Chapter 30
Induction and inductance
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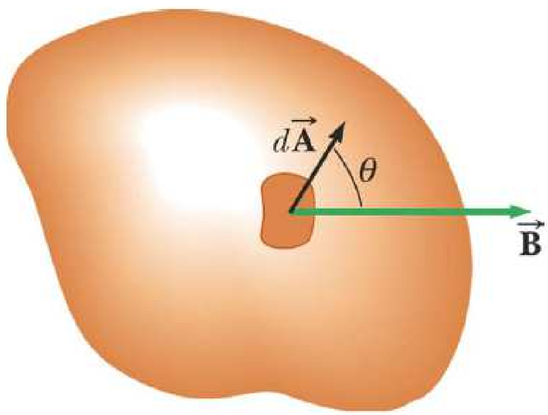
1. Faraday's law of induction

1.1 Definition of magnetic flux

An induced current is produced by a changing magnetic field. There is an induced emf associated with the induced current. A current can be produced without a battery present in the circuit. Faraday's Law of Induction describes the induced emf.

To express Faraday's finding mathematically, the magnetic flux is used. The flux depends on the magnetic field and the area:



The magnetic flux Φ through a surface A is given by

$$\Phi = \int_A \boldsymbol{B} \cdot d\boldsymbol{a}$$

in the units of

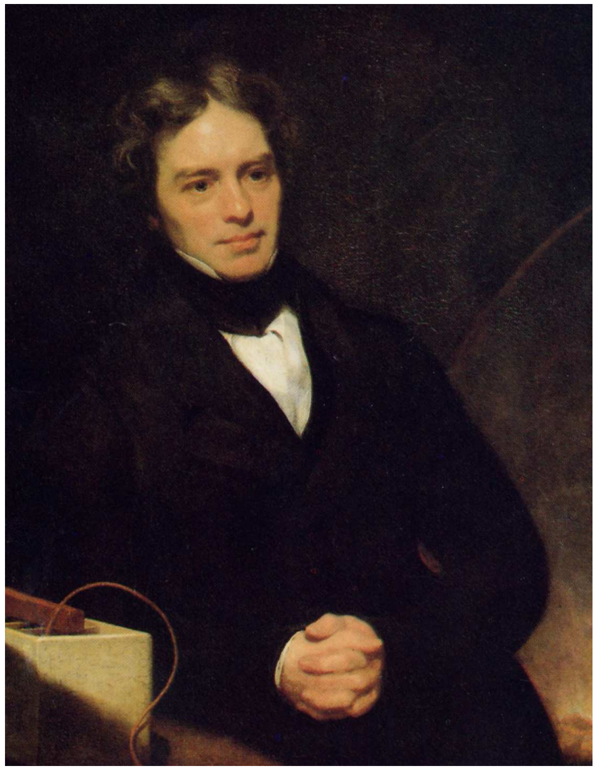
Wb (Weber) = 1 T m^2

This is just the number of magnetic field lines passing through the surface. Suppose that the surface is bounded by a closed loop. Faraday has demonstrated that an emf is induced around a closed loop, if the magnetic flux through the loop changes with time. This is the Faraday's law.

$$V = -\frac{d\Phi}{dt}$$

Here V is an emf measured in V (volts).

Michael Faraday (1791-1867).



1.2. Gauss' law for magnetism

The Gauss' law for magnetism states that the magnetic flux out of every closed surface vanishes. Thus we have

$$\oint \boldsymbol{B} \cdot d\boldsymbol{a} = 0$$

Comparison with the Gauss' law for the electric field leads to the conclusion that there is no magnetic analogue of electric charge. We often describe this by saying that there are no magnetic monopole. The validity of this equation has been established by a vast of experiments. Despite continuing research no one has as yet ever detected the presence of a magnetic monopole.

Using the Gauss' theorem (mathematics), we have

$$\oint_S \boldsymbol{B} \cdot d\boldsymbol{a} = \oint_V \nabla \cdot \boldsymbol{B} d^3\boldsymbol{r} = 0$$

or

$$\nabla \cdot \boldsymbol{B} = \text{div}\boldsymbol{B} = 0$$

in general.

This expression is one of the four Maxwell's Equations. Every field line that enters the surface also exits from it. The net flux through the surface is zero. It also follows that the magnetic field lines always form a closed loop. The magnetic field lines never have end points. Such a point would indicate the existence of a magnetic monopole.

((Note)) Experimental arrangement for the Faraday's law (H.E. White, 1940)

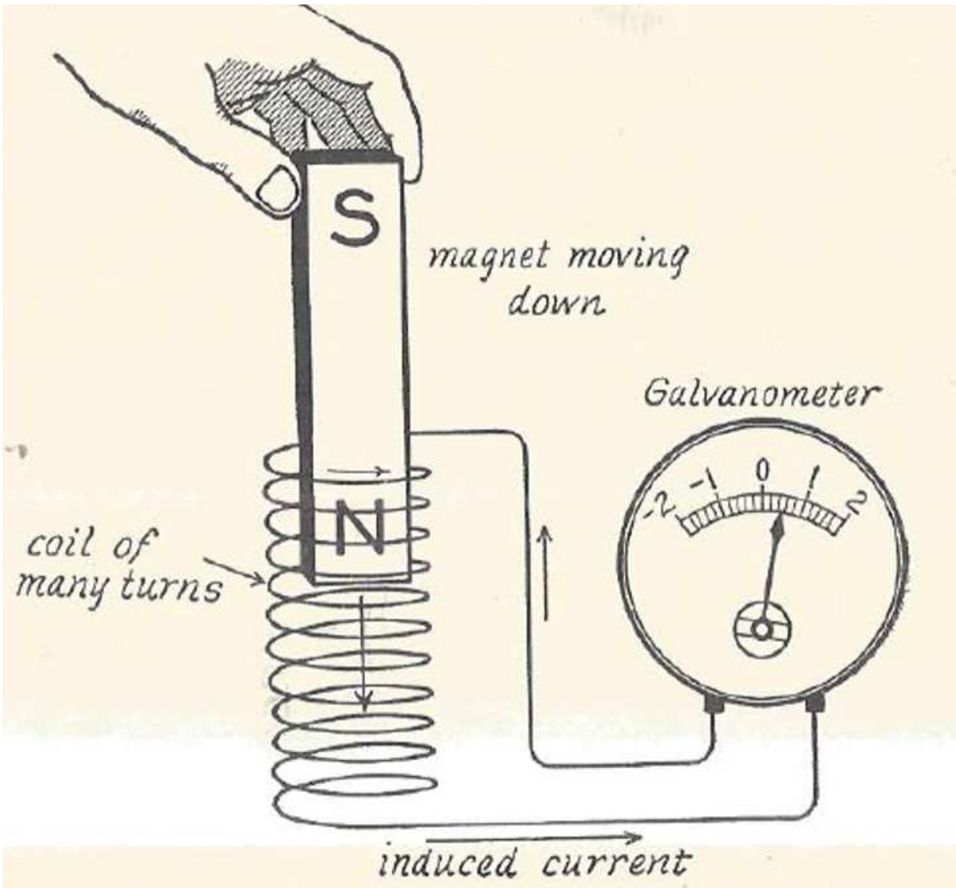


Fig. Schematic diagram of Faraday's experimental discovery of induced electric currents. (H.E. White, Classical and Modern Physics: A Descriptive Introduction (D. Van Nostrand, 1940). When the magnetic flux tends to increase, the induced current is generated according to the Lenz law.

2. Ways of inducing an emf

(a) The loop can be moved from one place to another place where ***B*** has a different strength, thereby changing the flux through the loop.

(b) The loop can be rotated, thereby changing the number of ***B*** lines passing through it.

(c) The shape of the loop can be changed, thereby changing its area.

(d) The magnetic field ***B*** passing through the loop can be changed (perhaps by changing the current in a solenoid that is creating the magnetic field).

3. Universal law of induction

If C is some closed curve, stationary in coordinates x, y, z, if S is a surface spanning C, and if $\boldsymbol{B}(x, y, z, t)$ is the magnetic field measured in x, y, z, at any time t, then

$$V = \int_C \boldsymbol{E} \cdot d\boldsymbol{s} = -\frac{d}{dt}\int_S \boldsymbol{B} \cdot d\boldsymbol{a} = -\frac{d}{dt}\Phi$$

Using the Stokes' law, we get

$$\int_C \boldsymbol{E} \cdot d\boldsymbol{s} = \int_S (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{a} = -\frac{d}{dt}\int_S \boldsymbol{B} \cdot d\boldsymbol{a} = -\frac{d}{dt}\Phi$$

Then it follows that

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \text{(Maxwell's equation)}$$

We will call this Faraday's law. It was discovered by Faraday but was first written in differential form by Maxwell, as one of these equations. Let's see how this equation gives the flux rule for circuits.

((**Note**))

The induced electric field is a non-conservative field that is generated by a changing magnetic field. The field cannot be an electrostatic field because if the field were electrostatic, and hence conservative, the line integral would be zero.

4. A conducting rod moving through a uniform magnetic field

We consider a metal rod (conductor) which moves at a constant velocity ($\boldsymbol{v}$) in a direction perpendicular to its length. Pervading the space through which the rod moves there is a uniform magnetic field $\boldsymbol{B}$ (//z) constant in time. There is no electric field in the reference frame F.

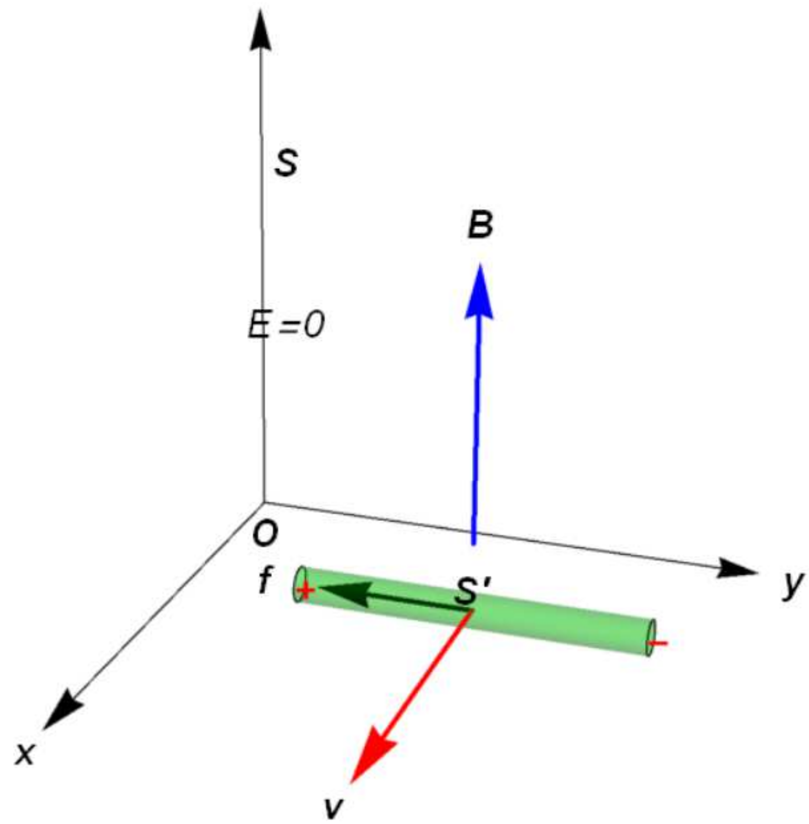
The rod contains charged particles that will move if a force is applied to them. Any charge that is carried along with the rod, such as the particle of charge q moves through the magnetic field $\boldsymbol{B}$ and thus experience a force.

$$\boldsymbol{f} = q(\boldsymbol{v} \times \boldsymbol{B})$$

The direction of the force is dependent on the sign of the charge q. When the rod is moving at constant speed and things have settled to a steady state, the force $\boldsymbol{f}$ must be balanced, at every point inside the rod, by an equal and opposite force. This can only arise from an electric field in the rod. The electric field develops in the following way. The force $\boldsymbol{f}$ pushes negative charges toward one end of the rod, leaving the other end positively charges. This goes on until these separated charges themselves cause an electric field $\boldsymbol{E}$ such that, everywhere in the interior of the rod,

$$q\boldsymbol{E} = \boldsymbol{f}\ .$$

Then the motion of charge relative to the rod ceases. This charge distribution causes an electric field outside the rod, as well as inside. Inside the rod, there has developed an electric field $\boldsymbol{E} = \boldsymbol{v}\times\boldsymbol{B}$, exerting a force $q\boldsymbol{E}$ which just balances the force $q(\boldsymbol{v}\times\boldsymbol{B})$.



((Special relativity)) See Chapter 37S for the detail

Let us observe the system from a frame S' that moves with the rod. What is the magnetic field $\boldsymbol{B}'$ and the electric field $\boldsymbol{E}'$? Note that there is no electric field ($\boldsymbol{E}$ = 0) in the frame S ($\boldsymbol{B}\neq 0$). The $\boldsymbol{E}'$ and $\boldsymbol{B}'$ in the frame S' are related to those in the frame S as (special relativity, see Chapter 37S)

$$E_1 = 0\,,\ E_2 = 0\,,\ E_3 = 0$$
$$B_1 = 0\,,\ B_2 = 0\,,$$

$$E_1' = E_1 = 0$$
$$E_2' = \gamma(E_2 - c\beta B_3) = -\gamma v B_3 \quad ,$$
$$E_3' = \gamma(E_3 + c\beta B_2) = \gamma v B_2 = 0$$

$$B_1' = B_1 = 0$$

$$B_2' = \gamma(B_2 + \frac{\beta}{c}E_3) = \gamma B_2 = 0$$

$$B_3' = \gamma(B_3 - \frac{\beta}{c}E_2) = \gamma B_3$$

or

$$B_3' = \gamma B_3, \qquad E_2' = -\gamma v B_3$$

or

$$\begin{aligned} \boldsymbol{E}' &= \gamma(\boldsymbol{v} \times \boldsymbol{B}) \\ &= \gamma \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ v & 0 & 0 \\ 0 & 0 & B_3 \end{vmatrix} \\ &= (0, -\gamma v B_3, 0) \end{aligned}$$

or

$$\boldsymbol{E}' \approx \boldsymbol{v} \times \boldsymbol{B}$$

where $\gamma = \dfrac{1}{\sqrt{1 - v^2 / c^2}} \approx 1$ for $v<<c$ and $\beta = v/c$. The magnetic field $\boldsymbol{B}'$ (= $\gamma\boldsymbol{B}$) is almost equal to $\boldsymbol{B}$. The electric field $\boldsymbol{E}'$ has only a component along the y' axis (the same as y axis). The presence of the magnetic field $\boldsymbol{B}'$ has no influence on the static charge distribution.

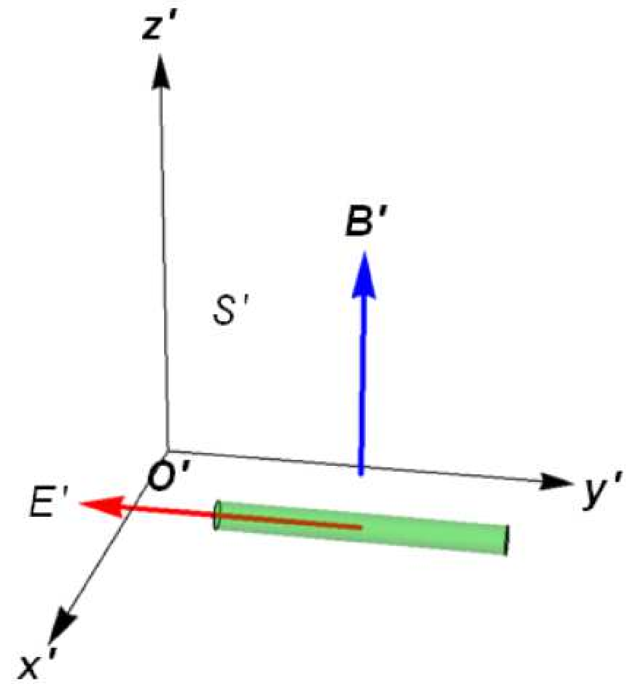
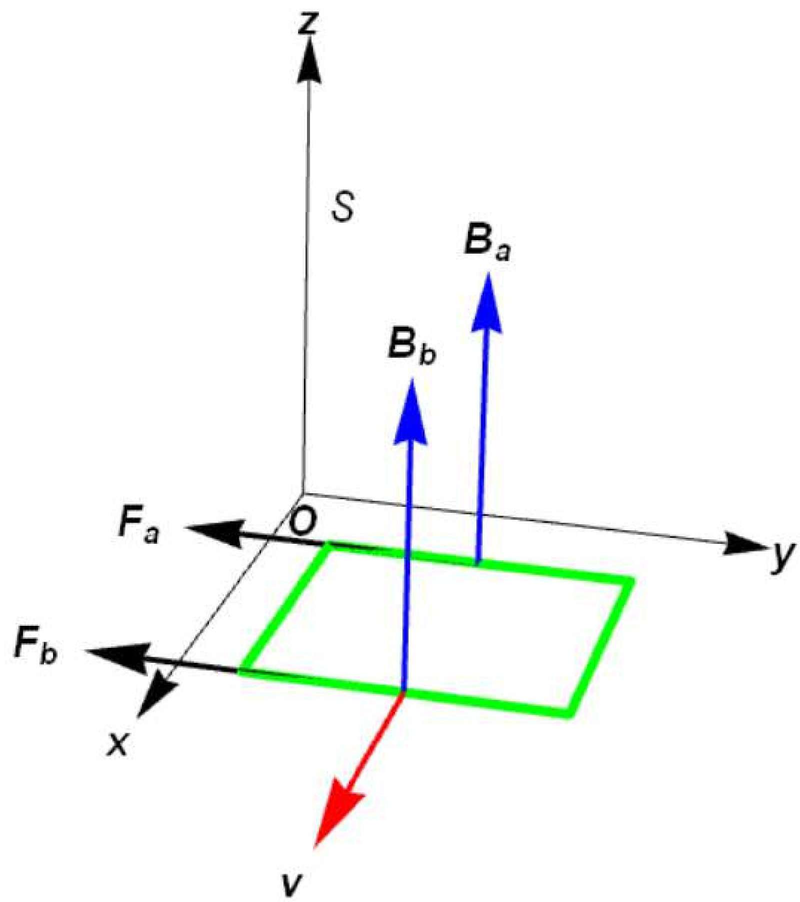


Fig.: The electric and magnetic field in the S' frame.

5. A loop moving through a nonuniform magnetic field; understanding the Faraday's law



F denotes the force which acts on a charge q that rides along with the loop. We evaluate the line integral of ***F***, taken around the whole loop. On the two sides of the loop which lie parallel to the direction of motion, ***F*** is perpendicular to the path element d***s***. So, there is no contribution to the line integral. Taking account of the contributions from the other two sides, each of length w, we have

$$W_{net} = \oint \boldsymbol{F} \cdot d\boldsymbol{s} = \oint q(\boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{s} = qv(B_a - B_b)w\,,$$

where

$$\boldsymbol{F}_a = q(\boldsymbol{v} \times \boldsymbol{B}_a)$$
$$\boldsymbol{F}_b = q(\boldsymbol{v} \times \boldsymbol{B}_b)$$

From the definition, we get

$$W_{net} = qV = q\oint \boldsymbol{E} \cdot d\boldsymbol{s} = qv(B_a - B_b)w\,,$$

or

$$V = \oint \boldsymbol{E} \cdot d\boldsymbol{s} = v(B_a - B_b)w .$$

The electromotive force V is related in a very simple way to the rate of change of magnetic flux through the loop. The magnetic flux through a loop is the surface integral of $\boldsymbol{B}$ over a surface which has the loop for its boundary.

$$\Delta\Phi = B_b vw\Delta t - B_a vw\Delta t = -(B_a - B_b)vw\Delta t$$

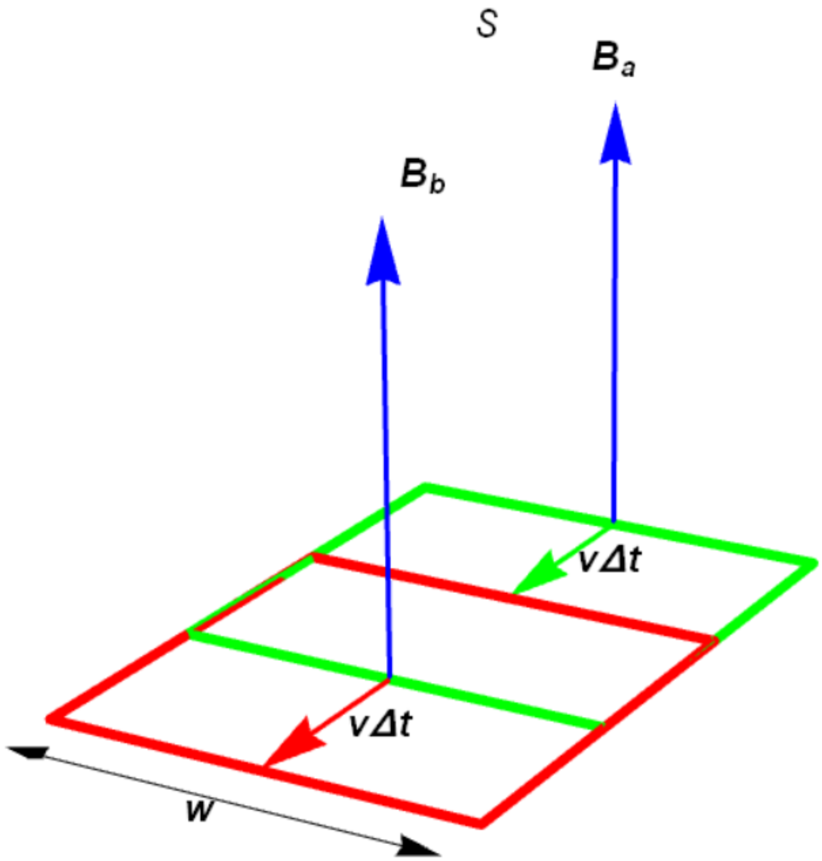


Fig. In the interval Δt, the loop gains an increment of the magnetic flux $B_b wv\Delta t$ and loses an increment $B_a wv\Delta t$.

or

$$V = \oint \boldsymbol{E} \cdot d\boldsymbol{s} = \oint (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{a} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int \boldsymbol{B} \cdot d\boldsymbol{a}$$
$$= (B_a - B_b)vw$$

((**Special relativity**)) See Chapter 37S for the detail

We now consider the frame S' attached to the loop.

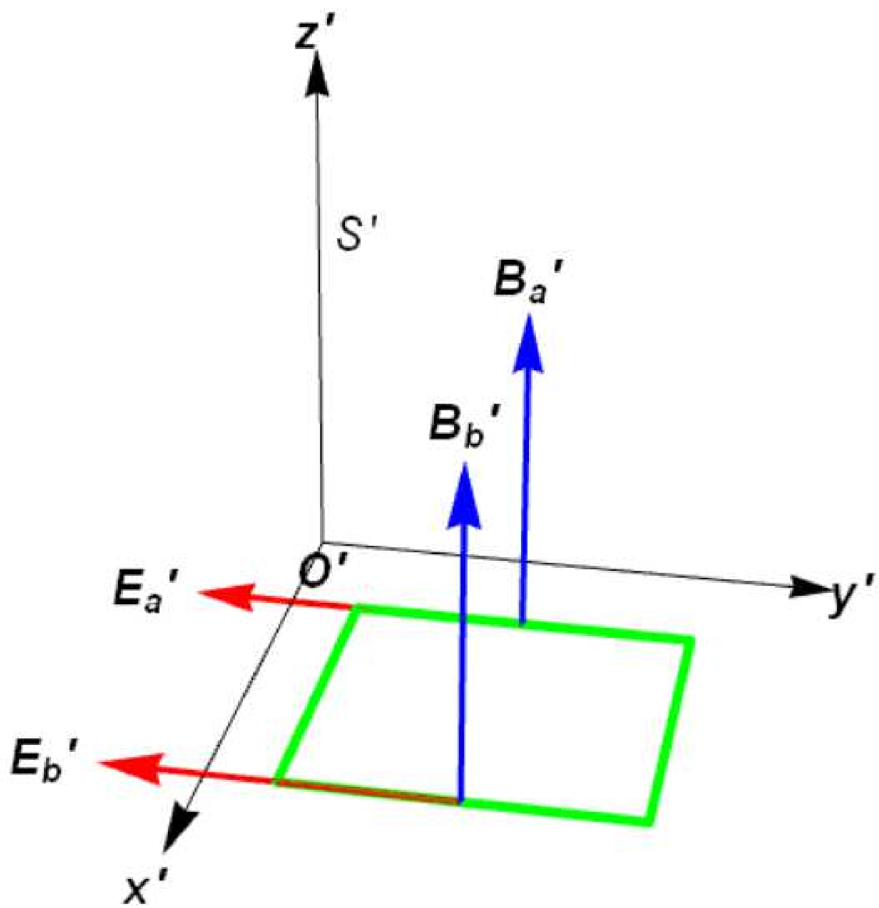


Fig.: The electric and magnetic field in the S' frame.

From the Lorentz transformation of ***E*** and ***B***,

$$E_1' = E_1 = 0 \qquad B_1' = B_1 = 0$$

$$E_2' = \gamma(E_2 - vB_3) = -\gamma v B_3 \quad , \qquad B_2' = \gamma(B_2 + \frac{\beta}{c}E_3) = \gamma B_2 = 0$$

$$E_3' = \gamma(E_3 + vB_2) = \gamma v B_2 = 0 \qquad B_3' = \gamma(B_3 - \frac{\beta}{c}E_2) = \gamma B_3$$

we have

$$E_a' = -\gamma v B_a \approx -vB_a \qquad E_b' = -\gamma v B_b = -vB_b$$

$$B_a' = \gamma B_a \approx B_a \quad , \qquad B_b' = \gamma B_b \approx B_b$$

For observers in the frame S', $\boldsymbol{E}_a$' and $\boldsymbol{E}_b$' are genuine electric field. It is not an electrostatic field. The integral of $\boldsymbol{E}$' around the loop, which is the electromotive force *V'*, is given by

$$V' = \oint \boldsymbol{E}\cdot d\boldsymbol{s} = vB_a w - vB_b w = vw(B_a - B_b)$$

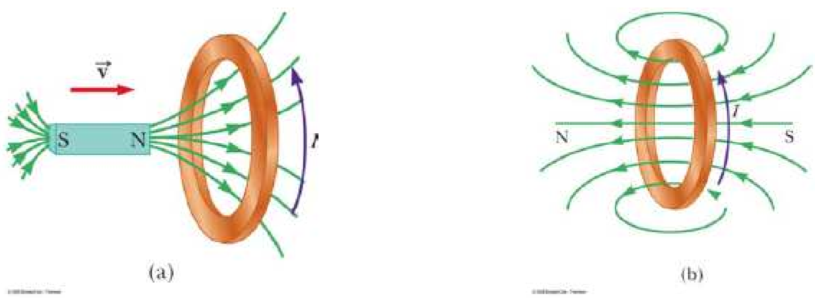
which is the same as that obtained for the frame S.

6. Lenz' law

6.1 Example

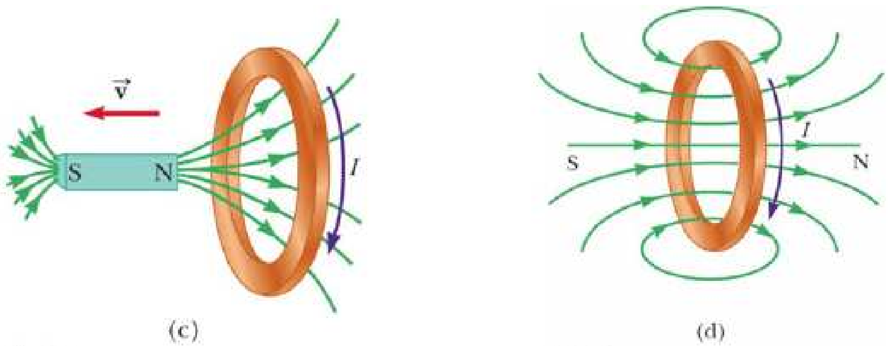
The induced current is always such as to oppose the change of flux (or the motion) that generated it. Any induced current resulting from an induced emf is in a direction such that the flux due to it will oppose the change in flux that caused the induced emf.

(i)



When the magnet is moved toward the stationary loop, a current is induced as shown in (a). This induced current produces its own magnetic field that is directed as shown in (b) to counteract the increasing external flux.

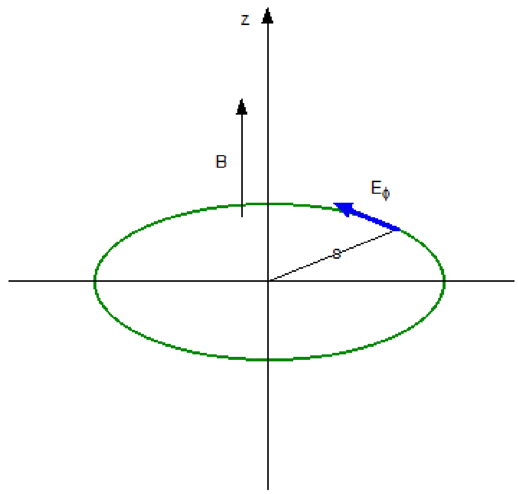
(ii)



When the magnet is moved away the stationary loop, a current is induced as shown in (c). This induced current produces its own magnetic field that is directed as shown in (d) to counteract the decreasing external flux.

Heinrich Lenz (1804 – 1865).

6.2 Mathematical formulation for the Lenz's law



$$V = \oint (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{a} = \oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d\Phi}{dt}$$

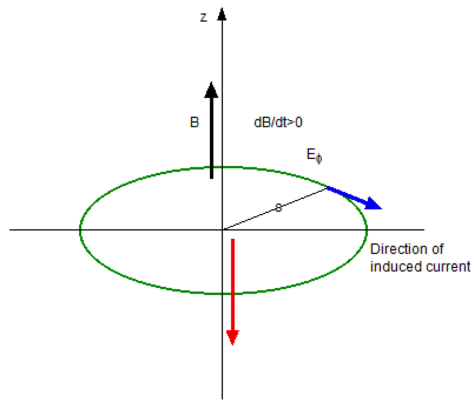
$$\Phi = BA = B(\pi s^2)$$

$$\oint \boldsymbol{E} \cdot d\boldsymbol{s} = E_\varphi(2\pi s) = -\frac{d\Phi}{dt} = -\pi s^2 \frac{dB}{dt}$$

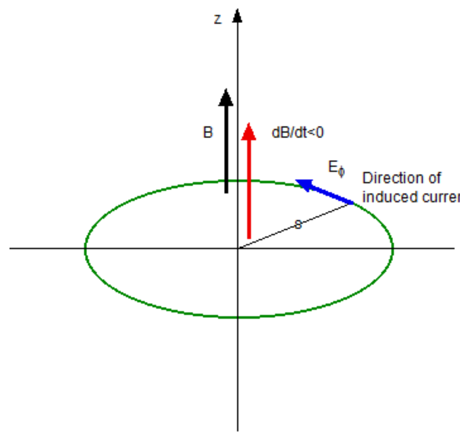
or

$$E_\phi = -\frac{\pi s^2}{2\pi s}\frac{dB}{dt} = -\frac{s}{2}\frac{dB}{dt},$$

where E_ϕ is the electric field along the orbit.



When the magnetic field B (along the z axis) increases with time ($dB/dt>0$), the induced current flows in a clock-wise. The magnetic field induced by the induced current is shown by the red arrow (the negative z axis) (Lenz's law).



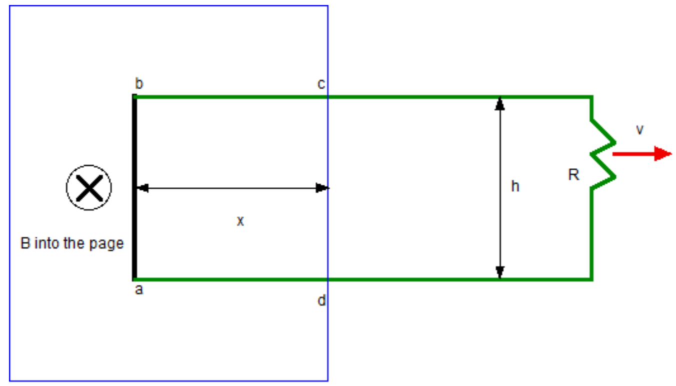
When the magnetic field B (along the z axis) decreases with time ($dB/dt<0$), the induced current flows in a counter clock-wise. The magnetic field induced by the induced current is shown by the red arrow (the positive z axis) (Lenz's law).

6.3 Direction of *v* in the form of $F = q(\boldsymbol{v}\times\boldsymbol{B})$ determined from the Lenz'law

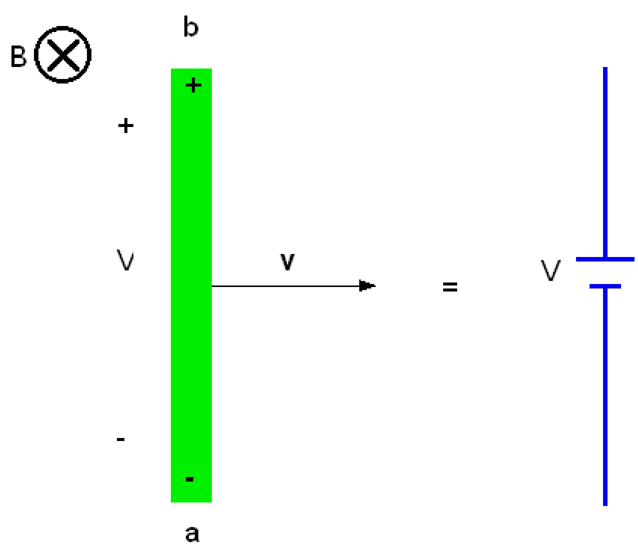
Here we consider how to determine the direction of v in the form of $\boldsymbol{F} = q(\boldsymbol{v}\times\boldsymbol{B})$, based on the Lenz's law. Suppose that a charged particle with the velocity v enters into the region in the presence of magnetic field $\boldsymbol{B}$ along the z axis (tending to increase). The particle undergoes a circular motion. The magnetic field is generated by this circular current. Because of the Lentz's law, this magnetic is antiparallel to the original magnetic field. The positive charge rotates in clock-wise, while the negative charge rotates in counter clock wise.

7. Moving loop in a uniform magnetic field: model of generator

7.1 Principle



In the region enclosed by blue line, there is a uniform magnetic field $\boldsymbol{B}$, pointing into the page. R is the resistor



The charges only on the segment ab experiences a magnetic force,

$$\boldsymbol{f}_{mag} = q(\boldsymbol{v} \times \boldsymbol{B})$$

The work done on the system is

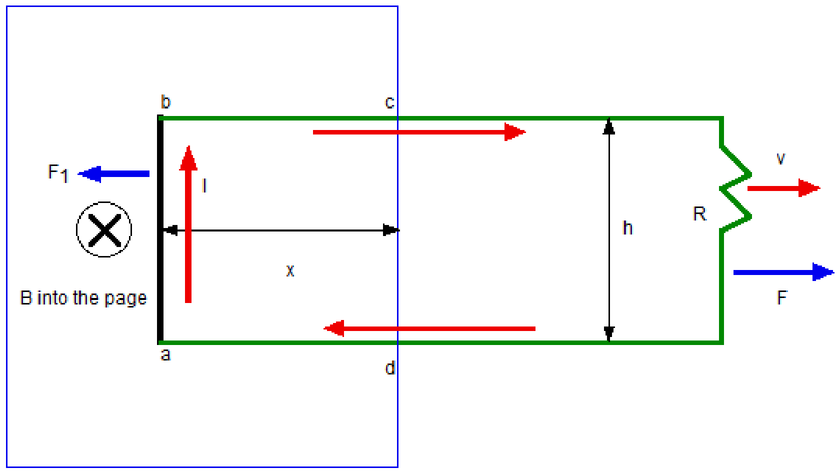
$$W_{net} = \int \boldsymbol{f}_{mag} \cdot d\boldsymbol{s} = qvBh$$

where d$\boldsymbol{s}$ ia along the wire (ab). The emf V is

$$V = \frac{W_{net}}{q} = vBh$$

This emf drives current around the loop in the clockwise direction. The current I is equal to

$$I = \frac{V}{R} = \frac{vBh}{R}$$



The force $\boldsymbol{F}_1$ due to the current is given by

$$F_1 = hIB = hB\frac{vBh}{R} = \frac{h^2B^2v}{R}$$

The agent that pulls the loop must exert a force $\boldsymbol{F}$ equal in magnitude to $\boldsymbol{F}_1$, if the loop is to move at constant speed. Thus the agent must therefore do work at the steady rate of

$$P = F_1 v = \frac{h^2B^2v^2}{R}$$

The rate at which energy is dissipated in the loop as a result of the Joule heating by the induced current,

$$P = i^2R = \frac{h^2B^2v^2}{R}$$

((**Note**))
$\boldsymbol{f}_{mag}$ contributes nothing to work because it is perpendicular to the motion of charge. $\boldsymbol{F}$ contributes nothing to the emf, because it is perpendicular to the wire.

7.2 Faraday's law and Lentz' law

The total magnetic flux Φ is given by

$$\Phi = \int \boldsymbol{B} \cdot d\boldsymbol{a} = Bhx$$

$$V = -\frac{d\Phi}{dt} = -Bh\frac{dx}{dt} = -Bhv$$

The current direction is determined from the Lenz' law.

When the loop moves to the x-direction, the total flux (into the page) tends to decrease. According to the Lentz's law, the induced current flows clockwise such that the total flux (into the page) tends to increase.

8. Sliding conducting bar

A metal bar of mass m slides frictionlessly on two parallel conducting rails a distance h apart. A resistor R is connected across the rails and a uniform magnetic field $\boldsymbol{B}$, pointing into the page, fills the entire region.

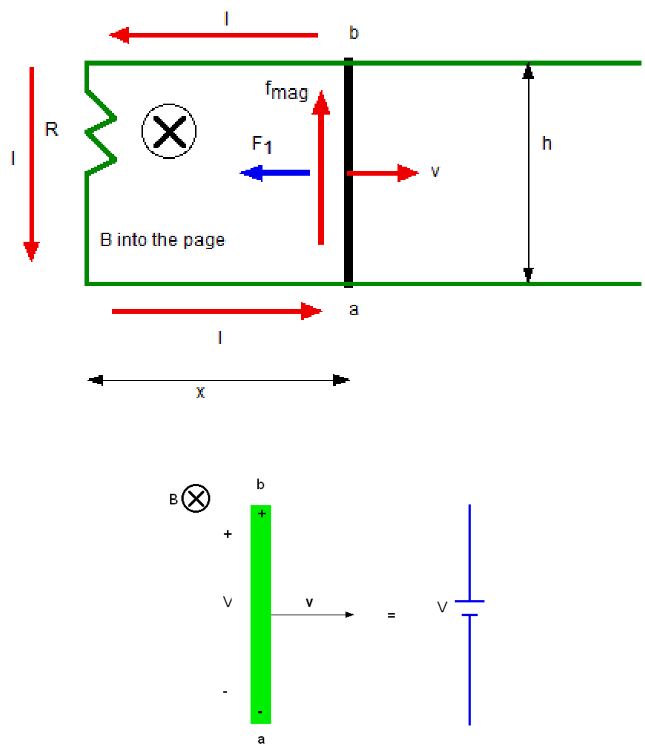


Fig. An emf is induced in a loop if the flux is changed by varying the area of the circuit.

When the bar moves to the right at the speed $\boldsymbol{v}$, The magnetic force $\boldsymbol{f}_{mag}$ on the bar is given by

$$\boldsymbol{f}_{mag} = q(\boldsymbol{v} \times \boldsymbol{B})$$

$$f_{mag} = qvB$$

The work done on the system is

$$W_{net} = \int \boldsymbol{f}_{mag} \cdot d\boldsymbol{s} = qvBh$$

where d$\boldsymbol{s}$ ia along the wire (ab). The emf V is

$$V = \frac{W_{net}}{q} = vBh$$

The current I flows in a counterclockwise direction.

$$I = \frac{V}{R} = \frac{vBh}{R}$$

Once the current flows in the loop, the force due to the current is exerted on the metal bar along the negative x direction.

$$F_1 = IBh = \frac{vB^2h^2}{R}$$

We set up an equation of motion for the metal bar with the mass m.

$$m\frac{dv}{dt} = -F_1 = -\frac{vB^2h^2}{R}$$

or

$$\frac{dv}{dt} = -\frac{F_1}{m} = -\frac{B^2h^2}{mR}v$$

We assume that the initial velocity of the metal bar is v_0 at $t = 0$. The solution of the differential equation is obtained as

$$v = v_0 \exp[-\frac{B^2h^2}{mR}t]$$

The rate of the energy dissipated in the resistor is

$$\frac{d\varepsilon}{dt} = RI^2 = R\frac{v^2B^2h^2}{R^2} = \frac{{v_0}^2B^2h^2}{R}\exp[-\frac{2B^2h^2}{mR}t]$$

or

$$\varepsilon = \frac{{v_0}^2 B^2 h^2}{R}\int_0^{\infty} \exp[-\frac{2B^2h^2}{mR}t]dt$$

$$= \frac{{v_0}^2 B^2 h^2}{R}\frac{mR}{2B^2h^2} = \frac{m{v_0}^2}{2}$$

((Note)) **Experimental arrangement for demonstrating an induced current produced in a wire (H.E. White, 1940)**

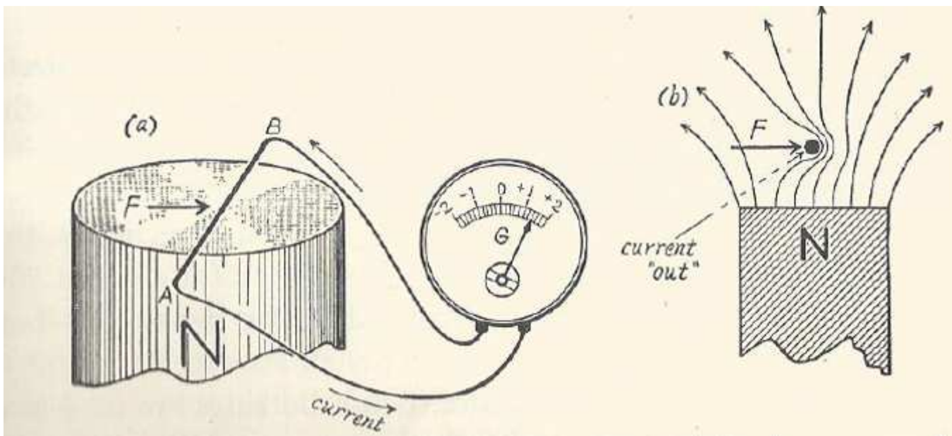


Fig. Experimental arrangement for demonstrating an induced current produced in a wire while it is cutting across magnetic lines of force. (H.E. White, Classical and Modern Physics; A Descriptive Introduction (D. Van Nostrand, 1940).

9. Self-inductance of solenoid

We assume a uniformly wound solenoid having N turns and length l. Assume that l is much greater than the radius of the solenoid. The magnetic field inside a solenoid is

$$B = \mu_0 n i$$

The magnetic flux through each turn is

$$\Phi = BA = \mu_0 n A i$$

Then the emf across the solenoid is

$$V = -N\frac{d\Phi}{dt} = -N\mu_0 n A\frac{di}{dt}$$

for the coil of the N turn, where $N = nl$. The self-inductance L is defined by

$$V = -L\frac{di}{dt}$$

leading to

$$L = \mu_0 n^2 Al$$

((**Example**))
For $N = 2800$, $l = 0.6$ m. $r = 0.05$ m = 5 cm, $L = 0.129$ H.

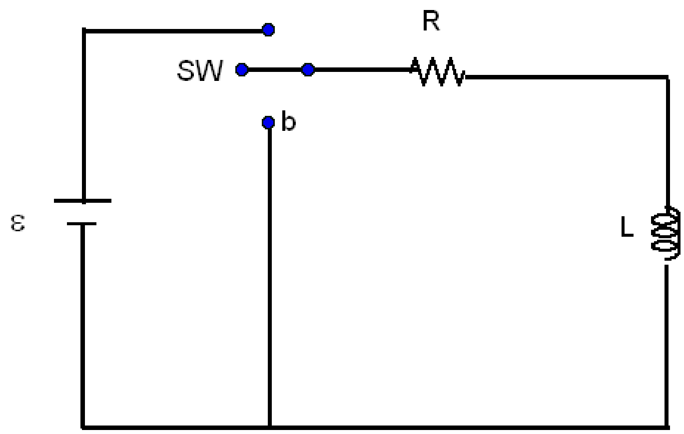
This shows that L depends on the geometry of the object. The SI unit of L is H (Henry), named for Joseph Henry (1797 – 1878).

1 H = V s/A = 1 T m^2/A

Born	December 17, 1797 Albany, New York, USA
Died	May 13, 1878 (aged 80) Washington, D. C., USA
Nationality	United States
Field	Physics
Institutions	Albany Academy, Princeton University Smithsonian Institution
Alma mater	Albany Academy
Known for	Electromagnetic induction

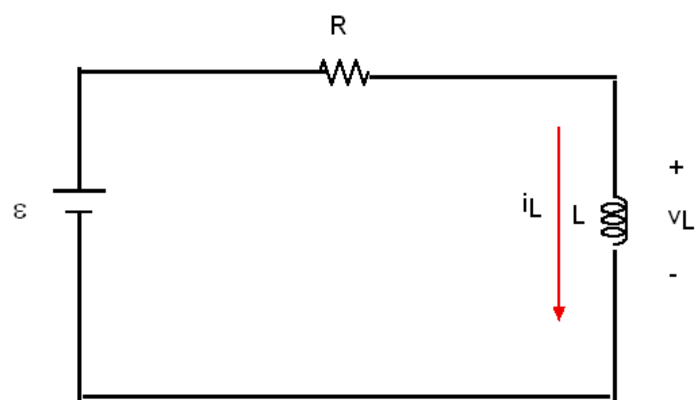
10. *RL* circuits
10.1 Example-1

An *RL* circuit contains an inductor *L* and a resistor *R*. When the switch is closed (at time $t = 0$), the current begins to increase. At the same time, a back emf is induced in the inductor that opposes the original increasing current.



(a) Switching of SW from neutral position to a

The switch SW is connected to the terminal a at $t = 0$,



In this circuit, we have

$$-\varepsilon + Ri_L + v_L = 0$$

$$v_L = L\frac{di_L}{dt}$$

Then the first-order differential equation is obtained as

$$L\frac{di_L}{dt} + Ri_L = \varepsilon$$

with an initial condition, $i_L(0) = 0$ and $v_L(0) = 0$.

The solution of this differential equation is

$$i_L(t) = \frac{\varepsilon}{R}(1 - e^{-\frac{Rt}{L}})$$

The current i_L is

$$v_L(t) = \varepsilon e^{-\frac{Rt}{L}}$$

Note that $\tau = L/R$ is a time constant.

We note that the current i_L is continuous at $t = 0$. The voltage v_L across the inductor L is discontinuous at $t = 0$.

((**Mathematica**))

Out[2]=

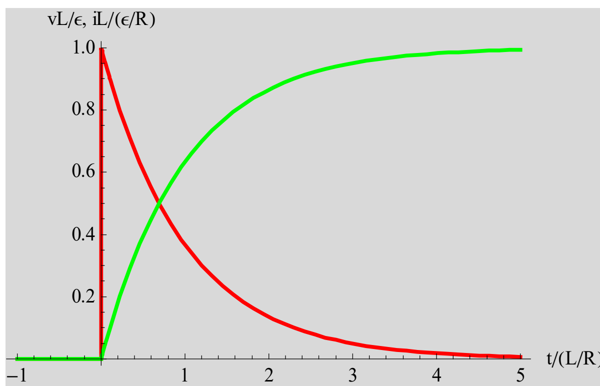
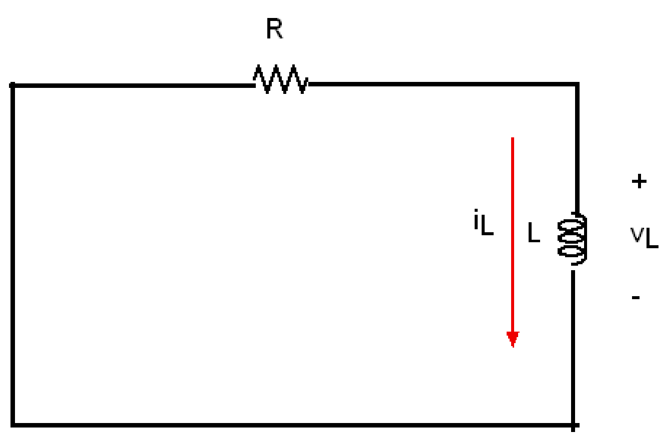


Fig. time dependence of v_L/ε (red, discontinuous at $t = 0$) and $i_L/(\varepsilon/R)$ (green, continuous at $t = 0$).

(b) Switching of SW from a to b



We have

$$Ri_L + v_L = 0$$

$$v_L = L\frac{di_L}{dt}$$

Then the first-order differential equation is obtained as

$$L\frac{di_L}{dt} + Ri_L = 0$$

with an initial condition, $i_L(0) = \frac{\varepsilon}{R}$ and $v_L(0) = 0$. The solution is

$$i_L(t) = \frac{\varepsilon}{R}e^{-\frac{Rt}{L}}$$

The voltage v_L is obtained as

$$v_L(t) = -\varepsilon e^{-\frac{Rt}{L}}$$

We note that the current i_L is continuous at $t = 0$. The voltage v_L across the inductor L is discontinuous at $t = 0$.

((**Mathematica**))

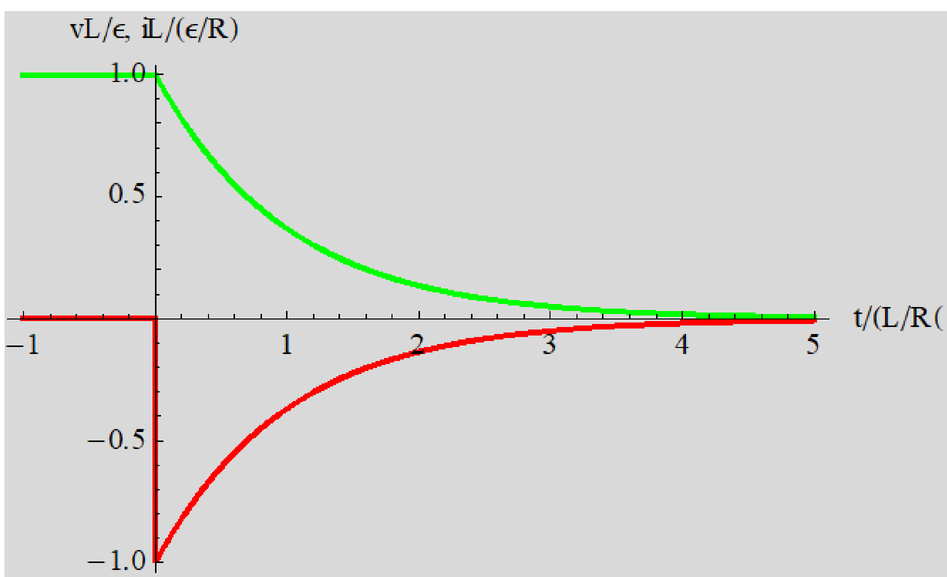


Fig. time dependence of v_L/ε (red, discontinuous at t = 0) and $i_L/(\varepsilon/R)$ (green, continuous at t = 0).

((**Note**))
The current (i) flowing through a inductance is a *good variable* for the calculation in the *RL* circuit, since $i(t)$ is continuous at t = 0. The reason is as follow.

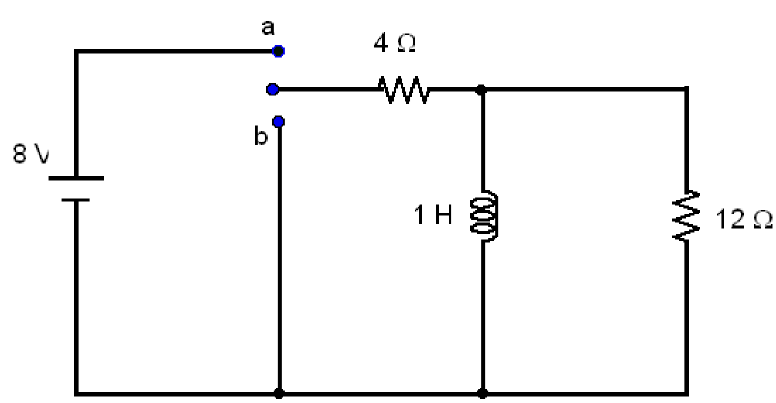
We now consider the difference between the current i_L passing through a inductor at a time $t = t_0$ and that at a time $t = t_0+\varepsilon$,

$$i_L(t_0+\varepsilon)-i_L(t_0)=\frac{1}{L}\int_{t_0}^{t_0+\varepsilon}v_L(t)dt$$

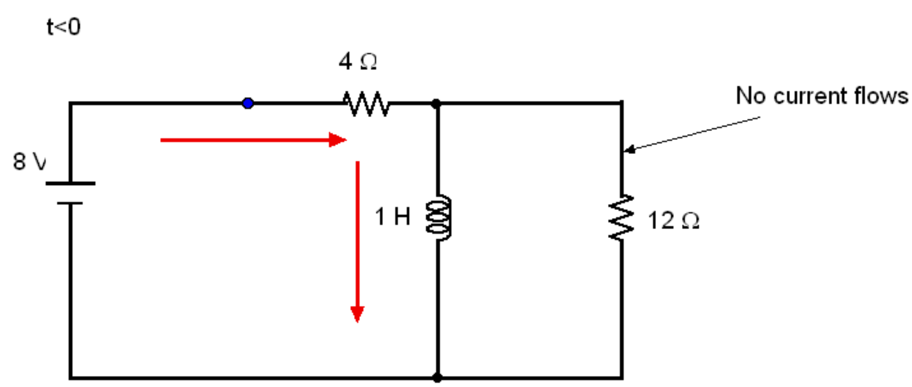
where $v_L(t)$ is the voltage across the inductor. This integral gets arbitrarily small if ε gets arbitrarily small. This means that $i_L(t)$ is continuous at any time t_0.

10.2 Example-2

We consider a *RL* circuit where $e_s(t)$ = 8 V for t<0 (switch to a) and 0 V for t>0 (switch to b)



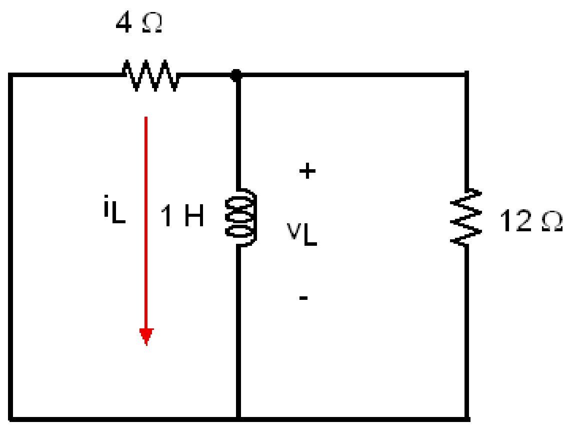
(a) For t<0 (SW to a), the current flows into the inductor, but not flows the resistance (12 Ω) (in the steady state). The inductance is in a short-circuit state.



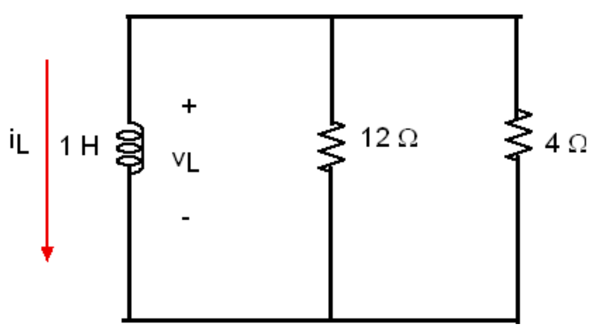
$$v_L(0) = 0$$

$$i_L(0) = \frac{8}{4} = 2A$$

(b) For t>0 (SW to b)



or



Noting that 12 Ω //4 Ω = 3 Ω, we have

$$v_L = L\frac{di_L}{dt} = -3i_L, \qquad \text{or} \qquad \frac{di_L}{dt} + 3i_L = 0$$

or

$$i_L(t) = i_L(0)e^{-3t} = 2e^{-3t}$$

$$v_L(t) = -6e^{-3t}$$

((**Mathematica**))

```
iL = If[t < 0, 2, 0] + If[t > 0, 2 e^(-3 t), 0];
vL = If[t < 0, 0, 0] + If[t > 0, -6 e^(-3 t), 0];
Plot[{iL, vL}, {t, -0.5, 1},
 PlotStyle → {{Thick, Red}, {Thick, Green}},
 AxesLabel → {"t(s)", "iL,vL"}]
```

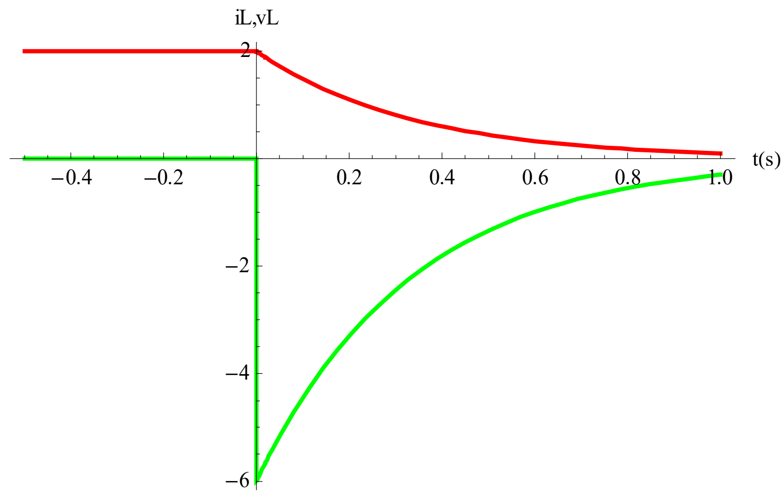


Fig time dependence of i_L (red) and v_L (green), Only $i_L(t)$ ic continuous at $t = 0$.

11. Energy storage in a magnetic field

We now consider the rate of energy

$$E_0 i = Li\frac{di}{dt} + Ri^2$$

where $E_0 i$ is the rate at which energy is being supplied by the battery, Ri^2 is the rate at which the energy is being delivered to the resistor, and $Li\, di/dt$ is the rate at which the energy is being delivered to the inductor.

Let U_B denote the energy stored in the inductor L at any time. The rate at which the energy is stored is

$$\frac{dU_B}{dt} = Li\frac{di}{dt} = \frac{1}{2}L\frac{d}{dt}i^2$$

or

$$U_B = \int dU_B = \int d(\frac{1}{2}Li^2) = \frac{1}{2}Li^2 \qquad \text{(magnetic energy)}$$

((**Work-energy theorem**))

We derive the form of magnetic energy using the work-energy theorem.

The energy of inductance is given by

$$\Delta W = VI\Delta t = -LI\frac{dI}{dt}\Delta t = -LIdI$$

Using the work-energy theorem, we have

$$\Delta U = -\Delta W = LIdI$$

or

$$U = \frac{1}{2}LI^2$$

12. Magnetic energy density

For the solenoid coil (the total number of turns; *N*, area; *A* and height; *l*),

$$B = \mu_0 nI$$

$$V = -N\frac{d\Phi}{dt} = -N\frac{d}{dt}(BA) = -NA\frac{d}{dt}(\mu_0 nI),$$

or

$$V = -NA\mu_0 n\frac{dI}{dt} = -Ah\mu_0 n^2\frac{dI}{dt} = -L\frac{dI}{dt}$$

leading to the self inductance L as $L = Al\mu_0 n^2$. Thus, we get

$$\frac{1}{2}LI^2 = \frac{1}{2}(Al\mu_0 n^2)\left(\frac{B^2}{\mu_0^2 n^2}\right) = Al\left(\frac{B^2}{2\mu_0}\right)$$

where Al is the volume of the system.

So the energy stored per unit volume, u_B, is

$$u_B = \frac{U_B}{Al} = \frac{1}{Al}\frac{1}{2}(\mu_0 n^2 Al)i^2 = \frac{1}{2}\mu_0 n^2 i^2$$

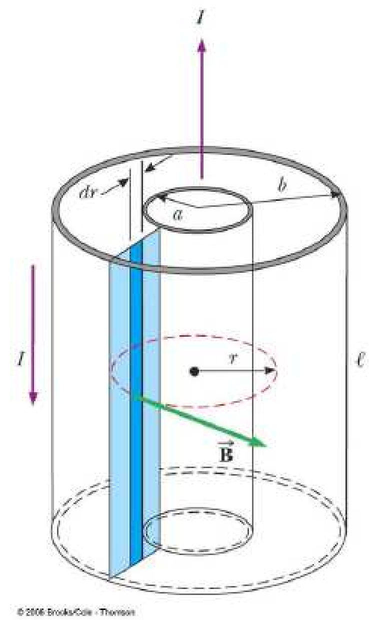
where $L = \mu_0 n^2 Al$. Since the magnetic field B is given by $B = \mu_0 ni$, we have

$$u_B = \frac{1}{2}\mu_0 n^2 i^2 = \frac{1}{2}\mu_0 n^2\frac{B^2}{\mu_0^2 n^2} = \frac{B^2}{2\mu_0}$$

This applies to any region in which a magnetic field exists (not just for the solenoid).

13. Self inductance example-coaxial cable

A long coaxial cable consists of two thin-walled concentric conducting cylinders with radii a and b. The inner cylinder carries a steady current i and the outer cylinder provides the return path for the current. We calculate (a) self inductance, and (b) the energy stored in the magnetic field for a length l of the cables.



From the Ampere's law, the magnetic field between radii a and b, is obtained as

$$B = \frac{\mu_0 i}{2\pi r}$$

The total magnetic flux Φ is

$$\Phi = \int \boldsymbol{B} \cdot d\boldsymbol{a} = \int_a^b l dr \frac{\mu_0 i}{2\pi r} = \frac{\mu_0 l i}{2\pi} \int_a^b dr \frac{1}{r} = \frac{\mu_0 l i}{2\pi} \ln(\frac{b}{a}) = Li$$

Then the self-inductance is obtained as

$$L = \frac{\Phi}{i} = \frac{\mu_0 l}{2\pi} \ln(\frac{b}{a})$$

The total magnetic energy is

$$U_B = \int u_B d\tau = \int_a^b l 2\pi r dr \frac{B^2}{2\mu_0} = \int_a^b l 2\pi r dr \frac{1}{2\mu_0}\left(\frac{\mu_0 i}{2\pi r}\right)^2$$
$$= \frac{\mu_0 l i^2}{4\pi}\int_a^b \frac{1}{r} dr = \frac{\mu_0 l i^2}{4\pi}\ln\left(\frac{b}{a}\right) = \frac{1}{2} L i^2$$

14. Mutual inductance

Suppose that coils in two different circuits are near each other. The current I_1 in coil 1 can create the magnetic flux Φ_{21} in coil 2 and induces an emf in the coil 2. The mutual inductance of coil 2 with respect to the coil 1 is defined as

$$\Phi_{21} = M_{21} I_1, \qquad V_2 = -\frac{d\Phi_{21}}{dt} = -M_{21}\frac{dI_1}{dt}$$

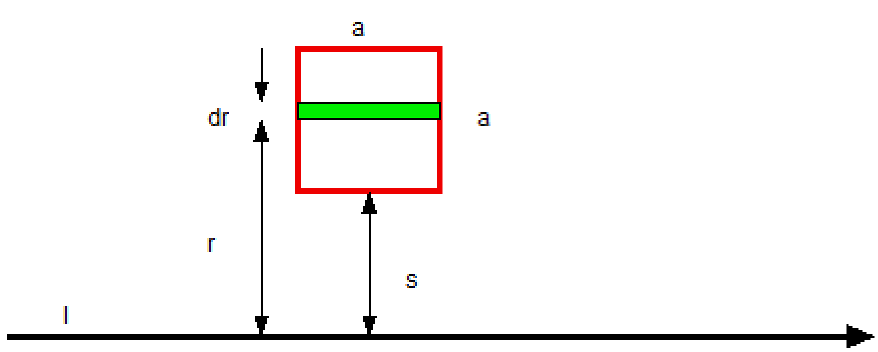
Similarly, the current I_2 in coil 2 can create the magnetic flux Φ_{12} in coil 1 and induces an emf in the coil 1. The mutual inductance of coil 1 with respect to the coil 2 is defined as

$$\Phi_{12} = M_{12} I_2, \qquad V_1 = -\frac{d\Phi_{12}}{dt} = -M_{12}\frac{dI_2}{dt}$$

We can show that $M_{12} = M_{21}$.

15. Mutual inductance: example

A square loop, side a, resistance R, lies a distance s from an infinite straight wire that carries current I. Now someone cuts the wire, so that the current I drops to zero. In what direction does the induced current in the square loop flow, and what total charge passes a given point in the loop during this current flows?



where the magnetic field points out of the page.

$$B(r) = \frac{\mu_0 I}{2\pi r}$$

$$d\Phi = B(r)da = \frac{\mu_0 I}{2\pi r} adr$$

$$\Phi = \int_s^{s+a} \frac{\mu_0 I}{2\pi r} adr = \frac{\mu_0 aI}{2\pi} \int_s^{s+a} \frac{1}{r} adr$$

$$= \frac{\mu_0 aI}{2\pi} \ln(\frac{s+a}{s})$$

The mutual inductance M is

$$M = \frac{\mu_0 a}{2\pi} \ln(\frac{s+a}{s})$$

The total charge Q is related to V through a relation,

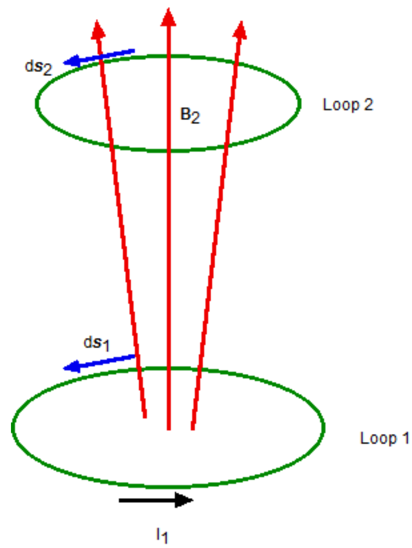
$$V = -\frac{d\Phi}{dt} = -\frac{\mu_0 a}{2\pi} \ln(\frac{s+a}{s}) \frac{dI}{dt} = RI = R\frac{dQ}{dt}$$

When the current I decreases, the magnetic flux Φ inside the square decreases. Lenz law says that the induced current flows counterclockwise, so that the field of the induced current points out of the page.

$$RdQ = -d\Phi = -\frac{\mu_0 a}{2\pi} \ln(\frac{s+a}{s}) dI$$

$$Q = -\frac{\mu_0 a}{2\pi R} \ln(\frac{s+a}{s}) \int_I^0 dI = \frac{\mu_0 aI}{2\pi R} \ln(\frac{s+a}{s})$$

16. Mutual inductance (formulation)



$$B_2 = \frac{\mu_0}{4\pi}\oint \frac{ds_1 \times (r_2 - r_1)}{|r_2 - r_1|^3}$$

$$\Phi_2 = \int B_2 \cdot da_2 = M_{21} I_1$$

where M_{21} is the mutual inductance of the two loops.

$$\Phi_2 = \int B_2 \cdot da_2 = \int (\nabla \times A_2) \cdot da_2 = \oint A_2 \cdot ds_2$$

Since

$$A_2 = \frac{\mu_0 I_1}{4\pi}\oint \frac{ds_1}{|r_2 - r_1|}$$

$$\Phi_2 = \oint A_2 \cdot ds_2 = \frac{\mu_0 I_1}{4\pi}\oint\oint \frac{ds_1 ds_2}{|r_2 - r_1|} \quad \text{(Neuman formula)}$$

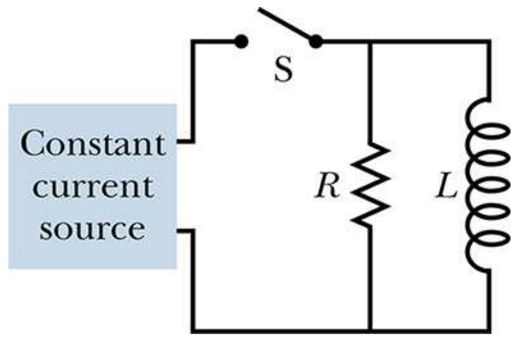
It is not very useful for practical calculations, but it does reveal two important things about mutual inductance.

(1) M_{21} is a purely geometrical quantity.
(2) $M_{21} = M_{12} = M$

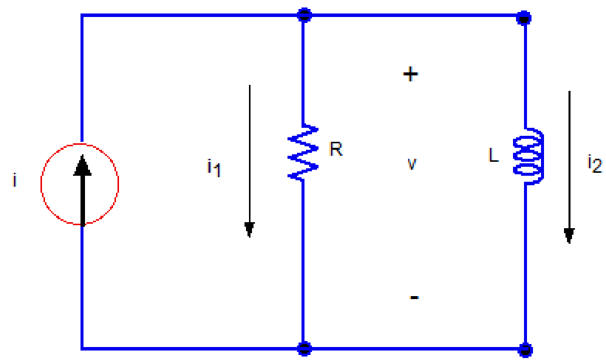
17. Typical problems

17-1 Problem 30-59 (SP-30)

In Fig. after switch S is closed at time $t = 0$, the emf of the source is automatically adjusted to maintain a constant current i through S. (a) Find the current through the inductor as a function of time. (b) At what time is the current through the resistor equal to the current through the inductor?



((**Solution**))



$$i = i_1 + i_2$$

$$v = Ri_1 = L\frac{di_2}{dt}$$

$$i_1 = \frac{L}{R}\frac{di_2}{dt}$$

Then we have

$$i = \frac{L}{R}\frac{di_2}{dt} + i_2$$

or

$$\frac{di_2}{dt} + \frac{R}{L}i_2 = \frac{R}{L}i$$

with the initial condition; $i_2(t = 0) = 0$. The solution of this first-order differential equation is as follows.

$$i_2 = i[1 - e^{-(R/L)t}]$$
$$i_1 = i - i_2 = ie^{-(R/L)t}$$

Suppose that $i_1 = i_2$ at $t = t_0$.

$$1 - e^{-(R/L)t_0} = e^{-(R/L)t_0}$$
$$e^{-(R/L)t_0} = \frac{1}{2}$$

or

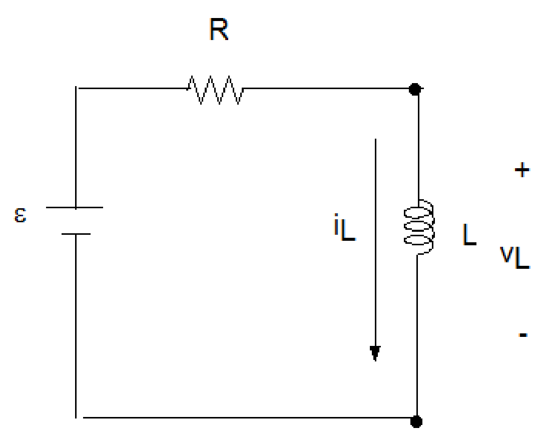
$$t_0 = \frac{L}{R}\ln 2$$

17.2 Problem 30-65 (SP-30)

For the circuit of Fig., assume that $\varepsilon = 10.0$ V, $R = 6.70\ \Omega$, and $L = 5.50$ H. The ideal battery is connected at time $t = 0$. (a) How much energy is delivered by the battery during the first 2.00 s? (b) How much of this energy is stored in the magnetic field of the inductor? (c) How much of the energy is dissipated in the resistor?

((**Solution**))

$\varepsilon = 10$ V, $R = 6.70\ \Omega$, $L = 5.50$ H.



$$-\varepsilon + Ri_L + v_L = 0$$
$$v_L = L\frac{di_L}{dt}$$

Then we have a first-order differential equation,

$$L\frac{di_L}{dt} + Ri_L = \varepsilon$$

with an initial condition, $i_L(0) = 0$

The solution of this differential equation is

$$i_L(t) = \frac{\varepsilon}{R}(1 - e^{-\frac{Rt}{L}})$$
$$= \frac{\varepsilon}{R}(1 - e^{-\frac{t}{\tau}})$$

where $\tau = \frac{L}{R}$.

(a) The energy delivered by the battery

$$U_\varepsilon = \int_0^t i_L \varepsilon dt$$
$$= \frac{\varepsilon^2}{R}\int_0^t (1 - e^{-\frac{t}{\tau}})dt$$
$$= \frac{\varepsilon^2}{R}(t + \tau e^{-t/\tau})|_0^t$$
$$= \frac{\varepsilon^2}{R}(t + \tau e^{-t/\tau} - \tau)$$

At $t = 2$ s, $U_\varepsilon = 18.67$ J

(b) The energy stored by the inductor

$$P_L = i_L v_L = Li_L \frac{di_L}{dt} = \frac{1}{2}L\frac{d}{dt}i_L^2$$

$$U_L = \int_0^t P_L dt = \frac{L}{2}i_L^2 |_0^t$$
$$= \frac{L\varepsilon^2}{2R^2}(1 - e^{-\frac{t}{\tau}})^2$$
$$= \frac{\varepsilon^2}{2R}\tau(1 - 2e^{-\frac{t}{\tau}} + e^{-\frac{2t}{\tau}})$$

At $t = 2$ s, $U_L = 5.10$ J

(c) The energy dissipated by the resistor

$$P_R = i_L v_R = R i_L{}^2$$

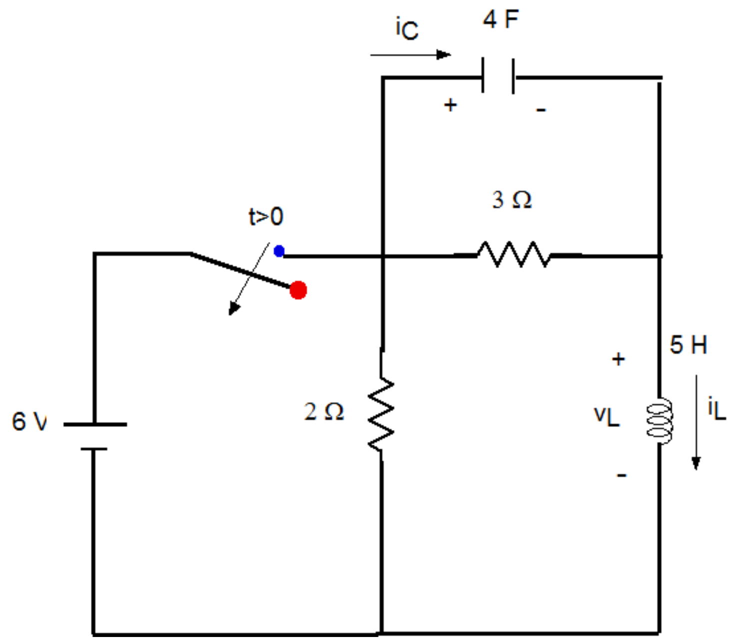
$$\begin{aligned}
U_R &= \int_0^t P_R dt = \int_0^t R i_L{}^2 dt \\
&= \int_0^t \frac{\varepsilon^2}{R}(1 - e^{-\frac{t}{\tau}})^2 dt \\
&= \frac{\varepsilon^2}{R}\int_0^t (1 - 2e^{-\frac{t}{\tau}} + e^{-\frac{2t}{\tau}})dt \\
&= \frac{\varepsilon^2}{R}[t + 2\tau e^{-\frac{t}{\tau}} - \frac{\tau}{2}e^{-\frac{2t}{\tau}}]_0^t \\
&= \frac{\varepsilon^2}{R}(t + 2\tau e^{-\frac{t}{\tau}} - \frac{\tau}{2}e^{-\frac{2t}{\tau}} - \frac{3}{2}\tau)
\end{aligned}$$

At t = 2 s, $U_R = 13.57$ J

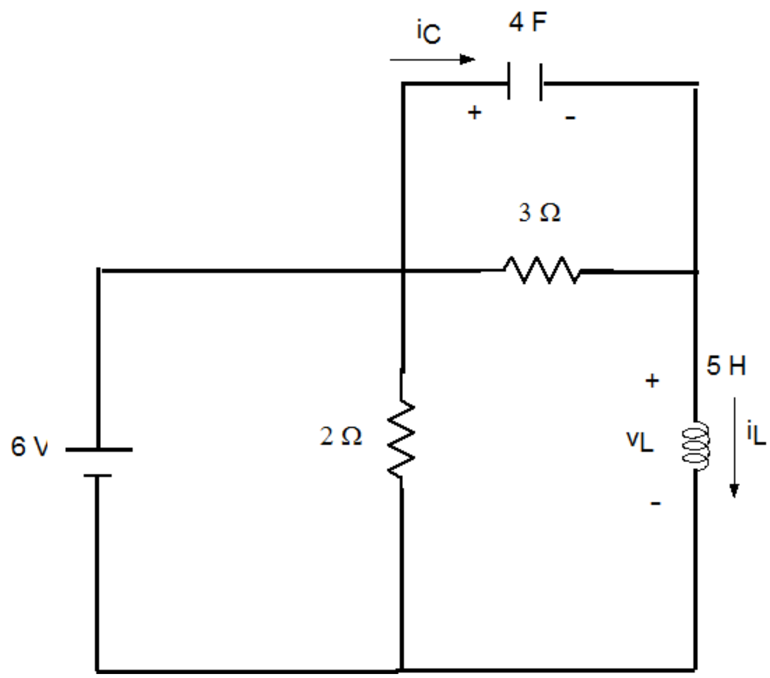
We note that the energy is conserved.

$$\begin{aligned}
U_L + U_R &= \frac{\varepsilon^2}{2R}\tau(1 - 2e^{-\frac{t}{\tau}} + e^{-\frac{2t}{\tau}}) + \frac{\varepsilon^2}{R}(t + 2\tau e^{-\frac{t}{\tau}} - \frac{\tau}{2}e^{-\frac{2t}{\tau}} - \frac{3}{2}\tau) \\
&= \frac{\varepsilon^2}{R}[\frac{\tau}{2}(1 - 2e^{-\frac{t}{\tau}} + e^{-\frac{2t}{\tau}}) + t + 2\tau e^{-\frac{t}{\tau}} - \frac{\tau}{2}e^{-\frac{2t}{\tau}} - \frac{3}{2}\tau] \\
&= \frac{\varepsilon^2}{R}(t + \tau e^{-\frac{t}{\tau}} - \tau)
\end{aligned}$$

17.3 ((**Example-3**)) *RLC* circuits



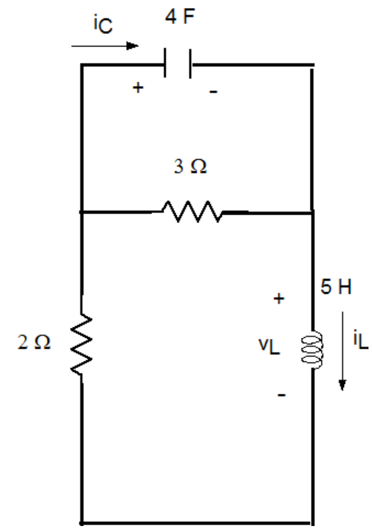
For $t<0$



$i_C(0) = 0$, $v_C(0) = 6$ V

$i_L(0) = 2$ A, $v_L(0) = 0$ V

For $t \geq 0$,



$$v_C = 3(i_L - i_C)$$

$$i_C = i_L - \frac{1}{3}v_C$$

From KVL we have

$$v_L + 2i_L + v_C = 0, \qquad \text{or} \qquad v_L = -2i_L - v_C$$

We also have

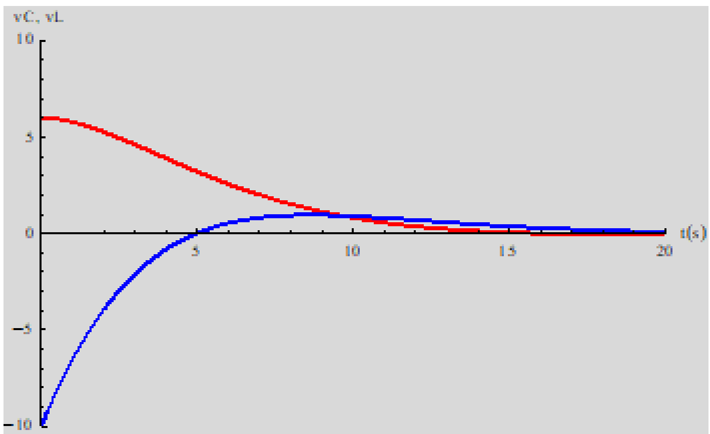
$$i_C = 4\frac{dv_C}{dt}, \qquad \text{or} \qquad \frac{dv_C}{dt} = \frac{i_C}{4} = \frac{1}{4}(i_L - \frac{1}{3}v_C)$$

$$v_L = L\frac{di_L}{dt} \qquad \text{or} \qquad \frac{di_L}{dt} = \frac{v_L}{5} = \frac{1}{5}(-2i_L - v_C)$$

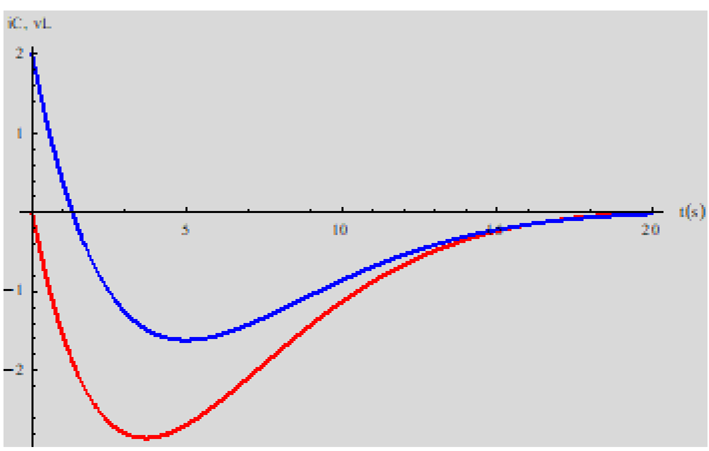
where $v_C(0)$= 6 V and $i_L(0)$= 2 A, $v_L(0)$ = -10V
The Mathematica is used to determine the time dependence of each voltage and current.

((**Mathematica**))

```
eq1 = {x'[t] == 1/4 (y[t] - 1/3 x[t]), y'[t] == 1/5 (-2 y[t] - x[t]),
   x[0] == 6, y[0] == 2}; eq2 = DSolve[eq1, {x[t], y[t]}, t] // Simplify;
vC = x[t] /. eq2[[1]]; iL = y[t] /. eq2[[1]]; iC = iL - 1/3 vC // Simplify;
vL = -2 iL - vC // Simplify;
Plot[{vC, vL}, {t, 0, 20},
 PlotStyle → {{Thick, Red}, {Thick, Blue}}, Background → LightGray,
 PlotRange → {{0, 20}, {-10, 10}}, AxesLabel → {"t(s)", "vC, vL"}]
```

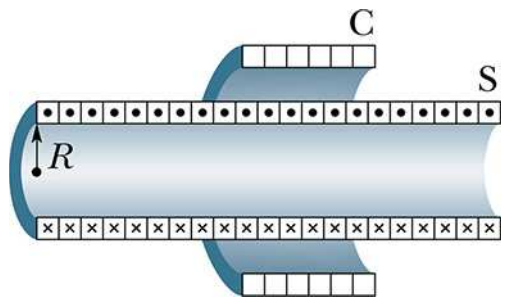



```
Plot[{iC, iL}, {t, 0, 20}, PlotStyle → {{Thick, Red}, {Thick, Blue}},
 Background → LightGray, AxesLabel → {"t(s)", "iC, vL"}]
```



17.4 Problem 30-76 (SP-30)

A coil of N turn is placed around a long solenoid S of radius R and n turn per unit length, as shown in Fig. (a) Show that the mutual inductance for the solenoid combination is given by $M = \mu_0 \pi R^2 nN$. (b) Explain why M does not depend on the shape, size, or possible lack of close packing of the coil.



((**Solution**))
(a)

$$B = \mu_0 ni \qquad \text{inside the solenoid C}$$

The magnetic flux:

$$\Phi = \pi R^2 B = \pi R^2(\mu_0 ni)$$

The emf induced in the coil S

$$V = -N\frac{d\Phi_c}{dt} = -N\pi R^2(\mu_0 n)\frac{di}{dt} = -M\frac{di}{dt}$$

(b)
M is the mutual inductance

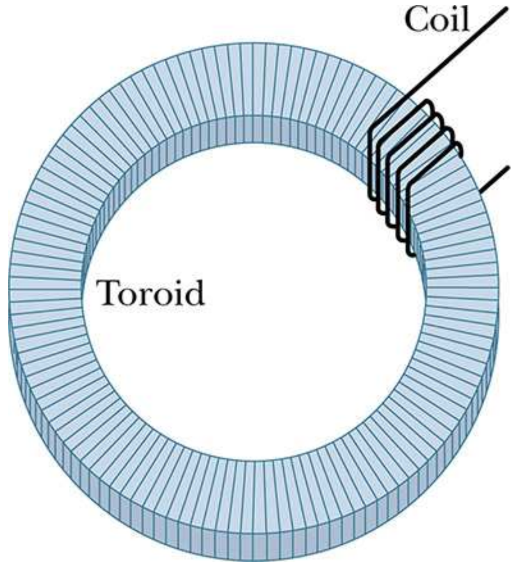
$$M = \mu_0 \pi n N R^2$$

which depends only on *R* and *N*.

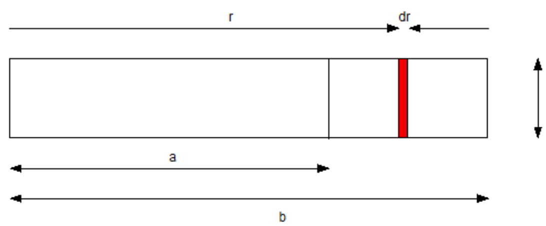
17.5 Problem 30-102 (SP-30)

Figure shows a coil of N_2 turns wound as shown around part of a toroid of N_1 turns. The toroid's radius is *a*, its outer radius is *b*, and its height is *h*. Show that the mutual inductance *M* for the toroid-coil combination is given by

$$M = \frac{1}{2\pi}\mu_0 N_1 N_2 h \ln(\frac{b}{a})$$



((**Solution**))



Ampere's law:

$$B(2\pi r) = \mu_0 (N_1 i_1)$$

or

$$B = \frac{\mu_0 N_1 i_1}{2\pi r}$$

The magnetic flux is obtained as

$$\Phi = \int_a^b Bhdr = \int_a^b \frac{\mu_0 N_1 i_1}{2\pi r} hdr$$

$$= \frac{\mu_0 N_1 i_1 h}{2\pi} \ln(\frac{b}{a})$$

The induced voltage is

$$V_2 = -N_2 \frac{d\Phi}{dt} = -\frac{\mu_0 N_1 N_2 h}{2\pi} \ln(\frac{b}{a}) \frac{di_1}{dt} = -M \frac{di_1}{dt}$$

with the mutual inductance

$$M = \frac{\mu_0 N_1 N_2 h}{2\pi} \ln(\frac{b}{a})$$

18. Faraday's law and Kirchhoff voltage law (revised on July 22, 2023)

During the class of MIT 802 Electricity and Magnetism, Walter Lewin showed an excellent discussion on the difference between the Faraday's and Kirchhoff voltage law using the following example of circuit.
https://www.youtube.com/watch?v=nGQbA2jwkWI&list=PLyQSN7X0ro2314mKyUiOILaOC2hk6Pc3j&index=18&t=0s

For the Faraday's law, we have

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = -N \frac{\partial \Phi}{\partial t}.$$

Note that

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = 0.$$

without solenoid. Using the Stokes theorem, we have

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = \int (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{a} = 0,$$

leading to

$$\boldsymbol{E} = -\nabla V.$$

Here we use the following relation

$$V(A_2) - V(A_1) = -\int_{A_2}^{A_2} \boldsymbol{E} \cdot d\boldsymbol{l}$$

where A_1 and A_2 on path considered.

For convenience, we use the following figure, instead of the figured used by Prof. Lewin.

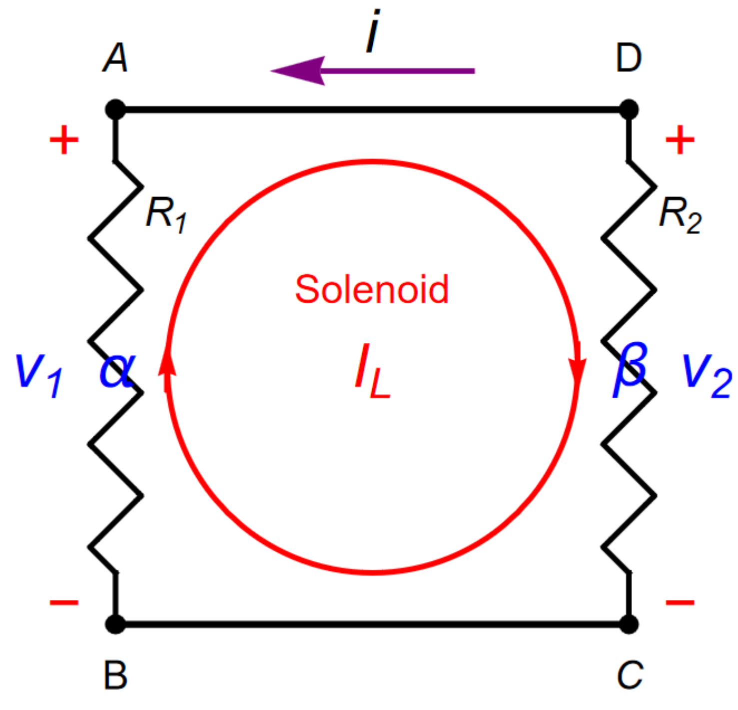


Fig. Loop circuit (DAαBCβD). A solenoid circuit with current i_L is located inside the loop current. The points B and C are grounded for convenience. $V_1 = R_1 i$. $V_2 = -R_2 i$. i is the loop current in the loop circuit (the direction is in counter clock-wise in this case after t = 0 when the solenoid current starts to flow).

We consider the Faraday's law for the above circuit. The circuit consists of two resistances R_1 and R_2 . There is no voltage source. Instead of it, the magnetic field due to the solenoid locating at the center of the circuit is abruptly applied to the plane of the circuit (into page), leading to the increase of the magnetic flux with time t. The induced loop current i simultaneously starts to flow in a counter clock-wise (CCW) (see the above **Fig.**). It tries to decrease the magnetic field, according to the Lentz law.

The path integral of electric field $\boldsymbol{E}$ along the closed path (DAαBCβD) is

$$\oint \boldsymbol{E}\cdot d\boldsymbol{l} = \oint (\nabla\times\boldsymbol{E})\cdot d\boldsymbol{a} = -\frac{\partial}{\partial t}\oint \boldsymbol{B}\cdot d\boldsymbol{a} = -N\frac{\partial\Phi}{\partial t} \qquad \text{(non-conservative case)}$$

where N is the total number of turns for solenoid coils. Concretely, we have

$$\oint \boldsymbol{E}\cdot d\boldsymbol{l} = \int_{DA}\boldsymbol{E}\cdot d\boldsymbol{l} + \int_{A\alpha B}\boldsymbol{E}\cdot d\boldsymbol{l} + \int_{BC}\boldsymbol{E}\cdot d\boldsymbol{l} + \int_{C\beta D}\boldsymbol{E}\cdot d\boldsymbol{l}\,.$$

Note that

$$\int_{DA} \boldsymbol{E} \cdot d\boldsymbol{l} = 0, \qquad \int_{BC} \boldsymbol{E} \cdot d\boldsymbol{l} = 0.$$

Because of the emf $(-N\frac{\partial \Phi}{\partial t})$ generated from the abrupt change of magnetic flux of the solenoid, there occurs a loop current i in the circuit (AαBCβDA). The direction of the loop current can be uniquely determined by the Lentz law. When the current in the solenoid increases, the magnetic field is produced inside the loop AαBCβDA (the direction of magnetic field is into the page). According to the Lenz law, the induced current flows in the circuit in the clock-wise, so that the direction of the magnetic field is out of page in **Fig**.

$$\int_{A\alpha B} \boldsymbol{E} \cdot d\boldsymbol{l} = -\int_{A\alpha B} \nabla V \cdot d\boldsymbol{l} = -V_B + V_A = V_A = R_1 i.$$

and

$$\int_{C\beta D} \boldsymbol{E} \cdot d\boldsymbol{l} = -\int_{C\beta D} \nabla V \cdot d\boldsymbol{l} = -V_D + V_C = -V_D = -R_2 i.$$

since the points B and C are grounded; $V_B = V_C = 0$. Thus, we have

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = -N\frac{\partial \Phi}{\partial t} = -(R_1 + R_2)i.$$

or

$$i = \frac{1}{R_1 + R_2} N \frac{\partial \Phi}{\partial t}.$$

The voltages at the point A and D are given by

$$V_A = R_2 i = \frac{R_2}{R_1 + R_2} N \frac{\partial \Phi}{\partial t}, \qquad V_D = -R_1 i = -\frac{R_1}{R_1 + R_2} N \frac{\partial \Phi}{\partial t}$$

We now evaluate the induced voltage associated with the change of magnetic flux,

$$
\begin{aligned}
V_L &= -N\frac{d\Phi}{dt} \\
&= -NA\frac{dB}{dt} \\
&= -NA\mu_0 n\frac{di_L}{dt} \\
&= -Al\mu_0 n^2\frac{di_L}{dt} \\
&= -L\frac{di_L}{dt}
\end{aligned}
$$

where A is the area and l is the length of the coil. Note that n is the number turns per unit length ($N = nl$) and L is the inductance of the solenoid,

$$L = Al\mu_0 n^2 .$$

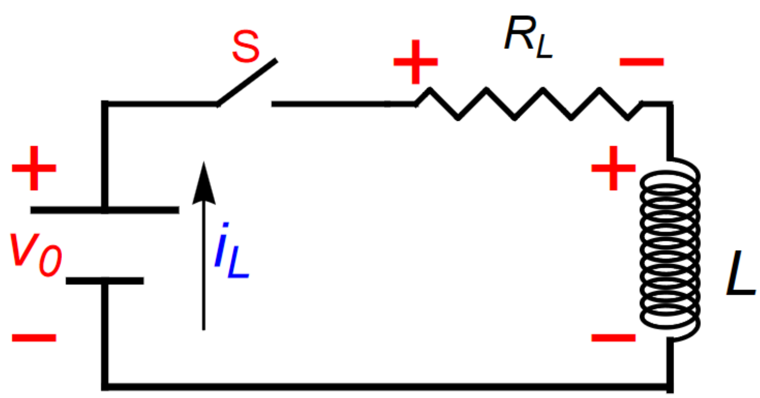


Fig. Solenoid circuit for the coil which generates a magnetic field inside the loop circuit. L and R_L are the inductance and resistance. V_0 is the voltage of DC battery. After the switch S is on, the current I_L flows through the solenoid, generating a time-dependent magnetic field.

The current flowing in the solenoid is determined as follows.

$$V_0 = R_L i_L + L\frac{di_L}{dt}, \qquad \text{(for } t > 0\text{)}$$

with initial condition $i_L = 0$, where i_L is the current flowing the inductance, R_L is the resistance, and L is the inductance of the solenoid. V_0 is the voltage source for the solenoid. Note that i_L is a good variable since it is continuous at $t = 0$. The solution for i_L is obtained as

$$i_L = \frac{V_0}{R_L}(1 - e^{-R_L t/L}).$$

Since

$$\frac{di_L}{dt} = \frac{V_0}{L} e^{-R_L t/L},$$

we have the final result for V_L

$$V_L = V_0 e^{-Rt/L}.$$

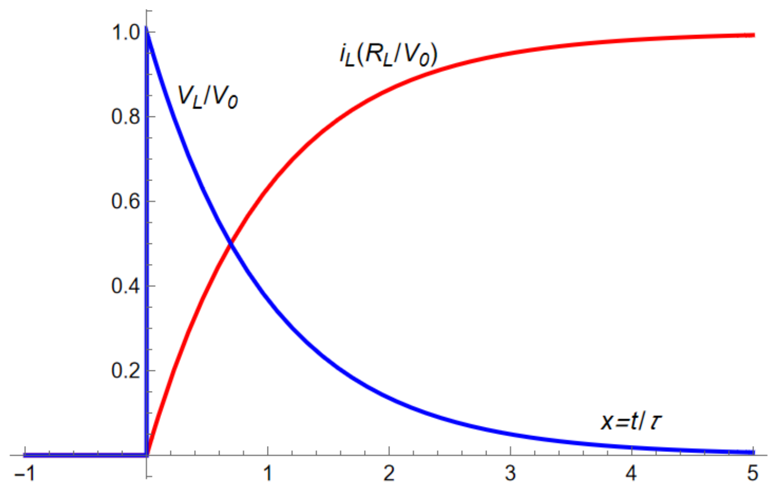


Fig. The plot of $i_L(R_L/V_0)$ and V_L/V_0 as a function of $x = t/\tau$ with the relaxation time $\tau = \frac{L}{R}$. While the normalized voltage V_L/V_0 undergoes a discontinuous jump at $t = 0$, and exponentially decreases, tending to zero after finite times. The normalized solenoid current $i_L(R_L/V_0)$ continuously starts to increase with t at $t = 0$, At $t = 0$, the solenoid circuit is switched to on.

Using this V_L we get the voltages V_A and V_D as a function of t.

$$V_A = R_1 i = \frac{R_1}{R_1 + R_2} V_0 e^{-Rt/L} = V_\alpha e^{-Rt/L},$$

$$V_D = -R_2 i = -\frac{R_2}{R_1 + R_2} V_0 e^{-R_L t/L} = -V_\beta e^{-R_L t/L}$$

where

$$V_\alpha = \frac{R_1}{R_1 + R_2} V_0 \ , \qquad V_\beta = \frac{R_2}{R_1 + R_2} V_0$$

When

$$R_1 = 0.9 \text{ k}\Omega\,,\ R_2 = 0.1 \text{ k}\Omega\,,\ V_0 = 1 \text{ V}\,.$$

we get

$$V_A = 0.9\, e^{-t/\tau} \text{ (V)}, \qquad V_D = -0.1\, e^{-t/\tau} \text{ (V)}$$

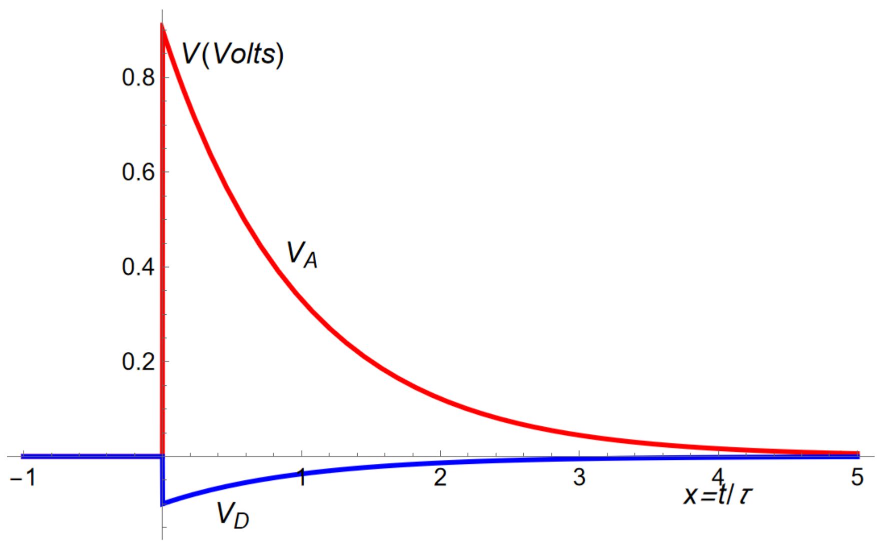


Fig. Plot of V_A and V_D as a function of *t*. When *t* = 0, the circuit of solenoid is switched to on. Note that $V_B = V_C = 0$ (grounded).

((**Demonstration by Walter Lewin**))

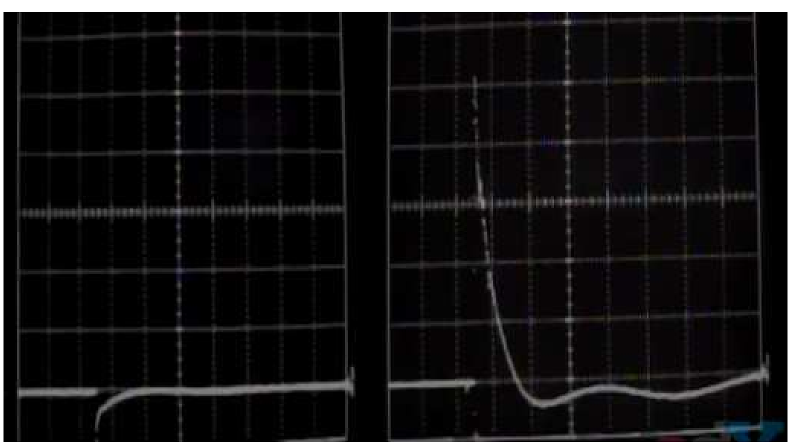


Fig. Experimental result of the time dependence of V_A (left) and V_D (right) on the oscilloscope. At $t = 0$, the solenoid circuit is switched to on. (result from the experiment by **Walter Lewin)**

((**Note**))

We note that similar problem is seen in **Problem 7-53** (p.349) in a text book:
David J. Griffiths, Introduction to Electrodynamics, 4th edition (Cambridge University Press, 2017).

((Problem 7.53))

The current in a long solenoid is increasing linearly with time, so the flux is proportional to *t*: $\Phi = \alpha t$. Two voltmeters are connected to diametrically opposite points (A and B), together with resistors (R_1 and R_2), as shown in **Fig**. 1. What is the reading on each voltmeter? Assume that these are ideal voltmeters that draw negligible current (they have huge internal resistance), and that a voltmeter registers $-\int_a^b \mathbf{E}\cdot d\mathbf{l}$ between the terminals and through the meter.

[Answer;

$$V_1 = \frac{\alpha R_1}{R_1 + R_2}, \qquad V_2 = -\frac{\alpha R_2}{R_1 + R_2}.$$

Notice that $V_1 \neq V_2$, even though they are connected to the same points.]

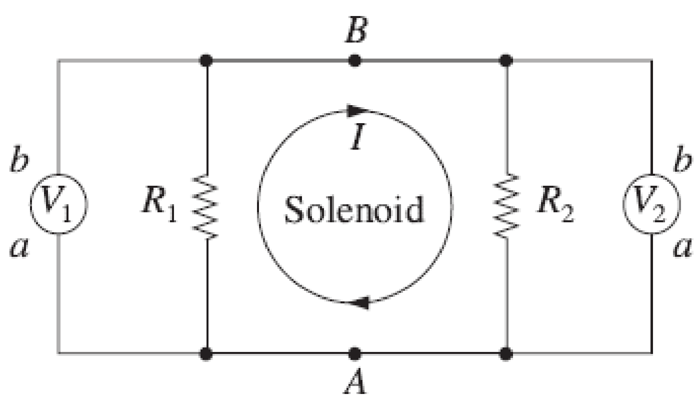
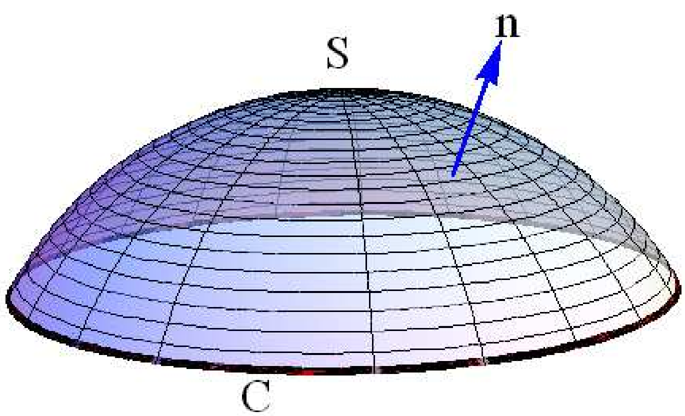


Fig.1 **A circuit from the book of D.J. Griffiths.**

APPENDIX-A Stokes' theorem

Let S be a surface of any shape bounded by a closed curve C. If F is a vector, then

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{l} = \int_S (\nabla\times\boldsymbol{F})\cdot d\boldsymbol{a} = \int_S (\nabla\times\boldsymbol{F})\cdot \boldsymbol{n}da\ .$$



((**Arfken**)) **Stokes' theorem**

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{l} = \int_S (\nabla\times\boldsymbol{F})\cdot d\boldsymbol{a}$$

Here C is the perimeter of S. This is Stokes' theorem. Note that the sign of the line integral and the direction of $d\boldsymbol{a}$ depend on the direction the perimeter is traversed, so consistent results will always be obtained. For the area and the line-integral direction shown in **Fig**, the direction of a for the shaded rectangle will be **out** of the plane of the paper.

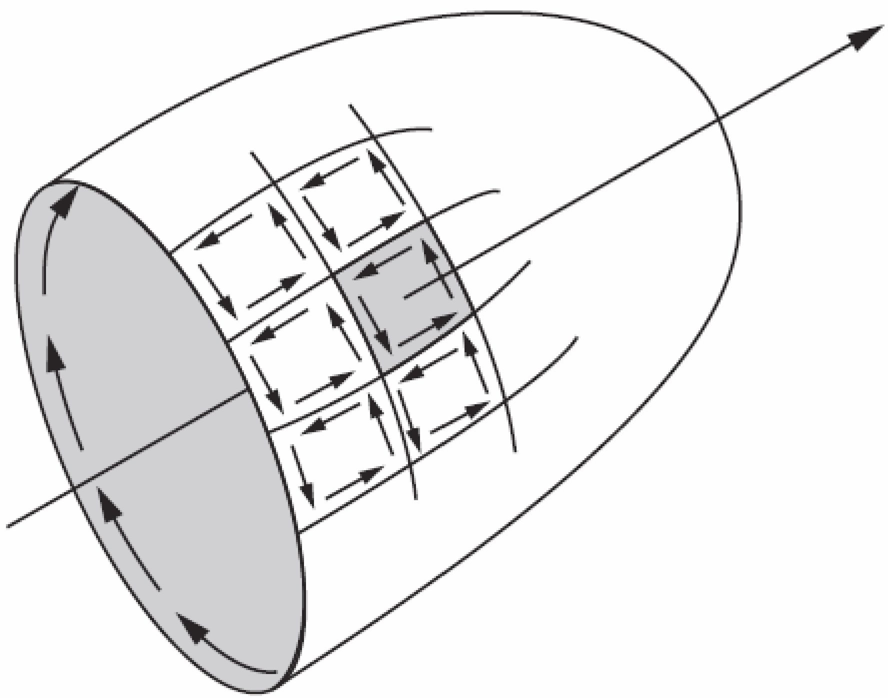


Fig. Direction of normal for the shaded rectangle when the perimeter of the surface is traversed as indicated. The direction of $d\boldsymbol{a}$ is out of paper, whilce the direction of dl is in counter clockwise.

REFERENCES

G.B. Arfken, H.J. Weber, and F.E. Harris, Mathematical Methods for Physicists 7-th edition, (Elsevier, 2013)

APPENDIX-B Mutual inductance

((Link))

In the Advanced Lab. (Senior Lab. and Graduate Lab), we developed an experiment on the measurement of mutual inductance. The URL of the related site is as follows.

M.J. Schauber, S.A. Newman, L.R. Goodman, I.S. Suzuki, and M. Suzuki
Am. J.Phys. **76**, 129 (2008)
"Measurement of mutual inductance from frequency dependence of AC coupled circuit using a digital lock in-amplifier."

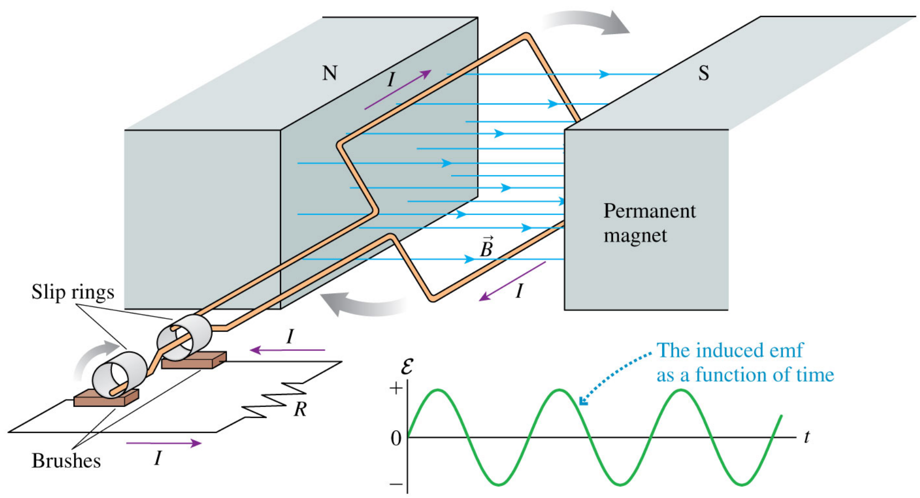
https://arxiv.org/abs/physics/0606124

APPENDIX-C Slip ring in AC generator and DC motor

The role of slip ring in AC generator is different from that in the DC motor (commutator).

(a) Slip ring in AC generator

In a version of the AC induction motor referred to as a wound rotor motor, slip rings are used not for transferring power, but for inserting resistance into the rotor windings. The slip ring brushes, made of graphite, are connected to a resistive device, such as a rheostat. As the slip rings turn with the rotor, the brushes maintain constant contact with the rings and transfer the resistance to the rotor windings.

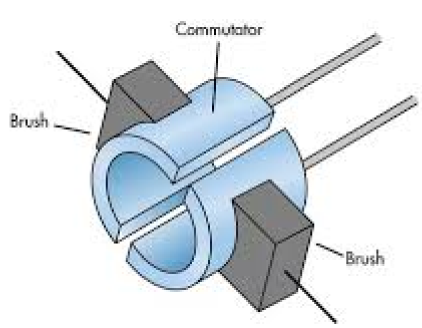


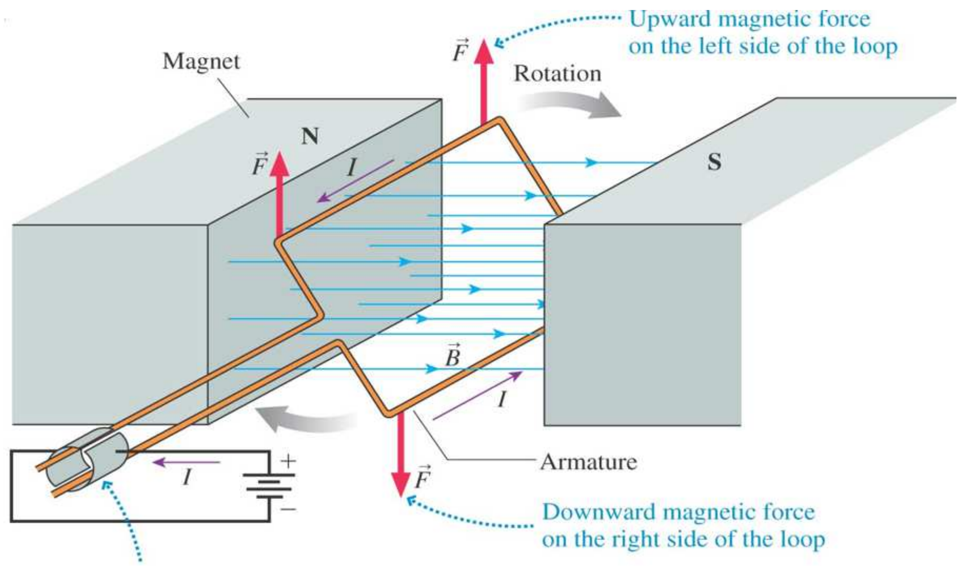
(b) Slip ring (commutator) in DC motor

Commutators are used in DC motors to reverse the polarity of current in the armature windings. The ends of each armature coil are connected to commutator bars located 180 degrees apart. As the armature spins, brushes supply current to opposing segments of the commutator and, therefore, to opposing armature coils.

$\boldsymbol{\tau} = \boldsymbol{\mu} \times \boldsymbol{B}$: basis for electric DC motor

Commutator: Split ring that changes the current direction to keep non-zero torque on coil.





((**Note**)) **DC motor** (H.E. White, 1940)

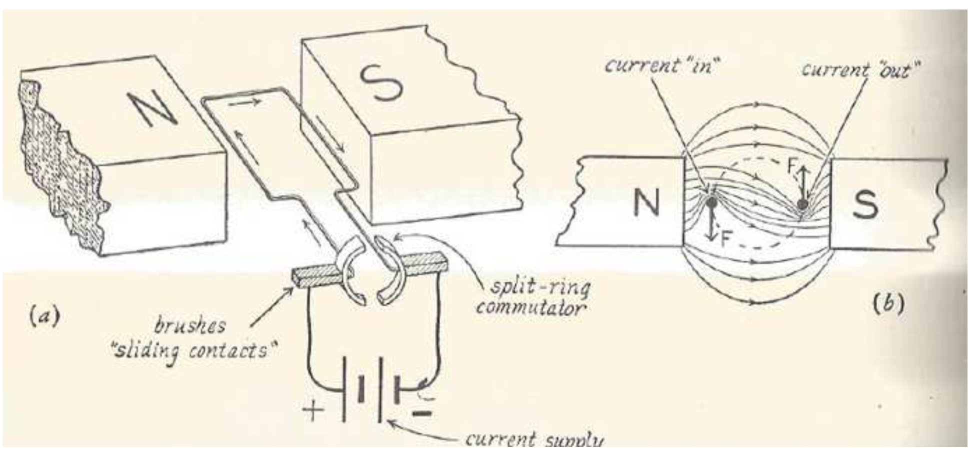


Fig. Illustration showing the principle of the DC electric motor (H.E. White, Classical and Modern Physics; A Descriptive Introduction (D. van Nostrand, 1940).

APPENDIX-D Magnetic quantum flux

https://en.wikipedia.org/wiki/Magnetic_flux_quantum

The magnetic flux, represented by the symbol Φ, threading some contour or loop is defined as the magnetic field $\boldsymbol{B}$ multiplied by the loop area $\boldsymbol{a}$, i.e.

$$\Phi = \boldsymbol{B} \cdot \boldsymbol{a}$$

Both $\boldsymbol{B}$ and $\boldsymbol{a}$ can be arbitrary and so is Φ. However, if one deals with the superconducting loop or a hole in a bulk superconductor, it turns out that the magnetic flux threading such a hole/loop is quantized. The **magnetic flux quantum**

$$\Phi_0 = \frac{h}{2e} = 2.06783348 \times 10^{-15} \text{ Wb.}$$

is a combination of fundamental physical constants: the Planck constant h and the electron charge e. Its value is, therefore, the same for any superconductor. The phenomenon of flux quantization was discovered experimentally by B. S. Deaver and W. M. Fairbank and, independently, by R. Doll and M. Näbauer, in 1961. The quantization of magnetic flux is closely related to the Little–Parks effect, but was predicted earlier by Fritz London in 1948 using a phenomenological model.

The inverse of the flux quantum, $1/\Phi_0$, is called the **Josephson constant**, and is denoted K_J. It is the constant of proportionality of the Josephson effect, relating the potential difference across a Josephson junction to the frequency of the irradiation. The Josephson effect is very widely used to provide a standard for high-precision measurements of potential difference, which (since 1990) have been related to a fixed, conventional value of the Josephson constant, denoted $K_{J\text{-}90}$. With the 2019 redefinition of SI base units, the Josephson constant will have an exact value of K_J = 483597.84841698...GHz·V^{-1}, which will replace the conventional value $K_{J\text{-}90}$.