Chapter 31
Electromagnetic oscillations and AC circuit
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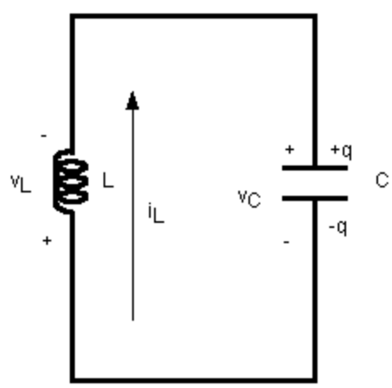
The phasor diagram is very useful in discussing the AC circuit formed of series connection of *R*, *L*, and *C*. The detail of this concept will be discussed below. We note that this method is not appropriate for the AC circuits which are formed of various kind of combinations with connections of *R*, *L*, and *C* elements. In this case we need to use the concept of impedance such as *R*, $i\omega L$, and $1/(i\omega C)$ in the frequency domain. We also use the transformations [$i(t) = \mathrm{Re}(Ie^{i\omega t})$ for the current and $v(t) = \mathrm{Re}(Ve^{i\omega t})$ for the voltage] between the frequency domain and time domain, The quantities I and V are complex numbers. In the frequency domain, the AC circuit can be simply solved by using the KCL and KVL laws. In general physics, in spite of convenience, this method is not adopted.

1. *LC* oscillations



In a resistanceless *LC* circuit, the total energy *U* is given by

$$U = \frac{1}{2}Li_L^{\ 2} + \frac{1}{2}Cv_C^{\ 2},$$

where $q = Cv_C$.

From the energy conservation, *U* remains constant with time.

$$\frac{dU}{dt} = Li_L \frac{d}{dt}i_L + Cv_C \frac{dv_C}{dt} = 0.$$

Since $i_L = \dfrac{dq}{dt}$ and $v_C = \dfrac{q}{C}$, we have

$$\frac{d^2q}{dt^2} + \omega^2 q = 0\,, \qquad\qquad (LC \text{ oscillation, simple harmonics}).$$

with

$$\omega = \frac{1}{\sqrt{LC}}\,.$$

The total energy U is rewritten as

$$U = \frac{1}{2}L\left(\frac{dq}{dt}\right)^2 + \frac{1}{2C}q^2\,.$$

We assume the initial condition as

At $t = 0$,

$q = Q$ and $dq/dt = 0$.

The solution of the second order differential equation is given by

$$q = Q\cos(\omega t)\,,$$

$$i_L = \frac{dq}{dt} = -Q\omega\sin(\omega t)\,.$$

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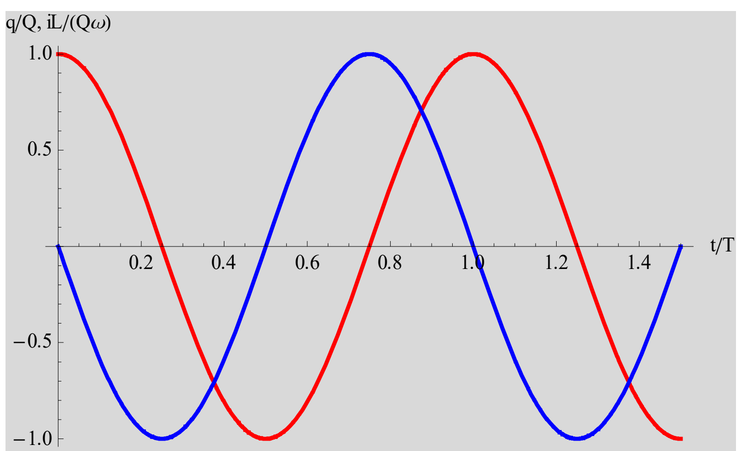


Fig. q/Q (red) and $i_L/(Q\omega)$ (blue) as a function of t/T.

Energy of U_E and U_B

$$U_E = \frac{q^2}{2C} = \frac{Q^2}{2C}\cos^2(\omega t)$$

$$U_B = \frac{1}{2}LQ^2\omega^2\sin^2(\omega t) = \frac{Q^2}{2C}\sin^2(\omega t)$$

$$U = U_E + U_B = \frac{Q^2}{2C}$$

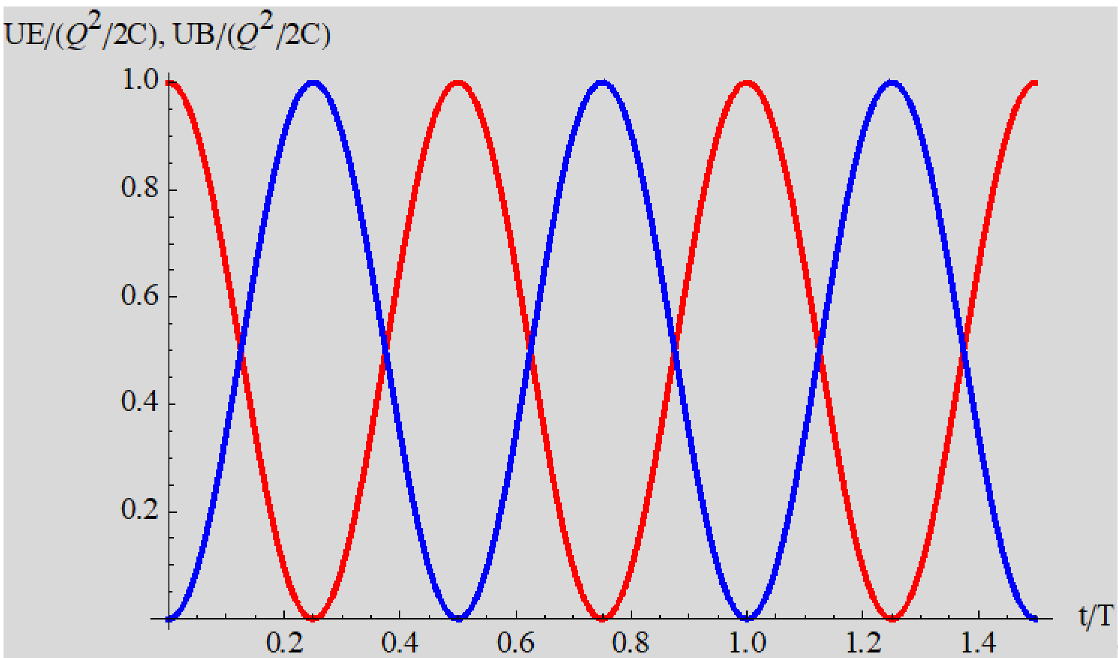
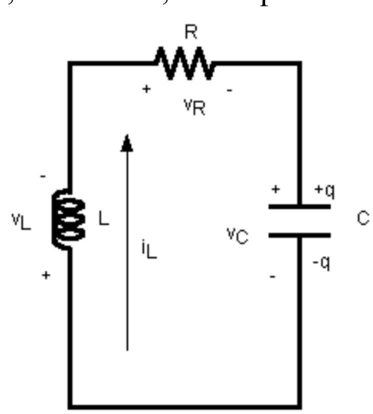


Fig. $U_E/(Q^2/2C)$ (red) and $U_B/(Q^2/2C)$ (blue) as a function of *t*/*T*.

2. Damped oscillations in a RLC circuit

A circuit containing resistance, inductance, and capacitance is called an *RLC* circuit.



$$v_L = L\frac{di_L}{dt}$$

$$v_R = Ri_L$$

$$q = Cv_C$$

$$i_L = \frac{dq}{dt}$$

From the KVL, we have

$$v_L + v_R + v_C = 0,$$

or

the second order differential equation

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0.$$

We assume that

$$q = x, \quad 2\beta = R/L, \text{ and } \omega_0^2 = 1/(LC)$$

Then we have

$$x''(t) + 2\beta x'(t) + \omega_0{}^2 x(t) = 0,$$

with the initial conditions

$$x'(0) = v_0 \text{ and } x(0) = x_0$$

The solution of this differential equation depends is classed into three types,

(1)	underdamping:	$\beta^2 - \omega_0^2 < 0$
(2)	critical damping	$\beta^2 - \omega_0^2 = 0$
(3)	overdamping	$\beta^2 - \omega_0^2 > 0$

((**Note**))
We assume that $x(t)$ is expressed by a solution given by $\exp(pt)$. Then we have

$$x''(t) + 2\beta x'(t) + \omega_0{}^2 x(t) = (p^2 + 2\beta p + \omega_0{}^2)e^{pt} = 0$$

The solution of the quadratic equation $p^2 + 2\beta p + \omega_0{}^2 = 0$ is

$$p^2 + 2\beta p + \beta^2 = -(\omega_0{}^2 - \beta^2)$$
$$(p+\beta) = \pm i\sqrt{\omega_0{}^2 - \beta^2}$$
$$p = -\beta \pm i\sqrt{\omega_0{}^2 - \beta^2} = -\beta \pm i\omega_1$$

with

$$\omega_1 = \sqrt{\omega_0{}^2 - \beta^2}\ .$$

The solution is given by

$$q(t) = x(t) = e^{-\beta t}[C_1\cos(\omega_1 t) + C_2\sin(\omega_1 t)]\ .$$

From the initial condition, we have

$$C_1 = x_0\ ,$$

$$C_2 = \frac{v_0 + \beta x_0}{\omega_1}\ .$$

The final form is given by

$$q(t) = e^{-\beta t}[x_0\cos(\omega_1 t) + \left(\frac{v_0 + \beta x_0}{\omega_1}\right)\sin(\omega_1 t)]\ .$$

3. AC circuit theory

3.1 phasor diagram

(a)

$$i = I_{\max}\sin(\omega t)$$

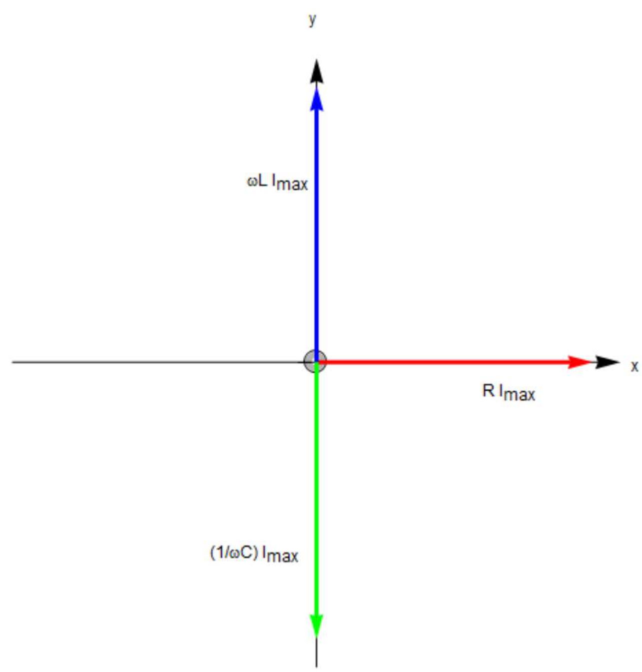
$$v_R = RI_{\max}\sin(\omega t)$$
$$v_C = \frac{1}{C}\int idt = -\frac{I_{\max}}{\omega C}\cos(\omega t) = \frac{I_{\max}}{\omega C}\sin(\omega t - \frac{\pi}{2})$$
$$v_L = L\frac{di}{dt} = \omega LI_{\max}\cos(\omega t) = \omega LI_{\max}\sin(\omega t + \frac{\pi}{2})$$

(b)

$$i = I_{\max}\cos(\omega t)$$

$$v_R = RI_{\max}\cos(\omega t)$$
$$v_C = \frac{1}{C}\int i dt = \frac{I_{\max}}{\omega C}\sin(\omega t) = \frac{I_{\max}}{\omega C}\cos(\omega t - \frac{\pi}{2})$$
$$v_L = L\frac{di}{dt} = -\omega L I_{\max}\sin(\omega t) = \omega L I_{\max}\cos(\omega t + \frac{\pi}{2})$$

The phasor diagram is given by the following figure.



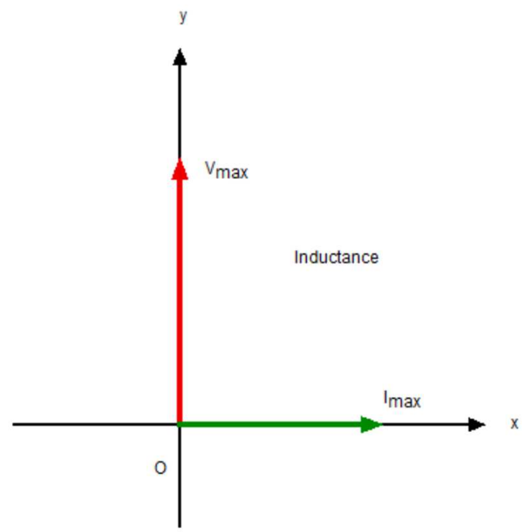
For more general case (more complicated AC circuits), we need to use the method of complex number. This will be discussed in the **Appendix**.

3.2. Impedance of single elements in AC circuit

3.2.1 Inductance *L*

$$V_{\max} = \omega L I_{\max}$$

The relation between $V_{\max}$ and $I_{\max}$ for the inductance is described in the following phasor diagram.

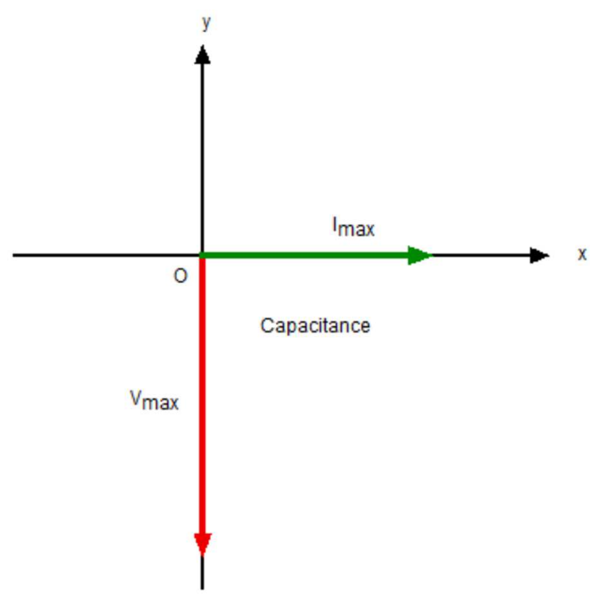


We say that V leads to I by 90° (or I lags V by 90°).

3.2.2 Capacitance *C*

$$V_{\max} = \frac{I_{\max}}{\omega C}$$

The relation between $V_{\max}$ and $I_{\max}$ for the capacitance is described in the following phasor diagram.



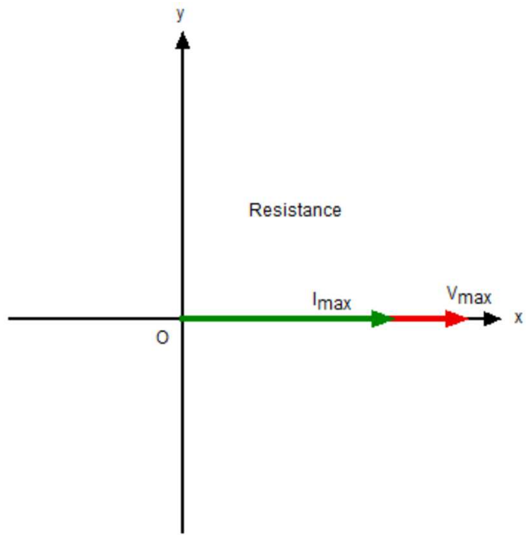
We say that V lags I by 90° (or I leads V by 90°).

3.2.3 Resistance

For resistance, one can write down a relation

$$V_{\max} = RI_{\max}$$

The relation between $V_{\max}$ and $I_{\max}$ for the resistance is described in the following phasor diagram.



3.3. Power dissipated in single elements

3.3.1 Definition

The time average of the power $P(t)$ over a period time T ($= 2\pi/\omega$) is given by

$$\begin{aligned}
P_{avg} &= \frac{1}{T}\int_0^T dt I_{\max}\sin(\omega t)V_{\max}\sin(\omega t+\phi) \\
&= \frac{1}{2T}I_{\max}V_{\max}\int_0^T dt[\cos\phi - \cos(2\omega t+\phi)] \\
&= \frac{1}{2}I_{\max}V_{\max}\cos\phi \\
&= I_{rms}V_{rms}\cos\phi
\end{aligned}$$

where $\cos\phi$ is the power factor, $I_{\max}$ and $V_{\max}$ are the amplitudes of current and voltage. Note that the definition of the root-mean squares $I_{\rm rms}$ and $V_{\rm rms}$ will be given later (Section 3.4), where

$$I_{rms} = \frac{1}{\sqrt{2}}I_{\max}, \qquad V_{rms} = \frac{1}{\sqrt{2}}V_{\max}$$

3.3.2 inductance

$$P_{avg} = 0$$

since $\phi = \pi/2$. No power is dissipated in an inductor.

3.3.3 Capacitance

$$P_{avg} = 0$$

since $\phi = -\pi/2$. No power is dissipated in a capacitor.

3.3.4 Resistance

$$P_{avg} = \frac{1}{2} R I_{\max}{}^2$$

Power is dissipated only in a resistor.

3.4. Root-mean square value of current and voltage

The root-mean square value of the current is defined as

$$\begin{aligned} i_{rms} &= \sqrt{\frac{1}{T}\int_0^T I_{\max}{}^2 \sin^2(\omega t) dt} \\ &= \sqrt{\frac{1}{2T}\int_0^T I_{\max}{}^2 [1-\cos(2\omega t)] dt} \\ &= \frac{I_{\max}}{\sqrt{2}} \end{aligned}$$

The root-mean square value of the voltage is defined as

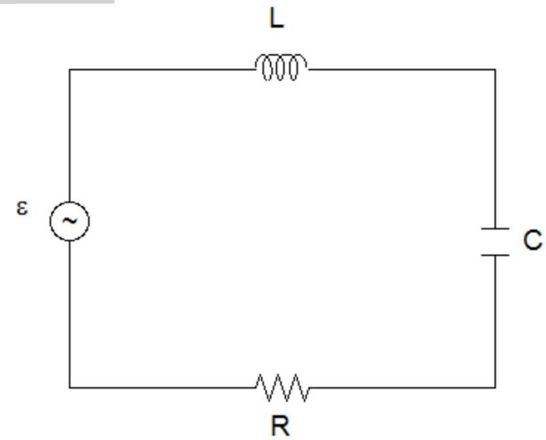
$$v_{rms} = \frac{V_{\max}}{\sqrt{2}}.$$

((**Note**))
Domestic electricity is provided at a frequency of 60 Hz in the United States, and a residential outlet provides 120 Vac. This means the rms voltage is $V_{\max} = 120\sqrt{2} = 170$ V.

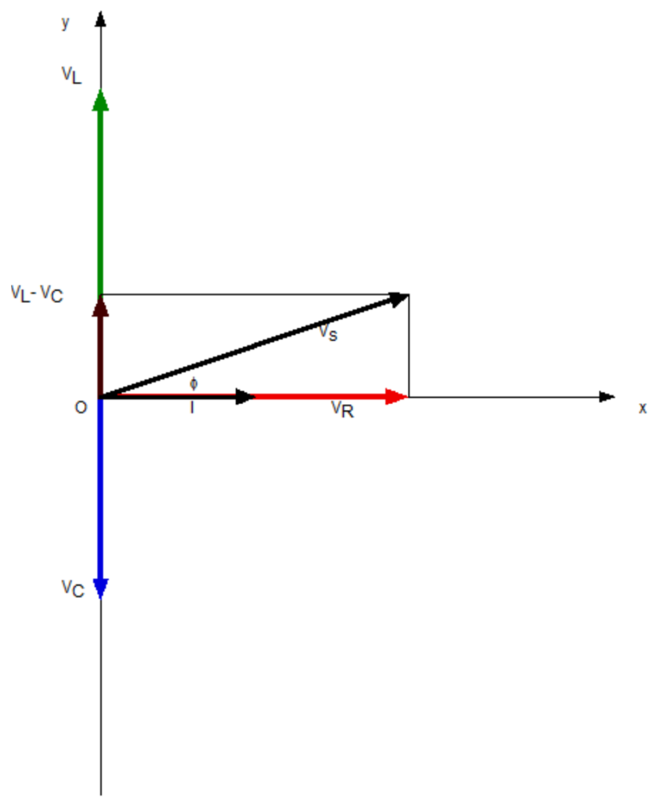
The power dissipated in the resistor is rewritten as

$$\boxed{P_{avg} = R(I_{rms})^2 = \frac{(V_{rms})^2}{R}}$$

3.5. The series RLC circuit



Phasor diagram of the RLC circuits is given in the following way, where we use I and V instead of $I_{\max}$ and $V_{\max}$ (or I_{rms} and V_{rms})



where

$$V_s = Z_s I$$
$$V_R = RI$$
$$V_L = \omega L I$$
$$V_C = \frac{1}{\omega C} I$$

where I is the amplitude of the current, V_s is the amplitude of the voltage source, Z_s is the total impedance defined by

$$Z_s = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2} \ ,$$

and the angle between V_s and I is defined as

$$\tan\theta = \frac{1}{R}(\omega L - \frac{1}{\omega C}) .$$

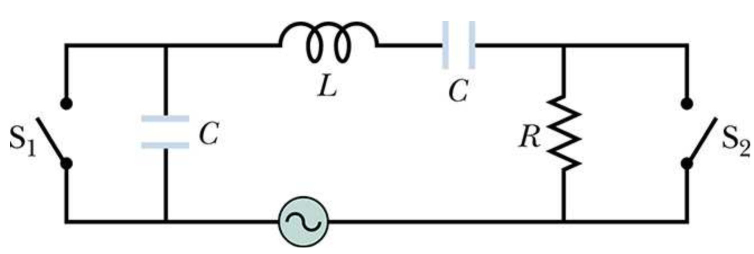
The average power dissipated in the system is

$$P = \frac{1}{2} V_s I \cos\phi = (V_s)_{rms} I_{rms} \cos\phi = R I_{rms}{}^2 .$$

3.6. Example (RLC circuit)

Problem 31-46 (SP-31)
31-46

Figure shows a driven *RLC* circuit that contains two identical capacitors and two switches. The emf amplitude is set at 12.0 V, and the driving frequency is set at 60.0 Hz. With both switches open, the current leads the emf by 30.9°. With switches S1 closed and switch S2 still open, the emf leads the current by 15.0°. With both switches closed, the current amplitude is 447 mA. What are (a) R, (b) C, and (c) L?

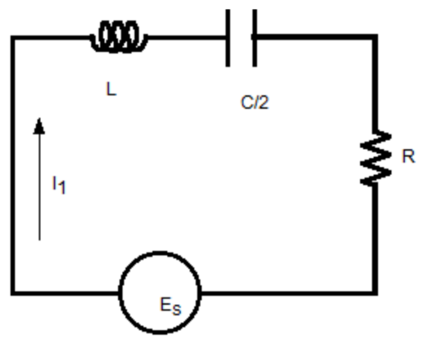


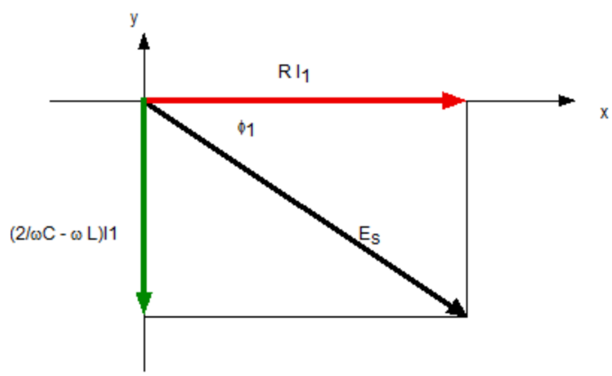
((**Solution**))

$E_0 = 12$ V

$f = 60$ Hz
$\omega = 2\pi f$

(1) S_1 and S_2 open





$$E_s = \sqrt{R^2 + (\frac{2}{\omega C} - \omega L)^2} I_1$$

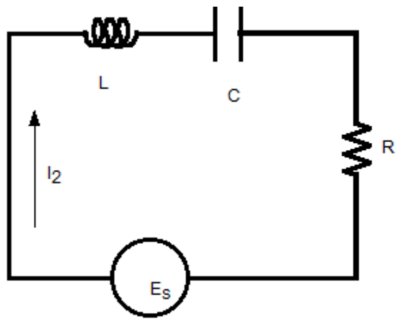
$$\tan\phi_1 = \frac{(\frac{2}{\omega C} - \omega L)}{R}$$

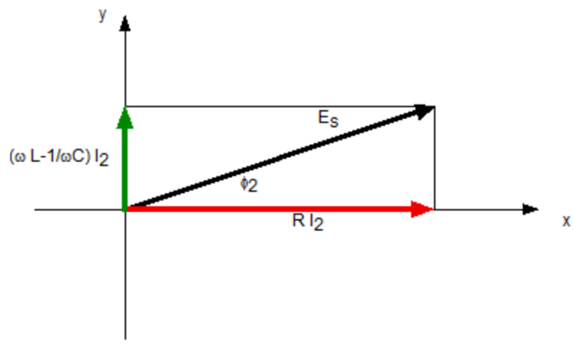
E_s lags I_1 by ϕ_1. The average power dissipated in the system is

$$P = \frac{1}{2} E_s I_1 \cos\phi_1 = \left(E_s\right)_{rms} (I_1)_{rms} \cos\phi_1 = R(I_1)_{rms}{}^2$$

where $E_s \cos\phi_1 = RI_1$

(2) S_1 closed and S_2 open





$$E_s = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}\, I_2$$

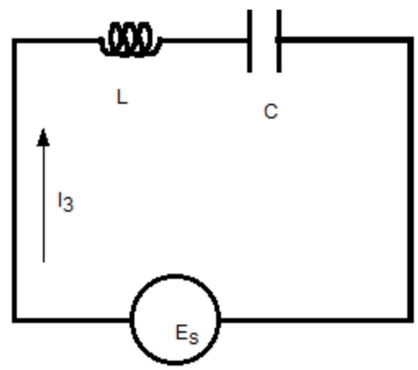
$$\tan\phi_2 = \frac{(\omega L - \frac{1}{\omega C})}{R}$$

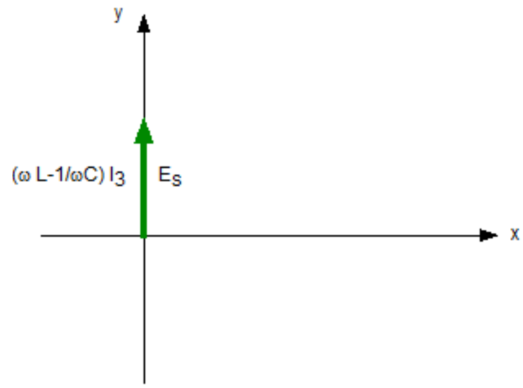
E_s leads I_2 by ϕ_2. The average power dissipated in the system is

$$P = \frac{1}{2} E_s I_2 \cos\phi_2 = \left(E_s\right)_{rms} (I_2)_{rms} \cos\phi_2 = R(I_2)_{rms}{}^2$$

where $E_s \cos\phi_2 = RI_2$

(3) S_1 closed and S_2 closed



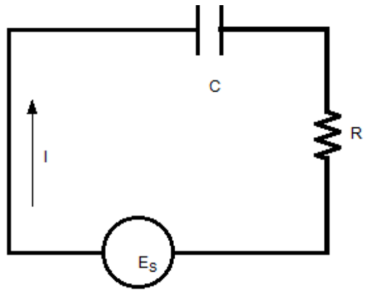


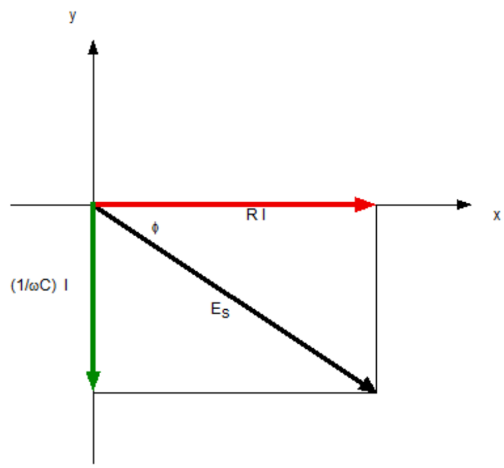
$$I_3 = \frac{E_s}{\left(\omega L - \frac{1}{\omega C}\right)}$$

3.7. Example (*RC* circuit)

Problem 31-44 (HW-31)) Hint

An alternating emf source with a variable frequency f_d is connected in series with a 50.0 Ω resistor and 20.0 μF capacitor. The emf amplitude is 12.0 V. (a) Draw a phasor diagram for phasor V_R (the potential across the resistor) and phasor V_C (the potential across the capacitor). (b) At what driving frequency f_d do the two phasors have the same length? At that driving frequency, what are (c) the phase angle in degrees, (d) the angular speed at which the phasors rotate, and (e) the current amplitude?





$$E_s = \sqrt{R^2 + (\frac{1}{\omega C})^2}\, I$$

$$\tan\phi = \frac{(1/\omega C)}{R}$$

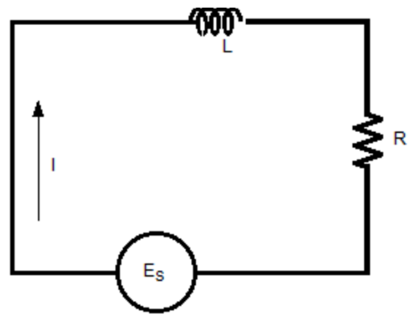
E_s lags I by the angle ϕ. The average power dissipated in the system is

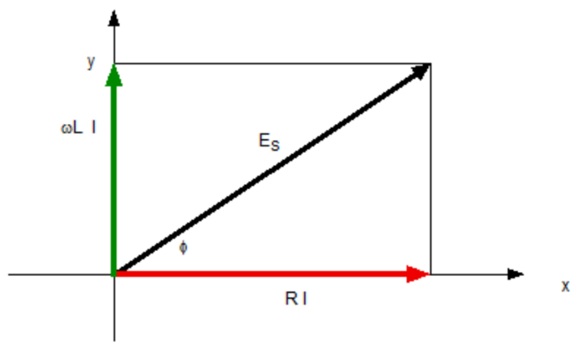
$$P = \frac{1}{2} E_s I \cos\phi = \left(E_s\right)_{rms} I_{rms} \cos\phi = R I_{rms}{}^2$$

where

$$E_s \cos\phi = RI$$

3.8. Example (*RL* circuit)





$$E_s = \sqrt{R^2 + (\omega L)^2}\, I$$

$$\tan\phi = \frac{\omega L}{R}$$

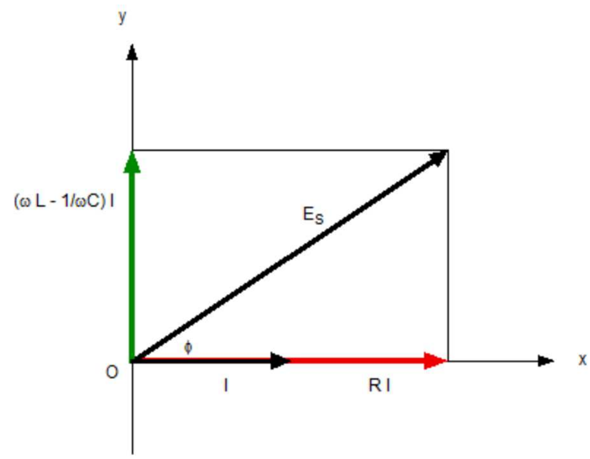
E_s leads I by the angle ϕ. The average power dissipated in the system is

$$P = \frac{1}{2} E_s I \cos\phi = (E_s)_{rms} I_{rms} \cos\phi = R I_{rms}{}^2$$

where

$$E_s \cos\phi = RI$$

3.9 Resonance of the *RLC* circuit



$$E_s = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}\, I$$

or

$$I = \frac{E_s}{Z_s} = \frac{E_s}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$$

where

$$Z_s = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2} \qquad \text{(impedance)}$$

The amplitude of the current has a maximum when

$$\omega = \frac{1}{\sqrt{LC}}$$

We define the quality factor of the circuit by

$$Q = \frac{L\omega_0}{R}$$

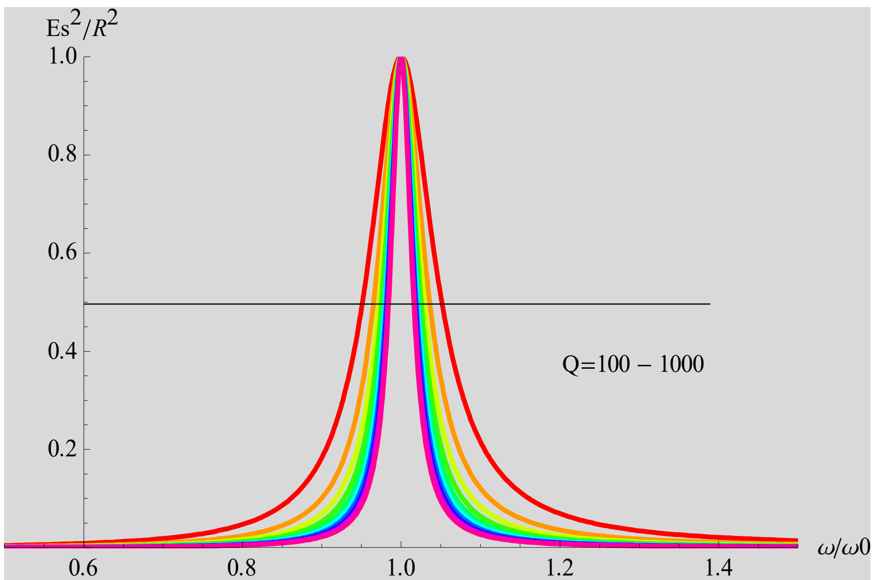
Then the amplitude of the current can be rewritten as

$$I = \frac{E_s}{R}\frac{1}{\sqrt{1 + Q^2(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega})^2}}$$

We now consider the frequency dependence of I^2, instead of I.

$$I^2 = \left(\frac{E_s}{R}\right)^2 \frac{1}{1+Q^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2}$$

We note that this is a scaling function of ω/ω_0. In the figures we show the plot of $\frac{I^2}{(Es^2/R^2)}$ vs $\frac{\omega}{\omega_0}$, where the quality factor Q is changes as a parameter.



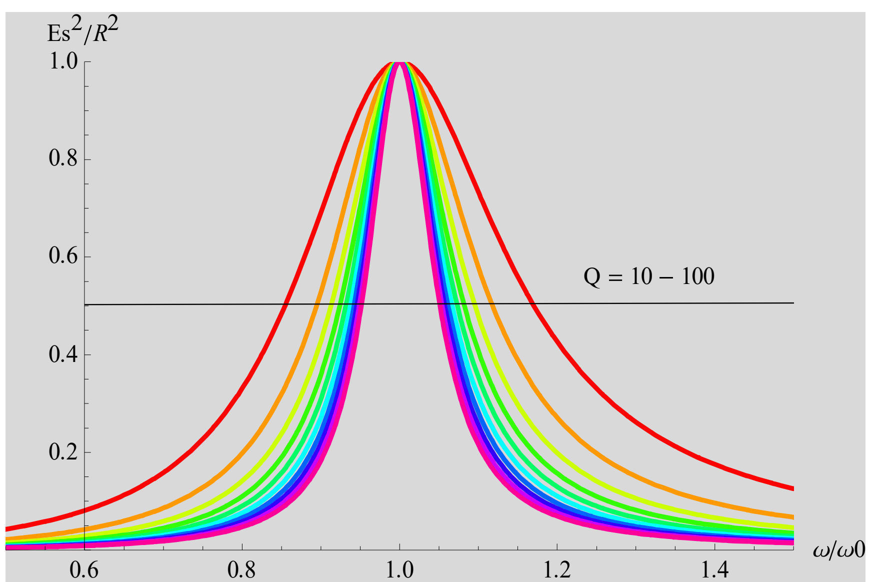


Fig. $\dfrac{I^2}{(Es^2/R^2)}$ vs $\dfrac{\omega}{\omega_0}$, where the quality factor Q is changes as a parameter.

Suppose that $\dfrac{\omega}{\omega_0} = 1+\delta$ (δ<<1).

$$\frac{\omega_0}{\omega} = (1+\delta)^{-1} \approx 1-\delta$$

Then we have

$$I^2 = \left(\frac{E_s}{R}\right)^2 \frac{1}{1+Q^2[1+\delta-(1-\delta)]^2}$$
$$= \left(\frac{E_s}{R}\right)^2 \frac{1}{1+4Q^2\delta^2} = \frac{I_{\max}{}^2}{1+4Q^2\delta^2}$$

When

$$2Q\delta = 1, \qquad \text{or} \qquad \delta = \frac{1}{2Q}$$

I^2 is equal to $\dfrac{I_{\max}{}^2}{2}$. Then the width is estimated as

$$2\Delta\omega = \omega_0(1+\delta) - \omega_0(1-\delta) = 2\omega_0\delta$$

or

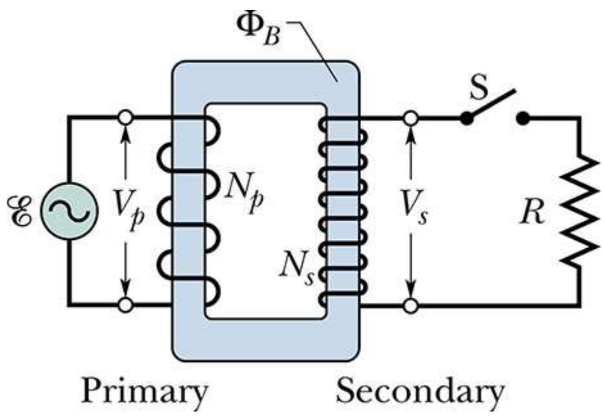
$$\frac{2\Delta\omega}{\omega_0} = 2\delta = \frac{1}{Q}$$

$$\boxed{Q = \frac{\omega_0}{2\Delta\omega}} \qquad \text{(definition of the quality factor)}$$

or

$$\boxed{2\Delta\omega = \frac{\omega_0}{Q}} \qquad \text{(full-width at half-maximum)}$$

4. Transformer



$$\varepsilon = V_p = -N_p \frac{d\Phi_B}{dt}$$

$$V_s = -N_s \frac{d\Phi_B}{dt}$$

Then we have

$$\frac{V_s}{V_p} = \frac{N_s}{N_p} \qquad \text{(transformation of voltage)}$$

If $N_s > N_p$, the transformer is called a step-up transformer. If $N_p > N_s$, the transformer is called a step-down transformer

From the energy conservation, we have

$$I_s V_s = I_p V_p,$$

or

$$\frac{I_s}{I_p} = \frac{V_p}{V_s} = \frac{N_p}{N_s}. \qquad \text{(transformation of current)}$$

The equivalent resistance R_{eq} is the value of the load resistance as seen by the generator.

$$R_{eq} = \frac{V_p}{I_p} = \frac{V_p}{V_s}\frac{I_s}{I_p}\frac{V_s}{I_s} = (\frac{N_p}{N_s})^2 R.$$

5. Typical examples

5.1 Problem 31-17 (SP-31)

In an oscillating LC circuit with C = 64.0 μF, the current is given by i = 1.60 sin(2500 t + 0.680), where t is in seconds, i in amperes, and the phase constant in radians. (a) How soon after $t = 0$ will the current reach its maximum value? What are (b) the inductance L and (c) the total energy?

((**Solution**))

$C = 64.0\ \mu$F
$i = 1.60\sin(2500t + 0.680)$
$I_{max} = 1.60$ A
$\omega = 2500$ rad/s

(a) $\sin(2500t + 0.680) = 1,\qquad 2500t + 0.680 = \dfrac{\pi}{2}$

or

$t = 356.30$ μs.

(b)

$$\omega = \frac{1}{\sqrt{LC}}$$
$$L = \frac{1}{C\omega^2} = 0.0025H$$

(c)

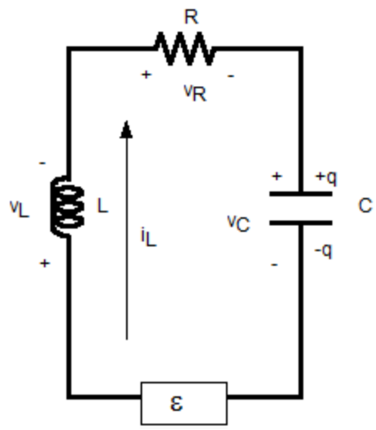
The total energy is conserved.

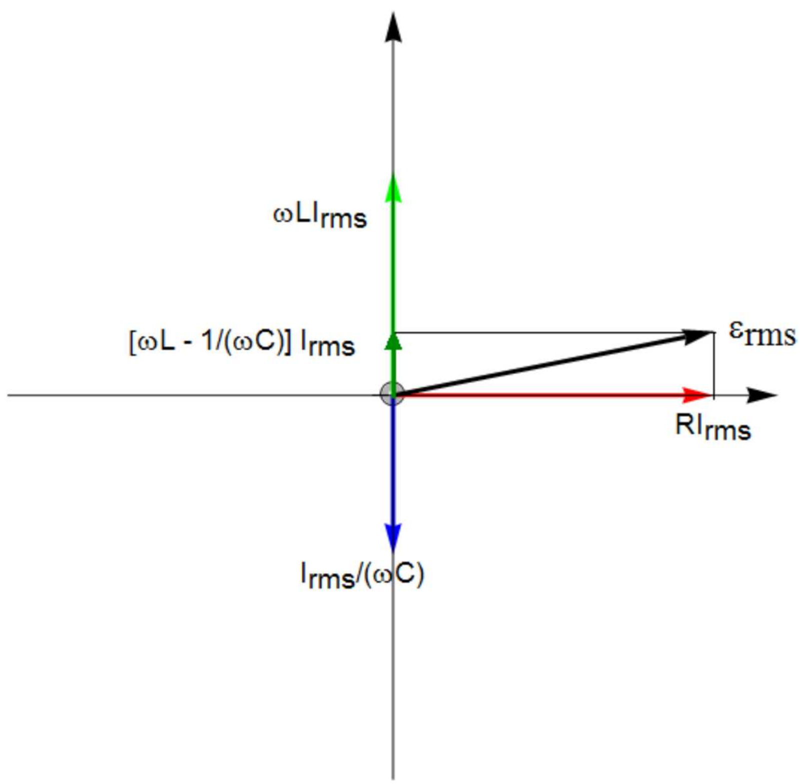
$$\varepsilon = \frac{1}{2}Li^2 + \frac{1}{2}Cv^2 = \frac{1}{2}Li_{\max}{}^2 = \frac{1}{2}Cv_{\max}{}^2 = \frac{1}{2}\times 0.0025\times 1.6^2 = 3.2\times 10^{-3}J$$

5.2 Problem 31-61 (SP-31)

In Fig. $R = 15.0\ \Omega$, $C = 4.70$ μF, and $L = 25.0$ mH. The generator provides an emf with rms voltage 75.0 V and frequency 550 Hz. (a) What is the rms current? What is the rms voltage across (b) R, (c) C, (d) L, (e) C, and L together, and (f) R, C, and L together? At what average rate is energy dissipated by (g) R, (h) C, and (i) L?

((**Solution**))





$R = 15\ \Omega$
$C = 4.70\ \mu F$
$L = 25.0\ mH$
$\varepsilon = 75.0\ V$
$f = 550\ Hz$

Phasor diagram

$$\varepsilon_{rms} = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2} I_{rms}$$

$$(V_L)_{rms} = \omega L I_{rms}$$

$$(V_R)_{rms} = R I_{rms}$$

$$(V_C)_{rms} = \frac{1}{\omega C} I_{rms}$$

$$(V_{LC})_{rms} = (\omega L - \frac{1}{\omega C}) I_{rms}$$

$$\varepsilon_{rms} = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2} I_{rms}$$

$$I_{rms} = \frac{\varepsilon_{rms}}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$$

(a) $I_{\text{rms}} = 2.58576$ A

(b) $(V_R)_{rms} = R I_{rms} = 38.7865$ V

(c) $(V_C)_{rms} = \frac{1}{\omega C} I_{rms} = 159.202$ V

(d) $(V_L)_{rms} = \omega L I_{rms} = 223.394$ V

(e) $(V_{LC})_{rms} = (\omega L - \frac{1}{\omega C}) I_{rms} = 64.192$ V

(f) $\varepsilon_{rms} = 75.0$ V

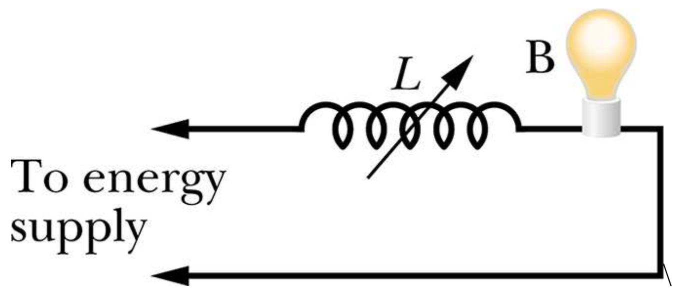
(g) $P_R = R I_{rms}^2 = 100.293$ W

(h) $P_C = 0$

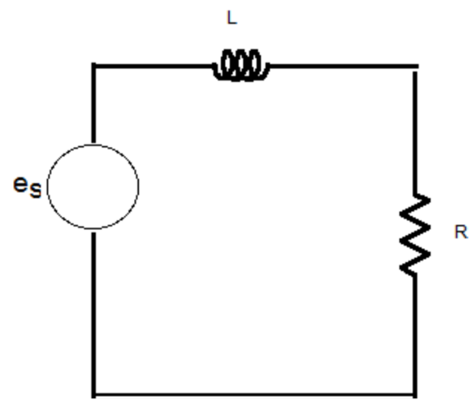
(i) $P_L = 0$

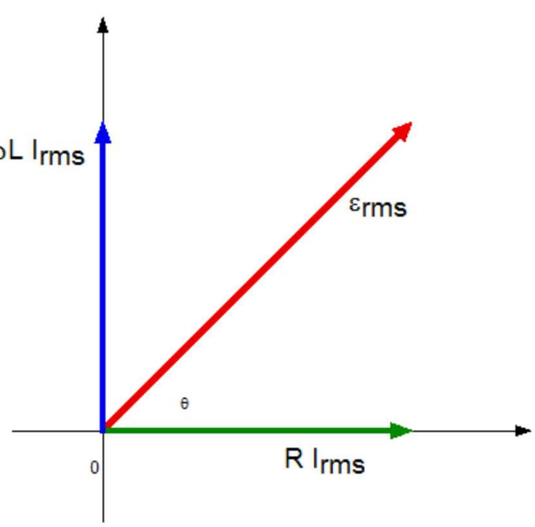
5.3 Problem 31-60 (SP-31)

A typical light dinner used to dim the stage lights in a theater consists of a variable inductor L (whose inductance is adjustable between 0 and $L_{\max}$) connected in series with a lightbulb B as shown in Fig. The electrical supply is 120 V (rms) at 60.0 Hz; the light-bulb is rated at 120 V, 1000 W. (a) What $L_{\max}$ is required if the rate of energy dissipation in the lightbulb is to be varied by a factor of 5 from its upper limit of 1000 W? Assume that the resistance of the lighbulb is indepenent of its temperature. (b) Could one use a variable resistor (adjustable between zero and R_{max}) instead of an inductor? (c) If so, what R_{max} is required? (d) Why isn't this done?



((**Solution**))





$L = 0 – L_{max}$
120 V (rms) at 60 Hz
Light bulb 1000 W at 120 V
$\varepsilon_{rms} = 120$ V
$f = 60$ Hz

$$\varepsilon_{rms} = \sqrt{R^2 + (\omega L)^2} I_{rms}$$

$$I_{rms} = \frac{\varepsilon_{rms}}{\sqrt{R^2 + (\omega L)^2}}$$

$$P_R = R I_{rms}{}^2 = R\left(\frac{\varepsilon_{rms}}{\sqrt{R^2 + (\omega L)^2}}\right)^2 = \frac{R\varepsilon_{rms}{}^2}{R^2 + (\omega L)^2}$$

(a) When $L = 0$,

$$P_R = \frac{\varepsilon_{rms}{}^2}{R} = 1000W = \frac{120^2}{R} \qquad \text{or} \qquad R = 14.4\ \Omega$$

$$P_R = \frac{R\varepsilon_{rms}{}^2}{R^2 + (\omega L)^2} = \frac{\varepsilon_{rms}{}^2}{R}\frac{1}{1 + \frac{(\omega L)^2}{R^2}} = \frac{1000W}{1 + \frac{(\omega L)^2}{R^2}} = \frac{1000W}{5}$$

or

$$1 + \frac{(\omega L)^2}{R^2} = 5$$

or

$$\frac{\omega L}{R} = 2 \qquad L = 2\frac{R}{\omega} = 76.39mH$$

(b) and (c) Yes.

The inductance is replaced by a variable resistance R_0.

$$P_R = \frac{R\varepsilon_{rms}{}^2}{(R + R_0)^2} = \frac{\varepsilon_{rms}{}^2}{R}\frac{1}{(1 + \frac{R_0}{R})^2}$$

$$(1 + \frac{R_0}{R}) = \sqrt{5}$$

$$\frac{R_0}{R} = \sqrt{5} - 1 = 1.2361$$

or

$$R_0 = 1.2361 \times 14.4\Omega = 17.799\Omega$$

(d) This is not done because we lose the energy in the variable resistance.

5.4 Phasor diagram of RLC circuit
G. Gladding, M. Selen, and T. Steltzer, SmartPhysics; Electricity and Magnetism (II) (W.H. Freeman, 2011)

Consider the driven LCR circuit. The source voltage has a maximum voltage of 10 V and oscillates at a frequency of 60 Hz, which converts to an angular frequency of 377 rad/s. The values for the circuit components are 20 Ω, 20 mH, and 150 μF, respectively.

Our goal is to determine $I_{\max}$, the maximum current, and ϕ, the phase angle between the current and the source voltage. Once we know these two quantities at any time. We can determine both of these quantities by simply constructing the phasor diagram.

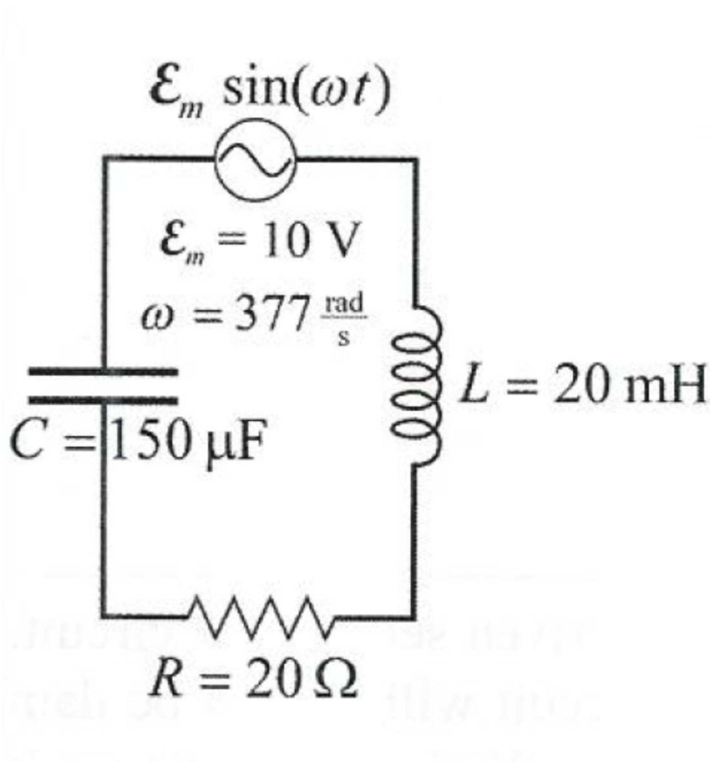


Fig. RLC circuit. $f = 60$ Hz. $L = 20$ mH. $C = 150$ μF. $\omega = 2\pi f = 376.991$ rad/s. $E_m = 10$ V.

((**Solution**))

Suppose that $I = I_{\max}\sin(\omega t)$. The voltages across the capacitance, inductance, and resistance are drawn in the x-y plane, as well as the current.

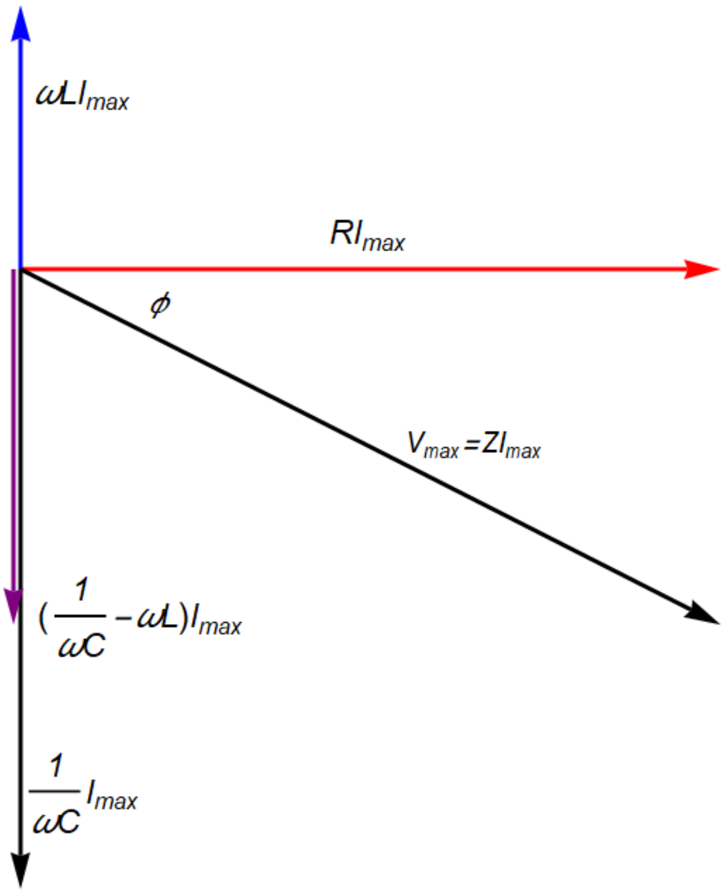


Fig. Phasor diagram for voltages of the circuit. We assume that $RI_{\max}$ is directed along the x axis. The voltage source is actually expressed by $\varepsilon_{\mathrm{m}} \sin(\omega t)$. So it is necessary to rotate the phasor diagram by appropriate angle at the last stage.

Note that

$$V_R = RI_{\max} = 20\ I_{\max}$$

$$V_C = (\frac{1}{\omega C})I_{\max} = 17.6839\ I_{\max}$$

$$V_L = (\omega L)I_{\max} = 7.53982\ I_{\max}$$

The source voltage;

$$V_Z = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2} I_{\max} = 22.4255\ I_{\max}$$

Since $V_Z = E_0 = 10.0\text{ V} = 22.4255\ I_{\max}$, we have the maximum current as

$$I_{\max} = \frac{E_0}{V_z} = 0.445921\ [\text{A}]$$

$$RI_{\max} = 8.91843\ [\text{V}]$$

$$\omega L I_{\max} = 3.36217\ [\text{V}]$$

$$\frac{1}{\omega C} I_{\max} = 7.88562\ \text{V}$$

The phase factor:

$$\phi = \arctan[\frac{V_C - V_L}{V_R}] = 26.89° .$$

Since $V_s = E_0 \sin(\omega t)$, we need to rotate the phasor diagram (in the *x*-*y* plane) around the origin by the angle ϕ in counter clockwise. So we get the final phasor diagram
The average power dissipated in the circuit;

$$\begin{aligned} P_{av} &= \frac{1}{2} R I_{\max}{}^2 \\ &= \frac{1}{2} Z \cos\phi I_{\max}{}^2 \\ &= \frac{1}{2} Z I_{\max} \cos\phi I_{\max} \\ &= \frac{1}{2} I_{\max} V_{\max} \cos\phi \end{aligned}$$

where $\cos\phi$ is the power factor.

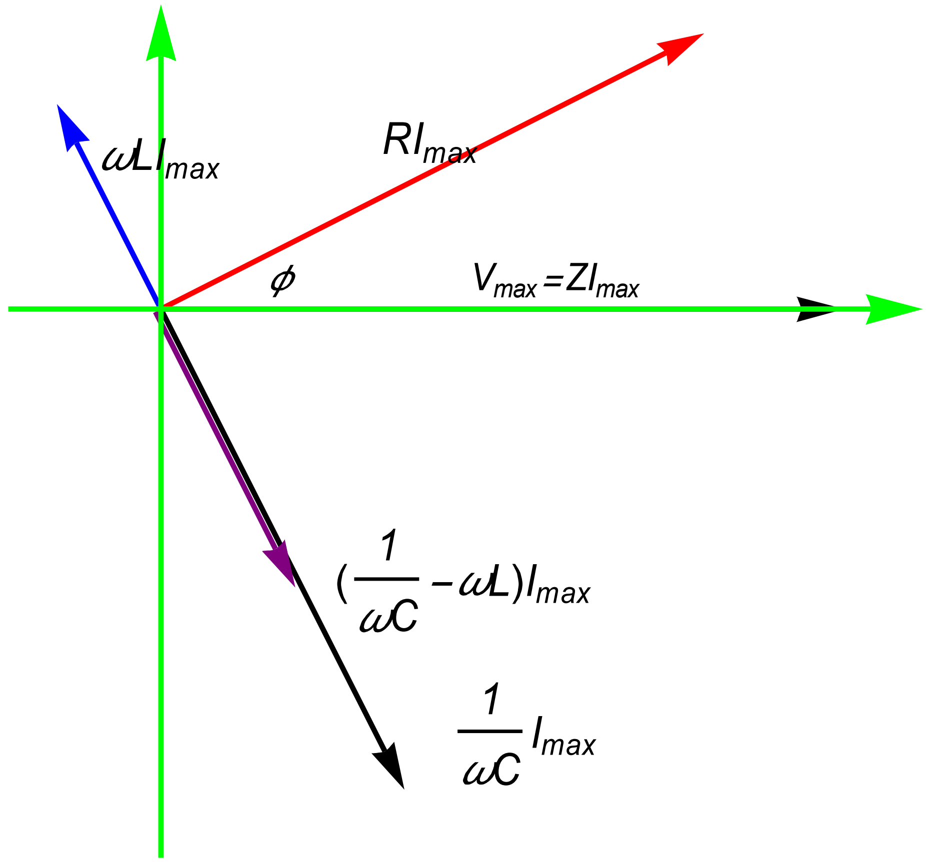


Fig. Phasor diagram of the RLC circuit. $\phi = 26.89°$.

For the voltage source	$10\ \sin(\omega t)$	[V]
For the voltage across the resistance:	$8.91834\ \sin(\omega t + \phi)$	[V]
For the voltage across the inductance	$3.36217\ \sin(\omega t + \phi + \frac{\pi}{2})$	[V]
For the voltage across the capacitance	$7.88562\ \sin(\omega t + \phi - \frac{\pi}{2})$	[V]
For the current	$0.445921\ \sin(\omega t + \phi)$	[A]

Appendix A AC circuit theory based on the complex plane

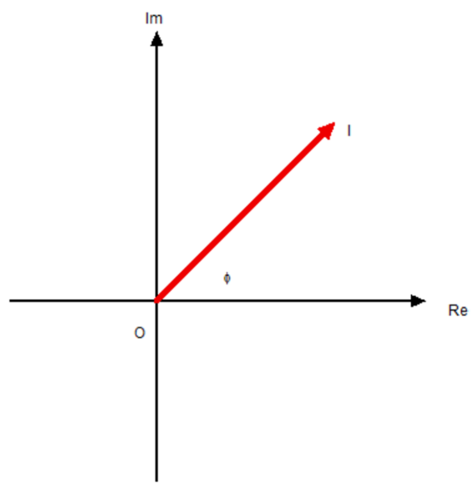
Here we discuss the AC circuit theory based on the complex number. This theory is mathematically correct.

A1. Impedance of single elements in AC circuit

We represent all voltages and current (sinusoidally change with time) by complex numbers, using the exponential notation. The time-varying current is written by

$$i(t) = \mathrm{Re}[Ie^{i\omega t}] = \frac{Ie^{i\omega t} + I^{*}e^{i-\omega t}}{2},$$

where ω (= $2\pi f$) is the angular frequency, I represents a complex number that is independent of t, and I^{*} is the complex conjugate. The complex number I is described by $= I_0 e^{i\phi}$ where I_0 is the amplitude and ϕ is the phase. For convenience, we use the phasor diagram in the complex plane, where for I is plotted as



$$i(t) = \mathrm{Re}[Ie^{i\omega t}] = \mathrm{Re}[I_0 e^{i\phi t} e^{i\omega t}] = I_0 \cos(\omega t + \phi).$$

The voltage is also described by

$$v(t) = \mathrm{Re}[Ve^{i\omega t}].$$

((**Note**))
The absolute values $|V|$ and $|I|$ are the amplitudes of the real voltage and current.

(a) Inductance *L*

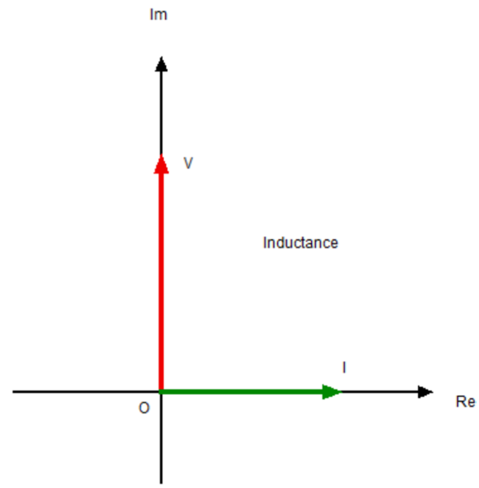
For inductance one can write down the relation

$$v(t) = \mathrm{Re}[Ve^{i\omega t}] = L\frac{di}{dt} = L\frac{d}{dt}\mathrm{Re}[Ie^{i\omega t}] = \mathrm{Re}[i\omega LIe^{i\omega t}]$$

or

$$V = i\omega LI = \omega Le^{i\pi/2}I$$

The relation between V and I for the inductance is described in the following phasor diagram.



We say that V leads to I by 90° (or I lags V by 90°). The impedance of the inductance is given by

$$\boxed{Z_L = \frac{\dot{V}}{\dot{I}} = i\omega L}$$

(b) Capacitance *C*

For capacitance, one can write down a relation

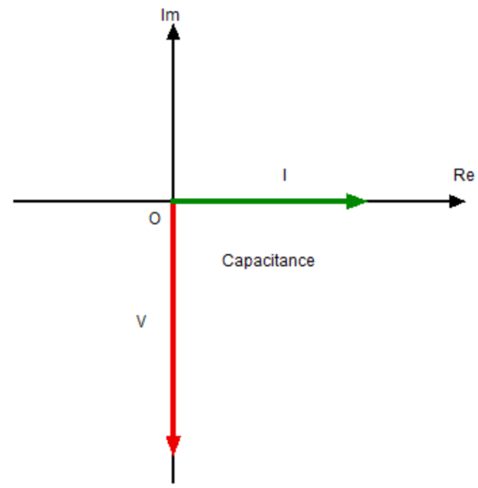
$$i(t) = \mathrm{Re}[Ie^{i\omega t}] = C\frac{dv(t)}{dt} = C\frac{d}{dt}\mathrm{Re}[Ve^{i\omega t}] = \mathrm{Re}[i\omega CVe^{i\omega t}]]$$

$$I = i\omega CV$$

or

$$V = \frac{I}{i\omega C} = \frac{1}{\omega C}e^{-i\pi/2}I$$

The relation between V and I for the capacitance is described in the following phasor diagram.



We say that V lags I by 90° (or I lags V by 90°). The impedance of the capacitance is given by

$$\boxed{Z_C = \frac{V}{I} = \frac{1}{i\omega C}}$$

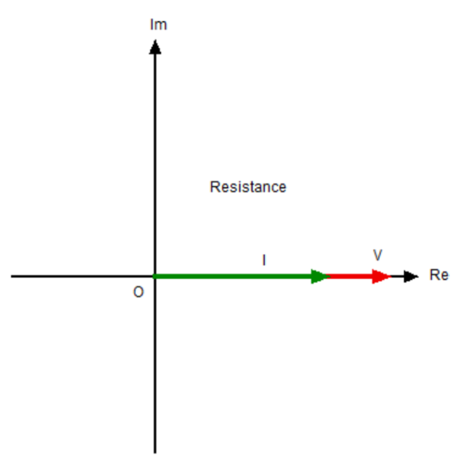
(c) Resistance

For resistance, one can write down a relation

$$v(t) = \mathrm{Re}[Ve^{i\omega t}] = Ri(t) = R\,\mathrm{Re}[Ie^{i\omega t}] = \mathrm{Re}[RIe^{i\omega t}]]$$

$$V = RI$$

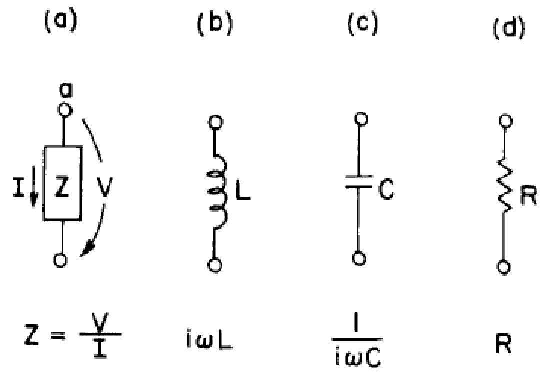
The relation between V and I for the resistance is described in the following phasor diagram.



The impedance of the resistance is given by

$$Z_R = R$$

In summary



((Note))
The analysis of the AC theory without the concept of the complex number will be given in the Appendix B.

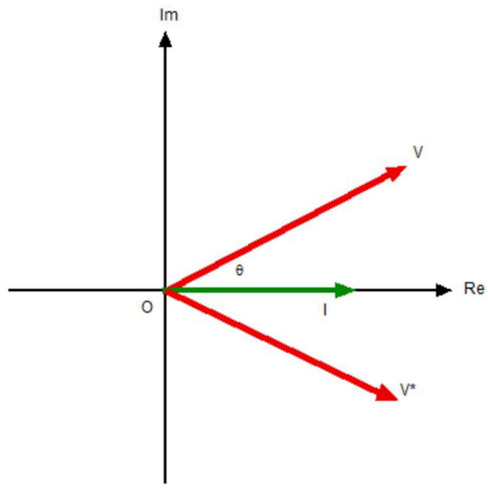
A2. Power dissipated in single elements

The power is given by

$$
\begin{aligned}
P(t) &= v(t)i(t) \\
&= \left(\frac{Ve^{i\omega t} + V^* e^{i-\omega t}}{2}\right)\left(\frac{Ie^{i\omega t} + I^* e^{i-\omega t}}{2}\right) \\
&= \frac{1}{4}(VIe^{2i\omega t} + V^* I^* e^{-2i\omega t} + V^* I + VI^*)
\end{aligned}
$$

The time average of the power $P(t)$ over a period time T (= $2\pi/\omega$) is given by

$$
P_{avg} = \frac{1}{T}\int_0^T P(t)dt = \frac{1}{4}(V^* I + VI^*) = \frac{1}{2}\mathrm{Re}[V^* I] = \frac{1}{2}\mathrm{Re}[I^* V]
$$



When $I = I_0$ and $V = V_0 e^{i\theta}$ (I_0 and V_0 are real numbers), then the power is rewritten as

$$P_{avg} = \frac{1}{2} I_0 V_0 \cos\theta = i_{rms} v_{rms} \cos\theta = R i_{rms}{}^2$$

where $\cos\theta$ is the power factor, and the definition of the root-mean squares $i_{\rm rms}$ and $v_{\rm rms}$ will given later.

(a) inductance

For the capacitance, we have $V^* = -i\omega L I^*$. Then $P_{\rm avg}$ is calculated as

$$P_{avg} = \frac{1}{2}\mathrm{Re}[V^* I] = \frac{1}{2}\mathrm{Re}(-i\omega L I^* I) = \frac{1}{2}\mathrm{Re}(-i\omega L |I|^2) = 0$$

No power is dissipated in an inductor.

(b) Capacitance

For the capacitance, we have $V^* = -\frac{I^*}{i\omega C}$. Then $P_{\rm avg}$ is calculated as

$$P_{avg} = \frac{1}{2}\mathrm{Re}[V^* I] = \frac{1}{2}\mathrm{Re}[(-\frac{I^*}{i\omega C})I] = \frac{1}{2}\mathrm{Re}[i\frac{1}{\omega C} I^* I] = \frac{1}{2}\mathrm{Re}[i\frac{1}{\omega C}|I|^2] = 0$$

No power is dissipated in a capacitor.

(c) Resistance

For the resistance, we have $V^* = RI^*$. Then $P_{\rm avg}$ is calculated as

$$P_{avg} = \frac{1}{2}\mathrm{Re}[V^* I] = \frac{1}{2}\mathrm{Re}[RI^* I] = \frac{1}{2}\mathrm{Re}[R|I|^2] = \frac{1}{2}R|I|^2 = \frac{1}{2}\frac{|V|^2}{R}$$

Power is dissipated in a resistor.

A3. Root-mean square value of current and voltage

What is the root-mean square of the current and voltage?

$$i(t) = \mathrm{Re}[Ie^{i\omega t}] = \mathrm{Re}[I_0 e^{i\phi} e^{i\omega t}] = I_0 \cos(\omega t + \phi)$$

The root-mean square value of the current is defined as

$$i_{rms} = \sqrt{\frac{1}{T}\int_0^T [i(t)]^2 dt} = \sqrt{\frac{I_0^{\ 2}}{T}\int_0^T \cos^2(\omega t + \phi) dt}$$
$$= \sqrt{\frac{I_0^{\ 2}}{2T}\int_0^T [1 - \cos(2\omega t + 2\phi)] dt} = \frac{I_0}{\sqrt{2}}$$

or

$$i_{rms} = \frac{|I|}{\sqrt{2}} = \frac{I_{\max}}{\sqrt{2}}$$

where $I_{\max}$ (=$|I|$ = $|I_0|$) is the maximum of the current.

Similarly, we have the root-mean value of the voltage as

$$v_{rms} = \frac{|V|}{\sqrt{2}} = \frac{V_{\max}}{\sqrt{2}}$$

The power dissipated in the resistor is rewritten as

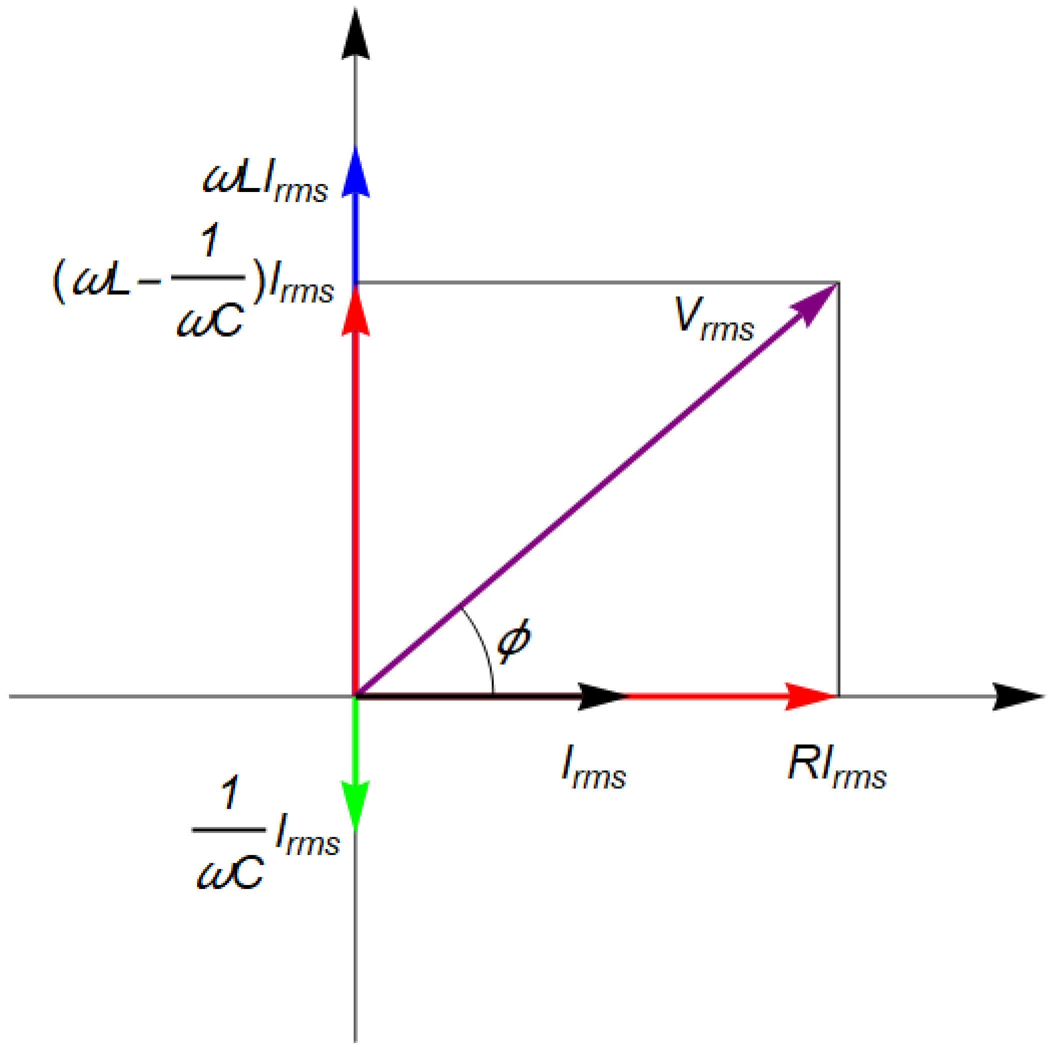
$$P_{avg} = R(i_{rms})^2 = \frac{(v_{rms})^2}{R}$$

((**Note**-1))

The rms value of any quantity that varies as sin(ωt) or cos(ωt) is always $1/\sqrt{2}$ times the maximum value, so

$v_{\mathrm{rms}} = 0.707 v_{\mathrm{max}}$.

((**Note-2**)) **Derivation of time averaged power**

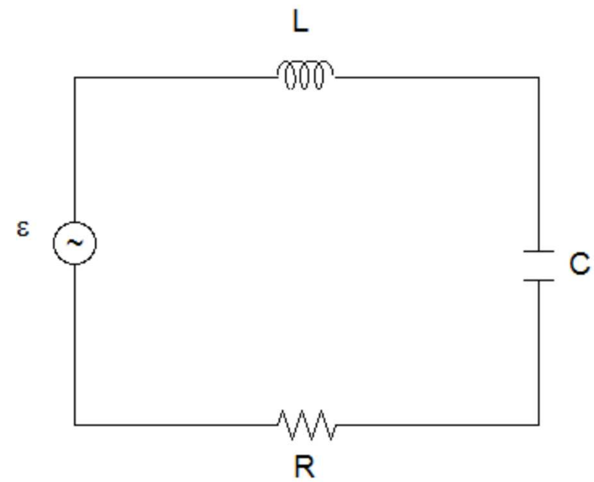


Power dissipated in resistor R

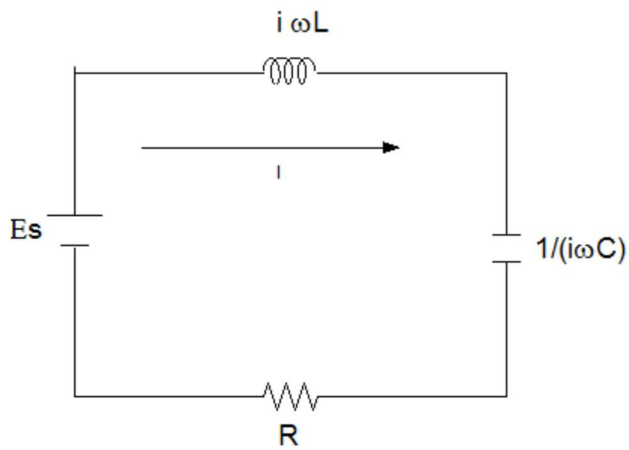
$$P = R[i(t)]^2 = RI_0^{\ 2} \cos^2(\omega t) = \frac{RI_0^{\ 2}}{2}[1 + \cos(2\omega t)]$$

$$P_{av} = \langle P \rangle = \frac{1}{2} RI_0^{\ 2} = RI_{rms}^{\ \ 2} = (RI_{rms}) I_{rms} = V_{rms} I_{rms} \cos\phi$$

A4. The series RLC circuit



Circuit in the time domain



AC circuit in the frequency domain

$$e_s(t) = v_R(t) + v_L(t) + v_C(t)$$

The relation of the expressions in the time domain and the frequency domain is given by

$$e_s(t) = \mathrm{Re}(E_s e^{i\omega t})$$
$$i(t) = \mathrm{Re}(I e^{i\omega t})$$
$$v_L(t) = \mathrm{Re}(V_L e^{i\omega t})$$
$$v_C(t) = \mathrm{Re}(V_C e^{i\omega t})$$
$$v_R(t) = \mathrm{Re}(V_R e^{i\omega t})$$

E_s, I, V_L, V_C, and V_R are complex numbers. The phasor diagram of these will be given later.

$$E_s = V_R + V_L + V_C = Z_s I$$
$$V_R = RI$$
$$V_L = i\omega L I$$
$$V_C = \frac{1}{i\omega C} I$$

or

$$I = \frac{E_s}{Z_s}$$
$$V_R = \frac{E_s R}{Z_s}$$
$$V_L = \frac{E_s i\omega L}{Z_s}$$
$$V_C = \frac{E_s (1/i\omega C)}{Z_s}$$

where Z_s is the total impedance defined by

$$\begin{aligned} Z_s &= R + i\omega L + \frac{1}{i\omega C} = R + i(\omega L - \frac{1}{\omega C}) \\ &= \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}(\cos\theta + i\sin\theta) \\ &= \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2} e^{i\theta} \\ &= |Z_s| e^{i\theta} \end{aligned}$$

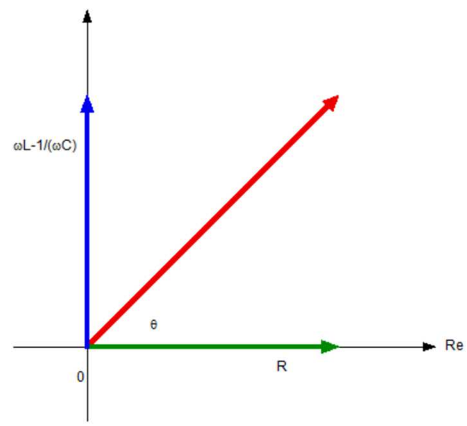
where θ is an angle and $e^{i\theta}$ is the phase factor,

$$\tan\theta = \frac{\omega L - \frac{1}{\omega C}}{R}$$

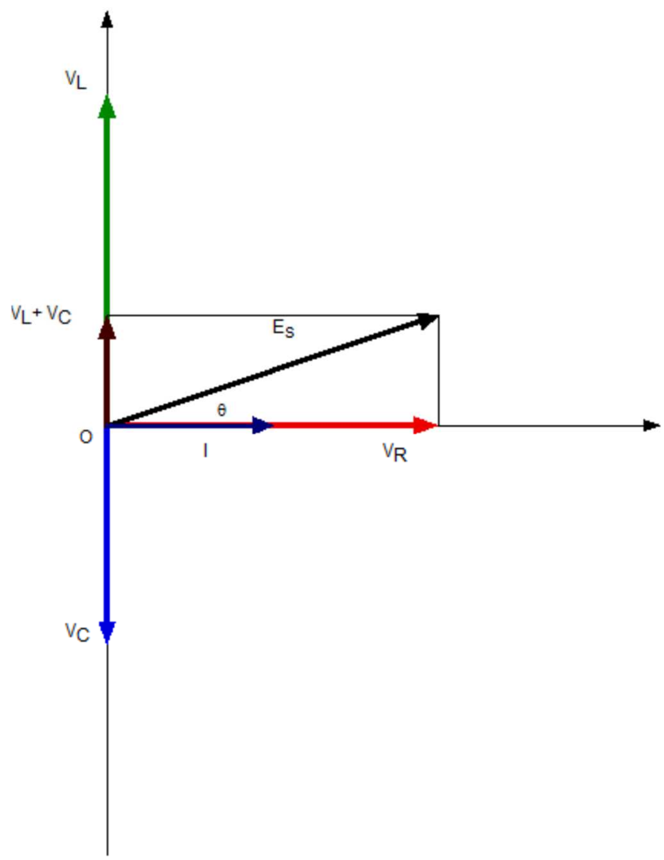
and

$$|Z_s| = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}$$

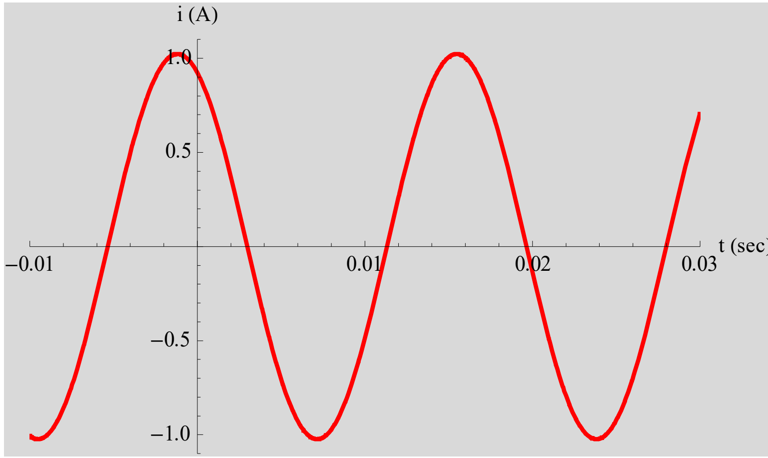
or

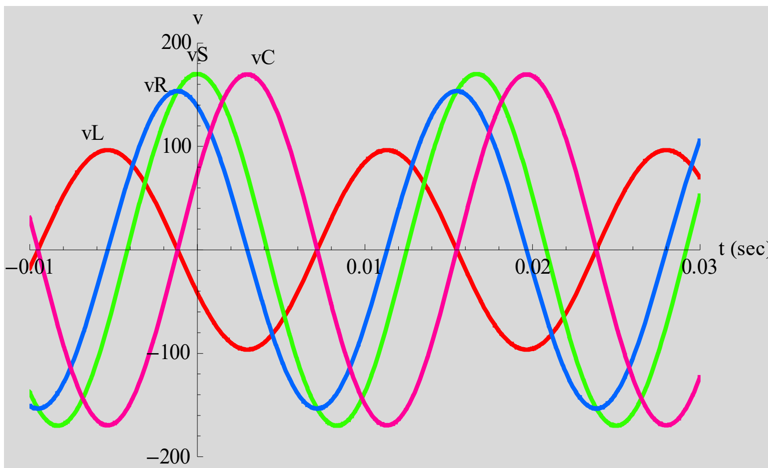


A5. Phasor diagram of the *RLC* circuits



((**Example**))
A series RLC circuit has R = 150 Ω, L = 0.25 H, and C = 16.0 μF. It is driven by 60 Hz voltage of peak value 170 V. Determine the time dependence of the voltages across each element.





A6. Infinite ladder network (from the book of Feynman)

What would happen if the following network we keep on adding a pair of the impedances (z_1 and z_2) forever, as we indicate by the dashed lines. Can we solve such an infinite network?

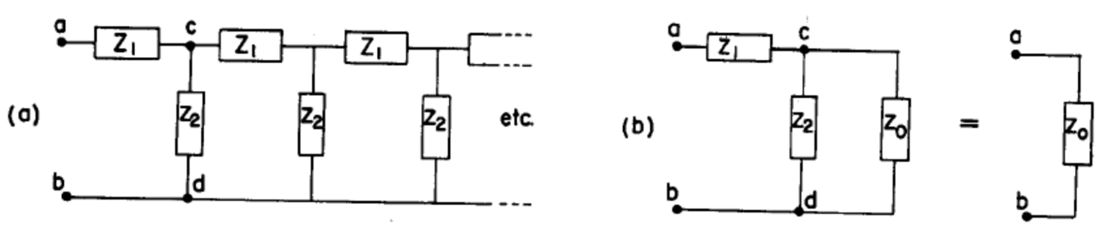


Fig. 22–19. The effective impedance of an infinite ladder.

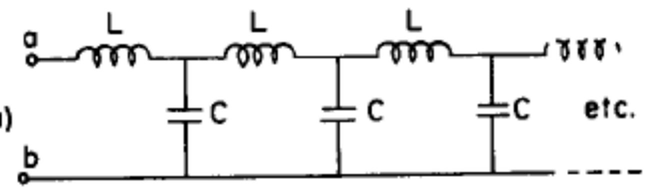
First, we notice that such an infinite network is unchanged even if we add one more section at the front end. If we add one more section to an infinite network it is still the same infinite network. Suppose we call the impedance between the two terminals a and b of the infinite network z_0. Then the impedance of all the stuff to the right of the two terminals c and d is also z_0. Taking into account of the similarity of the infinite network, we get an equation to determine the value of z_0 as

$$z_0 = z_1 + \frac{z_2 z_0}{z_2 + z_0}$$

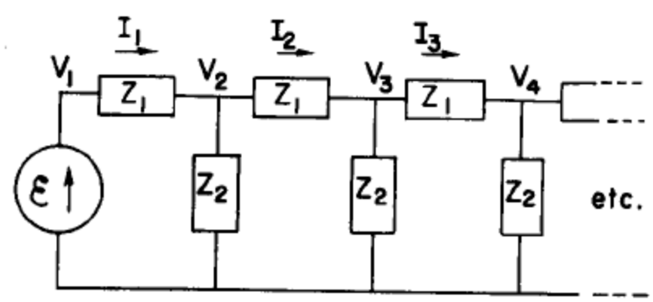
We can solve for z_0 to get

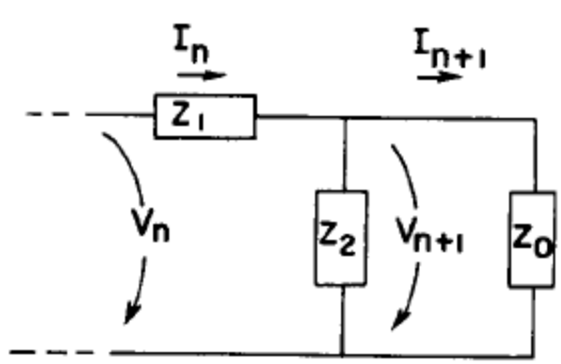
$$z_0 = \frac{z_1}{2} + \sqrt{\frac{z_1^{\ 2}}{4} + z_1 z_2}$$

This impedance z_0 is called the characteristic impedance of the infinite network.

(a)


Now we consider a specific example where an AC source is connected between the terminals a and b, and $z_1 = i\omega L$ and $z_2 = 1/(i\omega C)$. We define the complex current I_n and voltage V_n.

(a)


(b)


In the above Fig. we get the following recursion relation,

$$V_n - V_{n+1} = z_1 I_n = z_1 \frac{V_n}{z_0}$$

or

$$\frac{V_{n+1}}{V_n} = 1 - \frac{z_1}{z_0} = \frac{z_0 - z_1}{z_0} = \alpha$$

or

$$V_n = \alpha^n \varepsilon$$

We can call this ratio the propagation factor for one section of the ladder; we will call it α. It is the same for all sections. Note that

$$\begin{aligned} z_0 &= \frac{z_1}{2} + \sqrt{\frac{z_1^{\ 2}}{4} + z_1 z_2} \\ &= \frac{i\omega L}{2} + \sqrt{\frac{L}{C} - \frac{\omega^2 L^2}{4}} \end{aligned}$$

Then we have

$$\begin{aligned} \alpha = \frac{z_0 - z_1}{z_0} &= \frac{\sqrt{\frac{L}{C} - \frac{\omega^2 L^2}{4}} - \frac{i\omega L}{2}}{\sqrt{\frac{L}{C} - \frac{\omega^2 L^2}{4}} + \frac{i\omega L}{2}} \\ &= \frac{-ix + \sqrt{1 - x^2}}{ix + \sqrt{1 - x^2}} \end{aligned}$$

where $x = \frac{\omega}{\omega_0}$ and $\omega_0 = \frac{2}{\sqrt{LC}}$

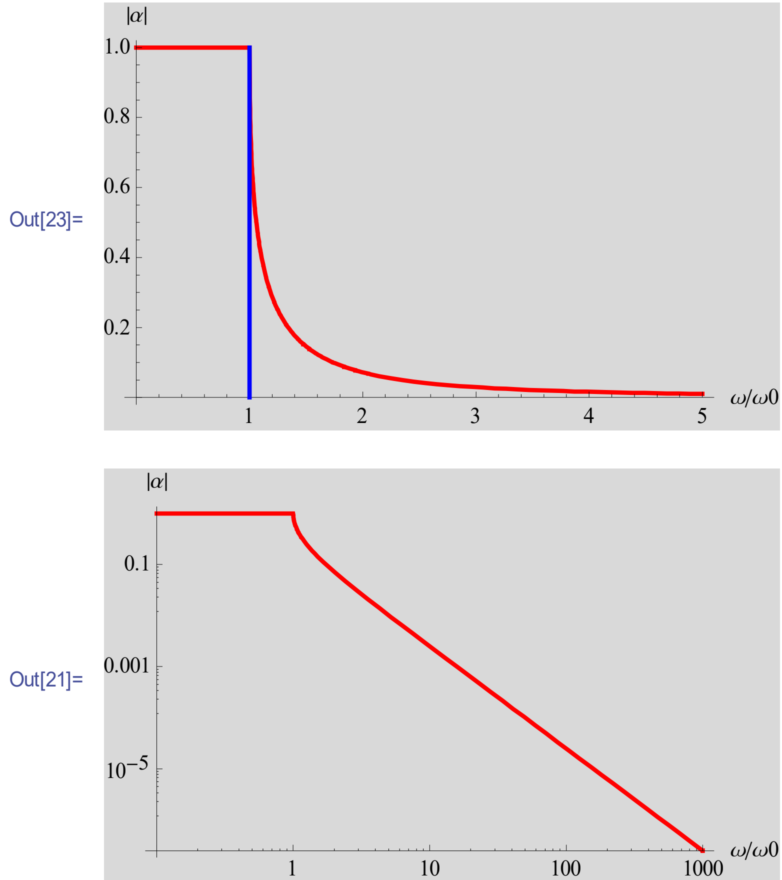


Fig. $|\alpha|$ as a function of $x = \dfrac{\omega}{\omega_0}$.

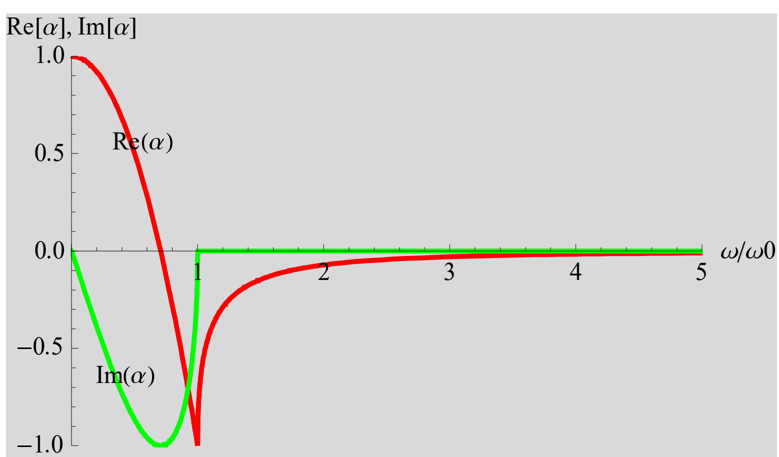
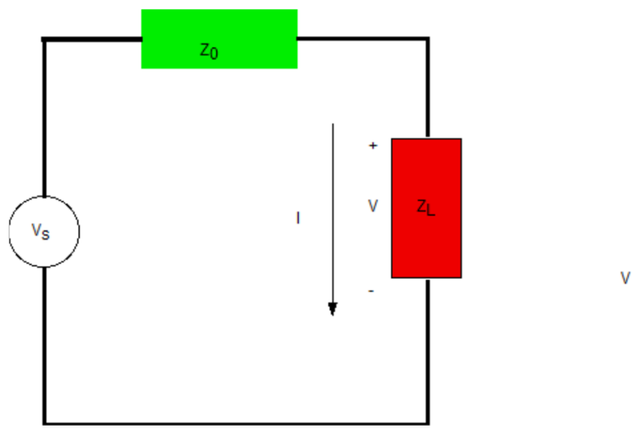


Fig. Re[α] and Im[α] as a function of $x = \dfrac{\omega}{\omega_0}$.

In summary, the network propagates energy for $\omega<\omega_0$ and blocks it for $\omega>\omega_0$. We say that the network passes low frequencies and reject or filters out the high frequencies. Any network designed to have its characteristics vary in a prescribed way wit frequency is called a filter. In this case, we have a low pass filter.

A7. Impedance matching



Given a circuit having a load impedance ($Z_L = R_L + iX_L$), what impedance will result in the maximum average power P being absorbed by the load?

$$I = \frac{V_s}{Z_0 + Z_L}$$

$$V = Z_L I = \frac{Z_L V_s}{Z_0 + Z_L}$$

The power P (absorbed by the lad) is given by

$$P = \frac{1}{2}\mathrm{Re}[VI^*] = \frac{1}{2}\mathrm{Re}[V^* I]$$

where

$$I^* V = \frac{V_s^*}{(Z_0 + Z_L)^*}\left(\frac{Z_L V_s}{Z_0 + Z_L}\right) = \frac{Z_L |V_s|^2}{|Z_0 + Z_L|}$$

and

$$Z_L = R_L + iX_L$$

$$Z_0 + Z_L = (R_0 + R_L) + i(X_0 + X_L)$$

Then P is given by

$$P = \frac{|V_s|^2 R_L}{2[(R_0 + R_L)^2 + (X_0 + X_L)^2]}$$

First, we assume that R_L is constant. Only X_L is a unknown parameter. P has a maximum when

$$X_L = -X_0$$

Then P obtained as

$$P = \frac{|V_s|^2 R_L}{2[(R_0 + R_L)^2}$$

has a maximum at $R_L = R_0$.

In conclusion, in order to get maximum power to Z_L, we select

$$Z_L = R_0 - iX_0 = Z_0^*$$

REFERENCES

E. Guillemin, Introductory Circuit Theory (John Wiley, 1953).

APPENDIX B. Complex number

B1. Imaginary unit

The imaginary unit is defined by $i = \sqrt{-1}$: $i^2 = -1$.

$$\sqrt{-5} = \sqrt{5}i$$
$$\sqrt{-20} = \sqrt{20}i = 2\sqrt{5}i$$

These numbers are called *a pure imaginary number*.

B2. Properties of complex number

For real numbers a, b, c, and d,

$$a + bi = c + di \qquad \rightarrow \qquad a = c \text{ and } b = d.$$

$$a + bi = 0 \qquad \rightarrow \qquad a = 0 \text{ and } b = 0.$$

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$
$$(a+bi)-(c+di)=(a-c)+(b-d)i$$
$$(a+bi)(c+di)=ac-bd+(ad+bc)i$$
$$(a+bi)(a-bi)=a^2-(bi)^2=a^2-(-1)b^2=a^2+b^2$$
$$\frac{a+bi}{c+di}=\frac{(a+bi)(c-di)}{(c+di)(c-di)}=\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

B3. Simplifying powers of *i*

The expression of i^n, where n is a positive whole number, can be reduced to either ±1 or $\pm i$.

$i^0 = 1$,	$i^1 = i$	$i^2 = -1$,	$i^3 = i^2 i = -i$
$i^4 = 1$,	$i^5 = i^4\, i = i$	$i^6 = i^4\, i^2 = i^2 = -1$	$i^7 = i^4 i^3 = -i$

In general, we have

$i^{4n} = 1$, $\quad i^{4n+1} = i \quad i^{4n+2} = -1, \quad i^{4n+1} = -i$

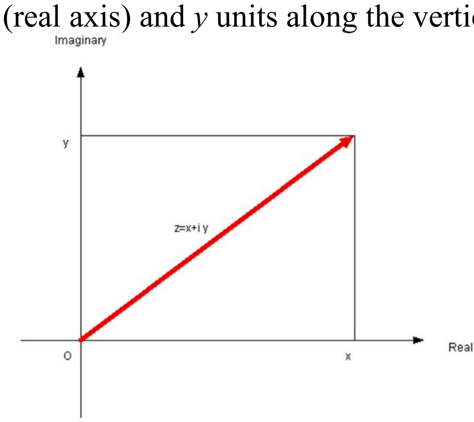
where n is an integer.

B4. Graphing complex number: complex plane

(a) Definition

$z = x + \mathrm{i}y$

A complex number in x+iy form can be graphed in the complex plane by measuring x units along the horizontal (real axis) and y units along the vertical (imaginary axis).



We define the absolute value of the complex number as

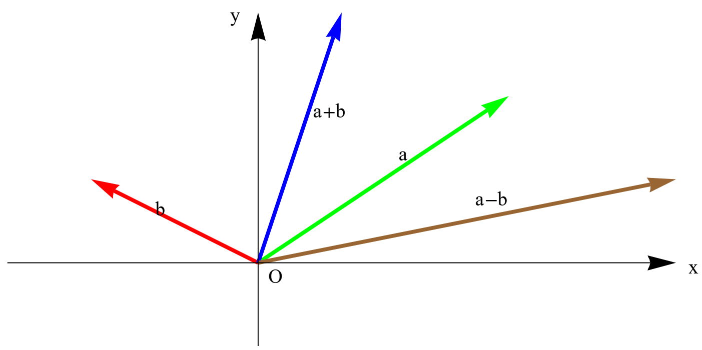
$$|z| = |x+iy| = \sqrt{x^2+y^2}$$

which is the distance between the ponits of z (= x+i y) and the origin.

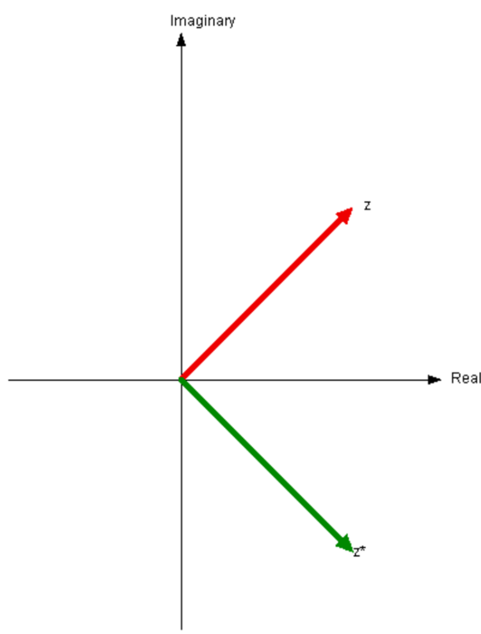
(b) Expression of the complex number in the real-imaginary plane

We make a plot of the complex numbers in the complex plane.

$a = 3+2i$, $\quad b = -2+i$
$a+b = 1+3i$, $\quad a - b=5+i$,



B5. Complex conjugate



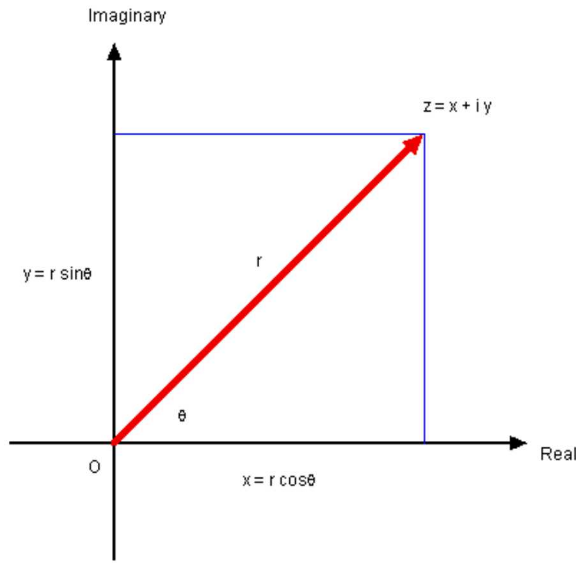
The complex number z and z^* are complex conjugate.

$$z = a + ib$$
$$z^* = a - ib$$

where a and b are real.

$$zz^* = (a+ib)(a-ib) = a^2 + b^2 .$$

B6. Angles and polar coordinates of complex number



We find the real (horizontal) and imaginary (vertical) components in terms of r (the length of the vector) and θ (the angle made with the real axis):

From Pythagoras, we have: $r^2 = x^2 + y^2$ and basic trigonometry gives us:

$$\tan\theta = \frac{y}{x}$$

$$x = r\cos\theta, \qquad y = r\sin\theta$$

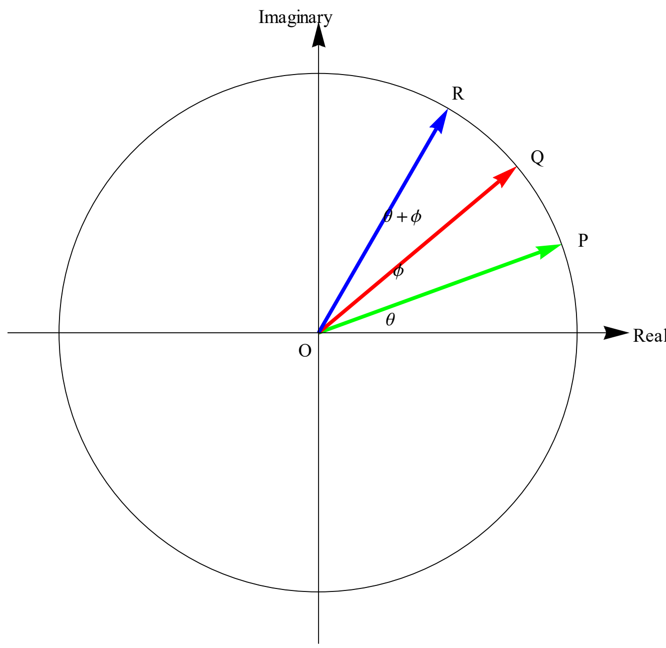
Multiplying the last expression throughout by i gives us:

$$yi = ir\sin\theta$$

So we can write the **polar form** of a complex number as:

$$z = x + iy = r(\cos\theta + i\sin\theta)$$

where r is the **absolute value** (or **modulus**) of the complex number, and θ is the **argument** of the complex number.



Points P, Q and R on the unit circle in the complex plane, are denoted by complex numbers z_1, z_2, and $z_1 z_2$.

$$
\begin{aligned}
z_1 &= \cos\theta + i\sin\theta \\
z_2 &= \cos\phi + i\sin\phi \\
z_1 z_2 &= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\
&= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi) \\
&= \cos(\theta+\phi) + i\sin(\theta+\phi)
\end{aligned}
$$

The complex conjugate of z_1 is

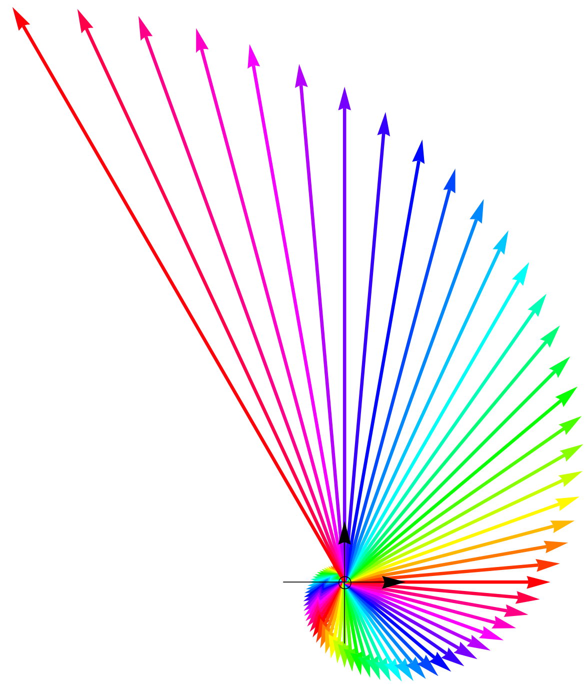
$$
\begin{aligned}
z_1^* &= \cos\theta - i\sin\theta \\
&= \cos(-\theta) + i\sin(-\theta)
\end{aligned}
$$

((Example))

We make a plot of z^n (n = 0, 1, 2, 3, …), where

$$z = r(\cos\theta + i\sin\theta).$$

We choose r = 1.05 and θ = 5º.



B7. Euler’s formula

Euler's formula states that, for any real number θ,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

where e is the base of the natural logarithm, i is the imaginary unit. Richard Feynman called Euler's formula "our jewel" and "the most remarkable formula in mathematics".

B8. de Moivre’s theorem

When $z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$$z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta} = r^n[\cos(n\theta) + i\sin(n\theta)].$$

where

$$r = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x}$$

((**Note**))

(a) The expression of z^{-n}

$$
\begin{aligned}
z^{-n} &= \left(re^{i\theta}\right)^{-n} \\
&= r^{-n}e^{-in\theta} \\
&= r^{-n}[\cos(-n\theta) + i\sin(-n\theta)] \\
&= r^{-n}[\cos(n\theta) - i\sin(n\theta)]
\end{aligned}
$$

(b) The expression of z^* (complex conjugate of z)

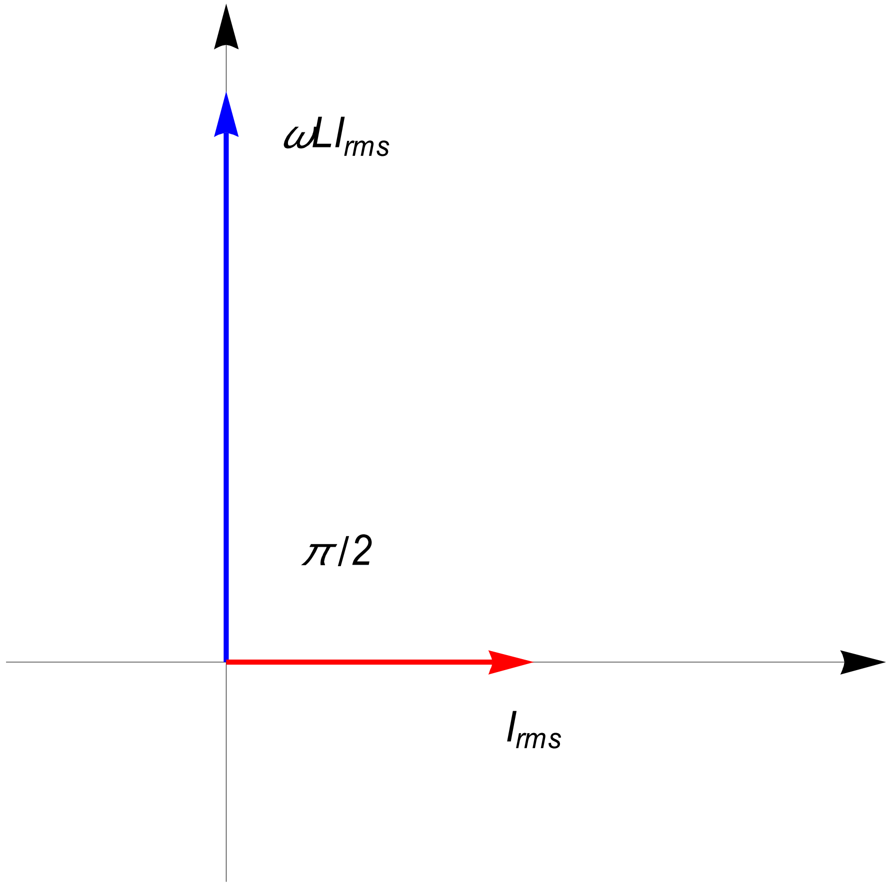
$$
\begin{aligned}
z^* &= r(\cos\theta - i\sin\theta) \\
&= r[\cos(-\theta) + i\sin(-\theta)] \\
&= re^{-i\theta}
\end{aligned}
$$

APPENDIX-C Summary of phasor diagrams in the AC circuit

Here, the phasor diagrams for typical AC circuits are summarized as follows. For convenience, the phase of I_{rms} is fixed as 0 (the positive x axis).

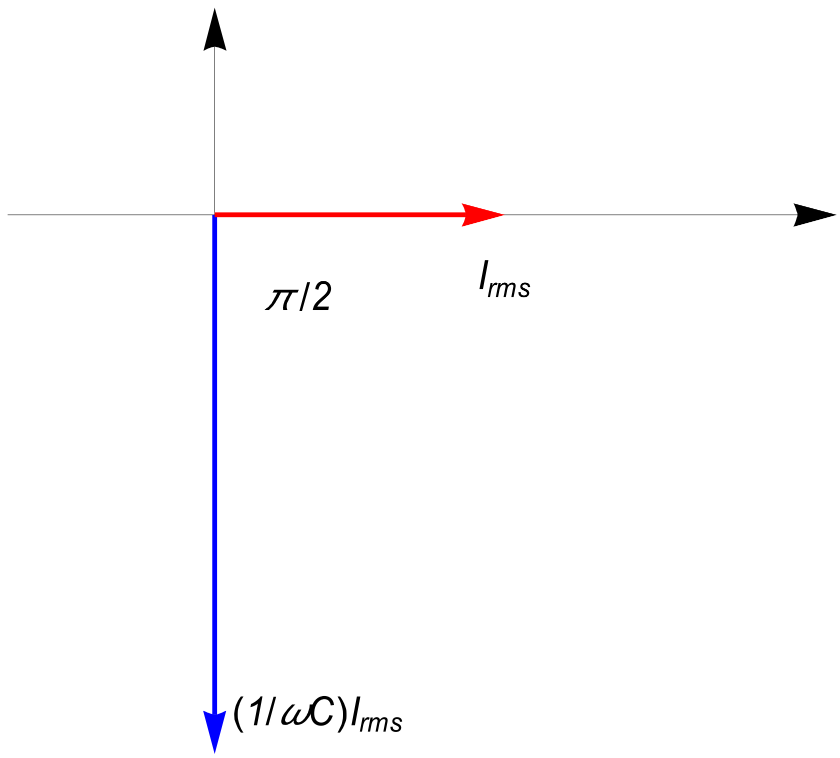
$$
I_{rms} = \frac{1}{\sqrt{2}} I_{\max}, \qquad V_{rms} = \frac{1}{\sqrt{2}} V_{\max}
$$

((Inductance))



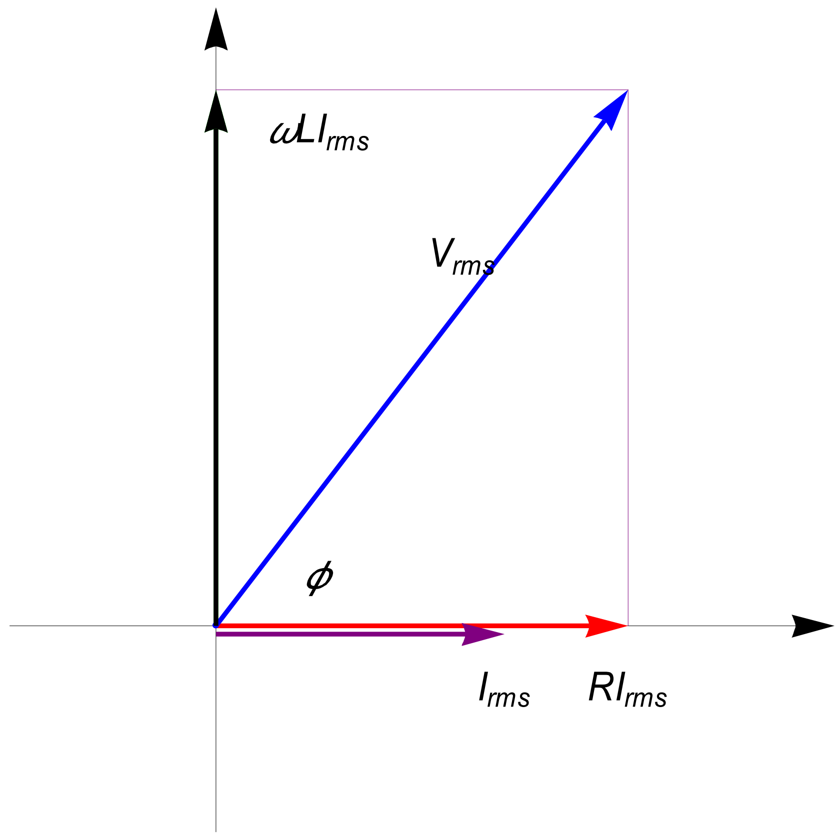
$$V_{rms} = \omega L I_{rms}$$

((**Capacitance**))



$$V_{rms} = \frac{1}{\omega C} I_{rms}$$

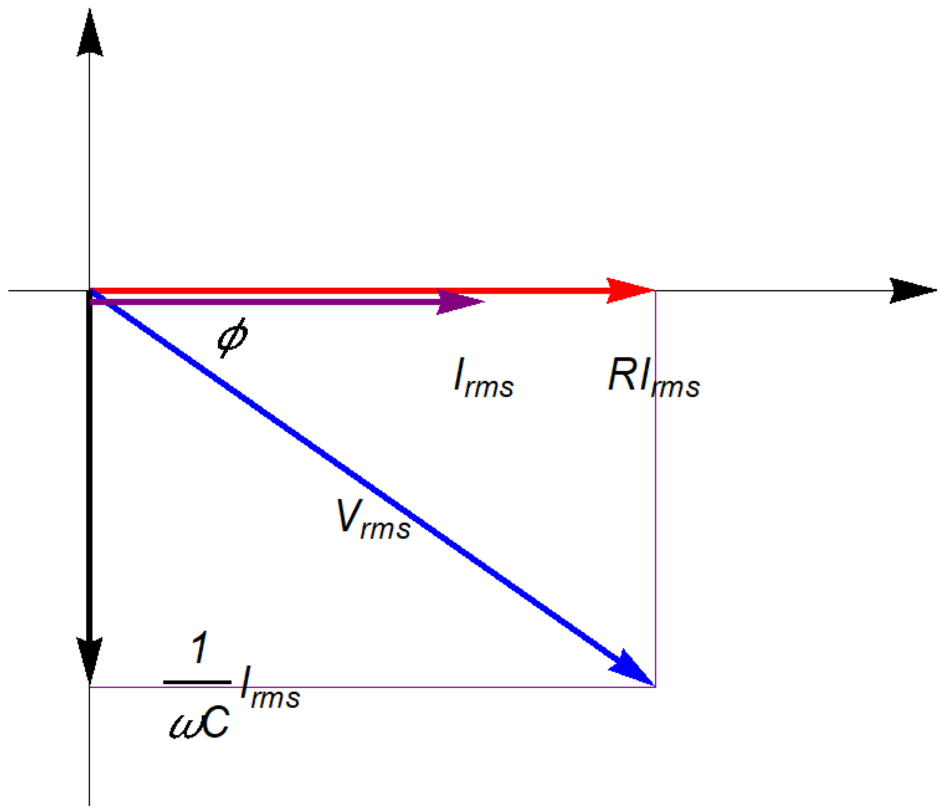
((Series of *R* and *L*))



$$V_{rms} = \sqrt{R^2 + (\omega L)^2}\, I_{rms}, \qquad \tan\phi = \frac{\omega L}{R}$$

$$V_{rms}\cos\phi = RI_{rms}$$

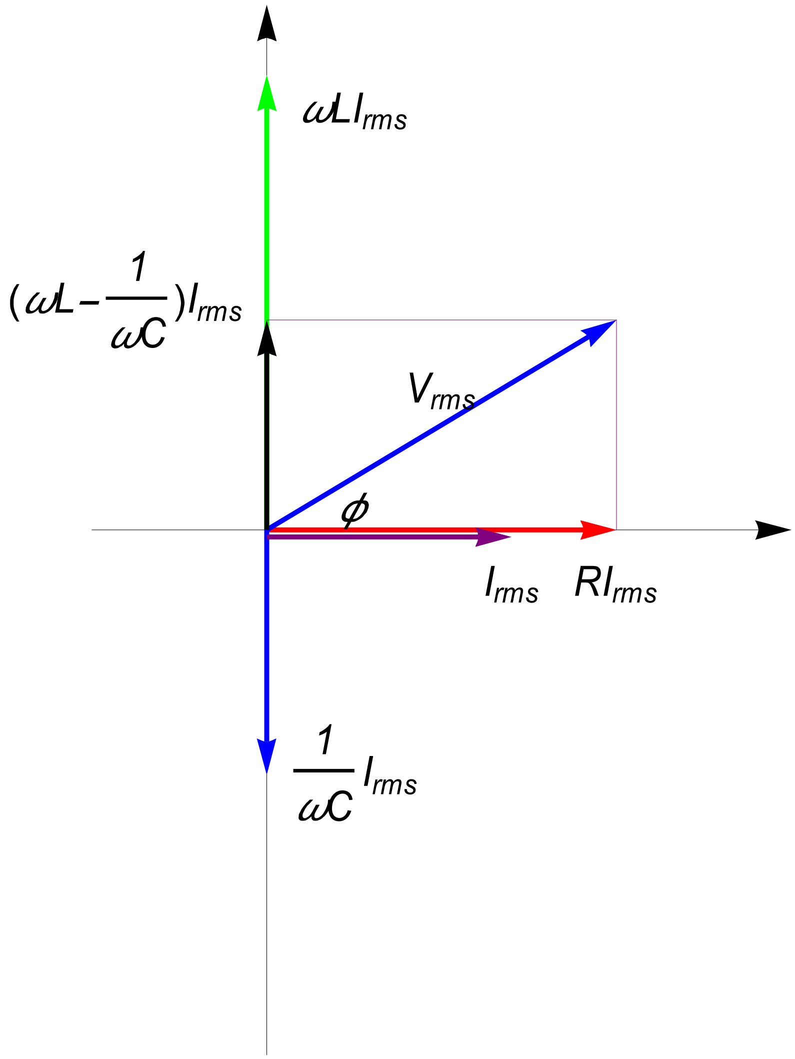
((Series of *R* and *C*))



$$V_{rms} = \sqrt{R^2 + (\frac{1}{\omega C})^2} I_{rms}, \qquad \tan\phi = \frac{1}{R\omega c}$$

$$V_{rms} \cos\phi = RI_{rms}$$

((Series of *R*, *L*, and *C*))



$$V_{rms} = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}\, I_{rms}\,,\ \tan\phi = \frac{1}{R}(\omega L - \frac{1}{\omega C})$$

$$V_{rms}\cos\phi = RI_{rms}$$

APPENDIX-D

Single-phase rms voltage and frequency over the world

Europe (Great Britain, France, Germany, Belgium, Netherland, Denmark, Sweden)

	230 V	50 Hz
Australia	230 V	50 Hz
U.S.A.	120 V	60 Hz
Equador	120 V	60 Hz
China	220 V	50 Hz
Malawi	220 V	50 Hz

Japan: 100 V 50 (East)/60Hz (West); regions of west and east separated by Fossa Magna.

In Japan, people (typically Osaka and Kyoto) west from Fossa Magna use 60 Hz, while people (typically Tokyo) east from Fossa Magna use 50 Hz.