mu_0 I, \qquad B_\phi = \frac{\mu_0 I}{2\pi r}$$

From the relation $\boldsymbol{B} = \nabla\times\boldsymbol{A}$

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} = \frac{1}{\rho}\begin{vmatrix} \boldsymbol{e}_\rho & \rho\boldsymbol{e}_\varphi & \boldsymbol{e}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\varphi & A_z \end{vmatrix}$$

$$\boldsymbol{B} = -\boldsymbol{e}_\phi \frac{\partial}{\partial\rho}A_z = \boldsymbol{e}_\phi \frac{\mu_0 I}{2\pi\rho}$$

$$\frac{\partial}{\partial\rho}A_z = -\frac{\mu_0 I}{2\pi\rho}$$

or

$$A_z(\rho) = -\frac{\mu_0 I}{2\pi}\ln\rho$$

29. Interaction between magnetic dipole moments.

We now consider the magnetic interaction between two magnetic moments $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$. The interaction is given by a Zeeman energy

$$U = -\boldsymbol{m}_2 \cdot \boldsymbol{B}_1$$

where $\boldsymbol{B}_1$ is the magnetic field due to the moment $\boldsymbol{m}_1$,

$$\boldsymbol{B}_1 = \frac{\mu_0}{4\pi r^5}[3(\boldsymbol{m}_1 \cdot \boldsymbol{r})\boldsymbol{r} - \boldsymbol{m}_1 r^2]$$

Then we have the dipole-dipole interaction

$$U = \frac{\mu_0}{4\pi}[\frac{\boldsymbol{m}_1 \cdot \boldsymbol{m}_2}{r^3} - 3\frac{(\boldsymbol{m}_1 \cdot \boldsymbol{r})(\boldsymbol{m}_2 \cdot \boldsymbol{r})}{r^5}]$$

Two dipoles have an attractive force between them if they are arranged end to end a repulsive force if they are oppositely arranged end to end. This result is the explanation of the familiar result that like magnetic poles repel and unlike attract.

Dipole configuration (see the figure below)

(a)

$$U_a = -2\frac{\mu_0 m_1 m_2}{4\pi r^3}$$ (attractive) antiferomagnetic arrangement

(b)

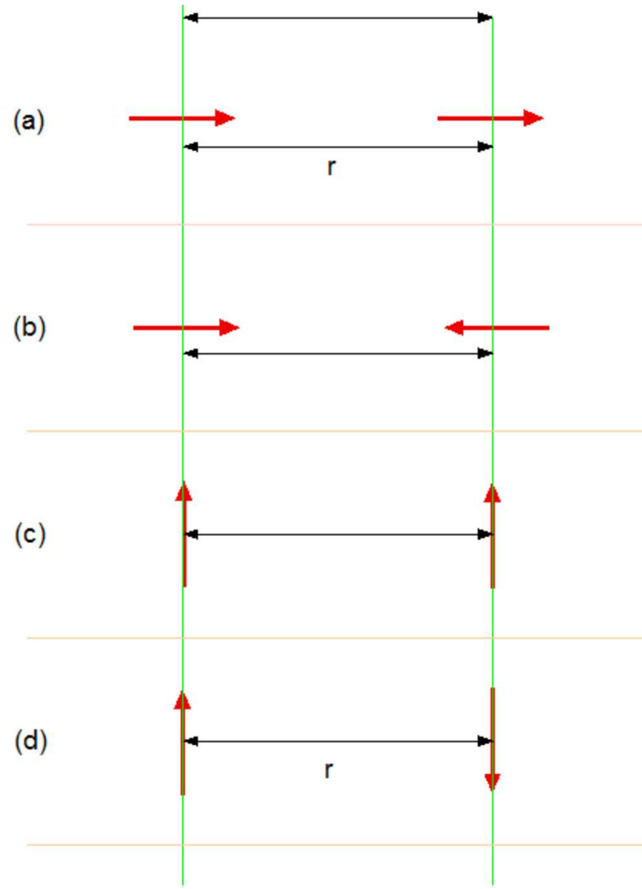
$$U_b = 2\frac{\mu_0 m_1 m_2}{4\pi r^3}$$ (repulsive)

(c)

$$U_c = \frac{\mu_0 m_1 m_2}{4\pi r^3}$$ (repulsive)

(d)

$$U_d = -\frac{\mu_0 m_1 m_2}{4\pi r^3}$$ (attractive) ferromagnetic arrangement



30. Torque due to the magnetic dipole moment

Here we show that the torque due to the magnetic moment m in the presence of a uniform magnetic field is given by

$$\boldsymbol{\tau} = \boldsymbol{m} \times \boldsymbol{B}.$$

The force on an infinitesimal element of a charge-carrying conductor in the presence of a magnetic field $\boldsymbol{B}$ is given by

$$d\boldsymbol{F} = Id\boldsymbol{s} \times \boldsymbol{B}$$

The infinitesimal torque is

$$d\boldsymbol{\tau} = \boldsymbol{r} \times d\boldsymbol{F}$$

If the circuit in question is represented by the contour C, then the torque on a complete closed circuit is

$$\boldsymbol{\tau} = \oint d\boldsymbol{\tau} = I\oint \boldsymbol{r} \times (d\boldsymbol{s} \times \boldsymbol{B})$$

Note that

$$\boldsymbol{r}\times(d\boldsymbol{s}\times\boldsymbol{B})+d\boldsymbol{s}\times(\boldsymbol{B}\times\boldsymbol{r})+\boldsymbol{B}\times(\boldsymbol{r}\times d\boldsymbol{s})=0\,,\quad \text{(formula)}$$

But we have

$$d[\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{B})]=d\boldsymbol{s}\times(\boldsymbol{r}\times\boldsymbol{B})+\boldsymbol{r}\times(d\boldsymbol{s}\times\boldsymbol{B})=0$$

where $\boldsymbol{B}$ is constant and $d\mathbf{r}=d\mathbf{s}$.

$$\boldsymbol{r}\times(d\boldsymbol{s}\times\boldsymbol{B})=d\boldsymbol{s}\times(\boldsymbol{B}\times\boldsymbol{r})$$

Then we have

$$\boldsymbol{r}\times(d\boldsymbol{s}\times\boldsymbol{B})=-\frac{1}{2}\boldsymbol{B}\times(\boldsymbol{r}\times d\boldsymbol{s})$$

leading to

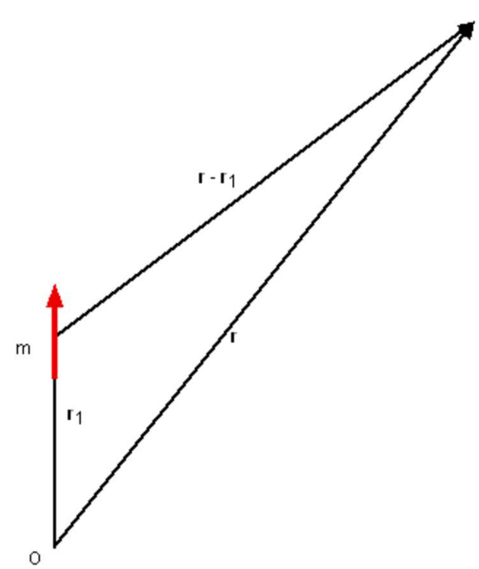
$$\boldsymbol{\tau}=I\oint\boldsymbol{r}\times(d\boldsymbol{s}\times\boldsymbol{B})=\frac{I}{2}\oint(\boldsymbol{r}\times d\boldsymbol{s})\times\boldsymbol{B}=\boldsymbol{m}\times\boldsymbol{B}$$

The magnetic moment is defined by

$$\boldsymbol{m}=I\boldsymbol{A}=\frac{I}{2}\oint(\boldsymbol{r}\times d\boldsymbol{s})$$

where $\boldsymbol{A}$ is the area enclosed by the curve. The direction of A is perpendicular to the plane.

31. The field of a magnetized object



The vector potential at the point P due to the magnetic dipole moment $\boldsymbol{m}$ is given by

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \frac{\boldsymbol{m} \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}$$

$\boldsymbol{M}$ is the magnetic dipole moment per unit volume.

$$\boldsymbol{m} = \boldsymbol{M}(\boldsymbol{r}')d\tau'$$

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{M}(\boldsymbol{r}') \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} d\tau'$$

$$\nabla' \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3}$$

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \boldsymbol{M}(\boldsymbol{r}') \times \nabla' \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} d\tau'$$

Using the formula

$$M(\boldsymbol{r}') \times \nabla' \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \nabla' \times M(\boldsymbol{r}') - \nabla' \times \frac{M(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}$$

we get

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{r}) &= \frac{\mu_0}{4\pi} \int \frac{\nabla' \times M(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} d\tau' - \frac{\mu_0}{4\pi} \int \nabla' \times \left(\frac{M(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \right) d\tau' \\ &= \frac{\mu_0}{4\pi} \int_V \frac{\nabla' \times M(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} d\tau' + \frac{\mu_0}{4\pi} \int_S \left(\frac{M(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \right) \times d\boldsymbol{a}' \end{aligned}$$

Here we use the formula

$$\int_V (\nabla \times \boldsymbol{v}) d\tau' = -\int_A (\boldsymbol{v} \times d\boldsymbol{a}')$$

Definition of current density

$$\boldsymbol{J}_b = \nabla \times \boldsymbol{M}$$

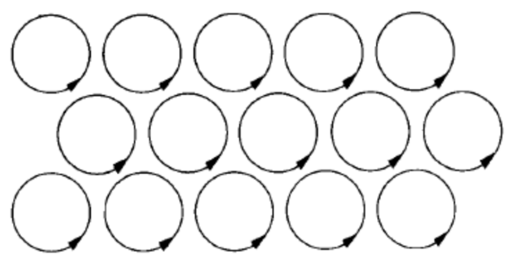
$$\boldsymbol{K}_b = \boldsymbol{M}(\boldsymbol{r}) \times \boldsymbol{n}$$

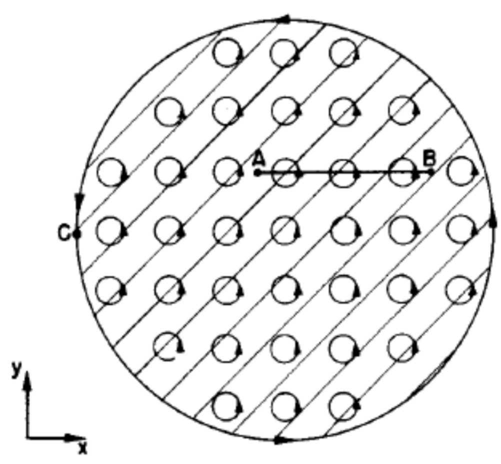
where $\boldsymbol{n}$ is the normal unit vector. The unit of $\boldsymbol{J}_b$ is $\frac{Am^2}{m^3}\frac{1}{m} = \frac{A}{m^2}$ (current density). The init of K_b is $\frac{Am^2}{m^3} = \frac{A}{m}$ (surface current density). The vector potential is now given by

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\boldsymbol{J}_b(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}d\tau' + \frac{\mu_0}{4\pi}\int_S \frac{K_b(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}da'$$

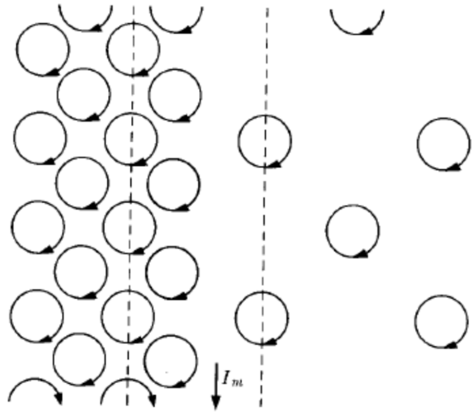
32. Physical meaning of magnetization current

The magnetization M provides us with a macroscopic description of the atomic currents inside matter. Each atomic current circuit produces a magnetic moment. The magnetization M is defined as the magnetic moments per unit volume.



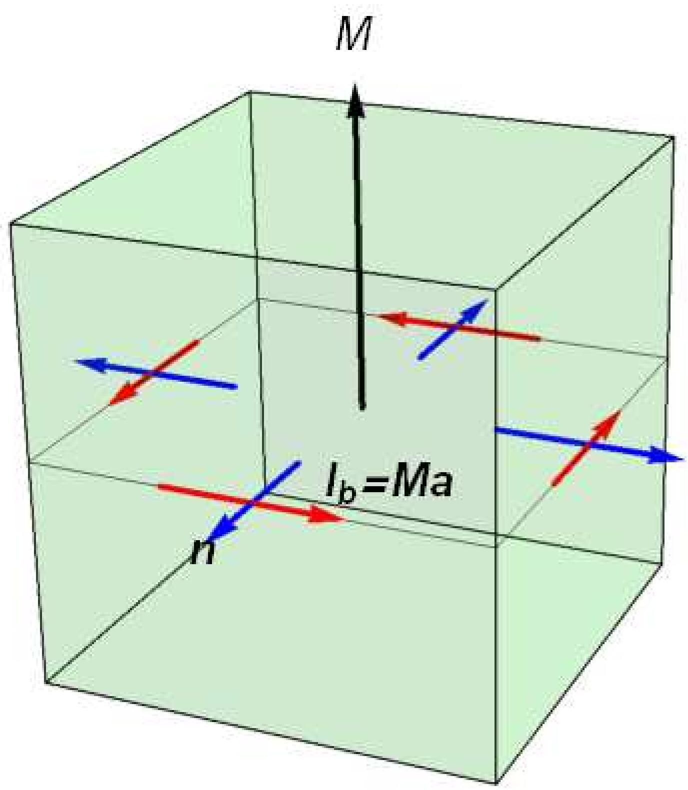


First we consider a simplified model of magnetized matter. It consisted of atomic loop currents circulating in the same direction, side by side. If the magnetization is uniform, the currents in the various loops tend to cancel each other out, and there is no net effective currents in the interior of the material. Only at the surface there is a net current always going in the same direction.



If the magnetization is non-uniform, the cancellation will not be complete. As an example of non-uniform magnetization, we consider the abrupt change in magnetization shown in the above figure. If we focus our attention on the region between the dotted lines, it is evident that there is more charge moving down than there is moving up. This we call the magnetization current. Thus, even though there is no charge transport, there is an effective motion of charge downward, and this current can produce a magnetic field.

33. Magnetic moment from surface current of cubic form



The volume of cube is given by

$$V = a^3$$

The magnetic moment *m* is given by

$$m = Ma^3 = I_b a^2$$

where *a* is the length of the side a for cube. Thus we have

$$I_b = Ma$$

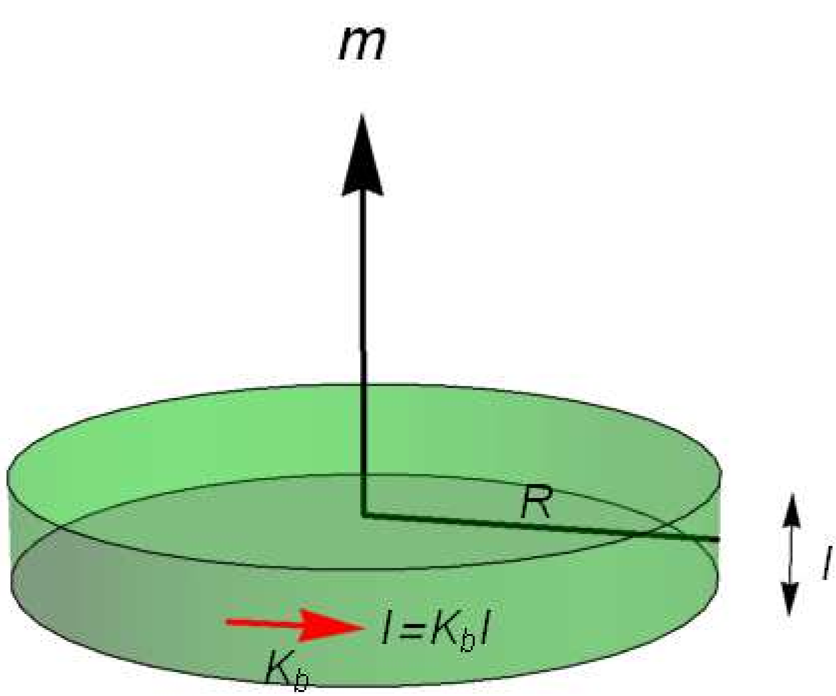
Note that

$$\boldsymbol{K}_b = \boldsymbol{M} \times \boldsymbol{n}, \qquad K_b = M$$

where the direction of *M* is perpendicular to that of the vector ***n***. The surface current is

$$I_b = K_b a = Ma$$

34. Magnetic moment from the surface current of disk-like form



The magnetic moment *m* is given by

$$m = M(\pi R^2 l) = I\pi R^2 = (K_b l)\pi R^2$$

leading to

$$I = Ml = K_b l$$

$$K_b = M,$$

$$\boldsymbol{K}_b = \boldsymbol{M} \times \boldsymbol{n}$$

where *M* is the magnetization, the magnetic moment per unit volume.

Note that the magnetic field at the center is

$$B = \frac{\mu_0 I}{2R^2} = \frac{\mu_0 M l}{2R^2}$$

When $R \to \infty$ and $l \to \infty$, $B \to 0$. Using the relation

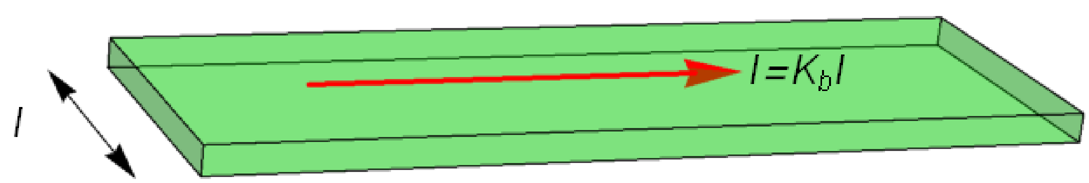
$$B = \mu_0 (H + M) = 0$$

we have

$$H = -M$$

which means that the demagnetization factor is $N_d = 1$.

35. Surface current and boundary of *B*-field



The charge contributing to the surface current

$$\Delta Q = \sigma (v \Delta t) l$$

The surface current:

$$I = \frac{\Delta Q}{\Delta t} = \sigma l v = K_b l$$

K_b is the unit of [A/m].

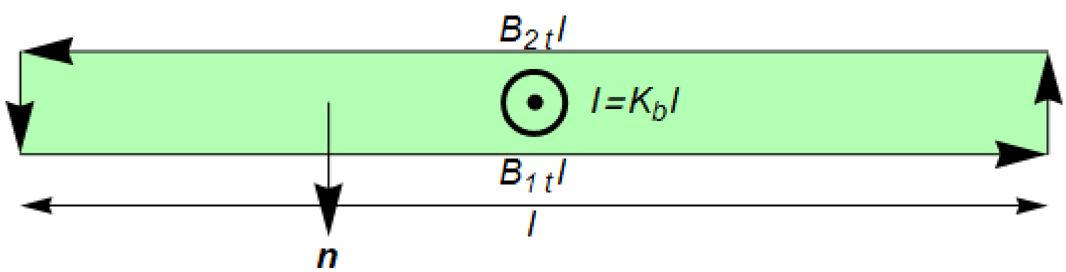
Boundary condition:

$$\oint (\nabla \times \boldsymbol{B}) \cdot d\boldsymbol{a} = \oint \boldsymbol{B} \cdot d\boldsymbol{s}$$

$$(\boldsymbol{B}_2 - \boldsymbol{B}_1)_t \Delta l = \mu_0 K_b \Delta l$$

or

$$(\boldsymbol{B}_2 - \boldsymbol{B}_1)_t = \mu_0 (\boldsymbol{K}_b \times \boldsymbol{n})$$



36. Equivalent current density of magnetization

We show that the volume current and surface current due to magnetic moment is

$$\boldsymbol{J}_b = \nabla \times \boldsymbol{M}, \qquad \boldsymbol{K}_b = \boldsymbol{M} \times \boldsymbol{n}$$

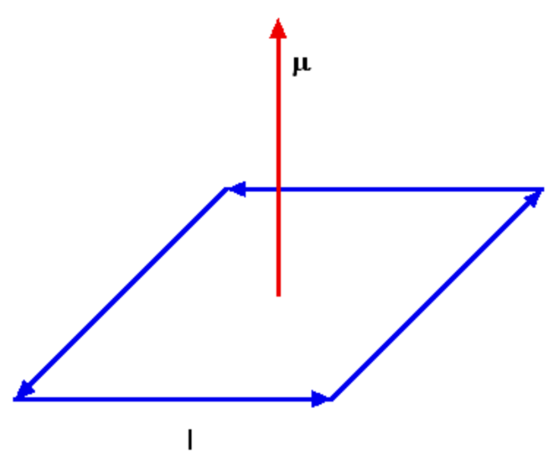


Fig. The magnetic dipole moment μ of a current loop with area A

A circular current I has a magnetic moment μ given by

$$\mu = IA$$

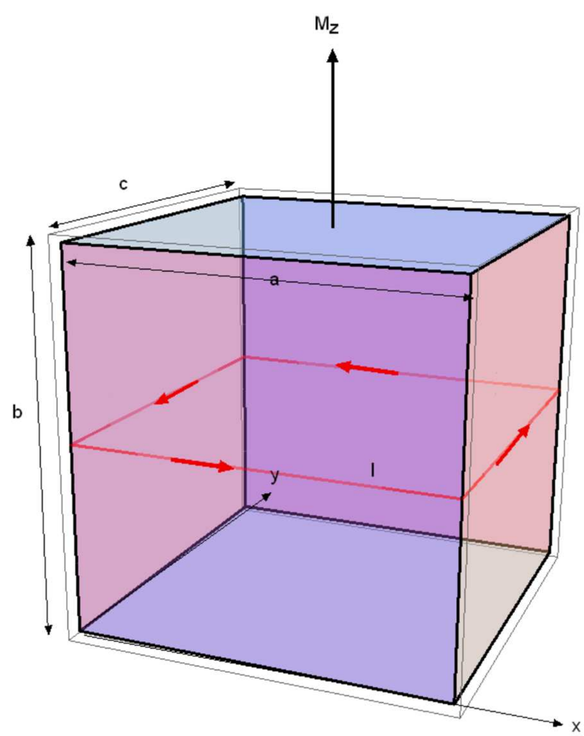
where A is the area of the current loop.

We now consider a small rectangular block inside of a magnetized material. We take the block so small that we can consider that the magnetization is uniform inside it. If this block has a magnetization M_z in the z direction. The total magnetic moment is

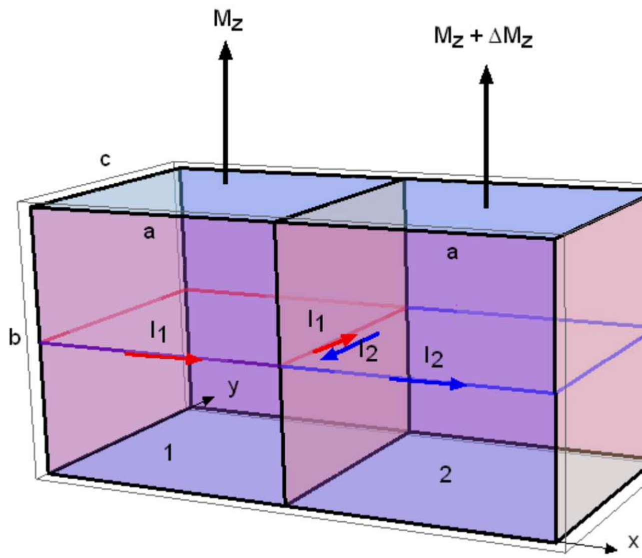
$$\mu = M_z(abc) = [M_z b]ac = Iac$$

So the surface current going around on the vertical faces is

$$I = M_z b$$



A small magnetized block is equivalent to a circulating surface current



If the magnetization of two neighboring blocks is not the same, there is a net surface current in between.

Now suppose that we imagine two such blocks (denoted by 1 and 2) next to each other. The block 1 has a magnetization M_z, while the block 2 has a magnetization $M_z + \Delta M_z$. The block 1 will produce a surface current I_1 flowing in the positive y direction. The block 2 will produce a surface current I_2 flowing in the negative y direction. The total surface current in the positive y-direction is given by

$$I = I_1 - I_2 = M_z b - (M_z + \Delta M_z)b = -\Delta M_z b = -\frac{\partial M_z}{\partial x} ab$$

Then the current density J_y is obtained as

$$J_y = -\frac{\partial M_z}{\partial x}$$

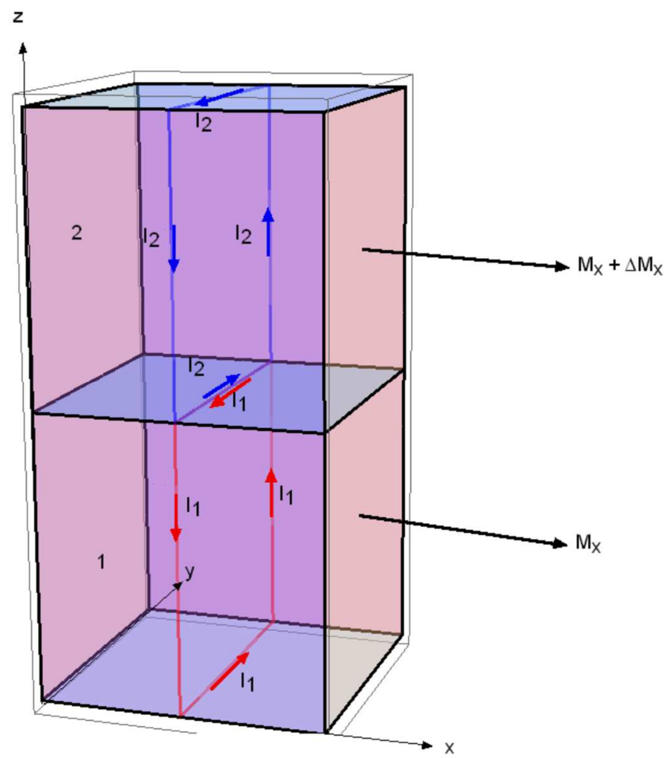


Fig. Two blocks, one above the other, may also contribute to J_y.

Next we imagine two such blocks (denoted by 1 and 2) next to each other. The block 1 has a magnetization M_x, while the block 2 has a magnetization $M_x + \Delta M_x$. The block 1 will produce a surface current I_1 flowing in the negative y direction. The block 2 will produce a surface current I_2 flowing in the positive y direction. The total surface current in the positive y-direction is given by

$$I = I_1 - I_2 = -M_x a + (M_x + \Delta M_x)a = \Delta M_x a = \frac{\partial M_x}{\partial z} ab$$

The current density J_y is obtained as

$$J_y = \frac{\partial M_x}{\partial z}$$

The resulting current density along the y direction is

$$J_y = \frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x} = (\nabla \times \boldsymbol{M})_y$$

or

$$\boldsymbol{J}_b = \nabla \times \boldsymbol{M}$$

which also means that

$$\nabla \cdot \boldsymbol{J}_b = 0$$

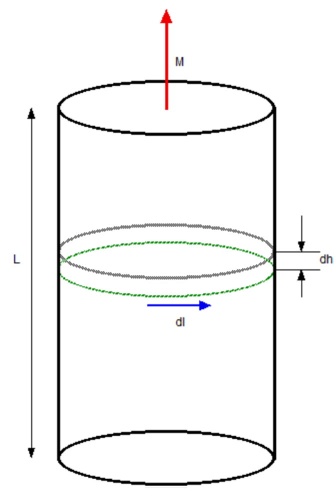
From the continuity of equation, this implies that there is no magnetic pole.

37. Magnetic field due to the uniformly magnetized cylinder

The magnetic field ***B*** due to the uniformly magnetized cylinder is given by

$$\boldsymbol{B} = \mu_0 \boldsymbol{M}$$

The proof is as follows.



Suppose that the cylinder is uniformly magnetized. The direction of the magnetization M is along the z axis. We now consider a small disk (area A and thickness dh) in the cylinder. The total magnetic moment inside the disk is

$$\Delta\mu = M(Adh) = dIA$$

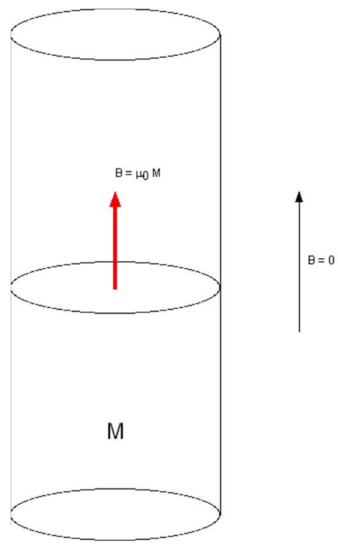
where dI is the equivalent surface current. From this we have

$$dI = Mdh$$

This current flowing on the surface of the cylinder forms a solenoid coil with the number of the coil's turns per unit length given by $n = 1/(\mathrm{d}h)$. The magnetic field B in side the long solenoid with sufficiently long length is

$$B = \mu_0 ndI = \mu_0 \frac{1}{dh} dI = \mu_0 \frac{1}{dh} Mdh = \mu_0 M$$

The magnetic field ***B*** outside the cylinder is equal to zero.

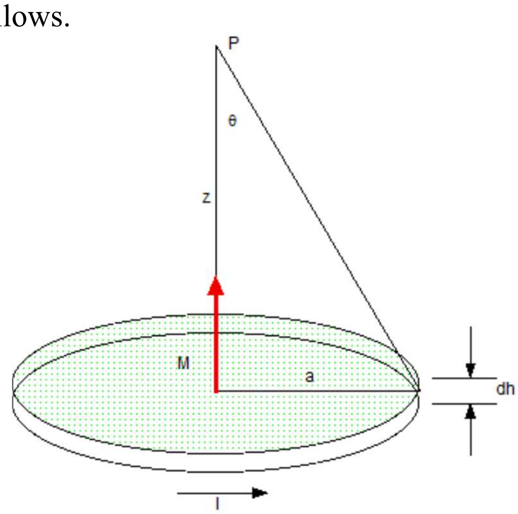


38. Magnetic field due to the uniformly magnetized disk

The magnetic field due to the uniformly magnetized disk is given by

$B = 0$.

The proof is given as follows.



We consider the small disk (radius a and thickness dh). The total magnetic moment inside the disk is

$$\Delta\mu = M(\pi a^2 dh) = I(\pi a^2)$$

where I is the surface current. From this we have

$$I = Mdh$$

((**Another method**)) The surface current density $\boldsymbol{K}_b$ is related by the magnetization $\boldsymbol{M}$ by the relation

$$\boldsymbol{K}_b = \boldsymbol{M} \times \boldsymbol{n} = M\boldsymbol{e}_\phi$$

Then the surface current is obtained as $K_b\, dh = M dh$

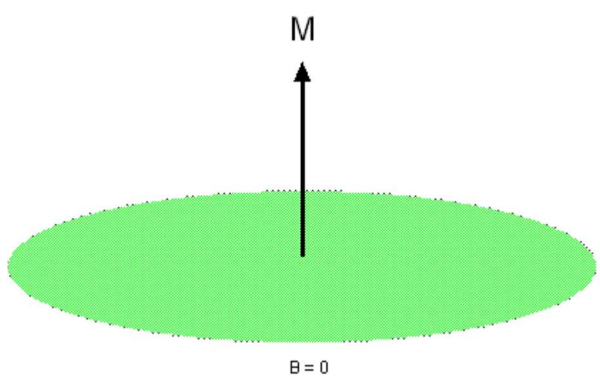
Using the Biot-Savart law, the magnetic field $\boldsymbol{B}$ at the point P can be calculated as

$$B_z(P) = \frac{\mu_0 M dh}{2} \frac{a^2}{(z^2 + a^2)^{3/2}}$$

When $z = 0$, we have

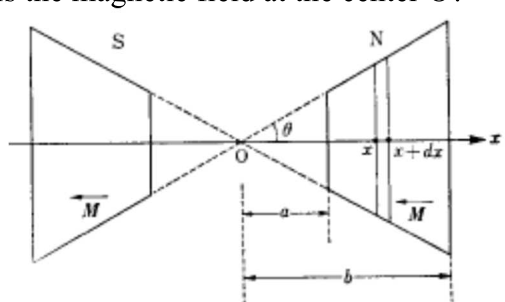
$$B_z(P) = \frac{\mu_0 M dh}{2a}$$

When $dh \to 0$ and $a \to \infty$, $B_z(P) = 0$.



39. Example

We now consider two of a part of circular cone which are magnetized with the magnetization M. What is the magnetic field at the center O?



$$dB_x(O) = \frac{\mu_0 M dx}{2} \frac{x^2 \tan^2\theta}{(x^2 + x^2\tan^2\theta)^{3/2}} = \frac{\mu_0 M dx}{2x}\sin^2\theta\cos\theta$$

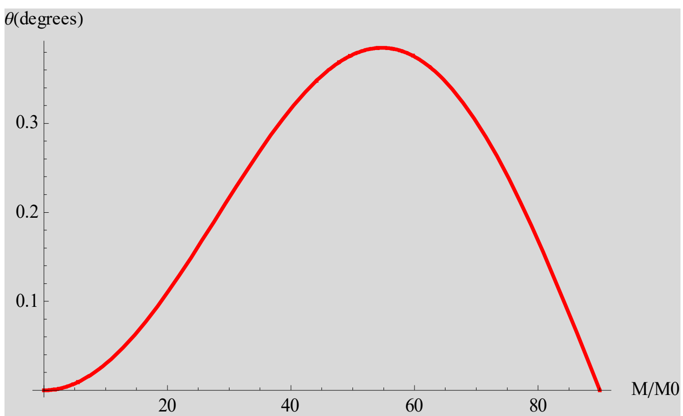
from one of the circular cone. Taking into account of the contributions from the two cones, we have

$$B_x(O) = 2\frac{\mu_0 M \sin^2\theta\cos\theta}{2}\int_a^b \frac{dx}{x} = \mu_0 M \sin^2\theta\cos\theta \ln(\frac{b}{a})$$
$$= \mu_0 M_0 \sin^2\theta\cos\theta$$

where

$$M_0 = M\ln(\frac{b}{a})$$

B_x(O) has a maximum (= 0.3849 $\mu_0 M_0$) at $\cos\theta = \frac{1}{\sqrt{3}}$ (θ= 54.6356º).



40. Magnetic field due to the uniformly magnetized sphere

The magnetic field due to the uniformly magnetized sphere is given by

$$\boldsymbol{B} = \frac{2}{3}\mu_0 \boldsymbol{M}$$

The proof is as follows. We consider the uniformly magnetized sphere with radius R. The surface current $\boldsymbol{K}_b$ is related by $\boldsymbol{M}$ by the relation

$$\boldsymbol{K}_b = \boldsymbol{M}\times\boldsymbol{n} = M\sin\theta\hat{\varphi}$$

In other words, the surface current in the surface region (θ– θ+dθ) is given by

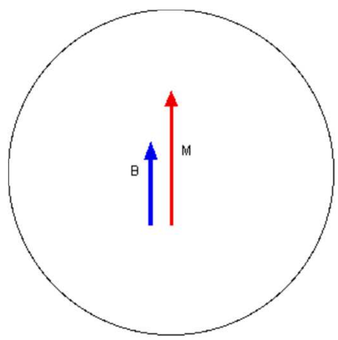
$$I = K_b (Rd\theta) = M \sin\theta (Rd\theta)$$

Using the Biot-Savart law, we can calculate the magnetic field ***B*** at the center O as

$$dB_z(O) = \frac{\mu_0 (M \sin\theta) Rd\theta}{2} \frac{R^2 \sin^2\theta}{R^3}$$
$$= \frac{\mu_0 M}{2} \sin^3\theta d\theta$$

Then we have

$$B_z(O) = \frac{\mu_0 M}{2} \int_0^{\pi} \sin^3\theta d\theta = \frac{\mu_0 M}{2} \frac{4}{3} = \frac{2}{3} \mu_0 M$$

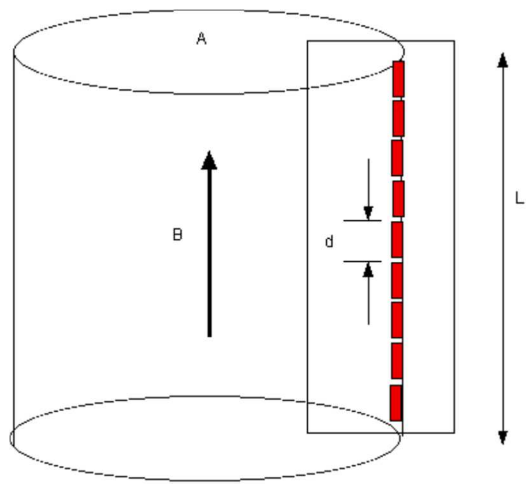


Note that in general the internal field is constant and is given by

$$\boldsymbol{B} = \frac{2\mu_0}{3} \boldsymbol{M}$$

any place inside the sphere.

41. Understanding of *M* using solenoid



We now consider the magnetic field of the solenoid. The total number of turns is N; $N = \frac{L}{d}$, where d is the length of the coil along the cylindrical axis. Using the Ampere's law, we have

$$BL = \mu_0 N i$$

or

$$B = \mu_0 \frac{N}{L} i = \mu_0 \frac{i}{d}$$

where i is the current flowing in the solenoid coil.

The magnetization M is given by

$$M = \frac{\mu_{tot}}{V} = \frac{NiA}{AL} = \frac{\frac{L}{d} iA}{AL} = \frac{i}{d}$$

Then we have the relation between B and M, given by

$$B = \mu_0 M$$

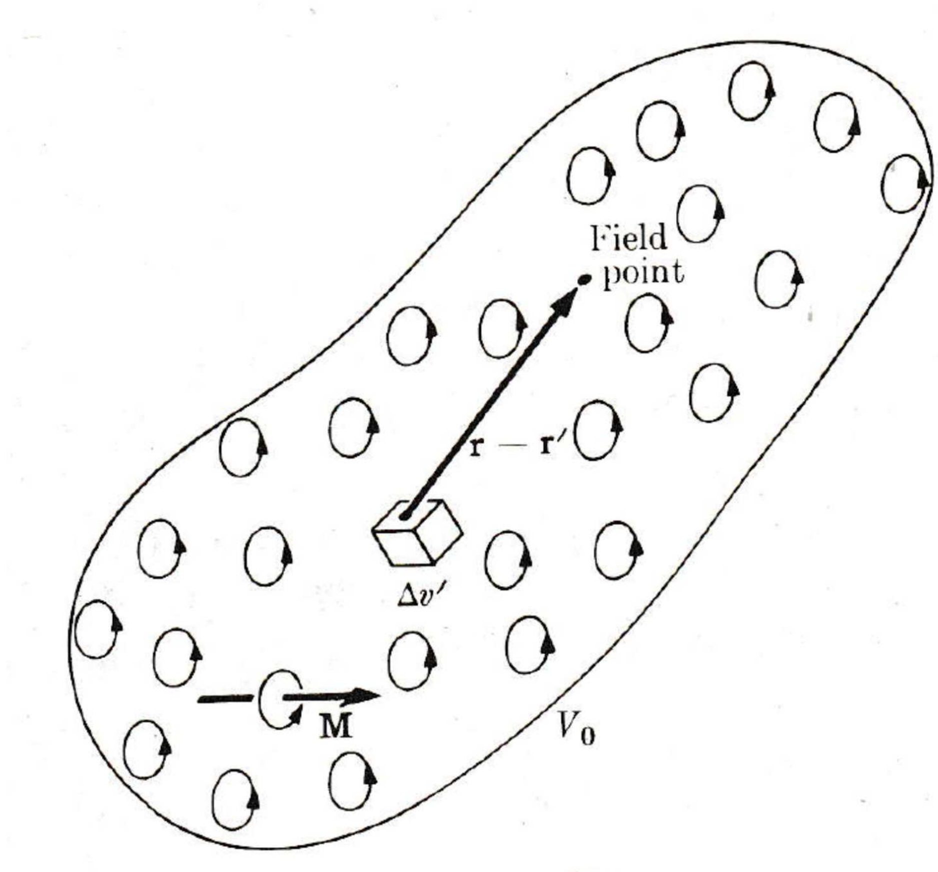
42. Vector potential due to the magnetic moment

The vector potential from the magnetic moment $\boldsymbol{m}$ is

$$\boldsymbol{A} = \frac{\mu_0}{4\pi} \frac{\boldsymbol{m} \times \boldsymbol{r}}{r^3}$$

The vector potential from the magnetization vector $\boldsymbol{M}$ is

$$A(r) = \frac{\mu_0}{4\pi}\int \frac{M(r')\times(r-r')}{|r-r'|^3} d^3r'$$



Using the relation

$$\nabla' \frac{1}{|r-r'|} = \frac{r-r'}{|r-r'|^3}$$

we get

$$A(r) = \frac{\mu_0}{4\pi}\int M(r')\times\nabla' \frac{1}{|r-r'|} d^3r'$$

We note that

$$\nabla'\times\frac{M(r')}{|r-r'|} = \frac{1}{|r-r'|}\nabla'\times M(r') - M(r')\times\nabla'\frac{1}{|r-r'|}$$

or

$$\boldsymbol{M}(\boldsymbol{r}')\times\nabla'\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}=\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\nabla'\times\boldsymbol{M}(\boldsymbol{r}')-\nabla'\times\frac{\boldsymbol{M}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

Thus we hav e

$$\boldsymbol{A}(\boldsymbol{r})=\frac{\mu_0}{4\pi}\int\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\nabla'\times\boldsymbol{M}(\boldsymbol{r}')d^3\boldsymbol{r}'-\frac{\mu_0}{4\pi}\int\nabla'\times\frac{\boldsymbol{M}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3\boldsymbol{r}'$$

Here we use the following formula

$$\int(\nabla\times\boldsymbol{v})d^3\boldsymbol{r}=\int\boldsymbol{v}\times d\boldsymbol{a}$$

((**Proof**))
We use the Stokes theorem,

$$\int\nabla\cdot(\boldsymbol{v}\times\boldsymbol{C})d^3\boldsymbol{r}=-\boldsymbol{C}\cdot\int\boldsymbol{v}\times d\boldsymbol{a}$$

where $\boldsymbol{C}$ is a constant vector.

$$\begin{aligned}\nabla\cdot(\boldsymbol{v}\times\boldsymbol{C})&=\boldsymbol{C}\cdot\left(\nabla\times\boldsymbol{v}\right)-\boldsymbol{v}\cdot(\nabla\times\boldsymbol{C})\\&=\boldsymbol{C}\cdot\left(\nabla\times\boldsymbol{v}\right)\end{aligned}$$

$$d\boldsymbol{a}\cdot(\boldsymbol{v}\times\boldsymbol{C})=\boldsymbol{C}\cdot(d\boldsymbol{a}\times\boldsymbol{v})=-\boldsymbol{C}\cdot(\boldsymbol{v}\times d\boldsymbol{a})$$

Then we have

$$\int\nabla\cdot(\boldsymbol{v}\times\boldsymbol{C})d^3\boldsymbol{r}=-\boldsymbol{C}\cdot\int\boldsymbol{v}\times\cdot d\boldsymbol{a}$$

Using this theorem, we get

$$\begin{aligned}\boldsymbol{A}(\boldsymbol{r})&=\frac{\mu_0}{4\pi}\int\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\nabla'\times\boldsymbol{M}(\boldsymbol{r}')d^3\boldsymbol{r}'+\frac{\mu_0}{4\pi}\int\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\boldsymbol{M}(\boldsymbol{r}')\times d\boldsymbol{a}'\\&=\frac{\mu_0}{4\pi}\int\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\nabla'\times\boldsymbol{M}(\boldsymbol{r}')d^3\boldsymbol{r}'+\frac{\mu_0}{4\pi}\int\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\boldsymbol{M}(\boldsymbol{r}')\times\boldsymbol{n}'da'\end{aligned}$$

The volume current is defined by

$$\boldsymbol{J}_b(\boldsymbol{r})=\nabla\times\boldsymbol{M}(\boldsymbol{r}),$$

The surface current is defined by

$$\boldsymbol{K}_b(\boldsymbol{r}) = \boldsymbol{M}(\boldsymbol{r}) \times \boldsymbol{n}$$

Thus we get the expression of the vector potential

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}_b(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} d^3\boldsymbol{r}' + \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{K}_b(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} da'$$

This is compared with the vector potential in the presence of a free current density $\boldsymbol{J}_f$

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}_f(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} d^3\boldsymbol{r}'$$

The current density is given by

$$\boldsymbol{J} = \boldsymbol{J}_f + \boldsymbol{J}_b .$$

43. Auxiliary field *H*

43.1 Definition

The current density is given by

$$\boldsymbol{J} = \boldsymbol{J}_b + \boldsymbol{J}_f$$

where $\boldsymbol{J}_f$ is a free current density and $\boldsymbol{J}_b$ is a current density due to the magnetization (bound).

$$\frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) = \boldsymbol{J} = \boldsymbol{J}_f + \boldsymbol{J}_b = \boldsymbol{J}_f + \nabla \times \boldsymbol{M}$$

or

$$\nabla \times (\frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}) = \boldsymbol{J}_f$$

We define the field ***H*** as

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_f$$

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}$$

where

$$\oint (\nabla \times \boldsymbol{H}) \cdot d\boldsymbol{a} = \oint \boldsymbol{H} \cdot d\boldsymbol{s} = I_f$$

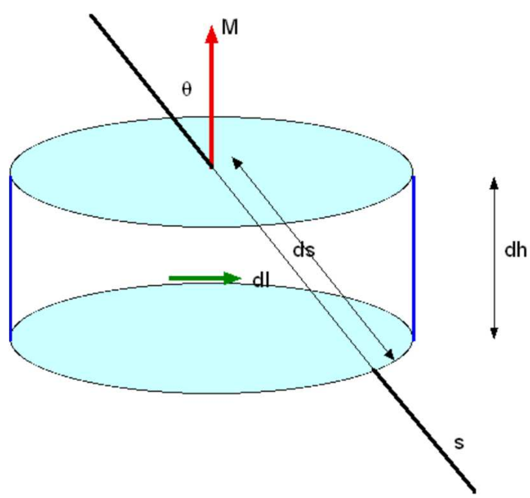
and I_f is the free currents enclosed inside the current path. The field ***H*** permits us to express Ampere's law in terms of the free current alone.

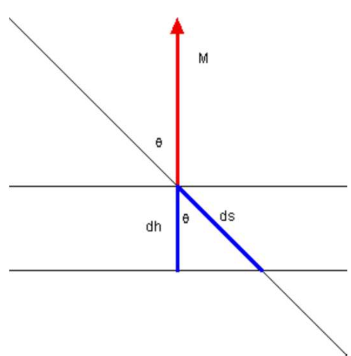
$$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M})$$

((**Note**)) The unit of H is as follows.

$$[H] = \frac{[B]}{[\mu_0]} = \frac{T}{T(m/A)} = [A/m]$$

43.2 Physical meaning of *H*





We now assume that the magnetization at the point P is given by ***M***. The equivalent current dI flowing the side face with the height dh, is obtained as

$$dI = Mdh = Mds\cos\theta = \boldsymbol{M}_s \cdot d\boldsymbol{s}$$

where $\boldsymbol{M}_s$ is the tangential component of $\boldsymbol{M}$ along the d$\boldsymbol{s}$ direction.

From the Ampere's law, we have

$$\oint \boldsymbol{B} \cdot d\boldsymbol{s} = \mu_0(i + I) = \mu_0 i + \mu_0 \oint \boldsymbol{M} \cdot d\boldsymbol{s}$$

where i is a true current.

Here we introduce a new vector $\boldsymbol{H}$ which is defined by

$$\mu_0 \boldsymbol{H} = \boldsymbol{B}' = \boldsymbol{B} - \mu_0 \boldsymbol{M}$$

Then we have

$$\oint \boldsymbol{H} \cdot d\boldsymbol{s} = i$$

This equation is valid even for the magnetic materials. Note that if $\boldsymbol{M} = 0$,

$$\boldsymbol{B} = \mu_0 \boldsymbol{H}$$

44. Magnetic susceptibility

44.1 Definition

In order to solve problems, it is essential to have a relationship between M and H, or equivalently, a relationship between M and one of the magnetic field vectors. These relationships depend on the nature of the magnetic materials and are usually obtained from experiment.

In a large class of materials there exists an approximately linear relationship between $\boldsymbol{M}$ and $\boldsymbol{H}$. If the material is isotropic as well as linear,

$$\boldsymbol{M} = \chi_m \boldsymbol{H}$$

where χ_m is called the magnetic susceptibility. If χ_m is positive, the material is called paramagnetic susceptibility, and if χ_m is negative, the material is diamagnetic. Then

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M} = \frac{\boldsymbol{B}}{\mu_0} - \chi_m \boldsymbol{H}$$

or

$$\boldsymbol{B} = \mu_0(1 + \chi_m)\boldsymbol{H} = \mu \boldsymbol{H}$$

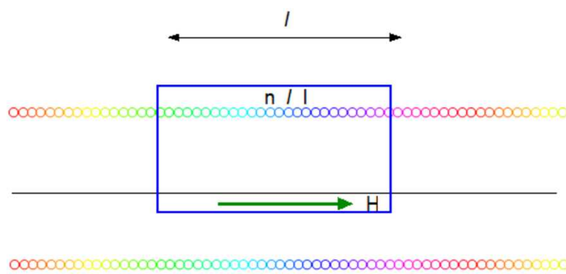
with

$$\frac{\mu}{\mu_0} = 1 + \chi_m$$

44.2 The method to determine *B* and *M* from *H*

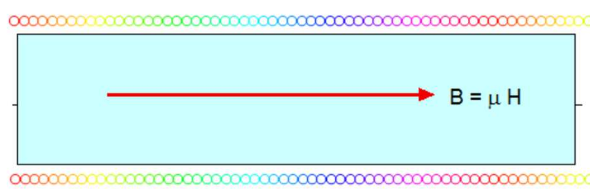
The method is similar to the discussion for the derivation of ***D***, ***E*** and q_b in the dielectric system. First we determine the auxiliary field *H* by using the formula

$$\oint \boldsymbol{H} \cdot d\boldsymbol{s} = I_f$$



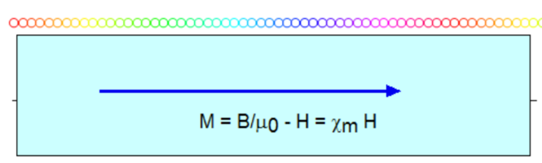
Next we calculate ***B*** using the relation

$$\boldsymbol{B} = \mu \boldsymbol{H}$$



Then we calculate ***M*** using the relation

$$\boldsymbol{M} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{H} = (\frac{\mu}{\mu_0} - 1)\boldsymbol{H} = \chi_m \boldsymbol{H}$$



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APPENDIX-A Spherical coordinates

$$\boldsymbol{e}_r = \frac{\partial \boldsymbol{r}}{\partial r} = \sin\theta\cos\phi\,\boldsymbol{e}_x + \sin\theta\sin\phi\,\boldsymbol{e}_y + \cos\theta\,\boldsymbol{e}_z\,,$$

$$\boldsymbol{e}_\theta = \frac{1}{r}\frac{\partial \boldsymbol{r}}{\partial \theta} = \cos\theta\cos\phi\,\boldsymbol{e}_x + \cos\theta\sin\phi\,\boldsymbol{e}_y - \sin\theta\,\boldsymbol{e}_z\,,$$

$$\boldsymbol{e}_\phi = \frac{1}{r\sin\theta}\frac{\partial \boldsymbol{r}}{\partial \phi} = -\sin\phi\,\boldsymbol{e}_x + \cos\phi\,\boldsymbol{e}_y\,.$$

This can be described using a matrix A as

$$\begin{pmatrix}\boldsymbol{e}_r\\ \boldsymbol{e}_\theta\\ \boldsymbol{e}_\phi\end{pmatrix} = \boldsymbol{A}\begin{pmatrix}\boldsymbol{e}_x\\ \boldsymbol{e}_y\\ \boldsymbol{e}_z\end{pmatrix} = \begin{pmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{e}_x\\ \boldsymbol{e}_y\\ \boldsymbol{e}_z\end{pmatrix}.$$

or by using the inverse matrix A^{-1} as

$$\begin{pmatrix}\boldsymbol{e}_x\\ \boldsymbol{e}_y\\ \boldsymbol{e}_z\end{pmatrix} = \boldsymbol{A}^{-1}\begin{pmatrix}\boldsymbol{e}_r\\ \boldsymbol{e}_\theta\\ \boldsymbol{e}_\phi\end{pmatrix} = \boldsymbol{A}^{T}\begin{pmatrix}\boldsymbol{e}_r\\ \boldsymbol{e}_\theta\\ \boldsymbol{e}_\phi\end{pmatrix} = \begin{pmatrix}\sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi\\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi\\ \cos\theta & -\sin\theta & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{e}_r\\ \boldsymbol{e}_\theta\\ \boldsymbol{e}_\phi\end{pmatrix}.$$

or

$$\boldsymbol{e}_x = \sin\theta\cos\phi\,\boldsymbol{e}_r + \cos\theta\cos\phi\,\boldsymbol{e}_\theta - \sin\phi\,\boldsymbol{e}_\phi\,,$$

$$\boldsymbol{e}_y = \sin\theta\sin\phi\,\boldsymbol{e}_r + \cos\theta\sin\phi\,\boldsymbol{e}_\theta + \cos\phi\,\boldsymbol{e}_\phi\,,$$

$$\boldsymbol{e}_z = \cos\theta\,\boldsymbol{e}_r - \sin\theta\,\boldsymbol{e}_\theta\,.$$

in the spherical coordinate systems.

$$\nabla = \boldsymbol{e}_r\frac{\partial}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi},$$

$$\boldsymbol{A} = A_r\boldsymbol{e}_r + A_\theta\boldsymbol{e}_\theta + A_\phi\boldsymbol{e}_\phi\,.$$

The divergence is given by

$$\nabla\cdot\boldsymbol{A}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}A_\phi\,.$$

$$\nabla\times\boldsymbol{A}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\boldsymbol{e}_r & r\boldsymbol{e}_\theta & r\sin\theta\boldsymbol{e}_\varphi\\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\varphi}\\ A_r & rA_\theta & r\sin\theta A_\varphi\end{vmatrix}$$

$$\nabla^2\psi=\frac{1}{r}\frac{\partial}{\partial r^2}(r\psi)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}$$

APPENDIX-B Cylindrical coordinates

The unit vectors are written as

$$\boldsymbol{e}_\rho=\frac{1}{h_\rho}\frac{\partial\boldsymbol{r}}{\partial\rho}=\frac{\partial\boldsymbol{r}}{\partial\rho}=\cos\varphi\boldsymbol{e}_x+\sin\varphi\boldsymbol{e}_y$$

$$\boldsymbol{e}_\varphi=\frac{1}{h_\varphi}\frac{\partial\boldsymbol{r}}{\partial\varphi}=\frac{1}{\rho}\frac{\partial\boldsymbol{r}}{\partial\varphi}=-\sin\varphi\boldsymbol{e}_x+\cos\varphi\boldsymbol{e}_y$$

$$\boldsymbol{e}_z=\frac{1}{h_z}\frac{\partial\boldsymbol{r}}{\partial z}=\frac{\partial\boldsymbol{r}}{\partial z}=\boldsymbol{e}_z$$

The above expression can be described using a matrix A as

$$\begin{pmatrix}\boldsymbol{e}_\rho\\ \boldsymbol{e}_\varphi\\ \boldsymbol{e}_z\end{pmatrix}=\boldsymbol{A}\begin{pmatrix}\boldsymbol{e}_x\\ \boldsymbol{e}_y\\ \boldsymbol{e}_z\end{pmatrix}=\begin{pmatrix}\cos\varphi & \sin\varphi & 0\\ -\sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\boldsymbol{e}_x\\ \boldsymbol{e}_y\\ \boldsymbol{e}_z\end{pmatrix}.$$

or by using the inverse matrix A^{-1} as

$$\begin{pmatrix}\boldsymbol{e}_x\\ \boldsymbol{e}_y\\ \boldsymbol{e}_z\end{pmatrix}=\boldsymbol{A}^{-1}\begin{pmatrix}\boldsymbol{e}_\rho\\ \boldsymbol{e}_\varphi\\ \boldsymbol{e}_z\end{pmatrix}=\boldsymbol{A}^T\begin{pmatrix}\boldsymbol{e}_\rho\\ \boldsymbol{e}_\varphi\\ \boldsymbol{e}_z\end{pmatrix}=\begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\boldsymbol{e}_\rho\\ \boldsymbol{e}_\varphi\\ \boldsymbol{e}_z\end{pmatrix}.$$

The differential operations involving ∇ are as follows.

$$\nabla\psi = \boldsymbol{e}_\rho \frac{\partial\psi}{\partial\rho} + \boldsymbol{e}_\varphi \frac{1}{\rho}\frac{\partial\psi}{\partial\varphi} + \boldsymbol{e}_z \frac{\partial\psi}{\partial z},$$

$$\nabla\cdot\boldsymbol{V} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho V_\rho) + \frac{1}{\rho}\frac{\partial}{\partial\varphi}V_\varphi + \frac{\partial}{\partial z}V_z,$$

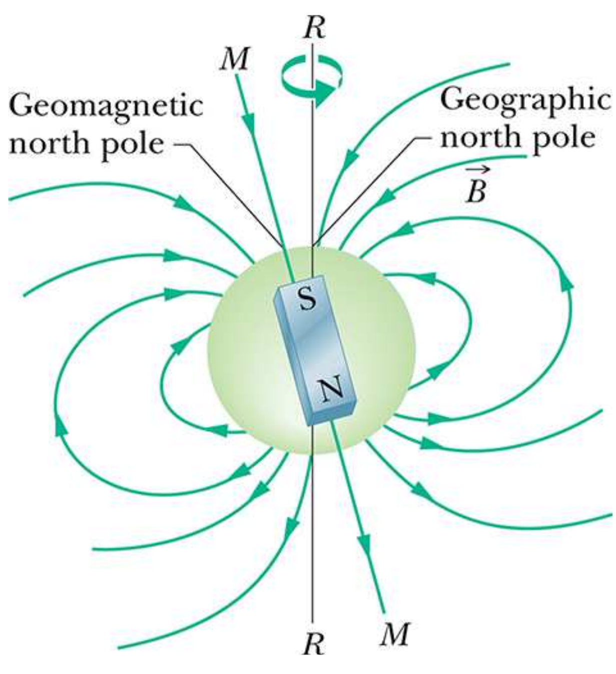
$$\nabla\times\boldsymbol{V} = \frac{1}{\rho}\begin{vmatrix} \boldsymbol{e}_\rho & \rho\boldsymbol{e}_\varphi & \boldsymbol{e}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ V_\rho & \rho V_\varphi & V_z \end{vmatrix},$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial\psi}{\partial\rho}) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}.$$

where $\boldsymbol{V}$ is a vector and ψ is a scalar.

APPENDIX-C Magnetic field of the Earth

C.1 Magnetic moment of the Earth



Halliday and Resnick;

$\mu = 8.00 \times 10^{22}$ A m^2

Table 2
Dipole Magnetic Moment Data
From Barnes Pages 33 & 61

Year	Dipole Moment ($\times 10^{22}$ amp-meter^2)
1835	8.558
1845	8.488
1880	8.363
1880	8.336
1885	8.347
1885	8.375
1905	8.291
1915	8.225
1922	8.165
1925	8.149
1935	8.088
1942.5	8.009
1945	8.065
1945	8.010
1945	8.066
1945	8.090
1955	8.035
1955	8.067
1958.5	8.038
1959	8.086
1960	8.053
1960	8.037
1960	8.025
1965	8.013

C.2 Magnetic axis and geograpic axis

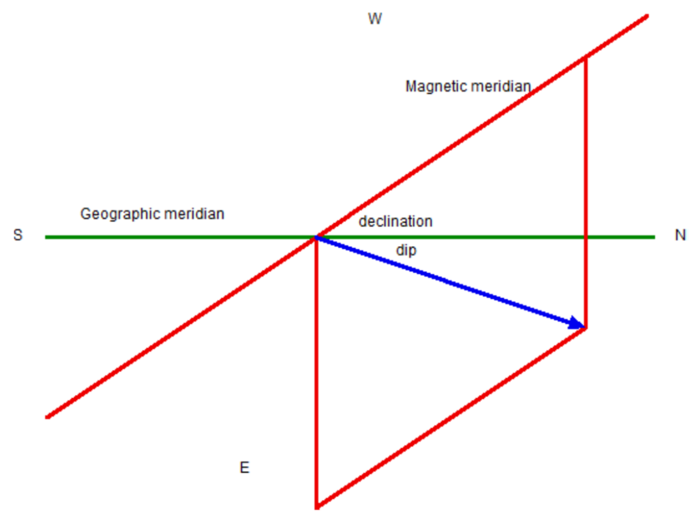
A freely suspended magnet always points in the North-South direction even in the absence of any other magnet. This suggests that the Earth itself behaves as a magnet which causes a freely suspended magnet (or magnetic needle) to point always in a particular direction: North and South. The shape of the Earth's magnetic field resembles that of a bar magnet of length one-fifth of the Earth's diameter buried at its center.

The South Pole of the Earth's magnet is in the geographical North because it attracts the North Pole of the suspended magnet and vice versa. Thus, there is a magnetic S-pole near the geographical North, and a magnetic N-pole near the geographical South. The axis of the Earth's magnet and the geographical axis do not coincide. The axis of the Earth's magnet is inclined at an angle of about 11.5° with the geographical axis.

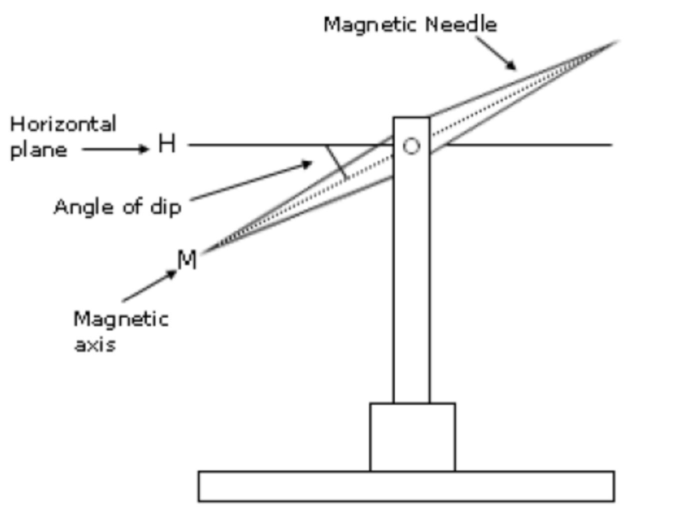
C.3 Cause of Earth's magnetism:

It is now believed that the Earth's magnetism is due to the magnetic effect of current which is flowing in the liquid core at the center of the Earth. Thus, the Earth is a huge electromagnet.

1. Declination
2. Angle of dip (or inclination)



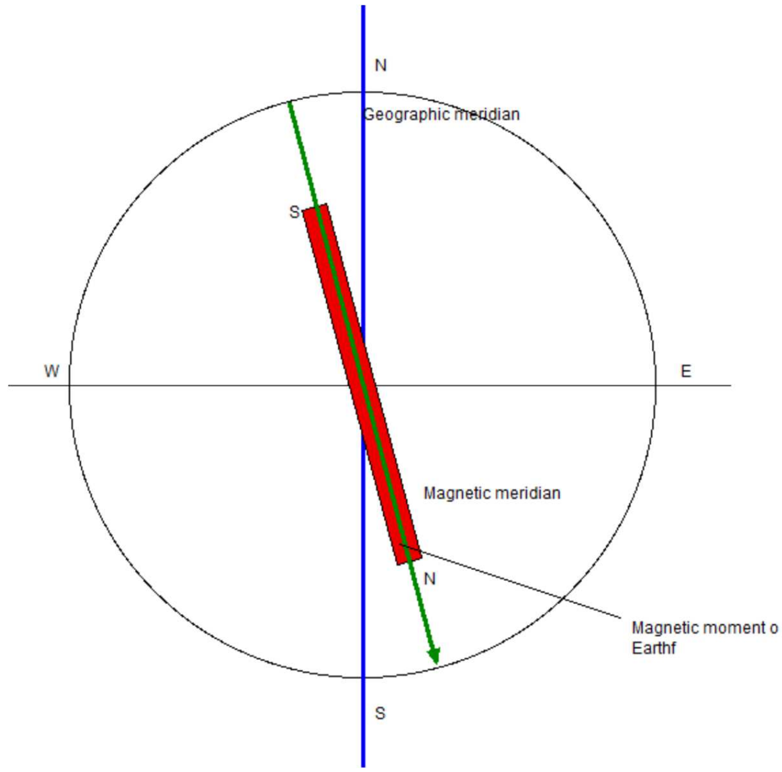
The vertical plane passing through the axis of a freely suspended magnet is called magnetic meridian. The direction of the Earth's magnetic field lies in the magnetic meridian and may not be horizontal. The vertical plane passing through the true geographical North and South (geographical axis of Earth) is called geographical meridian. The angle between the magnetic meridian and the geographical meridian at a place is called declination at that place.



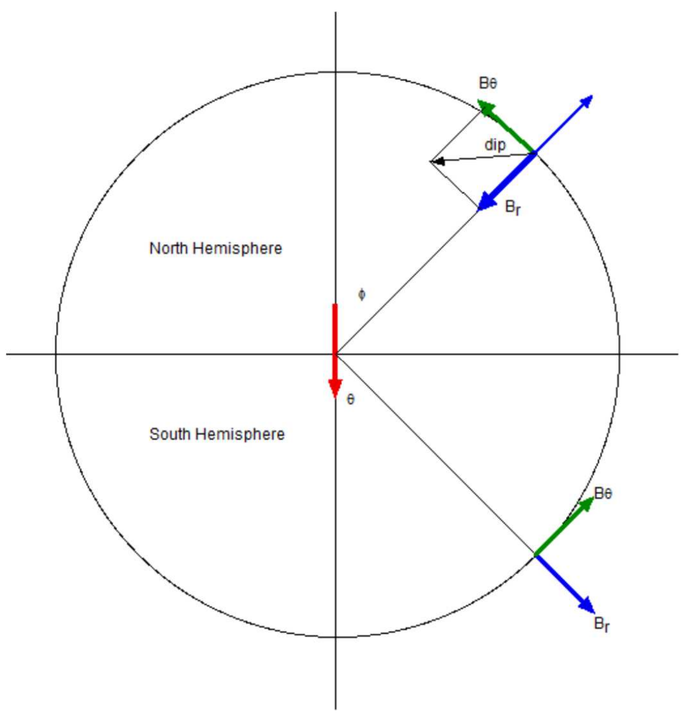
The angle HOM betw een the horizontal plane HO and the axis of freely suspended magnetic needle MO is Angle of Dip or Inclination.

C.4 Angle of dip (or inclination)

If we take a magnetic needle which is free to rotate in the vertical plane, then it will not remain perfectly horizontal. The compass needle makes a certainangle with the horizontal direction. In fact, in the Northern Hemisphere of Earth, the North Pole of the magnetic needle dips below the horizontal line. At any place, the magnetic needle points in the direction of the resultant intensity of the Earth's magnetic field at the place.



C.5 Magnetic field due to the magnetic moment



$$B_r = \frac{2\mu_0 m \cos\theta}{4\pi r^3} = \frac{2\mu_0 m \cos(\pi - \phi)}{4\pi r^3} = -\frac{2\mu_0 m \cos\phi}{4\pi r^3}$$

$$B_\theta = \frac{\mu_0 m \sin\theta}{4\pi r^3} = \frac{\mu_0 m \sin(\pi - \phi)}{4\pi r^3} = \frac{\mu_0 m \sin\phi}{4\pi r^3}$$

Note that the vertical component (B_r) is directed inward in the North Hemisphere.

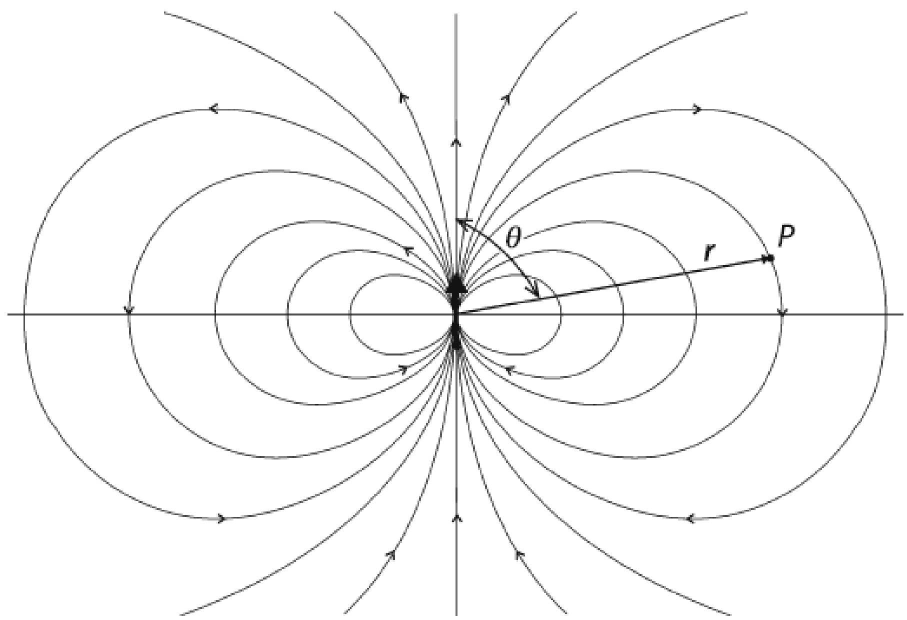
The angle between the magnetic field and the surface of the Earth is called the dip (inclination). We define the angle of dip as

$$\tan I = \frac{-B_r}{B_\varsigma} = \frac{2\cos\phi}{\sin\phi} = \frac{2\cos(\frac{\pi}{2} - \lambda)}{\sin(\frac{\pi}{2} - \lambda)} = 2\frac{\sin\lambda}{\cos\pi} = 2\tan\lambda$$

where λ is the latitude.

D. Earth magnetic field

The magnetic field of the Earth is due to a magnetic dipole.



The magnetic field is given by

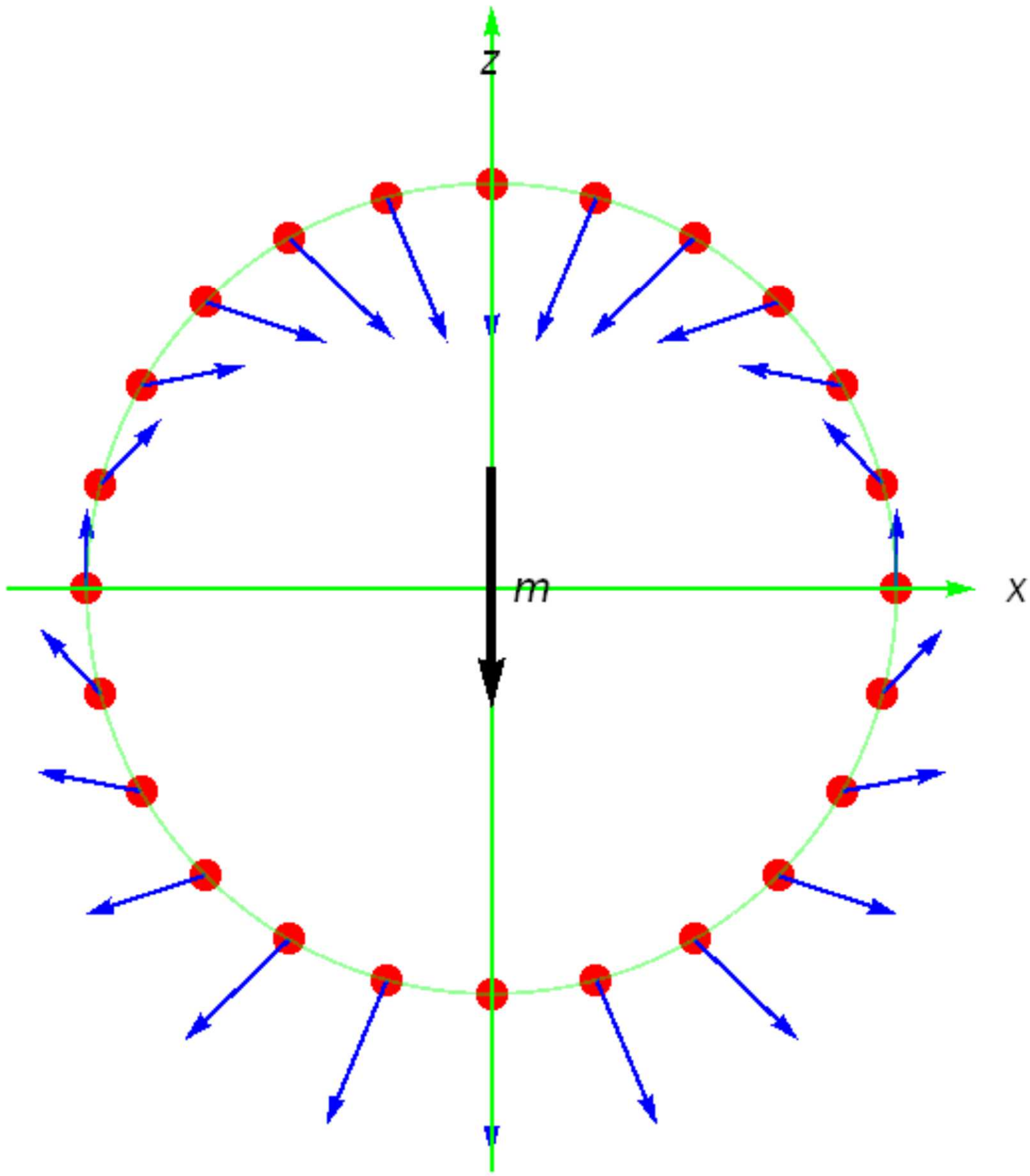
$$B_r \boldsymbol{e}_r = \frac{2\mu_0 m}{4\pi r^3}\cos\theta(\boldsymbol{e}_x \sin\theta + \boldsymbol{e}_z \cos\theta)$$

$$B_\theta \boldsymbol{e}_\theta = \frac{\mu_0 m}{4\pi r^3} \sin\theta(\boldsymbol{e}_x \cos\theta - \boldsymbol{e}_z \sin\theta)$$

and

$$\begin{aligned}
\boldsymbol{B} &= B_r \boldsymbol{e}_r + B_\theta \boldsymbol{e}_\theta \\
&= \frac{2\mu_0 m}{4\pi r^3} \cos\theta(\boldsymbol{e}_x \sin\theta + \boldsymbol{e}_z \cos\theta) + \frac{\mu_0 m}{4\pi r^3} \sin\theta(\boldsymbol{e}_x \cos\theta - \boldsymbol{e}_z \sin\theta) \\
&= \frac{\mu_0 m}{4\pi r^3}[2\cos\theta(\boldsymbol{e}_x \sin\theta + \boldsymbol{e}_z \cos\theta) + \sin\theta(\boldsymbol{e}_x \cos\theta - \boldsymbol{e}_z \sin\theta)] \\
&= \frac{\mu_0 m}{4\pi r^3}[\frac{3}{2}\sin 2\theta\ \boldsymbol{e}_x + (3\cos^2\theta - 1)\ \boldsymbol{e}_z]
\end{aligned}$$

where $\phi = 0$.



3D case

$$\begin{pmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\theta \\ \boldsymbol{e}_\phi \end{pmatrix} = \boldsymbol{A} \begin{pmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{pmatrix}$$

$$\begin{aligned}
\boldsymbol{B} &= B_r \boldsymbol{e}_r + B_\theta \boldsymbol{e}_\theta \\
&= \frac{2\mu_0 m}{4\pi r^3}\cos\theta(\boldsymbol{e}_x \sin\theta\cos\phi + \boldsymbol{e}_y \sin\theta\sin\phi + e_z \cos\theta e_z) \\
&+ \frac{\mu_0 m}{4\pi r^3}\sin\theta(\boldsymbol{e}_x \cos\theta\cos\phi + \boldsymbol{e}_y \cos\theta\sin\phi - \boldsymbol{e}_z \sin\theta) \\
&= \frac{\mu_0 m}{4\pi r^3}[2\cos\theta(\boldsymbol{e}_x \sin\theta\cos\phi + \boldsymbol{e}_y \sin\theta\sin\phi + e_z \cos\theta e_z) \\
&+ \sin\theta(\boldsymbol{e}_x \cos\theta\cos\phi + \boldsymbol{e}_y \cos\theta\sin\phi - \boldsymbol{e}_z \sin\theta)] \\
&= \frac{\mu_0 m}{4\pi r^3}[\ 3\cos\theta\sin\theta\cos\phi\boldsymbol{e}_x + 3\cos\theta\sin\theta\sin\phi\ \boldsymbol{e}_y + (3\cos^2\theta - 1)\boldsymbol{e}_z]
\end{aligned}$$

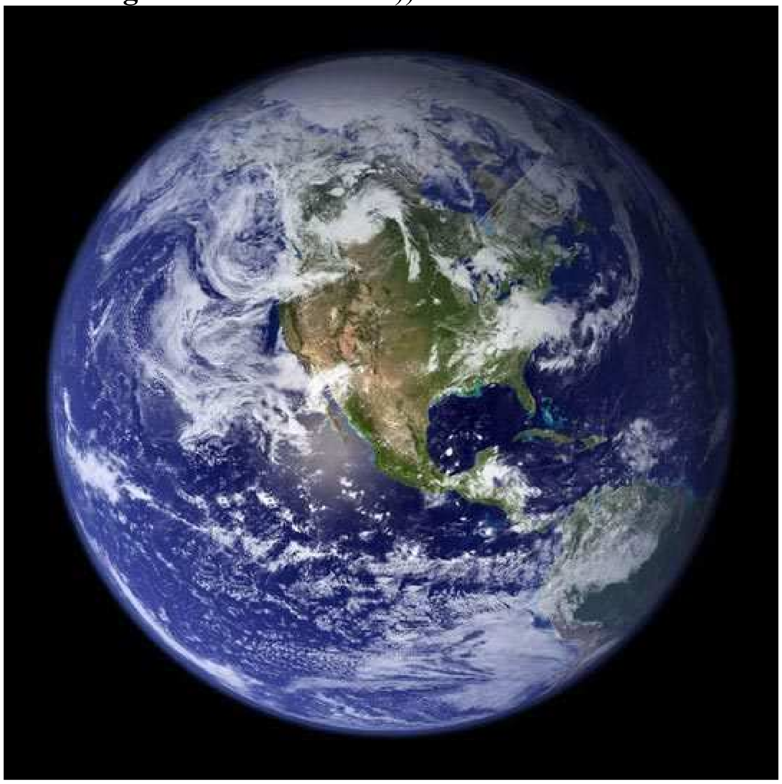
Note that

$$\frac{B_r}{B_\theta} = \frac{2\cos\theta}{\sin\theta} = 2\cot\theta = 2\tan\varphi$$

where

$$\varphi = \frac{\pi}{2} - \theta\ .$$

((**Evaluation of the magnetic field of Earth**))



https://cdn.mos.cms.futurecdn.net/yCPyoZDQBBcXikqxkeW2jJ-970-80.jpg

$$B_0 = \frac{\mu_0 m_E}{4\pi r^3} = 0.309217 \text{ x } 10^{-4} \text{ T} = 0.309217 \text{ Oe}$$

where

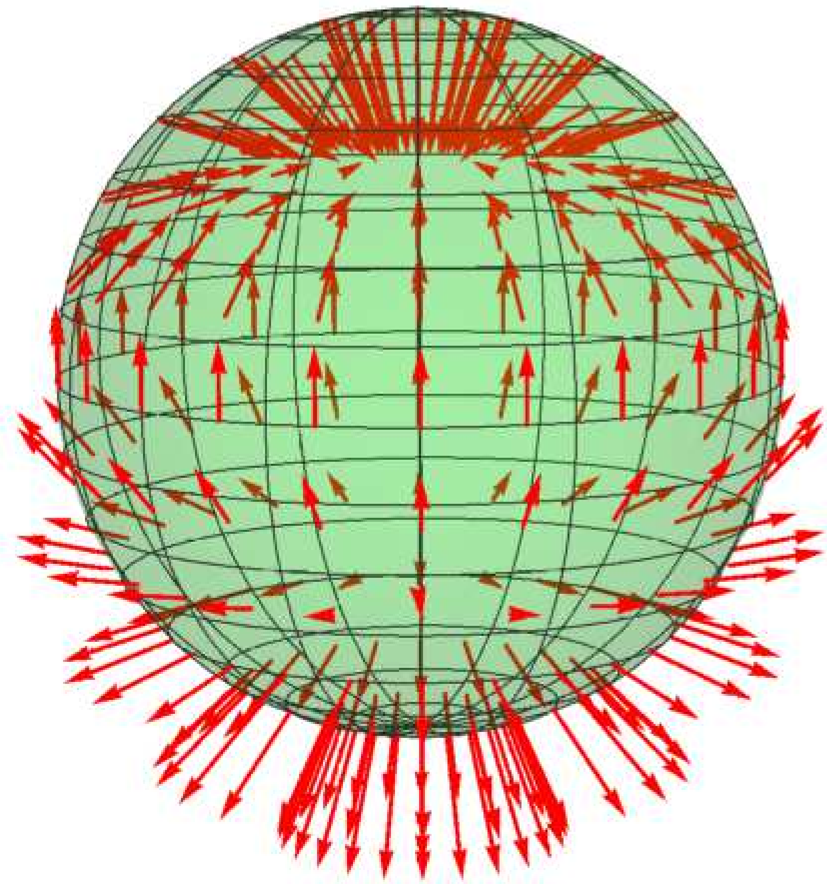
$\mu_0 = 4\pi \times 10^{-7}$ Tm/A	(permeability of free space)
$m_E = 7.79 \times 10^{22}$ Am^2.	(magnetic moment od the Earth)
$r = R = 6372.797$ km.	(radius of the Earth)

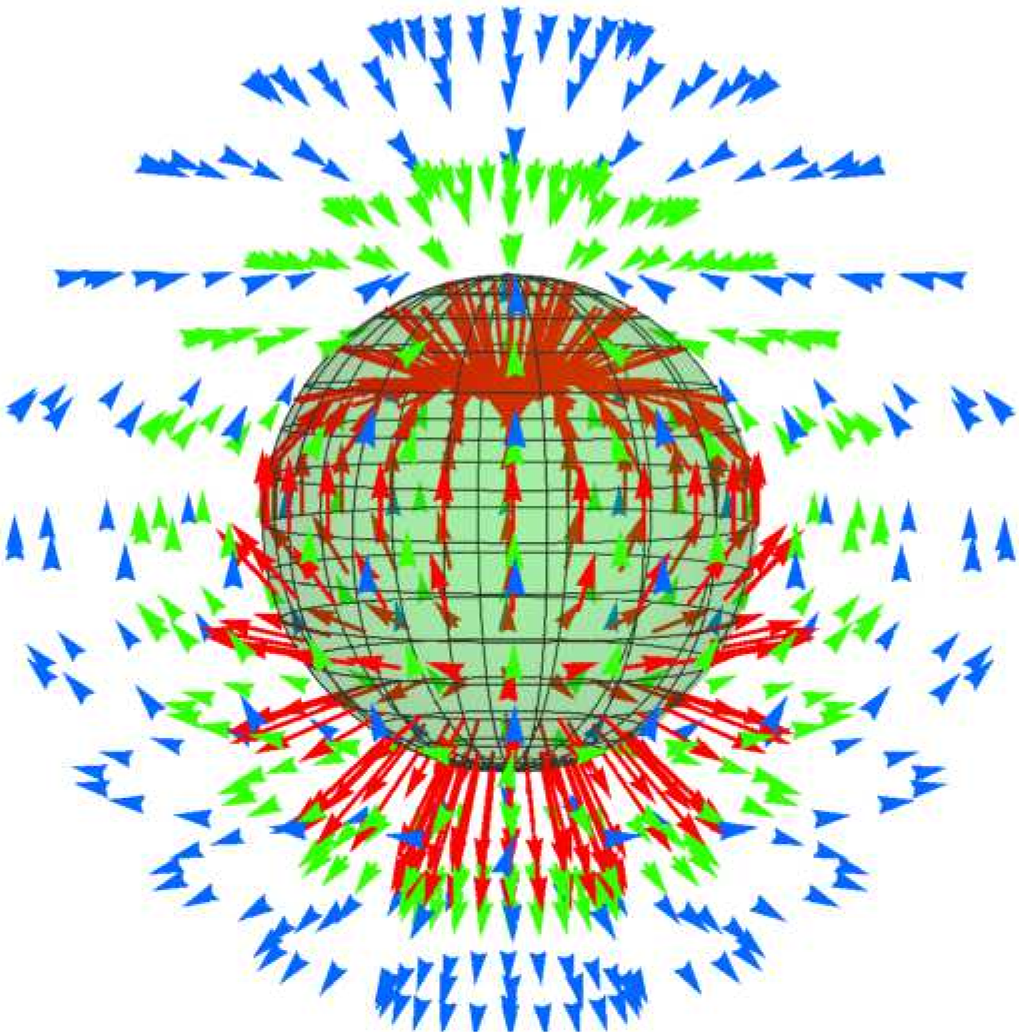
The magnetic moment is expressed by

$$m = I(\pi R)^2$$

or

$$I = \frac{m}{\pi R^2} = 6.27 \text{ x } 10^8 \text{ A.}$$





APPENDIX Analogy between the electric dipole moment and magnetic moment

(a) Electric dipole moment

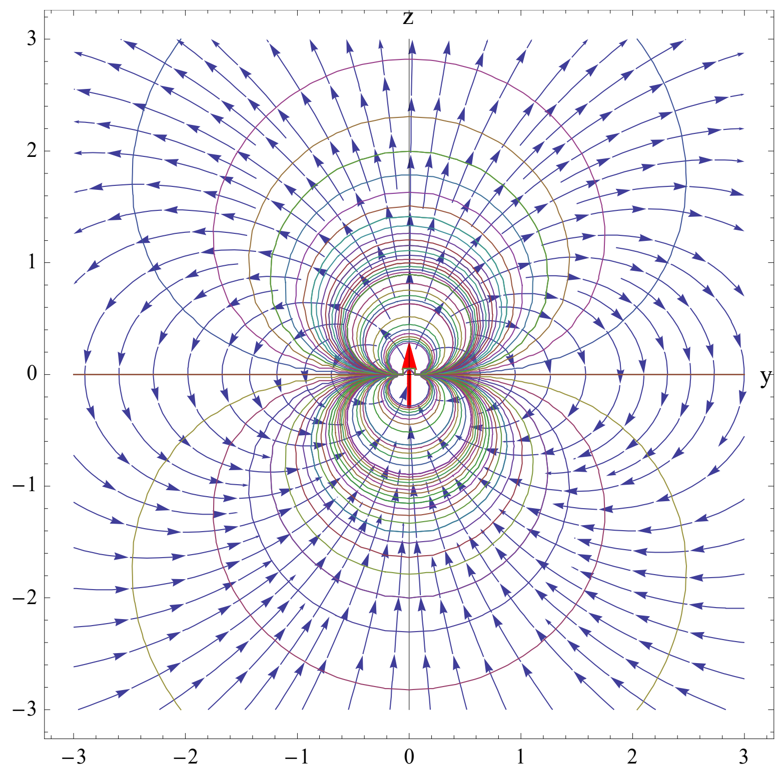


Fig. The electric field due to the electric dipole moment $\boldsymbol{p}$ (along the z axis) at the origin.

Electric dipole moment, $\boldsymbol{p}$: $p = ql$

Torque: $\boldsymbol{\tau} = \boldsymbol{p} \times \boldsymbol{E}$

Potential energy: $U_E = -\boldsymbol{p} \cdot \boldsymbol{E}$

The electric field:

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0 r^3}[3(\boldsymbol{p} \cdot \boldsymbol{e}_r)\boldsymbol{e}_r - \boldsymbol{p}].$$

(b) Magnetic dipole moment

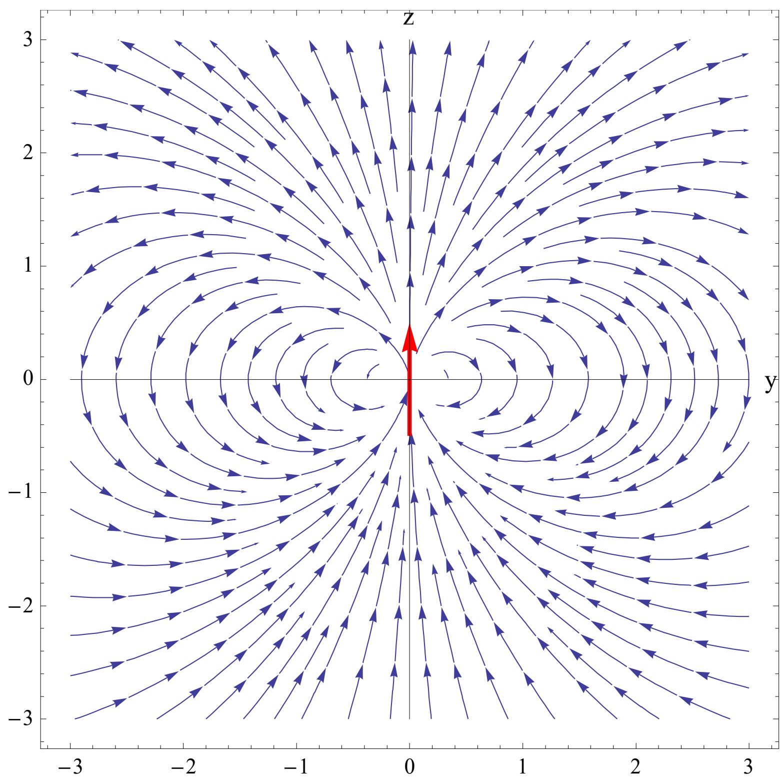


Fig. The magnetic field of a magnetic moment $\boldsymbol{\mu}$ (along the z axis) at the origin.

Magnetic dipole moment, $\boldsymbol{\mu}$; $\mu = IA$

Torque: $\boldsymbol{\tau} = \boldsymbol{\mu} \times \boldsymbol{B}$

Potential energy: $U_B = -\boldsymbol{\mu} \cdot \boldsymbol{B}$

The magnetic field:

$$\boldsymbol{B} = \frac{\mu_0}{4\pi r^3}[3(\boldsymbol{\mu} \cdot \boldsymbol{e}_r)\boldsymbol{e}_r - \boldsymbol{\mu}]$$