Chapter 32
Maxwell's equations; Magnetism of Matter

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Here we discuss how Maxwell can derive his famous equation.

1. Something is missing; electrodynamics before Maxwell

A statement equivalent to the Coulomb's law is the differential relation,

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} \qquad \text{(Gauss' law)}$$

connecting the electric charge density ρ and the electric field $\boldsymbol{E}$. This holds for moving charges as well as stationary charges.

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \text{(no magnetic monopole)}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \text{(Faraday's law)}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \qquad \text{(Ampere's law)}$$

Using the above relations, we calculate

$$\nabla \cdot (\nabla \times \boldsymbol{E}) = -\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{B}) = 0 \qquad \text{(OK)}$$

This is consistent. However,

$$\nabla \cdot (\nabla \times \boldsymbol{B}) = \mu_0 (\nabla \cdot \boldsymbol{J})$$

The left-hand side of this equation must be zero. But the right-hand side, in general, is not. For steady state, $\nabla \cdot \boldsymbol{J} = 0$ is OK. Otherwise, the Ampere's law cannot be right. This contradiction shows that the expression for the Ampere's law cannot be correct for a system in which the charge density is varying in time.

2. Complete Maxwell's equation

Maxwell's equation

The complete Maxwell's equations are given as follows.

(I)

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}. \qquad \text{(Gauss' law)} \qquad (1)$$

(Flux of $\boldsymbol{E}$ through a closed surface) = -(Charge inside)/ε_0. In dynamics as well as in static fields, Gauss' law is always valid.

(II)

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}. \qquad (2)$$

(Line integral of $\boldsymbol{E}$ around a loop) = $-\frac{d}{dt}$(Flux of B through the loop). This is a Faraday's law. It is generally true.

(III)

$$\nabla \cdot \boldsymbol{B} = 0. \qquad (3)$$

(Flux of $\boldsymbol{B}$ through a closed surface) = 0. This equation is the corresponding general law for magnetic fields. Since there are no magnetic charges, the flux of $\boldsymbol{B}$ through any closed surface is always zero

(IV)

$$\nabla \times \boldsymbol{B} = \mu_0(\boldsymbol{J} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}) \qquad (4)$$

(Integral of $\boldsymbol{B}$ around a loop) = μ_0 (current through the loop) + μ_0 ε_0 (Flux of $\boldsymbol{E}$ through the loop). This equation has something new. The correct general equation has a new part that was discovered by Maxwell. $\boldsymbol{J}_d = \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$ is called a displacement current.

Conservation of charge

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t}, \qquad \text{(equation of continuity)} \qquad (5)$$

(Flux of current through a closed surface)=-$\frac{\partial}{\partial t}$(Charge inside)

Force law

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

From the equation of continuity Eq.(5), we have

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\varepsilon_0 \nabla \cdot \boldsymbol{E}) = -\nabla \cdot (\varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t})$$

or

$$\nabla \cdot \boldsymbol{J} + \nabla \cdot (\varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}) = 0$$

From Eq.(4)

$$\nabla \cdot (\nabla \times \boldsymbol{B}) = \mu_0 [\nabla \cdot \boldsymbol{J} + \nabla \cdot (\varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t})] = 0$$

((**Note**)) From MIT Physics 8.02: Electricity and Magnetism, Course Notes 2004.

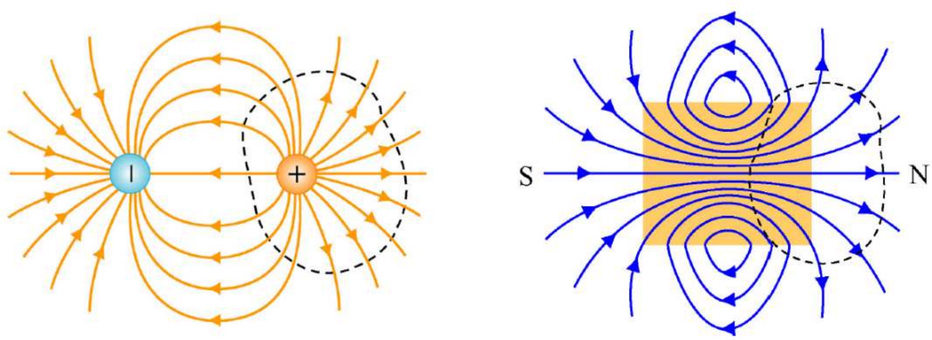
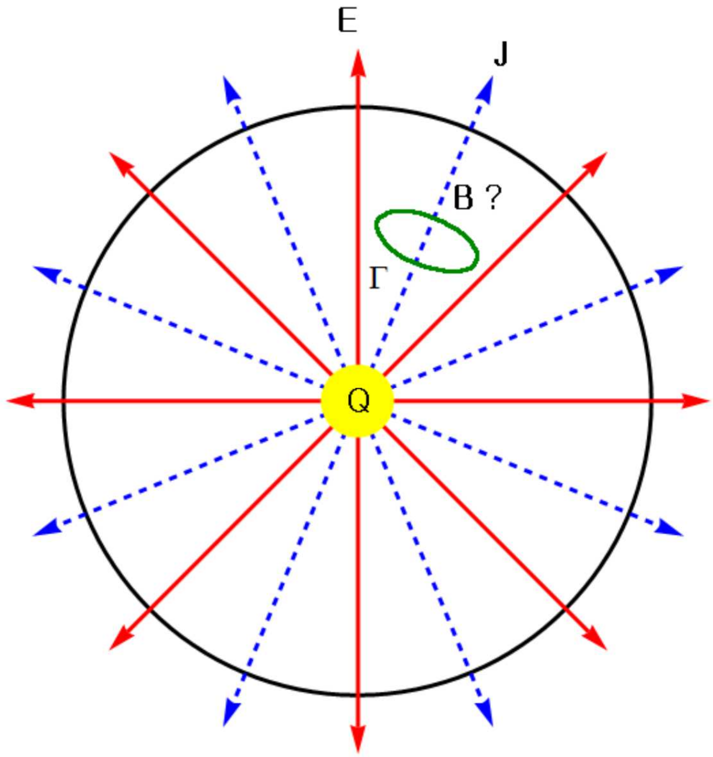


Fig. (a) $\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$. (b) $\nabla \cdot \boldsymbol{B} = 0$. The number of magnetic field lines entering a closed surface is equal to the number of field lines leaving the surface. There is no source or sink. In addition, the lines must be continuous with no starting or end points. For a bar magnet, the field lines that emanate from the north pole to the south pole outside the magnet return within the magnet and form a closed loop.

3. How the new term works; example-1 (Feynman)

((**Feynman**))



We consider what happens with a spherical symmetric radial distribution of current. Suppose we imagine a little sphere with radioactive material on it. This radioactive material is squirting out some charged particles. We could have a current that is everywhere radially outward.

Let the total charge inside any radius r be $Q(r)$. If the radial current density at the same radius is $\boldsymbol{J}(r)$, Eq.(5) requires that

$$\int_V \nabla \cdot \boldsymbol{J}(r) d\tau = \int_A \boldsymbol{J}(r) \cdot d\boldsymbol{a} = 4\pi r^2 J(r) = -\frac{\partial}{\partial t}\int_V \rho(r) d\tau = -\frac{\partial}{\partial t} Q(r)$$

$$\frac{\partial}{\partial t} Q(r) = -4\pi r^2 J(r)$$

We now ask about the magnetic field produced by the current in this situation. Suppose we draw loop Γ on a sphere of radius r. There is some current through this loop. So we might expect to find a magnetic field circulating in the direction shown. However, the correct answer is that there is no magnetic field, $\boldsymbol{B} = 0$ everywhere. Why is that? This result can be derived from Eq.(4).

$$\oint(\nabla\times\boldsymbol{B})\cdot d\boldsymbol{a}=\oint\boldsymbol{B}\cdot d\boldsymbol{s}$$
$$=\oint\mu_0(\boldsymbol{J}+\varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t})\cdot d\boldsymbol{a}$$
$$=\mu_0[4\pi r^2 J(r)+\varepsilon_0 4\pi r^2\frac{\partial}{\partial t}E(r)]$$

Here we note that

$$E(r)=\frac{Q(r)}{4\pi\varepsilon_0 r^2}$$

Then we have

$$\oint\boldsymbol{B}\cdot d\boldsymbol{s}=\mu_0(4\pi r^2 J(r)+\frac{\partial}{\partial t}Q)=0$$

The circulation of ***B*** depends not only on the total current through Γ but also on the rate of change with time of ***E*** through it. These two sources cancel and $\nabla\times B$ is always zero. This implies that there is no magnetic field ***B*** everywhere.

4. How the new term works; example of capacitance: Stokes' theorem

First, we show the definition of the Stokes' theorem between the surface integral and the path integral.

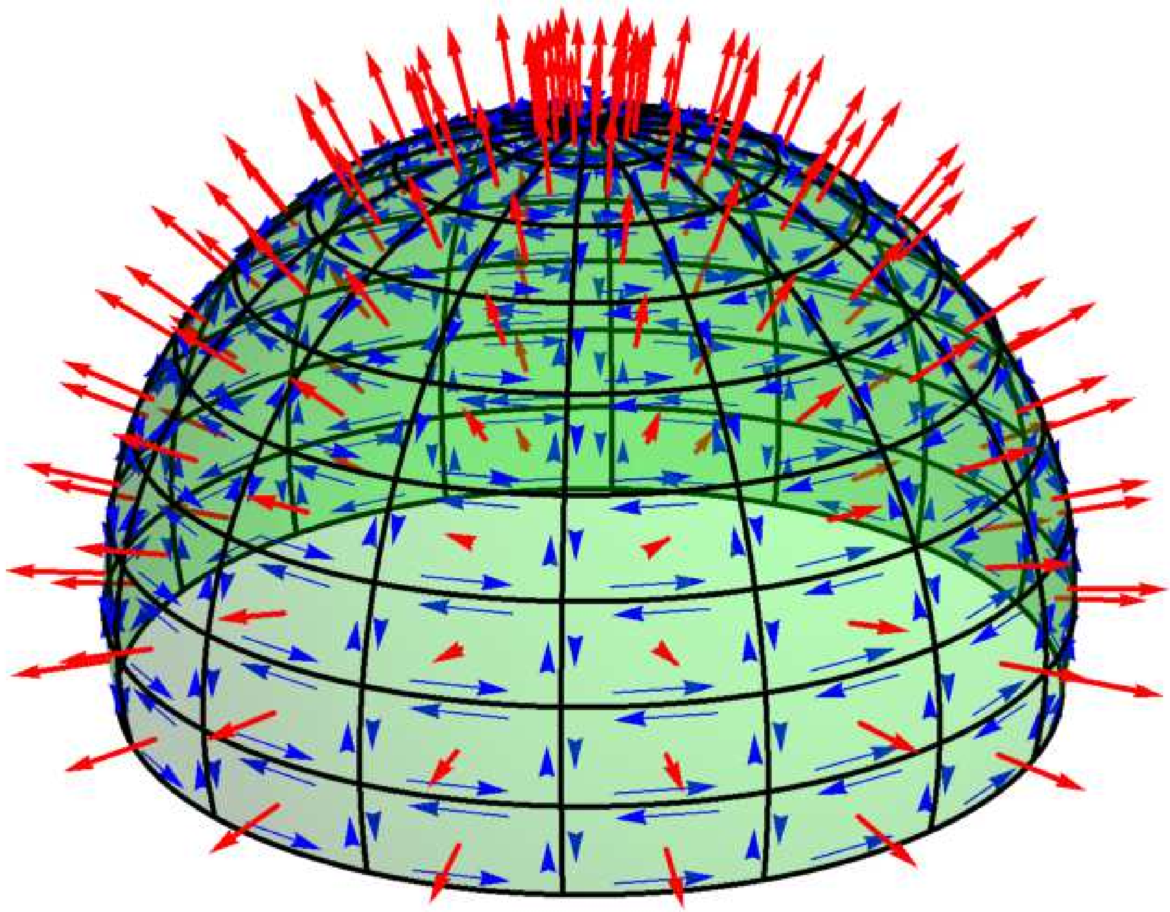
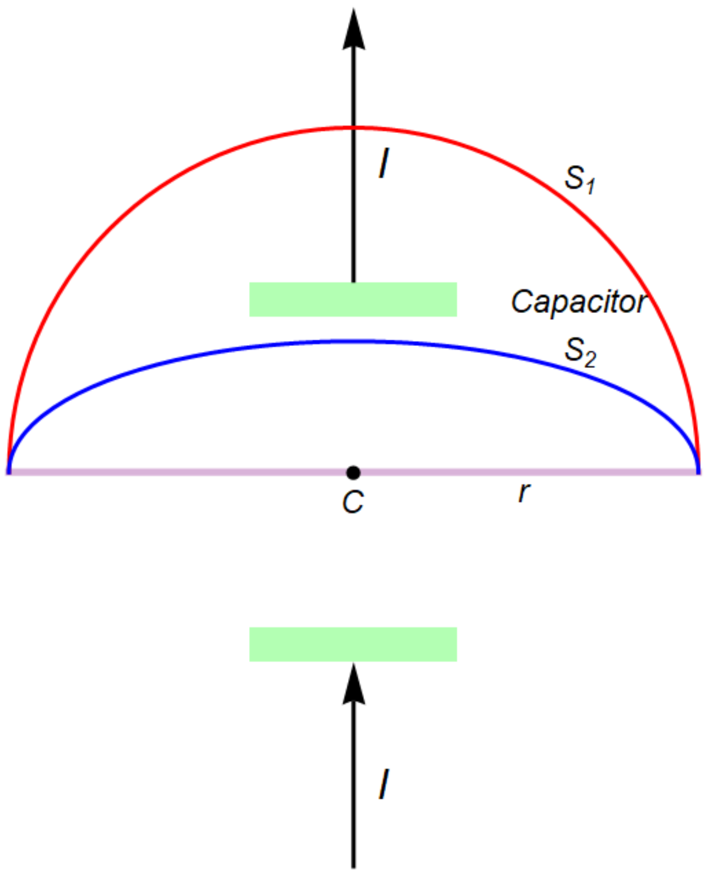


Fig. Stokes' theorem. The surface integral is replaced by the path integral around the perimeter. $\oint_S (\nabla \times \boldsymbol{B}) \cdot d\boldsymbol{a} = \oint_C \boldsymbol{B} \cdot d\boldsymbol{l}$. The red arrow denotes the direction of d$\boldsymbol{a}$ for each area element. S is the open surface. C is the closed path around the perimeter.



We use the Stokes' theorem.

$$\oint_C \boldsymbol{B}\cdot d\boldsymbol{l} = 2\pi r B = \mu_0 I_{enc} .$$

For the surface S_1,

$$I_{enc} = I .$$

For the surface S_2, the current is the displacement current, but not a current flowing along the wire.

$$I_{enc} = I = \frac{dQ}{dt} = A\frac{d\sigma}{dt} = A\varepsilon_0 \frac{dE}{dt} ,$$

or

$$I_d = I = \oint_{S_2} \boldsymbol{J}_d \cdot d\boldsymbol{a} ,$$

with

$$\boldsymbol{J}_d = \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}.$$

The Ampere's law can be corrected by Maxwell as

$$\nabla \times \boldsymbol{B} = \mu_0 (\boldsymbol{J} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}).$$

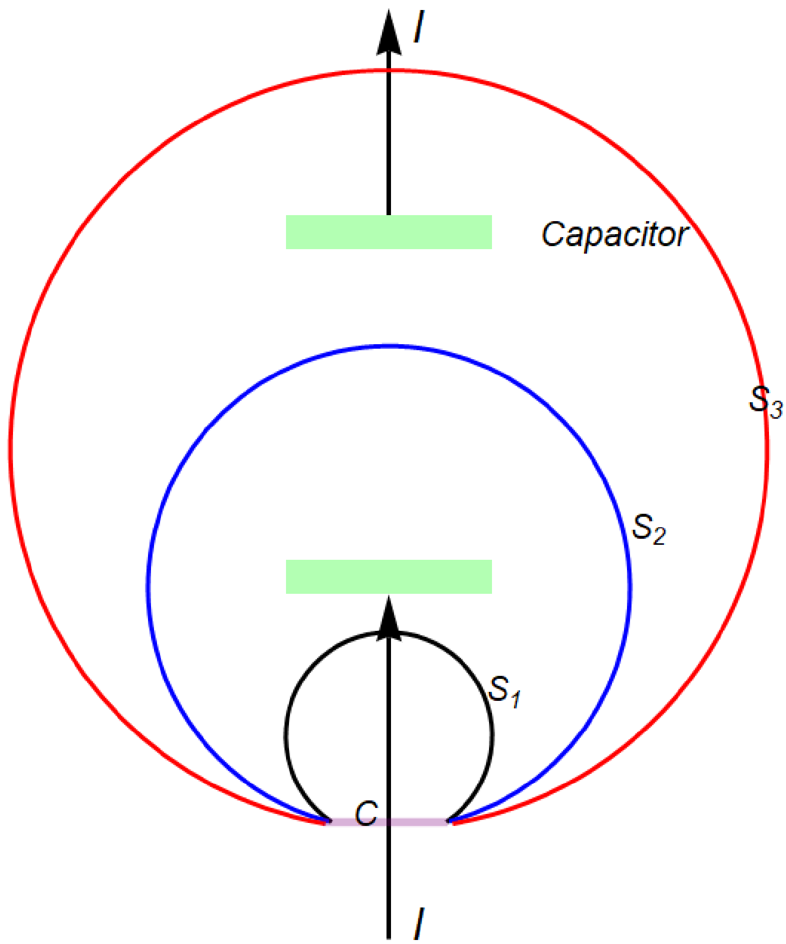


Fig. Ampere-Maxwell law for the capacitance. *S*: surface. *C*: path.

The Stokes' theorem

$$\oint_S (\nabla \times \boldsymbol{B}) \cdot d\boldsymbol{a} = \oint_C \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 \oint_S \boldsymbol{J} \cdot d\boldsymbol{a} = \mu_0 I_{enc}.$$

We apply the Stokes' theorem to the Ampere's law. We consider the three cases for the surfaces S_1, S_2, and S_3 for the surface integral, while the path C is the same (fixed).

S_1: no capacitance is included
S_2: one of the electrodes of the capacitance is included.
S_3: both electrodes of the capacitance is included.

For the surfaces S_1 or S_3, we have

$$B(2\pi r) = \mu_0 I \,, \qquad \text{or} \qquad B = \frac{\mu_0 I}{2\pi r} \,,$$

since

$$\oint_S \boldsymbol{J} \cdot d\boldsymbol{a} = I \,,$$

and

$$\oint_C \boldsymbol{B} \cdot d\boldsymbol{l} = 2\pi r B \,.$$

For the surface S_2, it seems that there is no current enclosed in the surface S_3. In order to get the same result for the magnetic field B, the current I is replaced by the displacement current.

$$I_d = \frac{dQ}{dt} = \varepsilon_0 A \frac{dE}{dt} = I \,.$$

The current density;

$$J_d = \frac{I_d}{A} = \varepsilon_0 \frac{dE}{dt} \,.$$

The ampere's law:

$$\nabla \times \boldsymbol{B} = \mu_0 (\boldsymbol{J} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}) \,.$$

where $\boldsymbol{J}$ is the conduction current density and $\boldsymbol{J}_d$ is the displacement current density.

((**Note**))
E.M. Purcell and D.J. Morin, Electricity and Magnetism 3rd edition (Cambridge 2013). p.433-434

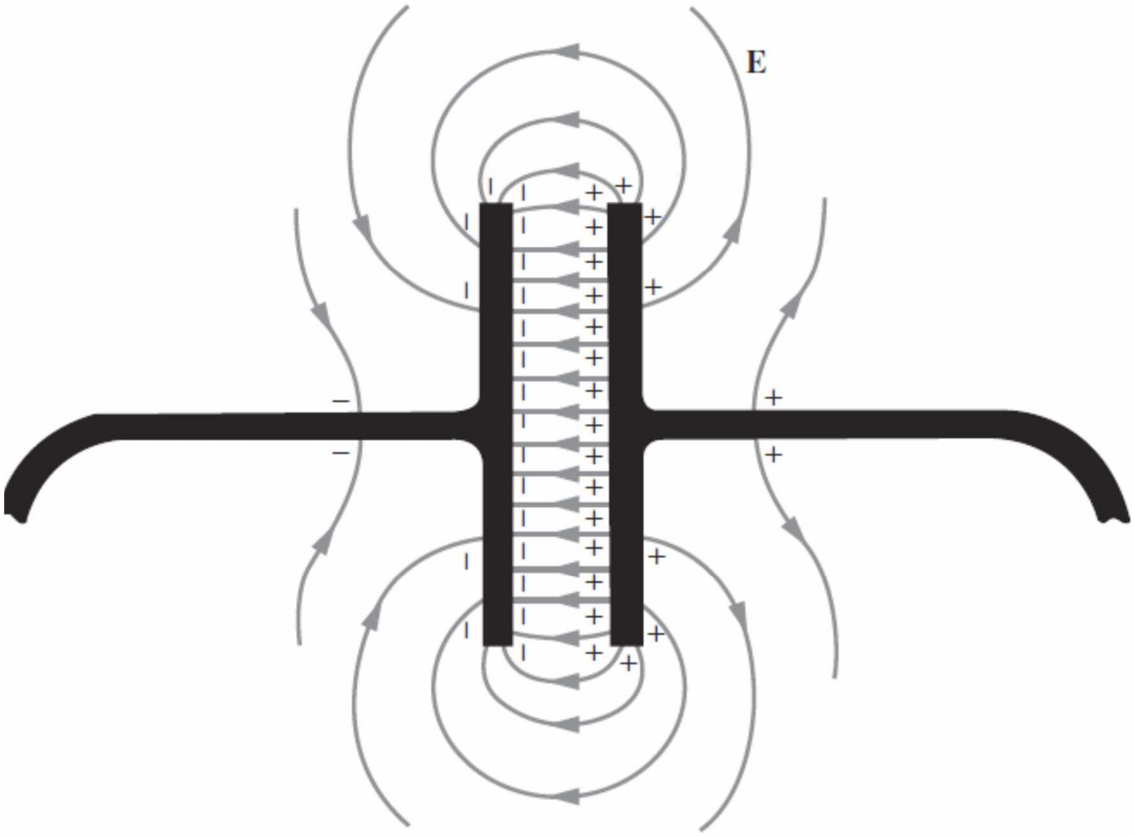


Fig. The electric field at a particular instant. The magnitude of $\boldsymbol{E}$ is decreasing everywhere as time goes on.

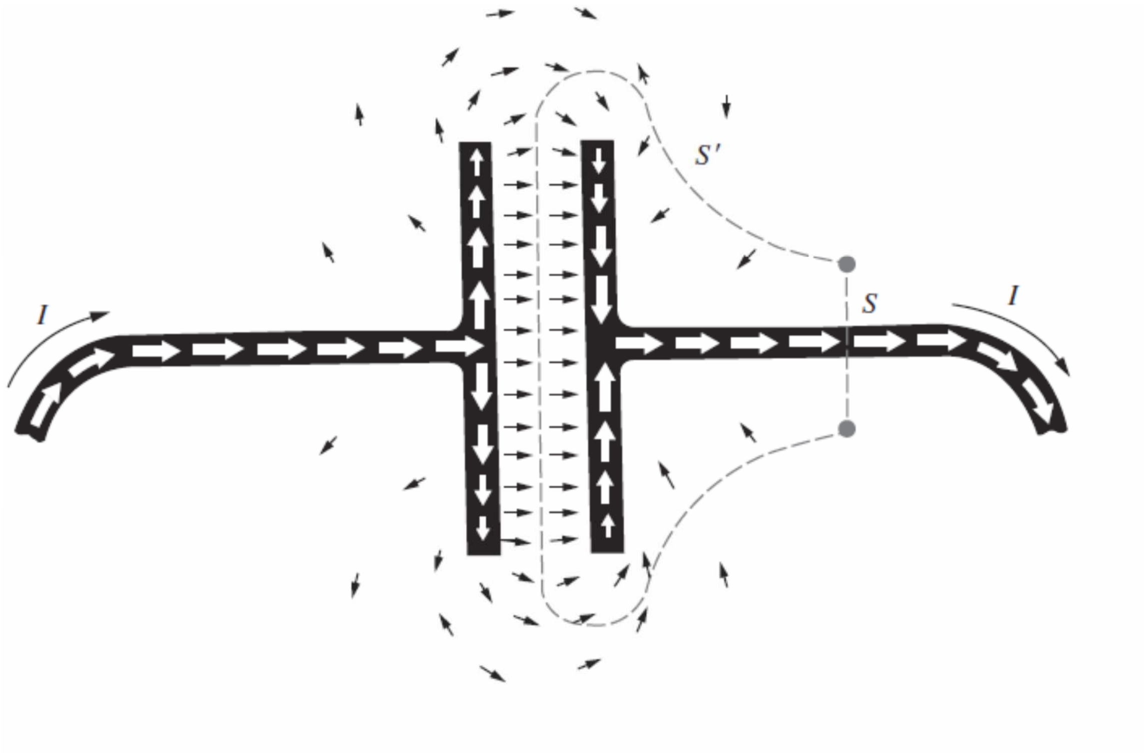
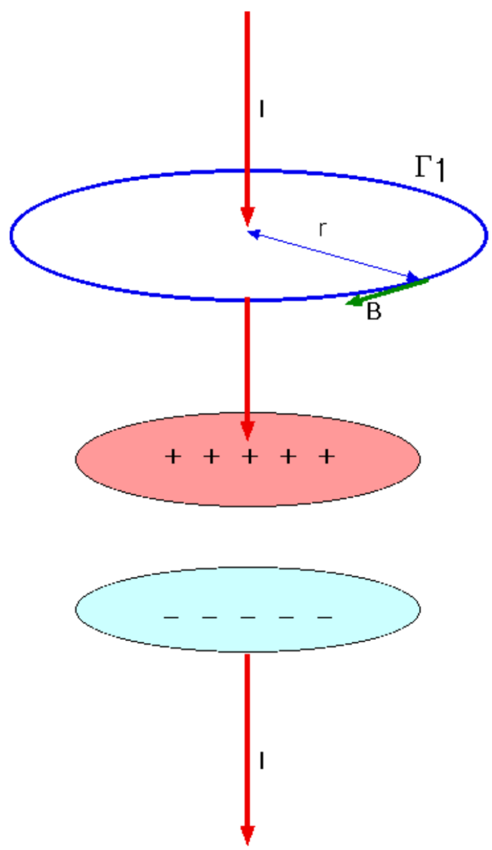


Fig. The conduction current (white arrows) and the displacement current (black arrows)

5. Displacement current: Ampere-Maxwell law



We consider the magnetic field of a wire used to charge a parallel-plate condenser. If the charge Q on the plate is charging with time, the current in the wires is equal to dQ/dt.

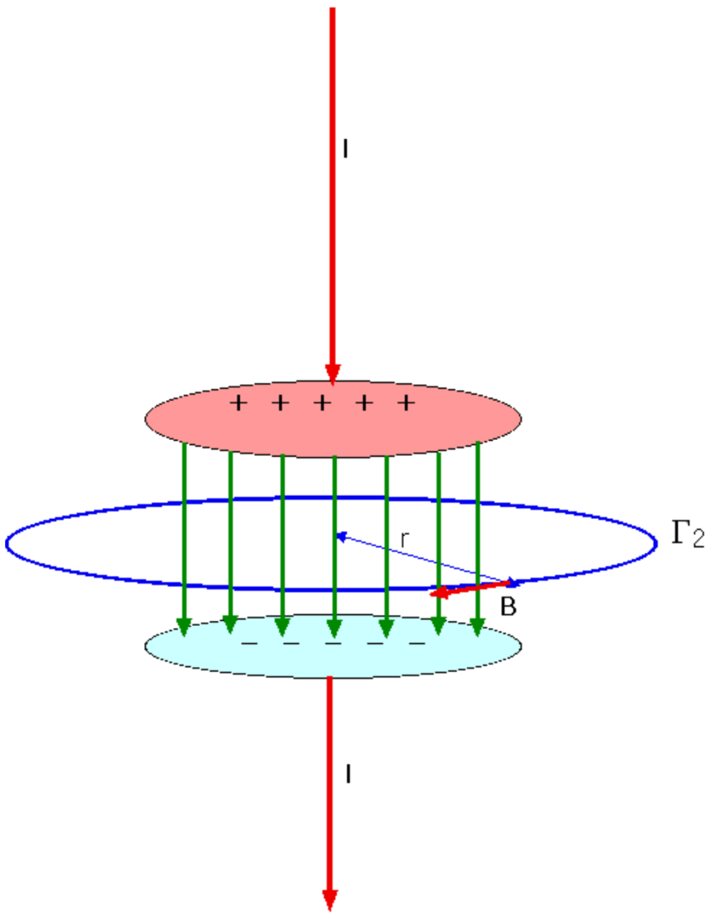
(a) Path Γ_1

Suppose we take a loop Γ_1 which is a circle with radius r.

$$\int_S (\nabla \times \boldsymbol{B}) \cdot d\boldsymbol{a} = \oint \boldsymbol{B} \cdot d\boldsymbol{s} = \int_S \mu_0 (\boldsymbol{J} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}) \cdot d\boldsymbol{a} = \mu_0 (I + \varepsilon_0 \int \frac{\partial \boldsymbol{E}}{\partial t} \cdot d\boldsymbol{a})$$

If we consider the appropriate plane surface S enclosed by the loop Γ_1, there are no electric fields on it (assuming the wire to be a very good conductor). The surface integral of $\int \frac{\partial \boldsymbol{E}}{\partial t} \cdot d\boldsymbol{a}$ is zero. Then the magnetic field is obtained as

$$B = \frac{\mu_0 I}{2\pi r}$$



Suppose, however, that we now slowly move the curve Γ downward. We get always the same result until we draw with the plates of the capacitor. The current I goes to zero. What happens to the magnetic field?

(b) Path Γ_2

Let's see what the Maxwell's equation says for the curve Γ_2, which is a circle of radius r whose plane passes between the capacitor plates.

$$\oint \boldsymbol{B} \cdot d\boldsymbol{s} = 2\pi r B$$

$$= \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \int_S \boldsymbol{E} \cdot d\boldsymbol{a}$$

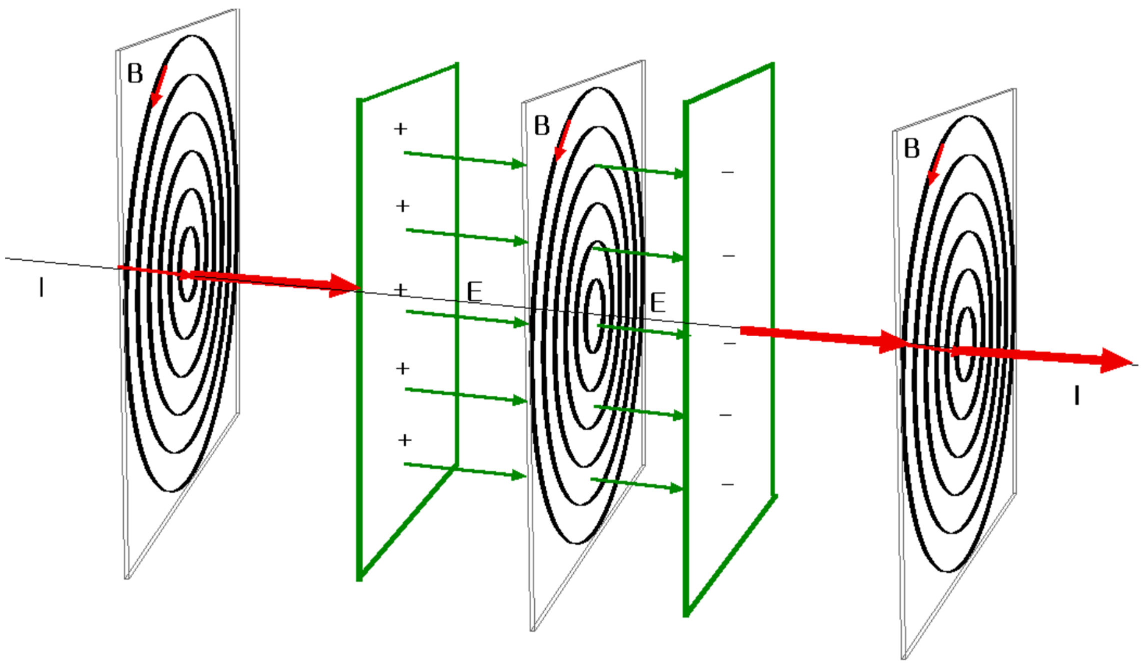
In other words, the line of integral of $\boldsymbol{B}$ around Γ_2 is equal to the time derivative of the flux of $\boldsymbol{E}$ through the appropriate plane circular surface S enclosed by the path Γ_2. From the Gauss' law, we know that the flux of $\boldsymbol{E}$ through the plane circular surface S is

$$\int_S \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{Q}{\varepsilon_0}$$

Note that the electric field inside the capacitor plate is equal to zero because of metal in applying the Gauss' law. Then we have

$$B = \frac{\mu_0 \varepsilon_0}{2\pi r}\frac{\partial}{\partial t}\int_S \boldsymbol{E}\cdot d\boldsymbol{a} = \frac{\mu_0 \varepsilon_0}{2\pi r}\frac{\partial}{\partial t}\frac{Q}{\varepsilon_0} = \frac{\mu_0}{2\pi r}\frac{\partial}{\partial t}Q = \frac{\mu_0 I}{2\pi r}$$

So we have the same result for B as described above. It is easy to see that this must always be so by applying the same arguments to the two circular surfaces S_1 and S_2 enclosed the paths Γ_1 and Γ_2, respectively. Through S_1 there is the current I, but no electric flux. Through S_2 there is no current, but an electric flux changing at the rate I/ε_0.



The displacement current flows in the separation gap of the capacitance,

$$\boldsymbol{J}_d = \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$
$$i_d = \varepsilon_0 A \frac{\partial E}{\partial t}$$

6. Magnetism and electrons

6.1. Orbital angular momentum and orbital magnetic moment

If an electron [charge $-e$ (e>0) and mass m] is moving in a circular orbit, there is a definite ratio between the magnetic moment and the angular momentum. Suppose that $\boldsymbol{L}$ is the orbital angular momentum and μ_{orb} is the orbital magnetic moment. The orbital angular momentum L is given by

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = mvr\hat{z}$$

The direction of L is perpendicular to the plane of the orbit. The orbital magnetic moment is given by

$$\boldsymbol{\mu}_L = -IA\hat{z} = -\frac{ev}{2\pi r}\pi r^2 \hat{z} = -\frac{evr}{2}\hat{z},$$

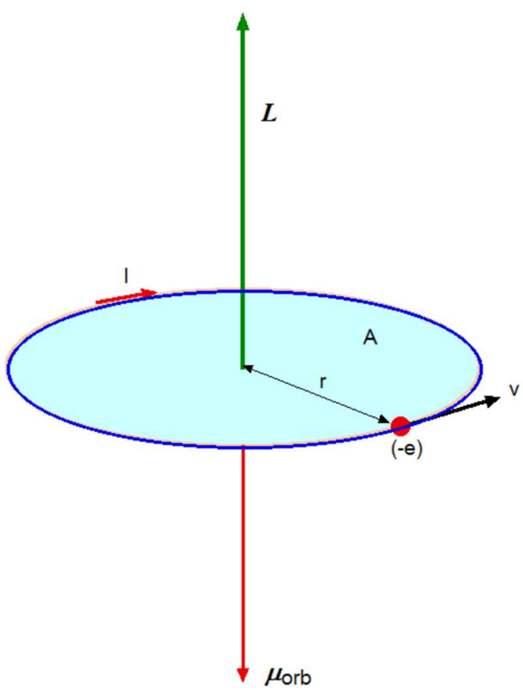
where A (= πr^2) is the area of the orbit and the current I is given by

$$I = ef = \frac{e}{T} = \frac{ev}{2\pi r},$$

where T (=$2\pi r/v$) is a period and f (=$1/T$) is the frequency. So we have the relation between the orbital angular momentum and the orbital magnetic moment as

$$\boldsymbol{\mu}_{orb} = -\frac{e(mvr)}{2m}\hat{z} = -\frac{e}{2m}\boldsymbol{L}.$$

The direction of the current is opposite to the direction of velocity of electron because the charge is negative. The orbital magnetic moment of the electron is antiparallel to the orbital angular momentum.



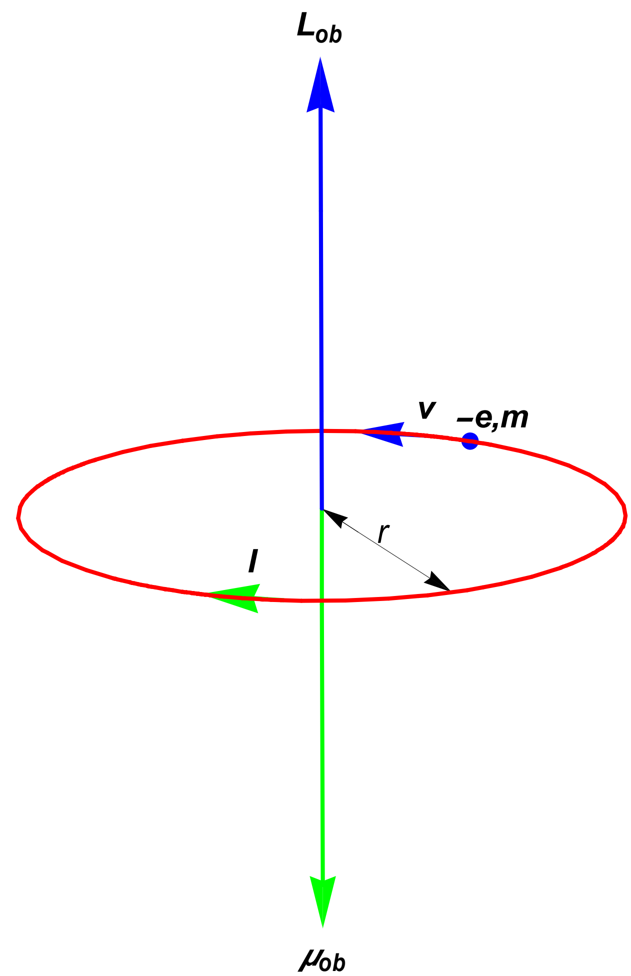


Fig. Orbital (circular) motion of electron with mass m and a charge $-e$. The direction of orbital angular momentum L is perpendicular to the plane of the motion (x-y plane). The orbital magnetic moment is antiparallel to the orbital angular momentum.

6.2 Physical meaning of the expression for the magnetic moment

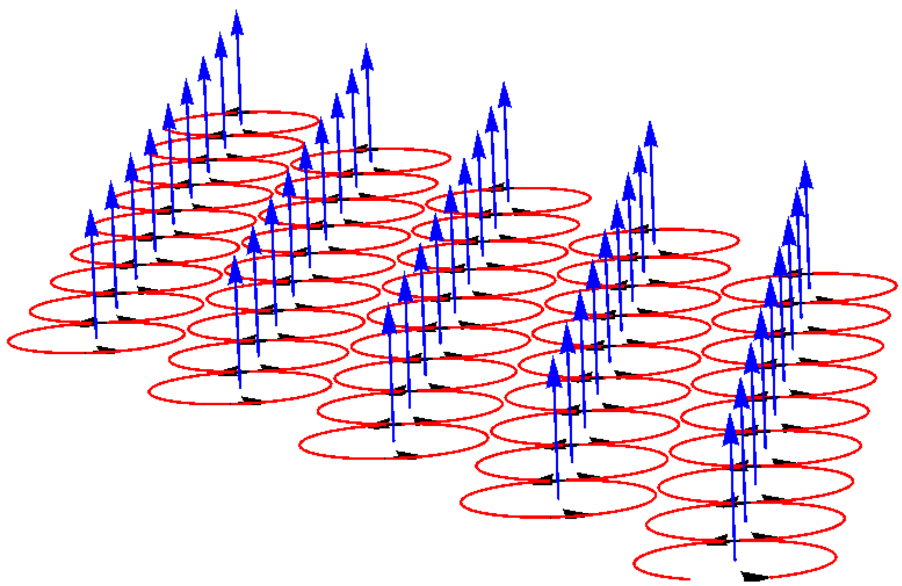
From the definition of the magnetic moment, a loop (orbital) current I flowing around a circle with a large radius R, produces a magnetic moment given by

$$\mu = IA,$$

where A is the area of the large circle. The magnetic moment can be rewritten as

$$\begin{aligned}\mu &= IA \\ &= I(A_1 + A_2 + A_3 + \ldots) \\ &= \sum_i IA_i \\ &= \sum_i \mu_i\end{aligned}$$

where the total area A is formed of area A_i ($i = 1, 2, 3, \ldots$) for the atomic-scale small circles with $A_i << A$. The above equation indicates that the same loop current I flowing around each small circle leads to the magnetic moment $\mu_i = IA_i$. Note that loop currents flowing inside the small circles cancel out completely. Effectively, the loop current I flows only around the outside boundary of the system.



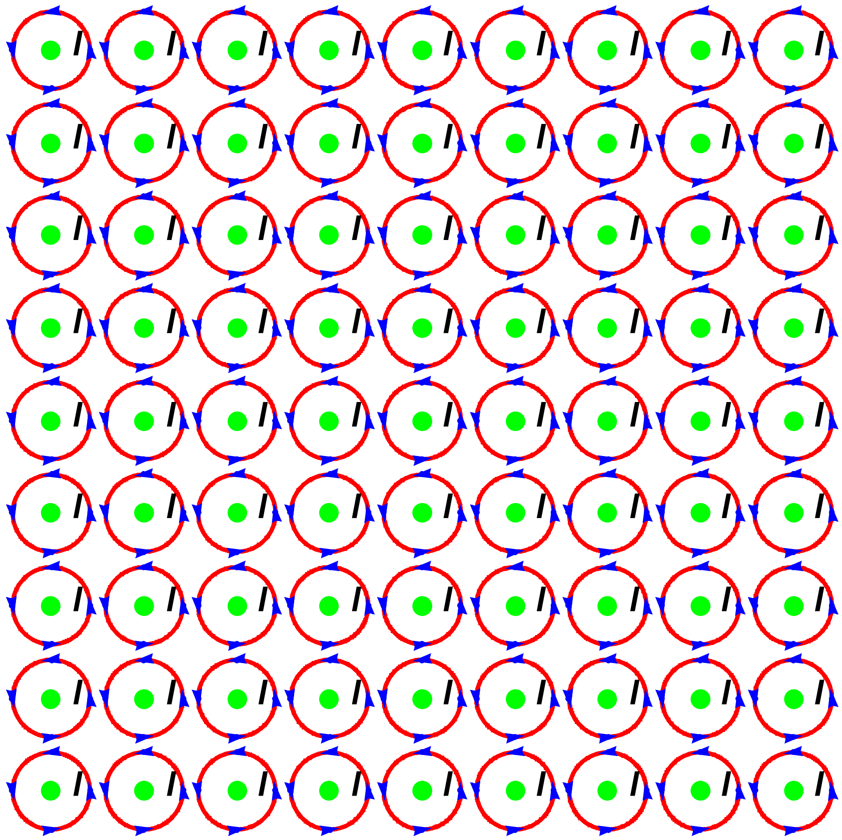


Fig. The magnetic moment of the total system is the collection of small magnetic moments arising from the atomic-scale loop currents I. Evidently, loop currents flowing inside the system cancel out completely. Only a loop current flowing the boundary of the system contributes to the resultant magnetic moment. Note that the direction of each magnetic moment vector is out of page.

6.3 de Broglie relation

Material particles, just like photons, can have a wavelike aspect. The various permitted energy levels appear as analogues of the normal modes of a vibrating string.

<u>Particle</u>:

E (energy), $\boldsymbol{p}$ (momentum)

<u>Wave</u>:

$\omega = 2\pi\nu$, $\boldsymbol{k}$ (wave vector)

Relation:

$$E = h\nu = \hbar\omega$$

$$\boldsymbol{p} = \hbar\boldsymbol{k}$$

The de Broglie relation between the momentum p and the wavelength λ

$$\lambda = \frac{2\pi}{k} = \frac{h}{p}, \qquad \text{or} \qquad p = \frac{h}{\lambda}$$

where λ is the de Broglie wavelength, h is the Planck's constant. For a circular orbit with the radius r, it is required from the quantum mechanics that

$$2\pi r = n\lambda$$

where n is integer. Since $p(2\pi r) = \frac{h}{\lambda} 2\pi r = nh$, the orbital angular momemtum L_z is obtained as

$$L_z = pr = \frac{nh}{2\pi} = n\hbar$$

where $\hbar = \frac{h}{2\pi}$ is the Dirac constant.

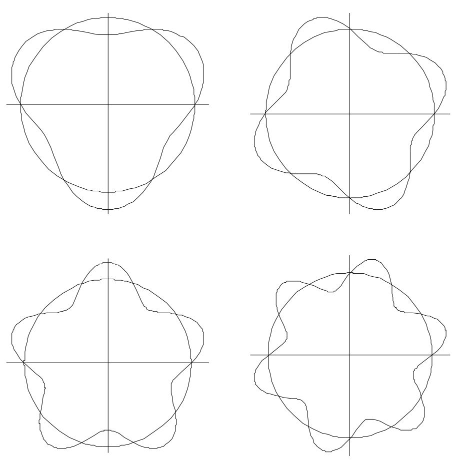


Fig. Acceptable wave on the ring (circular orbit). The circumference should be equal to the integer n (=1, 2, 3,…) times the de Broglie wavelength λ. The picture of fitting the de Broglie waves onto a circle makes clear the reason why the orbital angular momentum is quantized. Only integral numbers of wavelengths can be fitted. Otherwise, there would be destructive interference between waves on successive cycles of the ring.

The orbital magnetic moment is given by

$$\boldsymbol{\mu}_L = -\frac{e\hbar}{2m}\frac{\boldsymbol{L}}{\hbar} = -\mu_B \frac{\boldsymbol{L}}{\hbar},$$

where μ_B is the Bohr magneton and is

$$\mu_B = 9.27400915\times10^{-24}\ \text{J/T} \qquad \text{(SI units)}$$

or

$$\mu_B = 9.27400915\times10^{-21}\ \text{emu} \qquad \text{(cgs units, emu=erg/Gauss=erg/Oe)}$$

((Note))

J/T = 10^7 erg/(10^4 Oe) = 10^3 emu
J/T^2 = 10^{-1} emu/Oe

The value of the orbital magnetic moment is given by

$$|\boldsymbol{\mu}_L| = \frac{e\hbar}{2m} n = \mu_B n \qquad (n = 1, 2, 3, \ldots)$$

6.4 Spin angular momentum and spin magnetic moment

The electron also has a spin rotation around its own axis, and as a result of that spin, it has both a spin angular momentum and a spin magnetic moment. But for reasons that are purely relativistic quantum-mechanical – there is no classical explanation – the relation between the spin magnetic moment and the spin angular momentum is different from that for the orbital motion. The spin magnetic moment is given by

$$\boldsymbol{\mu}_S = -\mu_B \frac{g_e \boldsymbol{S}}{\hbar}$$

where g_e is the electron g-factor; g_e = 2.0023193043622 (NIST). The component of the spin angular momentum $\boldsymbol{S}$ is measured along the z axis. Then the measured component S_z can have only the two values given by

$$S_z = \pm\frac{1}{2}\hbar \qquad (|+\rangle\text{; spin up state and } |-\rangle\text{; spin down state}).$$

Then the value of spin magnetic moment is $\pm\mu_B$.

6.5 Periodic table of iron group elements

The Pauli principle produces any two electrons being in the same state (i.e., having the set of (n, l, m_l, m_s).

For fixed n, l = n-1, n-2, …, 2, 1

$m_l = l, l\text{-}1, \ldots, -l$ $(2l+1)$.
Therefore there are n^2 states for a given n.

```
n-1
 ∑  (2 l + 1) // Simplify
l=0
```

n^2

There are two values for m_s $(= \pm 1/2)$.
Thus, corresponding to any value of n, there are $2n^2$ states.

K shell

n	l	m	s	m_s	
1	0	0	1/2	±1/2	$(1s)^2$

L shell

n	l	m	s	m_s	
2	0	0	1/2	±1/2	$(2s)^2$
2	1	1,0,-1	1/2	±1/2	$(2p)^6$

M shell

n	l	m	s	m_s	
3	0	0	1/2	±1/2	$(3s)^2$
3	1	1,0,-1	1/2	±1/2	$(3p)^6$
3	2	2,1,0,-1,-2	1/2	±1/2	$(3d)^{10}$

N shell

n	l	m	s	m_s	
4	0	0	1/2	±1/2	$(4s)^2$
4	1	1,0,-1	1/2	±1/2	$(4p)^6$
4	2	2,1,0,-1,-2	1/2	±1/2	$(4d)^{10}$
4	3	3,2,1,0,-1,-2,-3	1/2	±1/2	$(4f)^{14}$

$(1s)^2|(2s)^2(2p)^6|(3s)^2(3p)^6(3d)^{10}|(4s)^2(4p)^6(4d)^{10}(4f)^{14}|(5s)^2(5p)^6\ ((5d)^{10}\ldots.$

Iron-group elements:

Ti^{3+}, V^{4+}	$(1s)^2\vert(2s)^2(2p)^6\vert(3s)^2(3p)^6\vert(3d)^1$
V^{3+}	$(1s)^2\vert(2s)^2(2p)^6\vert(3s)^2(3p)^6\vert(3d)^2$
Cr^{3+}, V^{2+}	$(1s)^2\vert(2s)^2(2p)^6\vert(3s)^2(3p)^6\vert(3d)^3$
Cr^{2+}, Mn^{3+}	$(1s)^2\vert(2s)^2(2p)^6\vert(3s)^2(3p)^6\vert(3d)^4$
Mn^{2+}, Fe^{3+}	$(1s)^2\vert(2s)^2(2p)^6\vert(3s)^2(3p)^6\vert(3d)^5$
Fe^{2+}	$(1s)^2\vert(2s)^2(2p)^6\vert(3s)^2(3p)^6\vert(3d)^6$
Co^{2+}	$(1s)^2\vert(2s)^2(2p)^6\vert(3s)^2(3p)^6\vert(3d)^7$
Ni^{2+}	$(1s)^2\vert(2s)^2(2p)^6\vert(3s)^2(3p)^6\vert(3d)^8$
Cu^{2+}	$(1s)^2\vert(2s)^2(2p)^6\vert(3s)^2(3p)^6\vert(3d)^9$

Atoms with filled n shells have a total angular momentum and a total spin of zero. Electrons exterior these closed shells are called valence electrons.

6.6 Magnetic moment of atom

We consider an isolated atom with incomplete shell of electrons. The orbital angular momentum $\boldsymbol{L}$ and spin angular momentum $\boldsymbol{S}$ are given by

$$\boldsymbol{L} = \boldsymbol{L}_1 + \boldsymbol{L}_2 + \boldsymbol{L}_3 + \ldots, \quad \boldsymbol{S} = \boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{S}_3 + \ldots \tag{1}$$

The total angular momentum $\boldsymbol{J}$ is defined by

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}. \tag{2}$$

The total magnetic moment $\boldsymbol{\mu}$ is given by

$$\boldsymbol{\mu} = -\frac{\mu_B}{\hbar}(\boldsymbol{L} + 2\boldsymbol{S}). \tag{3}$$

The Landé g-factor is defined by

$$\boldsymbol{\mu}_J = -\frac{g_J \mu_B}{\hbar}\boldsymbol{J}, \tag{4}$$

where

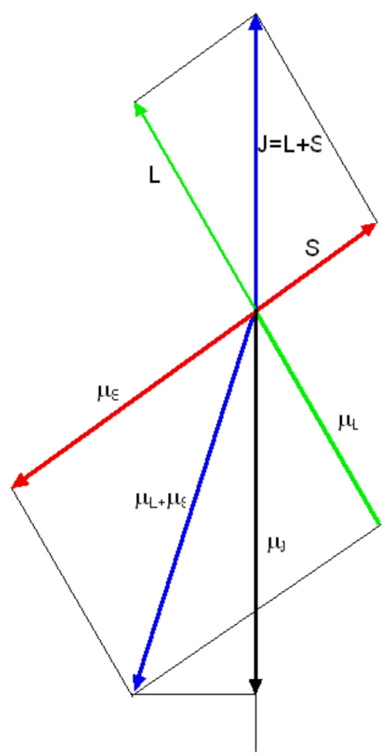


Fig. Basic classical vector model of orbital angular momentum (***L***), spin angular momentum (***S***), orbital magnetic moment (μ_L), and spin magnetic moment (μ_S). ***J*** (= ***L*** + ***S***) is the total angular momentum. μ_J is the component of the total magnetic moment ($\mu_L + \mu_S$) along the direction (-*J*).

Suppose that

$$\boldsymbol{L} = a\boldsymbol{J} + \boldsymbol{L}_\perp \quad \text{and} \quad \boldsymbol{S} = b\boldsymbol{J} + \boldsymbol{S}_\perp, \quad (5)$$

where a and b are constants, and the vectors $\boldsymbol{S}_\perp$ and $\boldsymbol{L}_\perp$ are perpendicular to $\boldsymbol{J}$.
Here we have the relation $a+b=1$, and $\boldsymbol{L}_\perp+\boldsymbol{S}_\perp=0$. The values of a and b are determined as follows.

$$a=\frac{\boldsymbol{J}\cdot\boldsymbol{L}}{\boldsymbol{J}^2},\ b=\frac{\boldsymbol{J}\cdot\boldsymbol{S}}{\boldsymbol{J}^2}. \tag{6}$$

Here we note that

$$\boldsymbol{J}\cdot\boldsymbol{S}=(\boldsymbol{L}+\boldsymbol{S})\cdot\boldsymbol{S}=\boldsymbol{S}^2+\boldsymbol{L}\cdot\boldsymbol{S}=\boldsymbol{S}^2+\frac{\boldsymbol{J}^2-\boldsymbol{L}^2-\boldsymbol{S}^2}{2}=\frac{\boldsymbol{J}^2-\boldsymbol{L}^2+\boldsymbol{S}^2}{2}, \tag{7}$$

or

$$\boldsymbol{J}\cdot\boldsymbol{S}=\frac{\boldsymbol{J}^2-\boldsymbol{L}^2+\boldsymbol{S}^2}{2}=\frac{\hbar^2}{2}[J(J+1)-L(L+1)+S(S+1)], \tag{8}$$

using the average in quantum mechanics. The total magnetic moment $\boldsymbol{\mu}$ is

$$\boldsymbol{\mu}=-\frac{\mu_B}{\hbar}(\boldsymbol{L}+2\boldsymbol{S})=-\frac{\mu_B}{\hbar}[(a+2b)\boldsymbol{J}+(L_\perp+2S_\perp)]. \tag{9}$$

Thus we have

$$\boldsymbol{\mu}_J=-\frac{\mu_B}{\hbar}(a+2b)\boldsymbol{J}=-\frac{\mu_B}{\hbar}(1+b)\boldsymbol{J}=-\frac{g_J\mu_B}{\hbar}\boldsymbol{J}, \tag{10}$$

with

$$g_J=1+b=1+\frac{\boldsymbol{J}\cdot\boldsymbol{S}}{\boldsymbol{J}^2}=\frac{3}{2}+\frac{S(S+1)-L(L+1)}{2J(J+1)}. \tag{11}$$

6.7 Hund's rule

6.7.1. Electron states in the atom

For a given l, the number m takes $2l+1$ values. The number s is restricted to only two values $\pm 1/2$. Hence there are altogether $2(2l+1)$ different states with the same n and l. There states are said to be equivalent.

According to Pauli's principle, there can be only one electron in each such state. Thus at most $2(2l+1)$ electrons in an atom can simultaneously have the same n and l.
Hund's rule is known concerning the relative position of levels with the same configuration but different L and S.

Hund'first law

(1) The maximum values of the total spin S allowed by the exclusion principle.

Hund's second law

(2) The maximum values of the total orbital angular momentum L consistent with this value of S.

Hund's third law

(i) $J = |L - S|$ for less than half full (spin-orbit interaction, the discussion will be made later)

(ii) $J = L + S$ for more than half full (spin-orbit interaction).

6.6.2. The electron configuration $(3d)^n$ (l = 2, n = 1 - 10)

A d shell corresponds to $l = 2$, with five values of m_l. Multiplying this by 2 for the spin states gives a total of 10. Then the configuration $(3d)^{10}$ represents a full shell. It is non-degenerate, and the state is $^1S_0$. This is a general rule for a full shell. It follows because each of electrons must have a different pair of m_l and m_s values.

$(3d)^1$: Ti^{3+}, V^{4+}

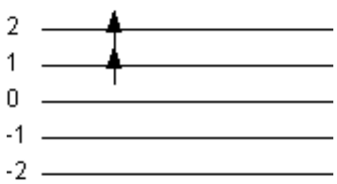
$^2D_{3/2}$ (ground state)

$L = 2$, $S = 1/2$, $J = 3/2$,

Fig.(a) Hund's law for the $(3d)^1$ electron configuration.

$(3d)^2$: V^{3+}

$^3F_2$

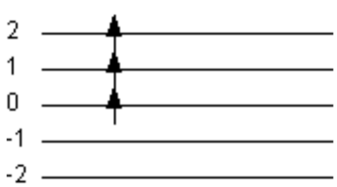


$L = 3$, $S = 1$, $J = 2$,

Fig.(b) Hund's law for the $(3d)^2$ electron configuration.

$(3d)^3$: Cr^{3+}, V^{2+}

$^4F_{3/2}$

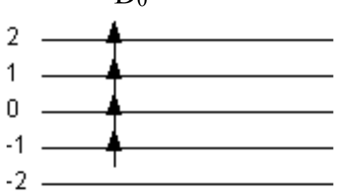


$L = 3$, $S = 3/2$, $J = 3/2$,

Fig.(c) Hund's law for the $(3d)^3$ electron configuration.

$(3d)^4$: Cr^{2+}, Mn^{3+}

$^5D_0$

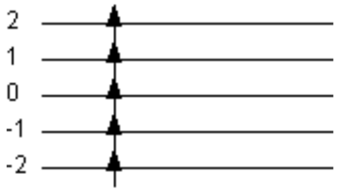


$L = 2$, $S = 2$, $J = 0$

Fig.(d) Hund's law for the $(3d)^4$ electron configuration.

$(3d)^5$: Fe^{3+}, Mn^{2+}

$^6S_{5/2}$

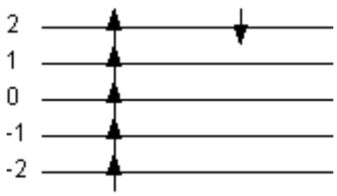


$L = 0, S = 5/2, J = 5/2$

Fig.(e) Hund's law for the $(3d)^5$ electron configuration.

$(3d)^6$: Fe^{2+}

$^5D_4$

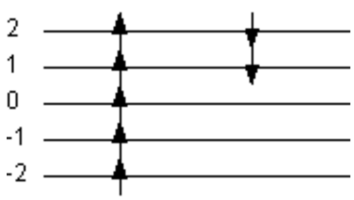


$L = 2, S = 2, J = 4$

Fig.(f) Hund's law for the $(3d)^6$ electron configuration.

$(3d)^7$: Co^{2+}

$^4F_{9/2}$

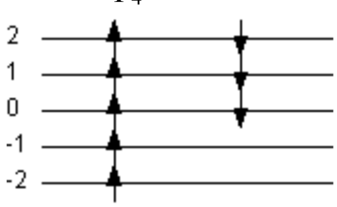


$L = 3, S = 3/2, J = 9/2$

Fig.1(g) Hund's law for the $(3d)^7$ electron configuration.

$(3d)^8$: Ni^{2+}

$^3F_4$

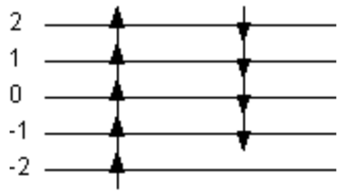


$L = 3, S = 1, J = 4$

Fig.(h) Hund's law for the $(3d)^8$ electron configuration.

$(3d)^9$: Cu^{2+}

$^2D_{5/2}$



$L = 2$, $S = 1/2$, $J = 5/2$

Fig.(i) Hund's law for the $(3d)^9$ electron configuration.

$(3d)^9$

This configuration represents a set of electrons one short of a full shell. Since a full shell has zero angular momentum (both orbital and spin), it follows that if one electron is removed from a full shell, the spin angular momentum of the remainder are minus those of the one that was removed. So the L, S, and J values of remainder are the same as if there were only one electron in the shell.

$(3d)^{10}$

A d shell corresponds to $l = 2$, with five values of m_l. Multiplying this by two for the spin states gives 10. Thus the configuration $(3d)^{10}$ represents a full shell. $L = 0$. $S = 0$. $J = 0$.

7. Definition of the magnetization

We define the magnetization $\boldsymbol{M}$ of a material as the net magnetic moment per unit volume. If there are n atoms per unit volume and their average magnetic moment is μ, then $\boldsymbol{M}$ can be written as

$$\boldsymbol{M} = n\boldsymbol{\mu}$$

The total magnetic moment of the system is μ_{tot}. The volume of the system is V. Then the magnetization of the system is defined by

$$\boldsymbol{M} = \frac{\boldsymbol{\mu}_{tot}}{V}$$

The unit of M is J/(T m^3) in SI units and emu/cm^3 in cgs units.

8. Paramagnetism: classical theory

We assume that a magnetic dipole moment μ of each molecule in the presence of a magnetic field $\boldsymbol{B}$. The potential energy is given by

$$U = -\boldsymbol{\mu}\cdot\boldsymbol{B} = -\mu B\cos\theta$$

where N is the number of magnetic dipole moments per unit volume and θ is the angle between $\boldsymbol{\mu}$ and $\boldsymbol{B}$. Then the magnetization M is given by

$$M = N\mu\langle\cos\theta\rangle$$

where

$$\langle \cos\theta \rangle = \frac{\int e^{-\frac{U}{k_B T}} \cos\theta d\Omega}{\int e^{-\frac{U}{k_B T}} d\Omega} = \frac{\int_0^{\pi} e^{\frac{\mu B \cos\theta}{k_B T}} \cos\theta (2\pi \sin\theta d\theta)}{\int_0^{\pi} e^{\frac{\mu B \cos\theta}{k_B T}} (2\pi \sin\theta d\theta)}$$

and k_B is the Boltzmann constant. For simplicity we put

$$x = \frac{\mu B}{k_B T} \qquad \text{and} \qquad s = \cos\theta .$$

Then we have

$$\langle \cos\theta \rangle = \frac{\int_{-1}^{1} e^{sx} s ds}{\int_{-1}^{1} e^{sx} ds} = \coth x - \frac{1}{x} = L(x)$$

where $L(x)$ is the Langevin function. For x<<1, the Langevin function is approximated as

$$L(x) = \frac{x}{3} - \frac{x^3}{45} + ... \approx \frac{x}{3}$$

and the derivative d$L(x)$/dx at $x = 0$ is equal to 1/3. Using this we have

$$M = N\mu L(x) = N\mu \frac{x}{3} = \frac{N\mu^2 B}{3k_B T}$$

9. Magnetization for the spin 1/2 system (quantum mechanics)

We discuss the magnetization for spin 1/2. The system consists of many spins (the number N per unit volume). There is no interaction between any two spins. The magnetic moment of spin (S = 1/2) is given by

$$\hat{\mu}_z = -2\mu_B \hat{S}_z / \hbar = -\mu \hat{\sigma}_z ,$$

where $\hat{\sigma}_z$ is a Pauli spin operator along the z axis. Then the Zeeman energy is described by

$$\hat{H} = -\hat{\mu}_z B = \mu \hat{\sigma}_z B , \qquad (1)$$

in the presence of a magnetic field B along the z axis.

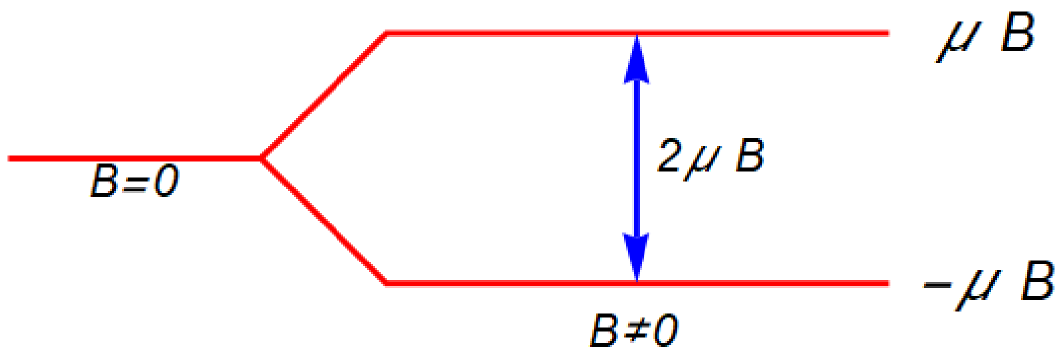


Fig. Zeeman splitting of the degenerate state under the application of magnetic field B.

The doublet state (degenerate) is split into two states:

1. lower energy level ($-\mu_B B$): state $\left|-\right\rangle$; μ_B (magnetic moment)
2. upper energy state ($\mu_B B$): state $\left|+\right\rangle$; - μ_B (magnetic moment)

The probability of finding the system in the lower state

$$\frac{N_1}{N}=\frac{e^x}{e^x+e^{-x}}$$

where $x = \mu/(k_B T)$. The probability of finding the system in the upper state is

$$\frac{N_2}{N}=\frac{e^{-x}}{e^x+e^{-x}}$$

The total magnetization M is

$$M=\mu(N_1-N_2)=N\mu\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)=N\mu\tanh x\,. \tag{2}$$

(a) For $x>>1$, $\tanh x \approx 1$,

$$M = M_{sat} = N\mu.$$

(b) For $x<<1$, $\tanh x \approx x$.

$$M = N\mu\left(\frac{\mu B}{k_B T}\right) = \frac{N\mu^2}{k_B T} B. \tag{3}$$

for $S = 1/2$.

((**Curie law**))

In 1895, Pierre Curie discovered this experimentally. The magnetization is directly proportional to the magnitude of the external magnetic field B and inversely proportional to the temperature T in K.

$$\frac{M}{B} = \frac{C}{T}$$

where C is called the Curie constant.

10. Ferromagnetism

10.1 Ferromagnetic order

A ferromagnet has a spontaneous magnetization – a magnetization even in zero applied magnetic field. The existence of a spontaneous magnetization suggests that the magnetic moments are arranged in a regular manner. Consider a paramagnet with ions having magnetic moments. Given an internal interaction tending to line up the magnetic moments parallel to each other, we shall have a ferromagnet. Let us postulate such an interaction and call it the exchange field. We treat the exchange field as equivalent to a magnetic field $\boldsymbol{B}_E$. The magnitude of the exchange field may be as high as 10^3 T. This field $\boldsymbol{B}_E$ is proportional to the magnetization $\boldsymbol{M}$, and is described by

$$B_E = \mu_0 \lambda M \quad \text{(mean field)}$$

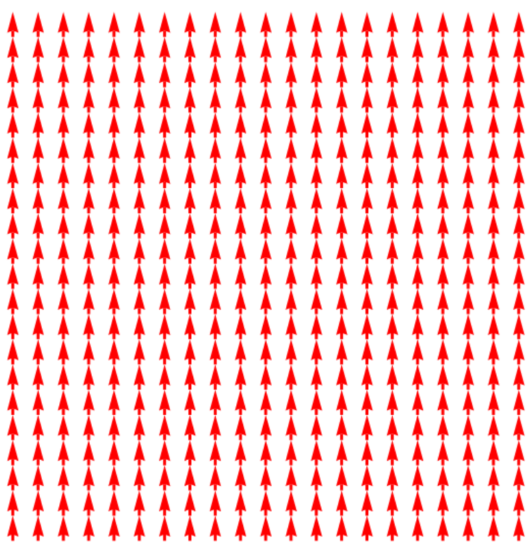
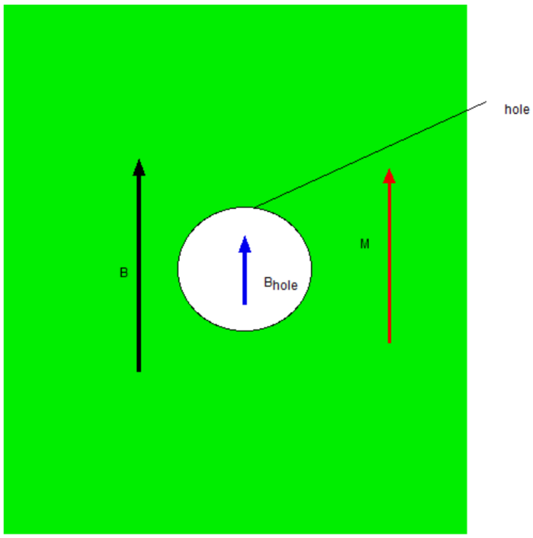


Fig. Ferromagnetic spin order

10.2 Curie temperature



((**Note**)) See Chapter 29S for the detail

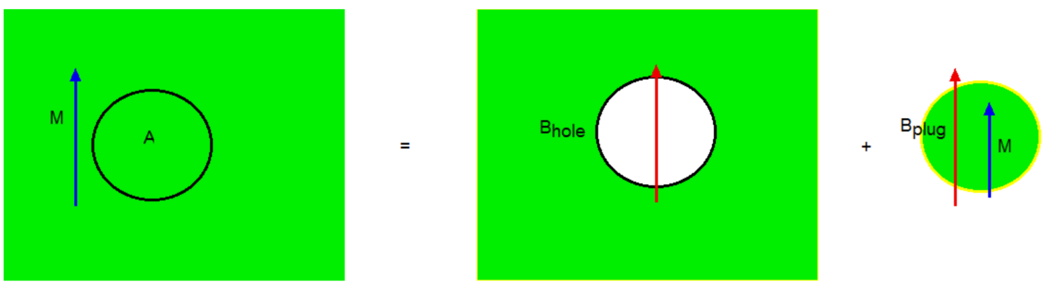
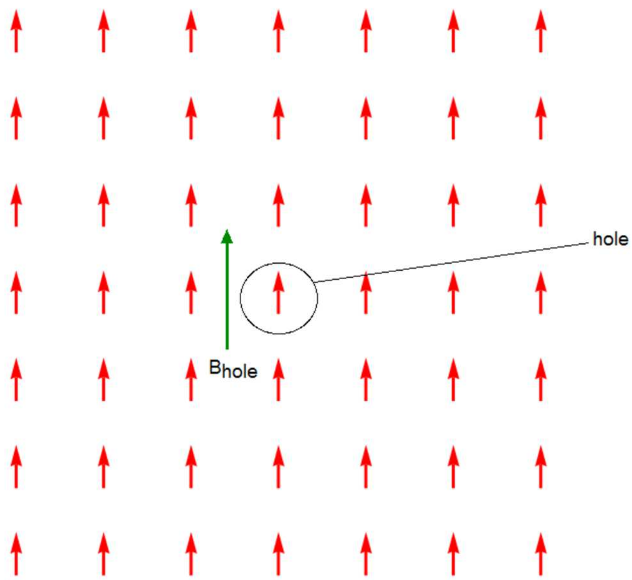


Fig. The field at any point A in a magnetic system can be considered as the sum of the field in a spherical hole plus the field due to a spherical plug.

For the spherical hole, the magnetic field is given by

$$\boldsymbol{B}_{plug} = \frac{2}{3}\mu_0 \boldsymbol{M}$$

$$\begin{aligned}\boldsymbol{B}_{hole} &= \boldsymbol{B} - \frac{2}{3}\mu_0 \boldsymbol{M} \\ &= \mu_0(\boldsymbol{H} + \boldsymbol{M}) - \frac{2}{3}\mu_0 \boldsymbol{M} \\ &= \mu_0(\boldsymbol{H} + \frac{1}{3}\boldsymbol{M})\end{aligned}$$



In Chapter 29, we show that the magnetic field in the hole (sphere) is given by

$$\boxed{B_{hole} = B_a + \lambda \mu_0 M}$$

where B_a ($= \mu_0 H$) is the external magnetic field, and λ is dependent on the shape of the hole and $\lambda = 1/3$ for sphere. For a spin $S = 1/2$, it is known that the magnetization M is given by

$$\frac{M}{M_s} = y = \tanh(x) \qquad \text{(a)}$$

where M_s ($= N\mu$) is the saturation magnetization. For x<<1, tanh(x) is approximated as

$$\tanh(x) = x - \frac{x^3}{3}.$$

The variable x is expressed by

$$\begin{aligned} x &= \frac{\mu B_{hole}}{k_B T} = \frac{\mu}{k_B T}(B_a + \lambda \mu_0 M) \\ &= \frac{\mu B_a}{k_B T} + \frac{\mu}{k_B T}\lambda \mu_0 \frac{M}{M_{sat}} M_{sat} \\ &= \frac{\mu B_a}{k_B T} + \frac{\lambda \mu_0 N \mu^2}{k_B T} y \end{aligned}$$

where M_{sat} ($= N\mu$) is the saturation magnetization. This equation can be rewritten as

$$y = \frac{k_B T}{\lambda \mu_0 N \mu^2}(x - \frac{\mu B_a}{k_B T}). \qquad \text{(b)}$$

For any given value of B_a, this is a straight-line relationship between y and x. The x intercept is at $x = \frac{\mu B_a}{k_B T}$ and the slope is $\frac{k_B T}{\lambda \mu_0 N \mu^2}$. For any particular B_a, we would have a line like the one denoted by (b). The intersection of curves (a) and (b) gives us the solution for M/M_{sat}.

We now look at how the solutions will go for various circumstances. We assume that $B_a = 0$.

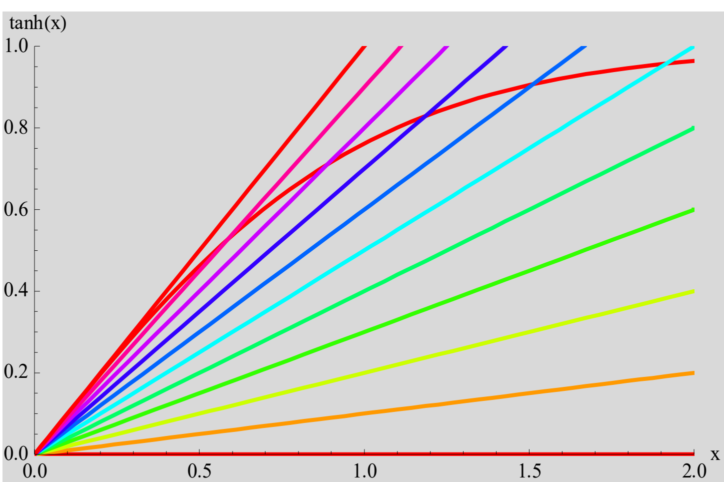
$$y = \tanh(x)$$

$$y = \frac{T}{T_c} x$$

where the characteristic temperature T_c is called a Curie temperature, and is given by

$$T_c = \frac{\lambda \mu_0 N \mu^2}{k_B}$$

The slope of the line (b) is proportional to T. When the slope is larger than 1, there is no solution for finite value of M. For $T>T_c$, we have a solution $M/M_{sat} = 0$ (paramagnetic phase). On the other hand, for $T<T_c$, we have a solution for the finite value of M/M_{sat} (ferromagnetic phase). Then the magnetic material should magnetize itself spontaneously below T_c.



10.3 Spontaneous magnetization M

When $B_a = 0$. y is given by

$$y = \frac{T}{T_c} x = tx\,,$$

with $t = T/T_c$. The reduced temperature (t) dependence of the spontaneous magnetization is a solution of $y = \frac{T}{T_c} x$ and $y = \tanh(x)$.

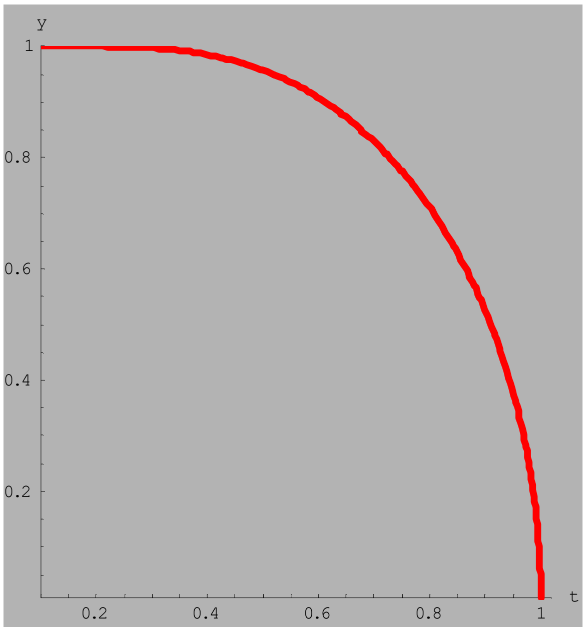
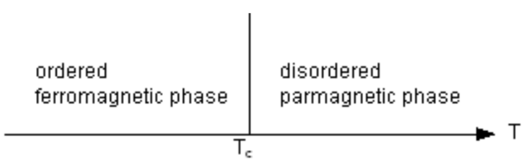


Fig. Spontaneous magnetization y as a function of t. The value of y is obtained from the equation $y = \tanh(y/t)$. $t = T/T_c$. $y = 1$ at $t = 0$.



10.4. Magnetic susceptibility of ferromagnetism

We now consider the magnetic susceptibility. The straight line is described by

$$y = \frac{T}{T_c}\left(x - \frac{\mu B_a}{k_B T}\right)$$

For $x \ll 1$, $y = x$. Then we have

$$\frac{T}{T_c}(x - \frac{\mu B_a}{k_B T}) = x$$

$$x(\frac{T}{T_c} - 1) = \frac{\mu B_a}{k_B T_c}$$

$$x = \frac{\frac{\mu B_a}{k_B}}{T - T_c}$$

Since $y = x$, we have

$$\frac{M}{B_a} = \chi = \frac{N\mu^2}{k_B}\frac{1}{T - T_c}$$

The susceptibility diverges as T approaches T_c from the high temperature side.

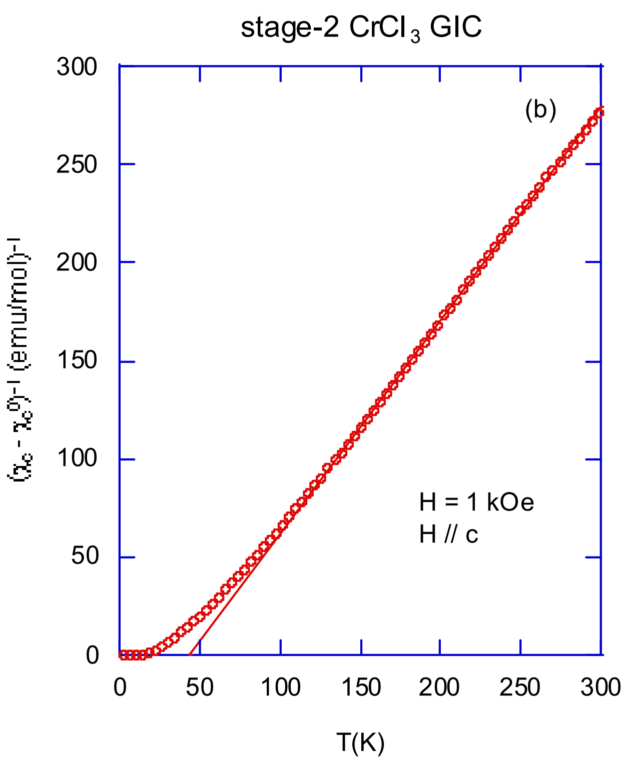
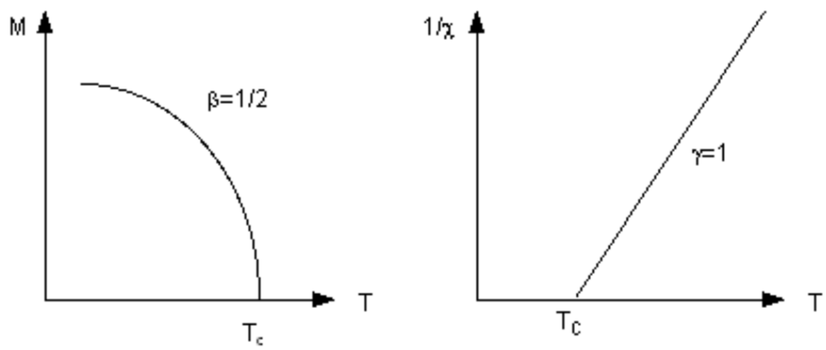


Fig. The reciprocal susceptibility as a function of T for stage-2 $CrCl_3$ GIC. $B//c$ (parallel to the c axis). B = 1 kOe. The straight line denotes the Curie-Weiss law.



11. A brief history on the modern magnetism

Magnetism is inseparable from quantum mechanics. A classical system in thermal equilibrium can display no magnetic moment, even in a magnetic field (Kittel). The magnetism is essentially the quantum phenomenon and is a property, reflecting the feature of quantum mechanics. In his talk titled the quantum mechanics, key to understanding magnetism (the Nobel lecture, December 8, 1977), Van Vleck pointed out that modern theories of magnetism have roots in two distinct traditions of theoretical developments. The first outstanding early attempt to understand magnetism at the atomic level was provided by the semi-empirical theories of Langevin and Weiss. These theories were able to explain experimental results on the magnetic properties of materials. Langevin assumed that an atomic or molecular magnet carries a permanent magnetic moment. He was quantizing the system without realizing it. If one applies classical dynamics and statistical mechanics consistently, one finds that the diamagnetic and paramagnetic contributions to the magnetic susceptibility exactly cancel. Thus there should be no magnetism. The break-through in understanding of magnetic phenomena at the atomic level occurred in 1913, when Niels Bohr introduced the significant concept of the quantization of the orbital angular momentum, as a part of his remarkable theory of the hydrogen spectrum. The quantization of electron orbits implied the existence of an elementary magnetic moment, the Bohr magneton. In 1922, Stern and Gerlach experimentally verified the quantized orbital angular momentum and hence the orbital magnetic moment.

The advent of quantum mechanics in 1926 furnished at last the key to the quantitative understanding of magnetism, (i) the discovery of the matrix form of quantum mechanics by Heisenberg and Born, (ii) the alternative but equivalent wave mechanical form by de Broglie and Schrödinger, and (iii) the introduction of electron spin by Uhlenbeck and Goulsmit. A quantum mechanics without spin and the Pauli's exclusion principle would not have been able to understand even the structure of the periodic table or most magnetic phenomena. Originally spin was a sort of the appendage to the mathematical framework, but in 1928, Dirac synthesized everything in his remarkable four first order simultaneous equations which is relativistically invariant under the Lorentz transformation. The electron spin and the factor of two came naturally out of the calculation. In 1928, Heisenberg has shown how the previously obscure Weiss molecular field could be attributed to a quantum mechanical exchange effect, arising from the Pauli's exclusion principle that no two electrons occupy the same state. The forces of interaction between neighboring atoms give rise to an exchange coupling between unpaired spinning electron. This leads to a scalar isotropic interaction of two spins with an exchange interaction constant.

12. Magnetization of magnetic systems (summary)

In paramagnetic and diamagnetic materials, the magnetization is maintained by the field. When the field B is removed, the magnetization M disappears. In Fact, for most substances, the magnetization is proportional to the field, provided the field is not too great. For notational consistency with the electrical case, one should express the proportionality thus:

$$\boldsymbol{M} = \frac{1}{\mu_0}\chi_m \boldsymbol{B} \qquad (1)$$

But custom dictates that it be written in terms of H instead of B

$$\boldsymbol{M} = \chi_m \boldsymbol{H} \qquad (2)$$

where χ_m is called the magnetic susceptibility. Here we use the notation (1) instead of (2), since the expression of $\boldsymbol{H}$ is not used in the standard textbook of general physics. Anyway, we are interested in the magnetization as a function of external magnetic field and temperature

Experimentally, one can measure the magnetization of samples using the SQUID (superconducting quantum interference device). Here we discuss the magnetization of three kinds of systems, paramagnetism, diamagnetism, and ferromagnet. The magnetization is the magnetic moment per unit volume. The magnetization M for the paramagnetic and diamagnetic systems can be expressed by

$$\mu_0 M = \mu_0 M(B) = \chi_m B$$

where the proportionality constant χ_m is dimensionless number called the magnetic susceptibility. B is the internal magnetic field. For diamagnetic materials, χ_m is a small negative constant independent of temperature. For paramagnetic materials, χ_m is positive and can be expressed by the Curie law.

$$\mu_0 M = \chi B_0 = \frac{C}{T} B_0 ,$$

where T is the temperature and C is the Curie constant and B is equal to the external magnetic field B_0. The Curie law arises from

$$\mu_0 M = n\mu \tanh(\frac{\mu B_0}{k_B T}) ,$$

for the two-level energy states of spin up and spin down, where n is the number density of magnetic atoms with magnetic moment μ. In the high temperature range, we get

$$\tanh(\frac{\mu B_0}{k_B T}) \approx \frac{\mu B_0}{k_B T},$$

leading to

$$\mu_0 M = \frac{n\mu^2 B_0}{k_B T} = \frac{C}{T} B_0 . \qquad \text{(Curie law)}$$

with the Curie constant

$$C = \frac{n\mu^2}{k_B}.$$

For the ferromagnet, the internal magnetic field B can be expressed by

$$B = B_0 + \lambda M(B),$$

where λ is constant and the second term is the mean field arising from the interaction with the neighboring magnetic moments. We note that the magnetization M is a function of B,

$$\mu_0 M = \mu_0 M(B) \approx \frac{C}{T} B = \frac{C}{T}(B_0 + \lambda M),$$

or

$$\mu_0 M = \frac{C}{T - \frac{C\lambda}{\mu_0}} B_0 = \frac{C}{T - T_c} B_0 .$$

Note that T_c is the Curie temperature and is defined by

$$T_c = \frac{C\lambda}{\mu_0} = \frac{n\mu^2 \lambda}{k_B \mu_0}.$$

For $T < T_c$, the system is in the ferromagnetic state where all the direction of spins. Suppose that there is no external magnetic field. it is expected that the spontaneous magnetization appears below T_c,

$$\mu_0 M = n\mu \tanh(\frac{\mu\lambda M}{k_B T}),$$

or

$$\frac{\mu_0 M}{n\mu} = \tanh(\frac{\mu_0 M}{n\mu}\frac{T_c}{T}).$$

We use the parameters y and x as

$$y = \frac{\mu_0 M}{n\mu}, \qquad x = \frac{T}{T_c}.$$

the above equation can be rewritten as

$$y = \tanh(\frac{y}{x}).$$

We make a plot of y vs x using the Mathematica.

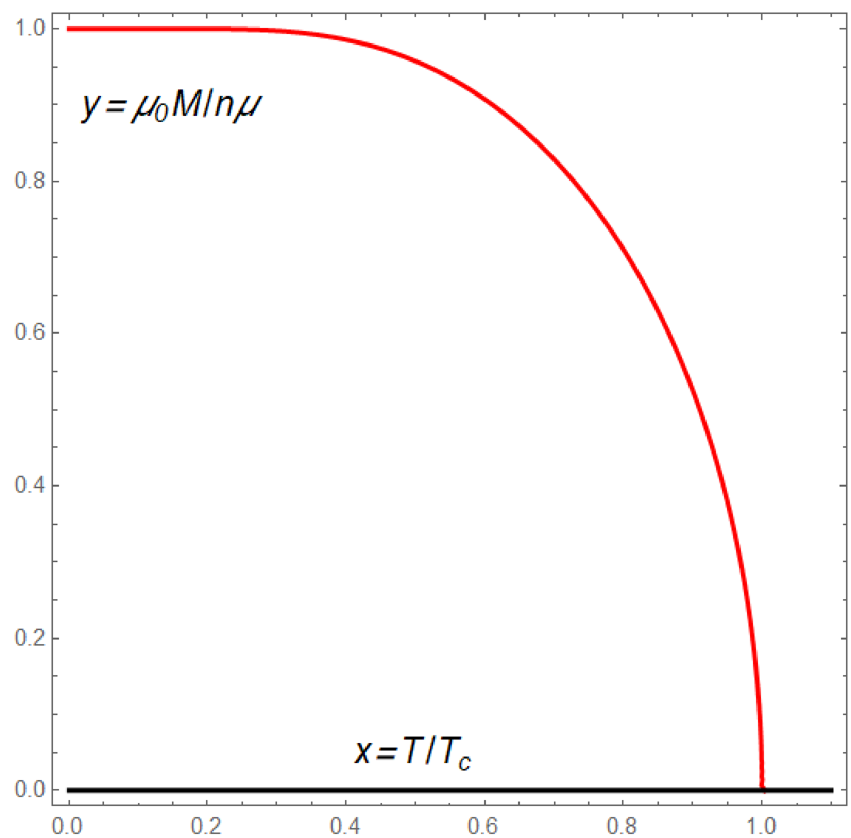


Fig. Spontaneous magnetization appears below the Curie temperature.

REFERENCES
D.J. Griffiths, Introduction to Electrodynamics, 2nd edition (Prentice Hall, 1981).

The magnetic susceptibility of typical materials

Material	χ_m
Aluminum	2.3×10^{-5}
Bismuth	-1.66×10^{-5}
Copper	-0.98×10^{-5}
Diamond	-2.2×10^{-5}
Gold	-3.6×10^{-5}
Magnesium	1.2×10^{-5}
Mercury	-3.2×10^{-5}
Silver	-2.6×10^{-5}
Sodium	-0.24×10^{-5}
Titanium	7.06×10^{-5}
Tungsten	6.8×10^{-5}
Hydrogen (1 atm)	-9.9×10^{-9}
Carbon dioxide (1 atm)	-2.3×10^{-9}
Nitrogen (1 atm)	-5.0×10^{-9}
Oxygen (1 atm)	2090×10^{-9}

((**Example**))

Graphite diamagnetism
levitation experiments
https://www.youtube.com/watch?v=rjBczjGQsdc

Bismuth diamagnetism
https://www.youtube.com/watch?v=A5pZZJ23rDM

Oxygen paramagnetism
https://www.youtube.com/watch?v=Lt4P6ctf06Q

13. Typical problems

13.1 Problem 32-3

A Gaussian surface in the shape of a right circular cylinder with end caps has a radius of 12.0 cm and a length of 80.00 cm. Through one end there is an inward magnetic flux of 25.0 μWb. At the other end there is a uniform magnetic field of 1.60 mT, normal to the surface and directed outward. What are the (a) magnitude and (b) direction (inward or outward) of the net magnetic flux through the curved surface?

((**Solution**))

Gauss' law for ***B***

$$\oint(\nabla\cdot\boldsymbol{B})d\tau=\oint\boldsymbol{B}\cdot d\boldsymbol{a}=\Phi_{total}=0$$

B = 1.60 mT (top surface)
r = 0.12 m
Φ_{bottom} = -25.0 mWb (bottom surface, the magnetic flux going inward)
L = 0.8 m

The total magnetic flux passing through a closed surface should be zero.

$$\Phi_{total}=\Phi_{top}+\Phi_{bottom}+\Phi_{side}=0$$

where

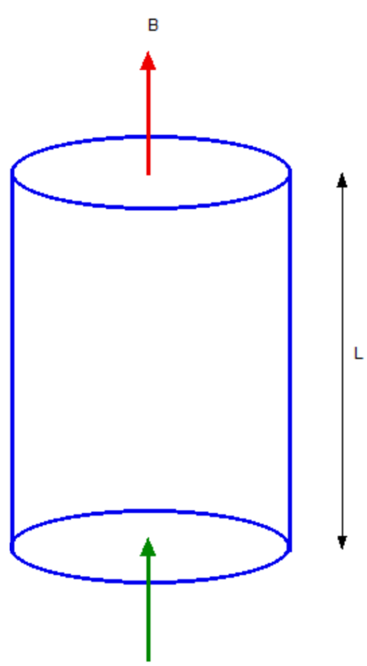
$$\Phi_{top}=B(\pi R^2)$$
$$\Phi_{bottomr}=-25.0\mu Wb$$

$$\Phi_{total}=B(\pi R^2)-25.0\mu Wb+\Phi_{side}=0$$

or

$$\Phi_{side}=-B(\pi R^2)+25.0\mu Wb=-4.74\times10^{-5}Wb$$

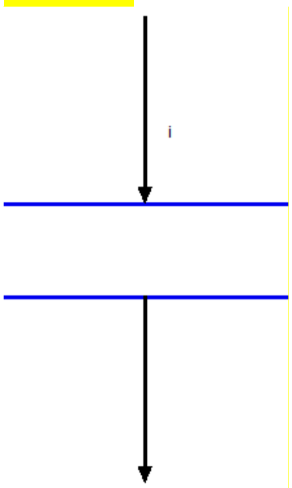
The magnetic flux passing the side surface goes inward.

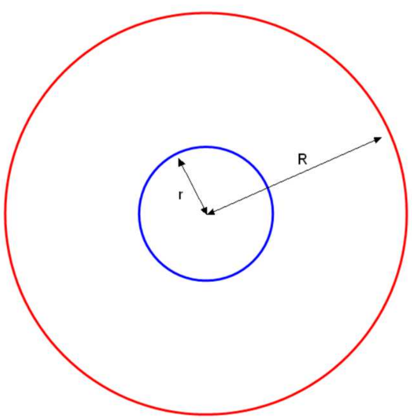


13.2 Problem 32-20

A capacitor with parallel circular plates of radius $R = 1.20$ cm is discharging via a current of 12.0 A. Consider a loop of radius $R/3$ that is centered on the central axis between the plates. (a) How much displacement current is encircled by the loop? The maximum induced magnetic field has a magnitude of 12.0 mT. At what radius (b) inside and (c) outside the capacitor gap is the magnitude of the induced magnetic field 3.00 mT?

((**Solution**))
$R = 1.20$ cm
$i = 12.0$ A





(a) $$i_d = i\frac{r^2}{R^2} = 1.333\,A$$

(b) and (c)

For $r<R$

$$\oint \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 i \frac{r^2}{R^2}$$

$$B(2\pi r) = \mu_0 i \frac{r^2}{R^2}$$

$$B = \frac{\mu_0 i}{2\pi r} \frac{r^2}{R^2} = \frac{\mu_0 i}{2\pi R^2} r$$

For $r>R$

$$\oint \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 i$$

$$B(2\pi r) = \mu_0 i$$

$$B = \frac{\mu_0 i}{2\pi r}$$

The maximum of B occurs at $r = R$.

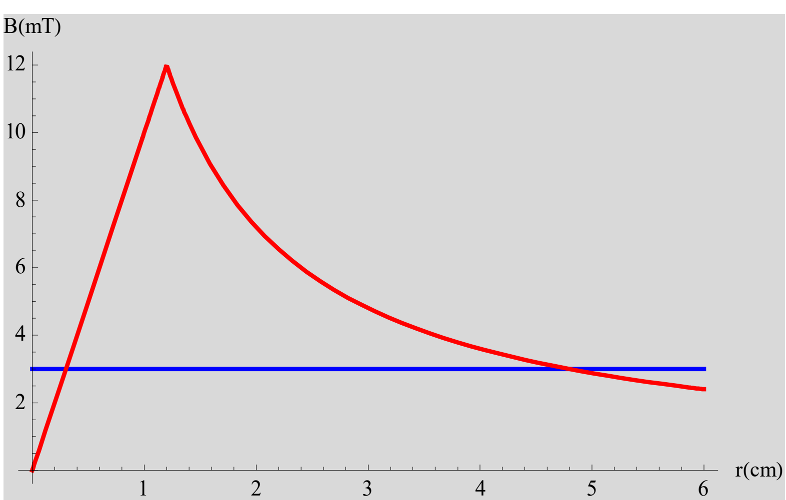
$$B_{\max} = \frac{\mu_0 i}{2\pi R}$$

Suppose that $B_{\max}$ = 12 mT. Then we have

$$B = B_{\max} \frac{r}{R} \qquad \text{for } r<R$$

$$B = B_{\max} \frac{R}{r} \qquad \text{for } r>R$$

where R = 1.20 cm. We make a plot of B (mT) as a function of r (cm).



When B = 3 mT,

r = 0.3 cm   or r = 4.8 cm

13.3 Problem 32-51

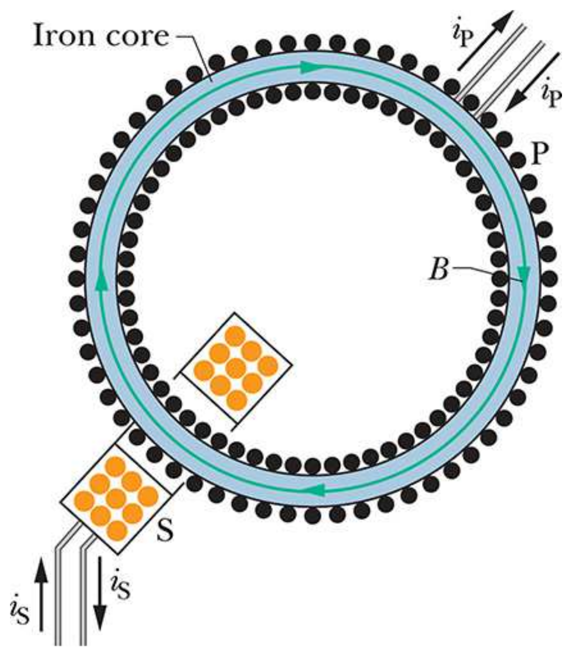
A Rowland ring is formed of ferromagnetic material. It is circular in cross section, with an inner radius of 5.0 cm and an outer radius of 6.0 cm, and is wound with 400 turns of wire, (a) what current must be set up in the windings to attain a toroidal field of magnitude B_0 = 0.2 mT? (b) A secondary coil wound around the toroid has 50 turns and resistance 8.0 Ω. If, for this value of B_0, we have B_M = 800 B_0, how much charge moves through the secondary coil when the current in the toroid windings is turned on?

((**My solution**))

Rowland's Ring

Rowland's Ring is a device for tracing out curves of magnetic induction as a function of magnetizing field for an iron ring; these curves are called *hystersis curves*. It was developed by Henry Augustus Rowland (1848-1901).

There was a lot of good physics in this experiment that I probably did last with my students in the late nineteen sixties. We did it with the aid of a ballistic galvanometer; today a modern integrating instrument would be used.



((**Solution**)) from the text book (Halliday and Resnick)

N_p = 400 turns, R_{avg} = 5.5 cm (in average) = 0.055 m.
Area of circular cross section; $A = \pi r^2$. r = 0.005 m.
N_s = 50. R_s = 8.0 Ω

The magnetization of a ferromagnetic material such as iron can be studied with an arrangement called a *Rowland ring*. The material is formed into a thin toroidal core of circular cross section. A primary coil P having n turns per unit length is wrapped around the core and carries current i_P. (The coil is essentially a long solenoid bent into a circle). If the iron core were not present, the magnitude of the magnetic field inside the coil would be,

$$B_0 = \mu_0 n_p i_p = \mu_0 i_p \frac{N_p}{2\pi R_{avg}}$$

$$i_p = B_0 \frac{2\pi R_{avg}}{\mu_0 N_p} = 0.1375 \text{ A}$$

However, with the iron core present, the magnetic field $\boldsymbol{B}$ inside the coil is greater than $\boldsymbol{B}_0$, usually by a large amount. We can write the magnitude of this field as

$$B = B_0 + B_M = 801 B_0$$

where B_M (= 800 B_0) is the magnitude of the magnetic field contributed by the iron core. This contribution results from the alignment of the atomic dipole moments within the iron, due to exchange coupling and to the applied magnetic field B_0, and is proportional to the magnetization M of the iron. That is, the contribution B_M is proportional to the magnetic dipole moment per unit volume of the iron.

To determine B_M we use a secondary coil S to measure B. The voltage generated across the secondary coil is given by

$$V_s = N_s \frac{d\Phi}{dt} = R_s i_s = R_s \frac{dq_s}{dt}$$

or

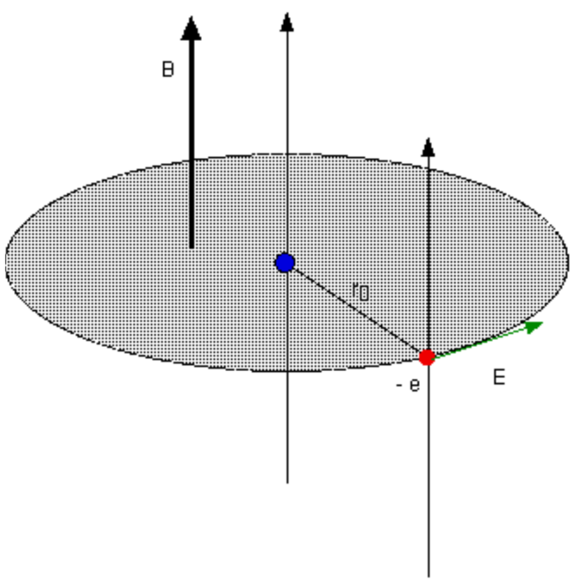
$$q_s = \frac{N_s}{R_s}\Phi = \frac{N_s}{R_s} A(B_0 + B_M) = 7.864 \text{ x } 10^{-5} \text{ C}$$

where

$$\Phi = A(B_0 + B_M) = 1.258 \text{ x } 10^{-5} \text{ Wb},$$
$$A = \pi r^2$$

Note: this method is very familiar to the experimentalists as a principle of the ballistic galvanometer.

APPENDIX-1 Core diamagnetism



We consider the diamagnetism from the classical point of view. Suppose that we slowly turn on the magnetic field. As the magnetic field changes an electric field is generated by magnetic induction, from the Faraday's law,

$$\int (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{a} = \oint \boldsymbol{E} \cdot d\boldsymbol{s} = -\int \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{a} = -\frac{\partial}{\partial t} \Phi_B$$

or

$$2\pi r E_\theta = -\frac{\partial}{\partial t} \Phi_B = -\pi r_0^{\ 2} \frac{dB}{dt}$$

or

$$E_\theta = -\frac{\pi r_0^{\ 2}}{2\pi r_0} \frac{dB}{dt} = -\frac{r_0}{2} \frac{dB}{dt}$$

The induced electric field acting on an electron in the atom produces a torque equal to

$$\boldsymbol{\tau} = \frac{d\boldsymbol{L}}{dt} = \boldsymbol{r} \times \boldsymbol{F} = (r_0 \hat{r}) \times (-e) E_\theta \hat{\theta} = -e r_0 E_\theta (\hat{r} \times \hat{\theta}) = \frac{e r_0^{\ 2}}{2} \frac{dB}{dt} \hat{z}$$

Integrating with respect to time t from zero field, we have

$$\Delta L_z = \frac{e r^2}{2} B$$

The magnetic moment $\Delta\mu$ is

$$\Delta\mu_z = -\frac{e}{2m} \Delta L_z = -\frac{e^2 B}{4m} r_0^{\ 2} = -\frac{e^2 B}{4m} [\langle x^2 \rangle + \langle y^2 \rangle]$$

The minus sign means that the added magnetic moment is opposite to the magnetic field. Here we note that r_0 is a radius from an axis through the atom parallel to $\boldsymbol{B}$. So if $\boldsymbol{B}$ is along the z axis, $r_0^2 = x^2 + y^2$. If we consider spherically symmetric atoms, the average of $x^2 + y^2$ is (2/3) of the average of the square of the true radial distance from the center point of the atom. Then we have

$$\Delta\mu_z = -\frac{e^2 B}{4m} \left(\frac{2}{3} \langle r^2 \rangle_{av}\right) = -\frac{e^2 B}{6m} \langle r^2 \rangle_{av}$$

When N_A is the Avogadro number and each atom has z electrons, the magnetic susceptibility is given by

$$\chi = \frac{N_A z (\Delta\mu_z)}{B} = -\frac{N_A z e^2}{6m} \langle r^2 \rangle_{avg} = -2.82834 \times 10^{15} z \langle r^2 \rangle_{avg} \ \text{J/T}^2$$

where r is in the units of m. Note that the unit of χ in the CGS unit is emu/Oe = (emu/Oe^2). The unit of χ in the SI units is J/T^2 = (1/10) emu/Oe.

The sign of χ is negative, which leads to the diamagnetism of matter. The graphite and bismuth has a large diamagnetic susceptibility.

((**Mathematica**))

```
NA qe^2
------- /. Physconst
 6 me

2.82834 × 10^15
```

((**Note**))
((Another method))

$$E_\theta = -\frac{\pi r_0^{\ 2}}{2\pi r_0}\frac{dB}{dt} = -\frac{r_0}{2}\frac{dB}{dt}$$

The force F_θ is related to E_θ as

$$F_\theta = m\frac{dv_\theta}{dt} = -eE_\theta = -e(-\frac{r_0}{2}\frac{dB}{dt}) = \frac{er_0}{2}\frac{dB}{dt}$$

$$\frac{dv_\theta}{dt} = \frac{er_0}{2m}\frac{dB}{dt}$$

or

$$\Delta v_\theta = \frac{er_0}{2m}\Delta B$$

Since

$$\Delta \boldsymbol{\mu}_L = -\frac{er_0}{2}\Delta v_\theta \hat{z} = -\frac{er_0}{2}\left(\frac{er_0}{2m}\Delta B\right)\hat{z} = -\left(\frac{er_0^{\ 2}}{4m}\Delta B\right)\hat{z}$$

APPENDIX-2 Maxwell's equation in the matter

2.1 Maxwell's equations

The equations governing electromagnetic phenomena are the Maxwell's equations, Maxwell's equation (in general)

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \boldsymbol{B} = \mu_0 (\boldsymbol{J} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t})$$

$\boldsymbol{B}$:	magnetic field
$\boldsymbol{E}$:	electric field
$\boldsymbol{H}$:	(here we call $\boldsymbol{H}$ field)
$\boldsymbol{D}$:	electric displacement vector
ε_0	permittivity of free space
μ_0	permeability of free space
ρ:	charge density
$\boldsymbol{J}$:	current density

$$\boldsymbol{B} = \mu_0 (\boldsymbol{M} + \boldsymbol{H}),$$

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$$

$$\nabla \cdot \boldsymbol{D} = \rho_f$$

$$\nabla \times \boldsymbol{H} = \frac{\partial}{\partial t} \boldsymbol{D} + \boldsymbol{J}_f$$

where

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

(a) Electric charge

$$\rho = \rho_f + \rho_b = \rho_f - \nabla \cdot \boldsymbol{P}$$

where ρ_f is the free charge density.

(b) Current density

$$\boldsymbol{J} = \boldsymbol{J}_f + \boldsymbol{J}_b + \boldsymbol{J}_M = \boldsymbol{J}_f + \nabla \times \boldsymbol{M} + \frac{\partial \boldsymbol{P}}{\partial t}$$

$\boldsymbol{P}$: electric polarization vector,
$\boldsymbol{J}_{\mathrm{P}}$: polarization current density
$\boldsymbol{M}$: magnetization vector
$\boldsymbol{J}_{\mathrm{M}}$: magnetization current density

$$\rho_b = -\nabla \cdot \boldsymbol{P}$$

$$\sigma_b = \boldsymbol{P} \cdot \boldsymbol{n}$$

$$\nabla \cdot \boldsymbol{J}_p = -\frac{\partial \rho_b}{\partial t}$$

$$\boldsymbol{J}_p = \frac{\partial \boldsymbol{P}}{\partial t}$$

$$\boldsymbol{J}_M = \nabla \times \boldsymbol{M}$$

2.2 Gauss; law, Ampere's law

(a) Gauss's law

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} = \frac{1}{\varepsilon_0}(\rho_f - \nabla \cdot \boldsymbol{P})$$

or

$$\nabla \cdot (\varepsilon_0 \boldsymbol{E} + \boldsymbol{P}) = \rho_f,$$

or

$$\nabla \cdot \boldsymbol{D} = \rho_f$$

where

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} \qquad \text{(electric displacement)}$$

(b) Ampere's law

$$\nabla \times \boldsymbol{B} = \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} + \mu_0 (\boldsymbol{J}_f + \nabla \times \boldsymbol{M} + \frac{\partial \boldsymbol{P}}{\partial t})$$

or

$$\nabla \times (\boldsymbol{B} - \mu_0 \boldsymbol{M}) = \mu_0 [\frac{\partial}{\partial t}(\varepsilon_0 \boldsymbol{E} + \boldsymbol{P}) + \boldsymbol{J}_f]$$

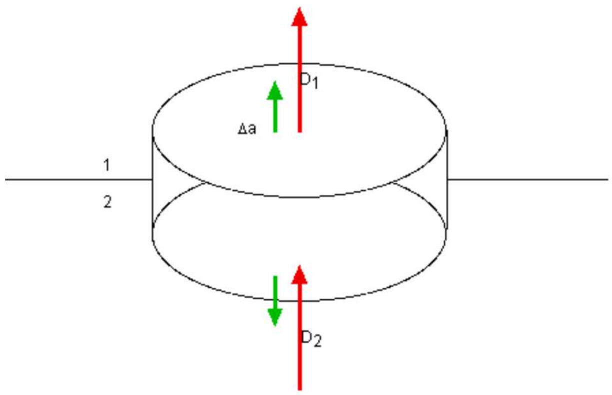
Here we define

$$\boldsymbol{B} = \mu_0(\boldsymbol{M} + \boldsymbol{H}), \qquad \text{and} \qquad \boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$$

Then we have

$$\nabla \times \boldsymbol{H} = \frac{\partial}{\partial t}\boldsymbol{D} + \boldsymbol{J}_f$$

2.3 Boundary conditions



(a) $\nabla \cdot \boldsymbol{D} = \rho_f$

$$(\boldsymbol{D}_1 - \boldsymbol{D}_2) \cdot \boldsymbol{n}\Delta a = \sigma_f \Delta a \qquad \text{or} \qquad (\boldsymbol{D}_1 - \boldsymbol{D}_2) \cdot \boldsymbol{n} = \sigma_f$$

where $\boldsymbol{n}$ is the unit vector perpendicular to the boundary surface (normal component).

$$D_1^{\perp} - D_2^{\perp} = \sigma_f \qquad \text{(normal component)}$$

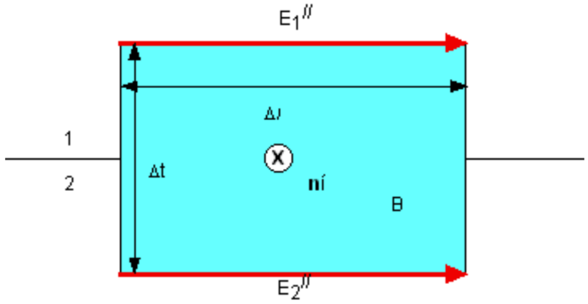
(b) $\nabla \cdot \boldsymbol{B} = 0$

$$(\boldsymbol{B}_1 - \boldsymbol{B}_2) \cdot \boldsymbol{n}\Delta a = 0 \qquad \text{or} \qquad (\boldsymbol{B}_1 - \boldsymbol{B}_2) \cdot \boldsymbol{n} = 0$$

where $\boldsymbol{n}$ is the unit vector perpendicular to the boundary surface (normal component).

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad \text{(normal component).}$$

(c) $\nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}$



$$\oint(\nabla\times\boldsymbol{E})\cdot d\boldsymbol{a}=\oint\boldsymbol{E}\cdot d\boldsymbol{l}=-\oint\frac{\partial\boldsymbol{B}}{\partial t}\cdot d\boldsymbol{a}=-\frac{\partial}{\partial t}\oint\boldsymbol{B}\cdot d\boldsymbol{a}\qquad\text{(Stokes theorem)}$$

$$(E_1^{//}-E_2^{//})\Delta l=-\left(\frac{\partial}{\partial t}\boldsymbol{B}\cdot\boldsymbol{n}'\right)\Delta t\Delta l$$

or

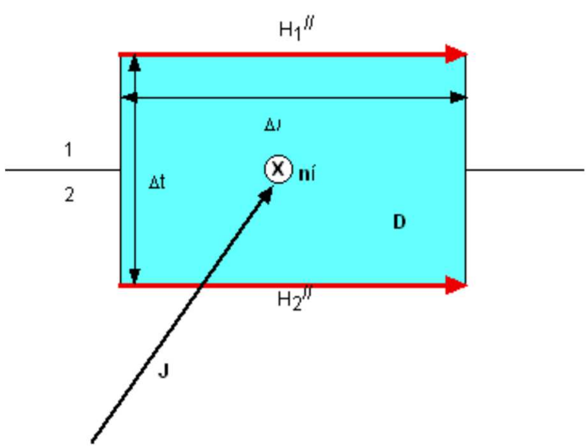
$$(E_1^{//}-E_2^{//})=-\left(\frac{\partial}{\partial t}\boldsymbol{B}\cdot\boldsymbol{n}'\right)\Delta t\to 0,\quad\text{when }\ \Delta t\to 0$$

or

$$E_1^{//}=E_2^{//}\qquad\text{(tangential component).}$$

(d) $\nabla\times\boldsymbol{H}=\frac{\partial}{\partial t}\boldsymbol{D}+\boldsymbol{J}_f$

There are two cases depending on the relation of directions of $\boldsymbol{J}$ and d$\boldsymbol{a}$.



$$\oint(\nabla\times\boldsymbol{H})\cdot d\boldsymbol{a}=\oint\boldsymbol{H}\cdot d\boldsymbol{l}=\oint\frac{\partial\boldsymbol{D}}{\partial t}\cdot d\boldsymbol{a}+\oint\boldsymbol{J}_f\cdot d\boldsymbol{a}$$

$$[H_1^{//}(\perp \boldsymbol{J}_f) - H_2^{//}(\perp \boldsymbol{J}_f)]\Delta l = \left(\frac{\partial}{\partial t}\boldsymbol{D}\cdot\boldsymbol{n}'\right)\Delta t\Delta l + J_f\Delta t\Delta l$$

or

$$K_f = J_f\Delta t$$

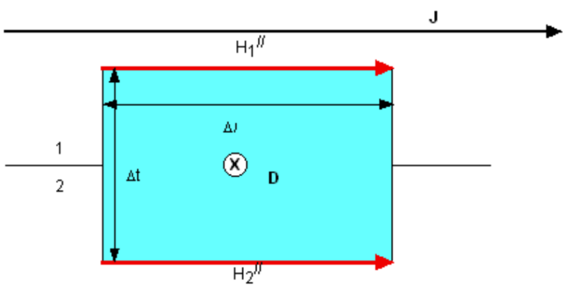
$$[H_1^{//}(\perp \boldsymbol{K}_f) - H_2^{//}(\perp \boldsymbol{K}_f)] = \left(\frac{\partial}{\partial t}\boldsymbol{D}\cdot\boldsymbol{n}'\right)\Delta t + K_f$$

where K_f is the line current density.

In the limit of $\Delta t \to 0$, we have

$$[H_1^{//}(\perp \boldsymbol{K}_f) - H_2^{//}(\perp \boldsymbol{K}_f)] = K_f$$

Similarly, we have



$$[H_1^{//}(// \boldsymbol{K}_f) - H_2^{//}(// \boldsymbol{K}_f)] = 0.$$

Thus we have the following boundary conditions.

$D_1^{\perp} - D_2^{\perp} = \sigma_f$	or	$\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_f$
$B_1^{\perp} - B_2^{\perp} = 0$		$B_1^{\perp} - B_2^{\perp} = 0$
$\boldsymbol{E}_1^{//} - \boldsymbol{E}_2^{//} = 0$		$\boldsymbol{E}_1^{//} - \boldsymbol{E}_2^{//} = 0$
$\boldsymbol{H}_1^{//} - \boldsymbol{H}_2^{//} = \boldsymbol{K}_f \times \boldsymbol{n}$		$\frac{1}{\mu_1}\boldsymbol{B}_1^{//} - \frac{1}{\mu_2}\boldsymbol{B}_2^{//} = \boldsymbol{K}_f \times \boldsymbol{n}$

APPENDIX-3 Definition of magnetic susceptibility in the standard textbook

The ***B***-field in the magnetic material can be expressed by

$$B = B_0 + \mu_0 M = (1 + \chi_M) B_0 = \kappa_M B_0$$

with

$$M = \chi_M \frac{B_0}{\mu_0}$$

where χ_M is the magnetic susceptibility and the ***B*** field (B_0) and ***M*** are in the same direction. In ferromagnetic materials, $\mu_0 M$ is often greater than the external applied field B_0 by a factor of several thousand.

Note that

$\chi_M = -1$	(complete diamagnetism such as superconductivity)
$\chi_M < 0$	(diamagnetism)
$\chi_M > 0$	($\chi_M \approx 0$, paramagnetism)
$\chi_M >> 1$	(ferromagnetism)

((**Note**))
Liquid oxygen (diamagnetism, 77 Ks). Liquid oxygen (paramagnetism, 90.19 K).

APPENDIX-4 Bohr model of hydrogen atom

Here we consider the Bohr model of hydrogen atom. The Newton's second law leads to

$$m\frac{v^2}{r} = \frac{e^2}{4\pi\varepsilon_0 r^2}, \qquad mv^2 r = \frac{e^2}{4\pi\varepsilon_0{}^2}$$

The quantization of orbital angular momentum:

$$L_z = mvr = n\hbar \qquad \text{(n: integer).}$$

From two equations, we have the velocity and radius.

$$v_n = \frac{e^2}{4\pi\varepsilon_0 n\hbar}, \qquad v_1 = \frac{e^2}{4\pi\varepsilon_0 \hbar} = 2.18769 \text{ x } 10^6 \text{ m/s}$$

$$r_n = \frac{n\hbar}{mv_n} = \frac{n\hbar}{\dfrac{me^2}{4\pi\varepsilon_0 n\hbar}} = \frac{4\pi\varepsilon_0\hbar^2}{me^2}n^2 \qquad r_1 = \frac{4\pi\varepsilon_0\hbar^2}{me^2} = 5.29177 \text{ x } 10^{-11} \text{ m}$$

which is the Bohr radius of the hydrogen atom. The period is

$$T_1 = \frac{2\pi r}{v_1} = 1.51983 \text{ x } 10^{-16} \text{ s}$$

The current:

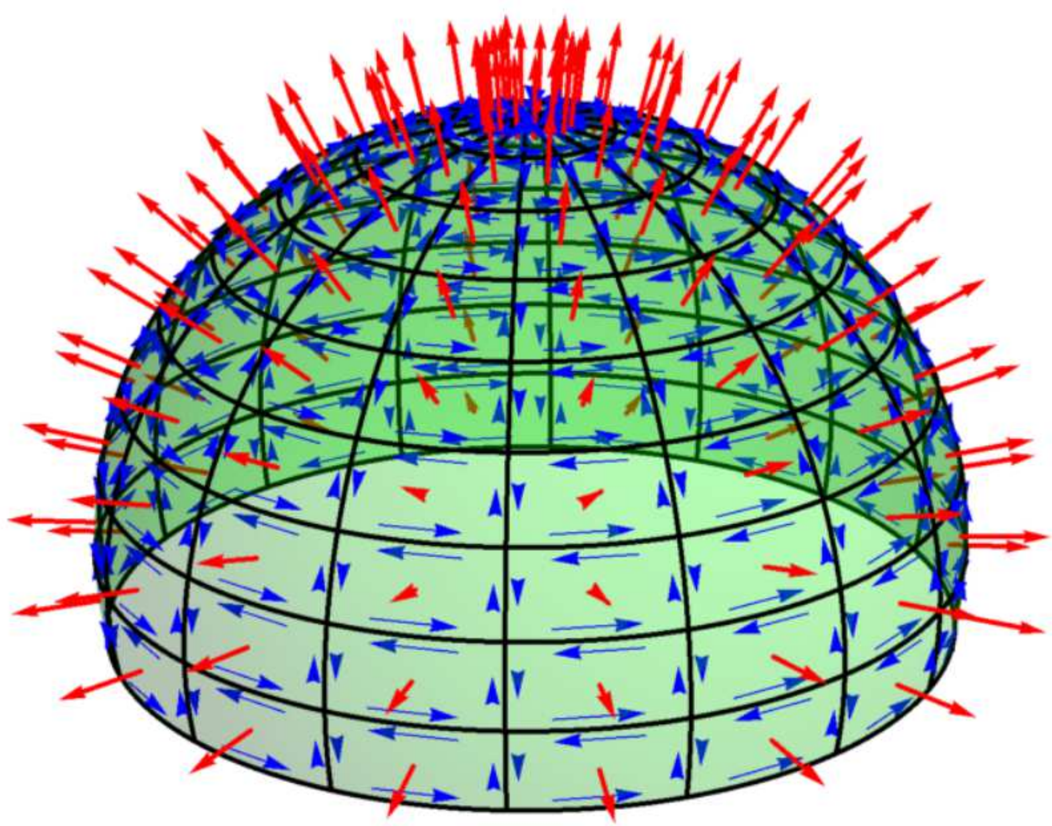
$$I_1 = \frac{e}{T_1} = 1.05418 \text{ mA}$$

The magnetic moment is

$$\mu_1 = I_1 A = 9.27401 \text{ x } 10^{-24} \text{ Am}^2$$

APPENDIX-5 Stokes' theorem

$$\oint_S (\nabla \times \boldsymbol{B}) \cdot d\boldsymbol{a} = \oint_C \boldsymbol{B} \cdot d\boldsymbol{l}$$



((Arfken)) **Stokes' theorem**

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{l} = \int_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{a}$$

Here C is the perimeter of S. This is Stokes' theorem. Note that the sign of the line integral and the direction of $d\boldsymbol{a}$ depend on the direction the perimeter is traversed, so consistent results will always be obtained. For the area and the line-integral direction shown in **Fig**, the direction of a for the shaded rectangle will be **out** of the plane of the paper.

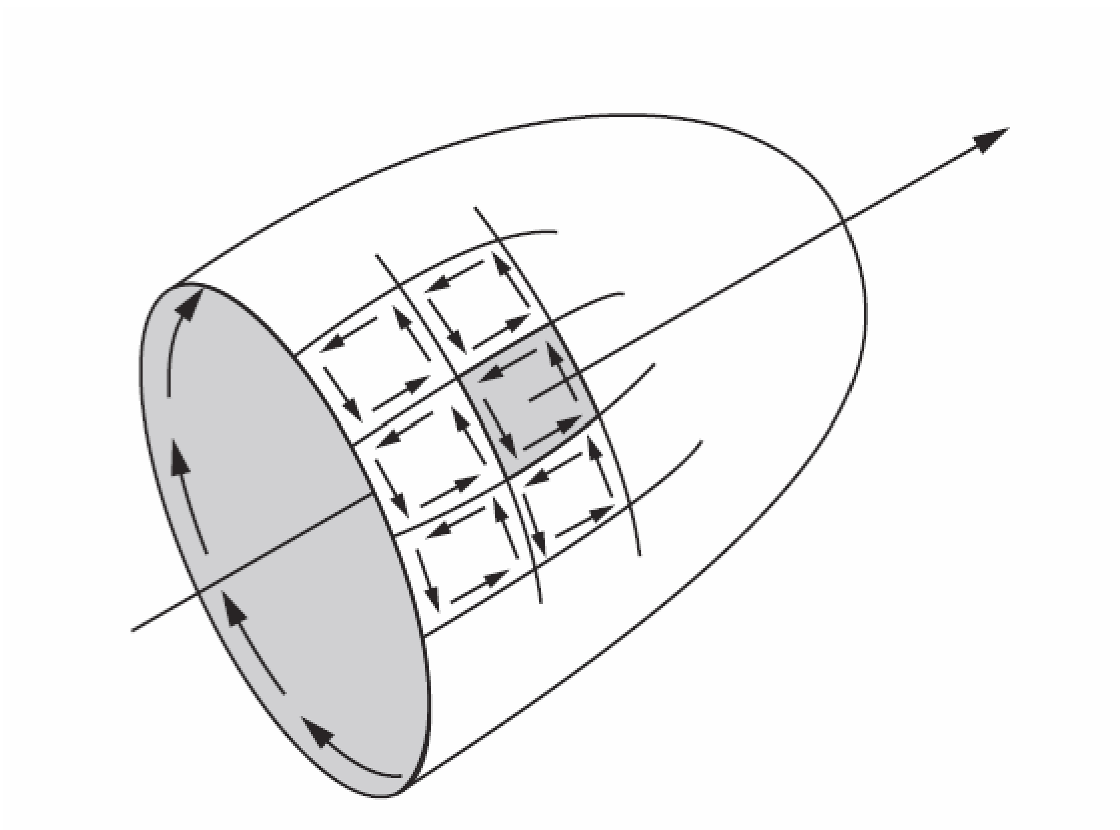


Fig. Direction of normal for the shaded rectangle when the perimeter of the surface is traversed as indicated. The direction of $d\boldsymbol{a}$ is out of paper, whilce the direction of dl is in counter clockwise.

REFERENCES

G.B. Arfken, H.J. Weber, and F.E. Harris, Mathematical Methods for Physicists 7-th edition, (Elsevier, 2013)

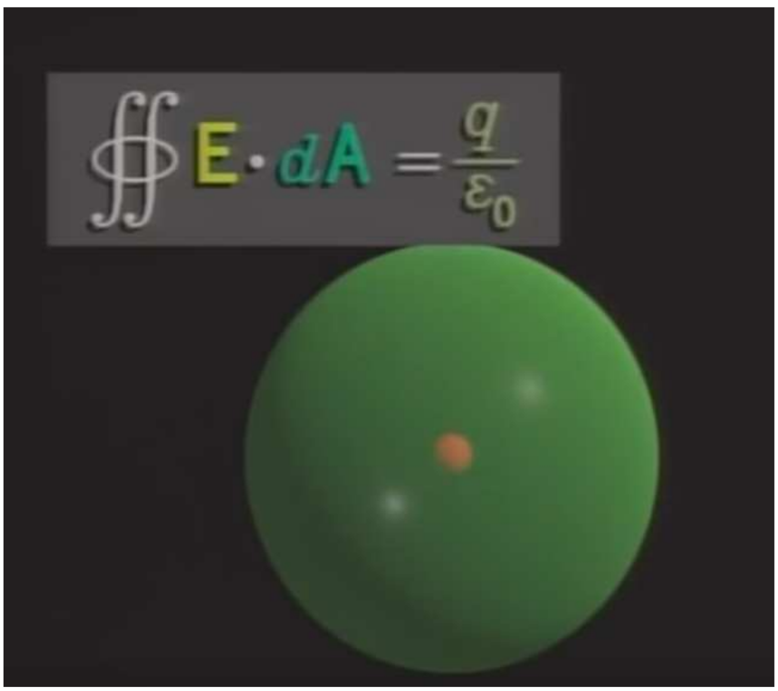
APPENDIX-6 **Yutube**

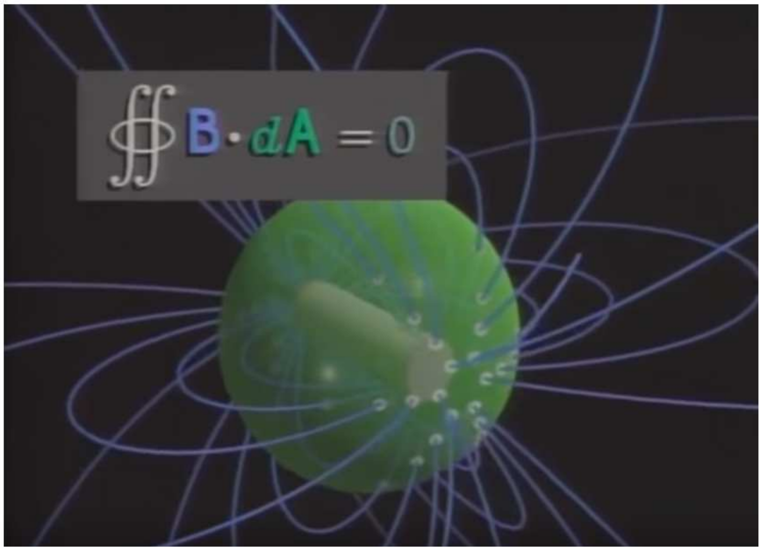
(a) **Lawrence Bragg**
https://www.youtube.com/watch?v=fa7GTW3486Y
Magnetism #3

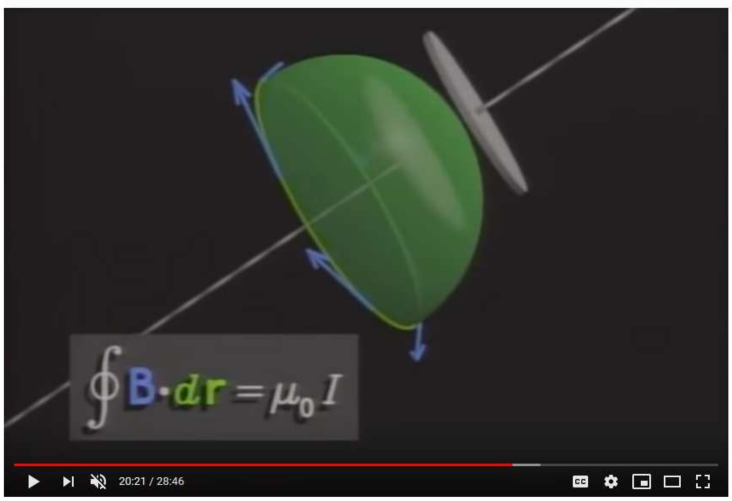
(b) **Walter Lewin**
https://www.youtube.com/watch?v=1xFRtdN5IJA&list=PLyQSN7X0ro2314mKyUiOILaOC2hk6Pc3j&index=23&t=0s
8.02x - Lect 21 - Magnetic Materials, Dia- Para- & Ferromagnetism
36.00 Ferromagnet phase transition

(c) **Mechanical Universe**
Episode 39 Maxwell's equation

Maxwell's equation




$\oiint \mathbf{B} \cdot d\mathbf{A} = 0$


$\oint \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$
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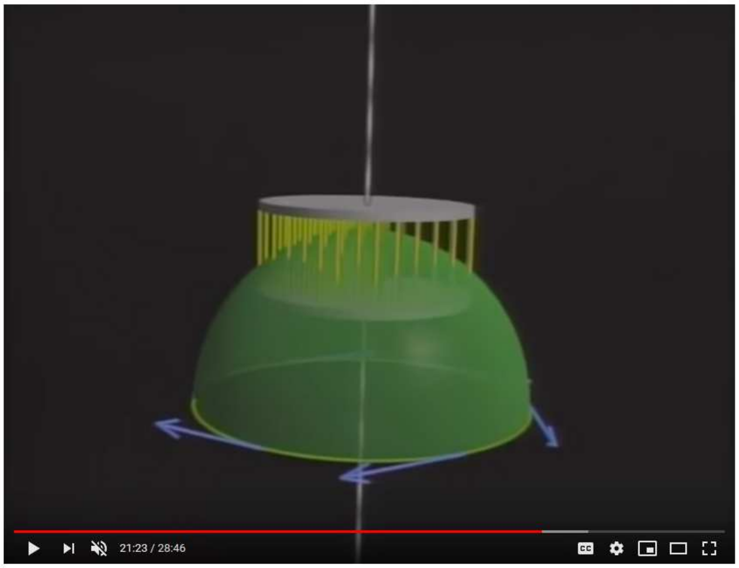
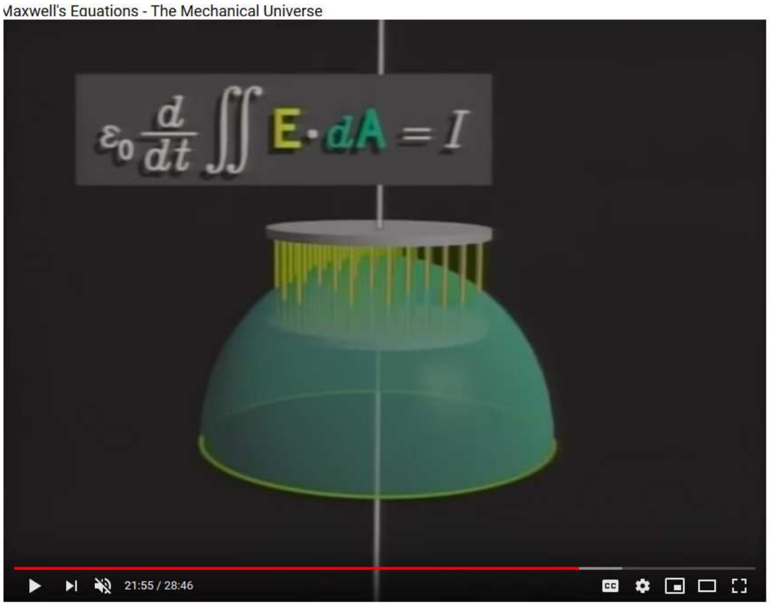


Maxwell's Equations - The Mechanical Universe



APPENDIX-7 Half wave antenna

We consider the production of electromagnetic waves by a *half-wave antenna.* In this arrangement, two conducting rods are connected to a source of alternating voltage (such as an *LC* oscillator). The length of each rod is equal to one-quarter the wavelength of the radiation emitted when the oscillator operates at frequency *f.* The oscillator forces charges to accelerate back and forth between the two rods. The configuration of the electric and magnetic fields at some instant when the current is upward. The separation of charges in the upper and lower portions of the antenna make the electric field lines resemble those of an electric dipole. (As a result, this type of antenna is sometimes called a *dipole antenna.*).

An oscillating electric dipole (an antenna) is used to generate electromagnetic radiation. It is a pair of electric charges that vary sinusoidally with time such that at any instant the two charges equal magnitude but opposite sign. One charge could be equal to

$q(t) = Q\cos(\omega t)$ and the other to $-q(t)$, where w is the angular frequency. One technique that works well for radio frequencies is to connect two straight conductors to the terminal of an ac source.

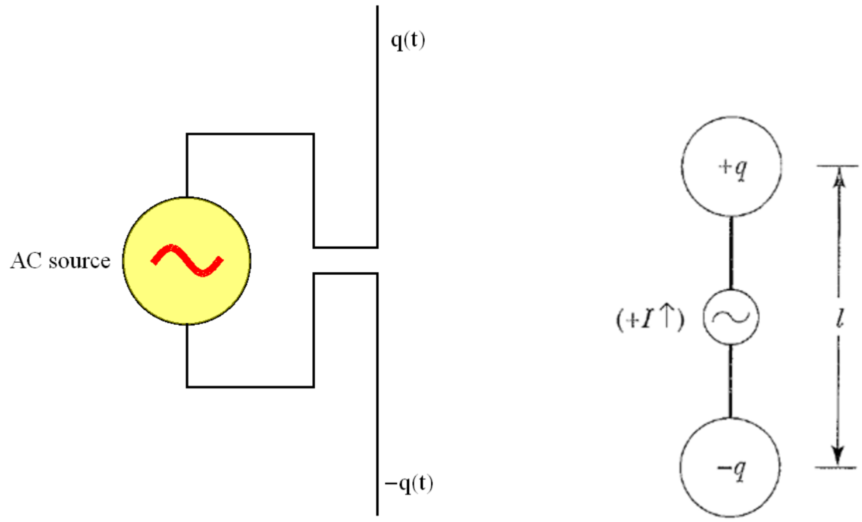


Fig. Half wave antenna. The total length of the antenna is $L = \lambda / 2$ (a half wavelength). An oscillating electric dipole antenna. Each terminal of an ac source is connected to a straight conductor; the two conductors together comprise the antenna. As the voltage across the source oscillates, the charges on the two conductors also oscillate. The charges are always equal in magnitude and apposite in sign.

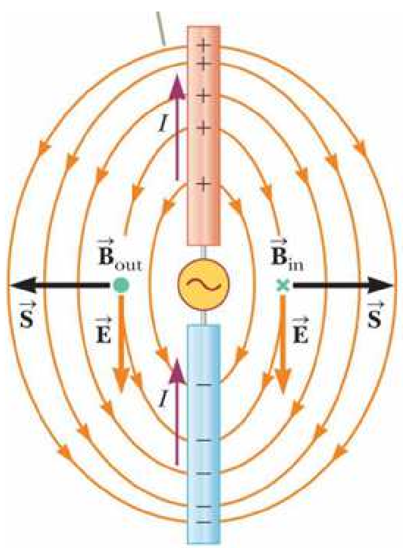


Fig. A half-wave antenna consisting of two metal rods connected to an alternating voltage source. This diagram shows ***E*** and ***B*** at an arbitrary instant when the current is upward.

((**Note**)) Typical values of f (frequency) and λ (wavelength)

$$c = \lambda f$$

where c is the velocity of light, c = 2.99792458 x 10^8 m/s. When f = 600 MHz, λ = 0.5 m.

The geometry of the antenna determines the geometrical properties of radiated electric and magnetic fields. We assume a dipole antenna, which can be considered simply as straight conductors. Charges surge back and forth in these two conductors at the angular frequency ω, driven by the oscillator.

The antenna can be regarded as an oscillating electric dipole, in which one branch carries an instantaneous charge $q(t)$ and the other branch carries -$q(t)$. The charge $q(t)$ varies sinusoidally with time and changes sign every half cycle. The charges certainly accelerated as they move back and force in antenna, and as a result the antenna is a source of electric dipole radiation. At any point in space there are electric and magnetic fields that vary sinusoidally with time.

At large distances the fields look locally to be plane. The only difference is that the amplitudes do not remain constant in the direction of propagation but fall off slowly as $1/r$, because the wave fronts are spherical rather than truly planar. The electric and magnetic field are in phase. The fields are mutually perpendicular. The pointing vector $\boldsymbol{S}$ (energy radiated) is obtained as

$$\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E}_1 \times \boldsymbol{B}_1 = \frac{{p_0}^2\omega^4}{16c^3\pi^2 r^2\varepsilon_0}\sin^2\theta\cos^2(kr-\omega t)\boldsymbol{u}_r \, .$$

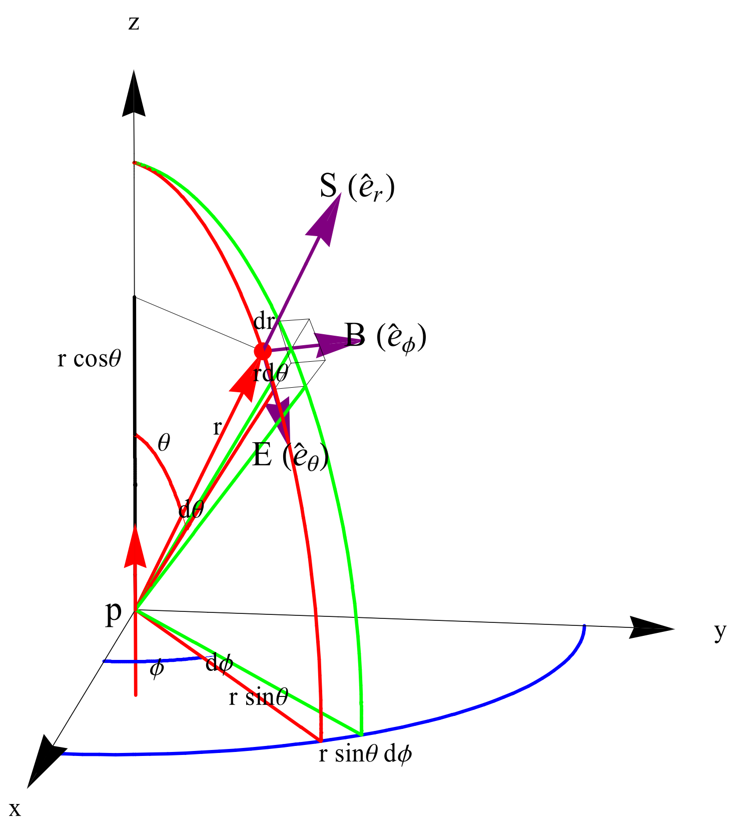


Fig. ***p***, ***E***, and ***B*** in the spherical co-ordinate system. ***p***, ***E***, and ***r***, are in the same plane. The vectors ***E*** and ***r***, are perpendicular to each other.

(1) The vectors ***p***, ***E*** and ***r,*** are in the same plane.
(2) ***E*** is always perpendicular to ***r***.

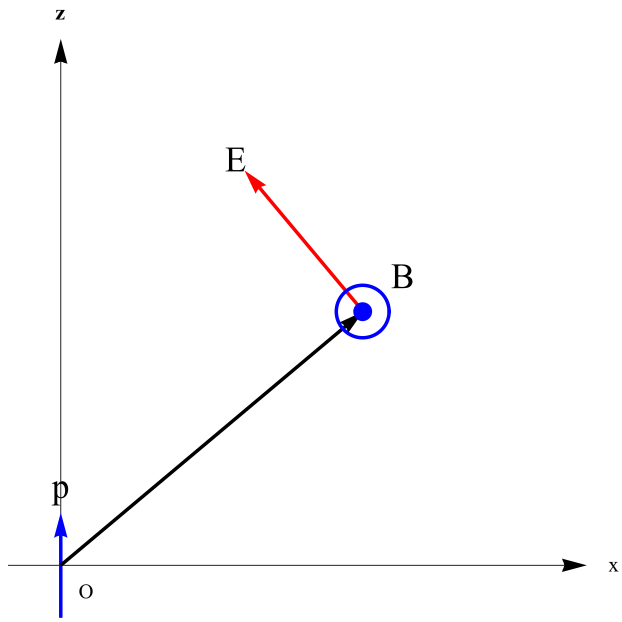


Fig. An oscillating electric dipole oriented along the z axis. The electric field $\boldsymbol{E}$, the electric dipole moment $\boldsymbol{p}$, and the position vector $\boldsymbol{r}$ lie in the same plane. The electric field $\boldsymbol{E}$ is perpendicular to r and the magnetic field $\boldsymbol{B}$.

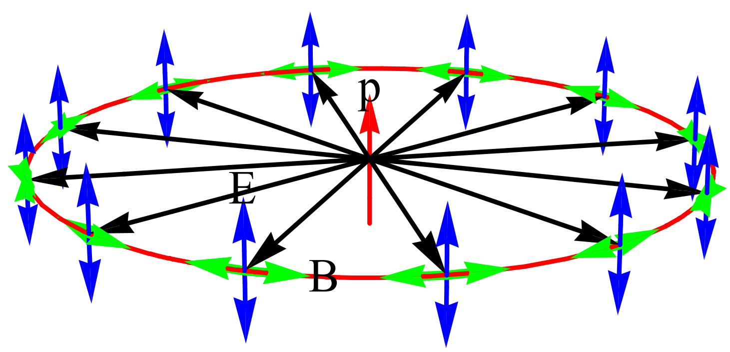


Fig. The instantaneous electric field on a sphere centered at a localized linearly oscillating charge. The electric field is along the z axis. The magnetic field $\boldsymbol{B}$ is tangential to the circle. $\boldsymbol{p}$ is the electrical dipole moment. $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric field and the magnetic field, respectively.

The magnitude of the time-averaged Poynting vector is obtained by averaging (in time) over a complete cycle ($T = 2\pi/\omega$)

$$P = \langle S \rangle = \frac{1}{T}\int_0^T S_r dt = \frac{p_0^{\,2}\omega^4}{32c^3\pi^2\varepsilon_0 r^2}\sin^2\theta$$
.

P is the total power radiated and is given by

$$P = \int_0^{\pi} < S > (2\pi r^2 \sin\theta) d\theta = \frac{p_0{}^2 \mu_0 \omega^4}{12 c\pi}.$$

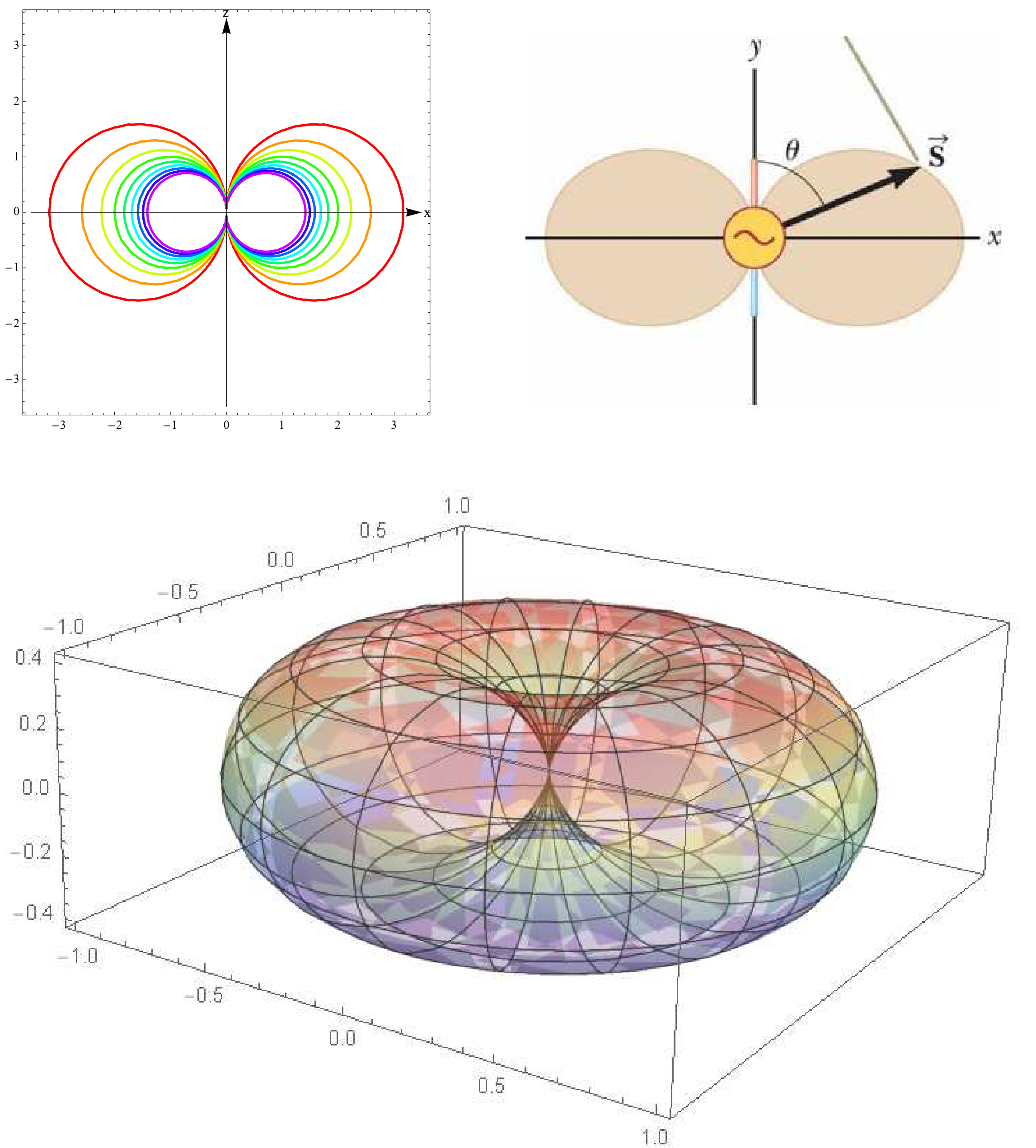


Fig. Dipole radiation pattern $P = \mathrm{A}\ \sin^2\theta/r^2$. A = 1. The distance r is changed as a parameter.

(1) StreamPlot of (E_x, E_z) in the z-x plane

The SteamPlot plots streamlines that show the local direction of the vector field at each point. The length of the arrow does not correspond to the magnitude of the vector. When $y = 0$, $E_y = 0$, we make a plot of the direction of (E_x, E_z) by using the Mathematica (StreamPlot). We assume that $A = 1$ ($\lambda = 1$ as normalization factor). The value α is changed as a parameter. Here we use $\alpha = 3$.

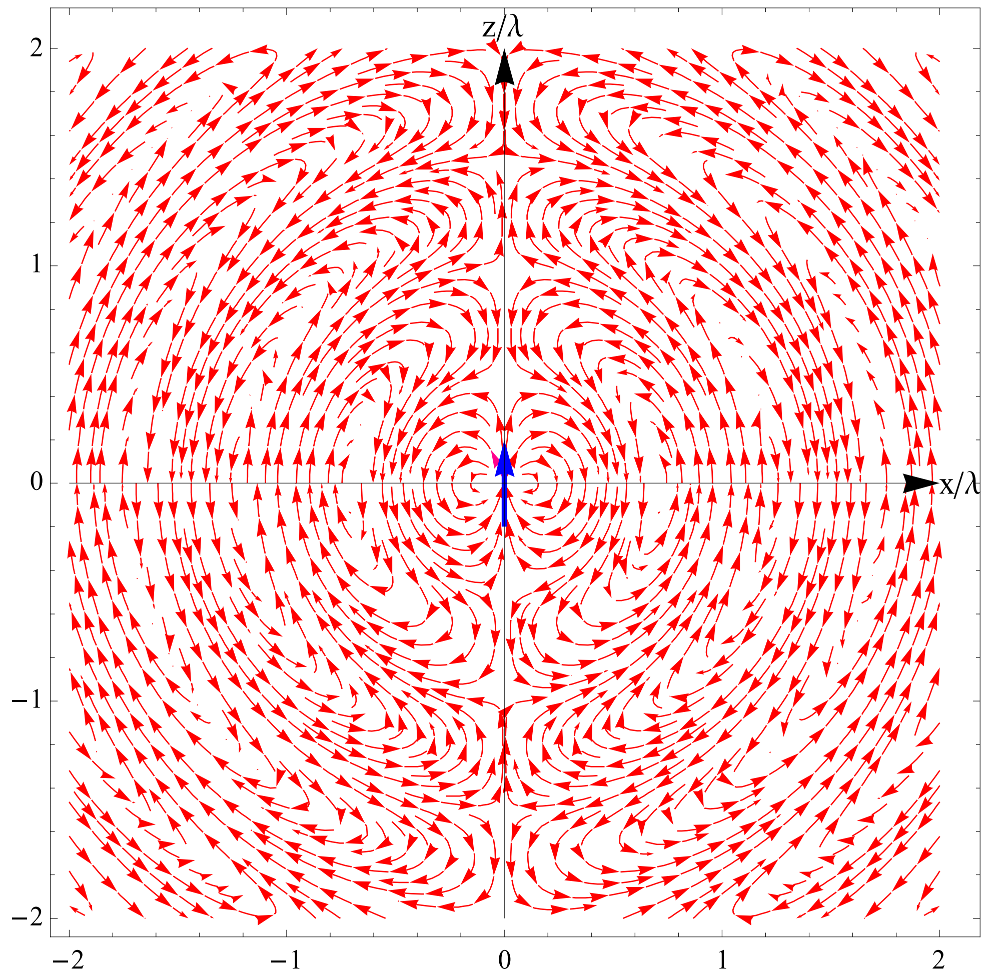


Fig. Direction of the electric field lines in the (x, z) plane. The magnitude of each arrow is the same, although the magnitude of $\boldsymbol{E}$ strongly depends on the position in the (x, z) plane. Note that $E_z = 0$ on the z axis. The direction of the oscillating electric dipole (located at the origin) is the z axis. $A = 1$. $\alpha = 3$. $\lambda = 1$ as a normalization factor.

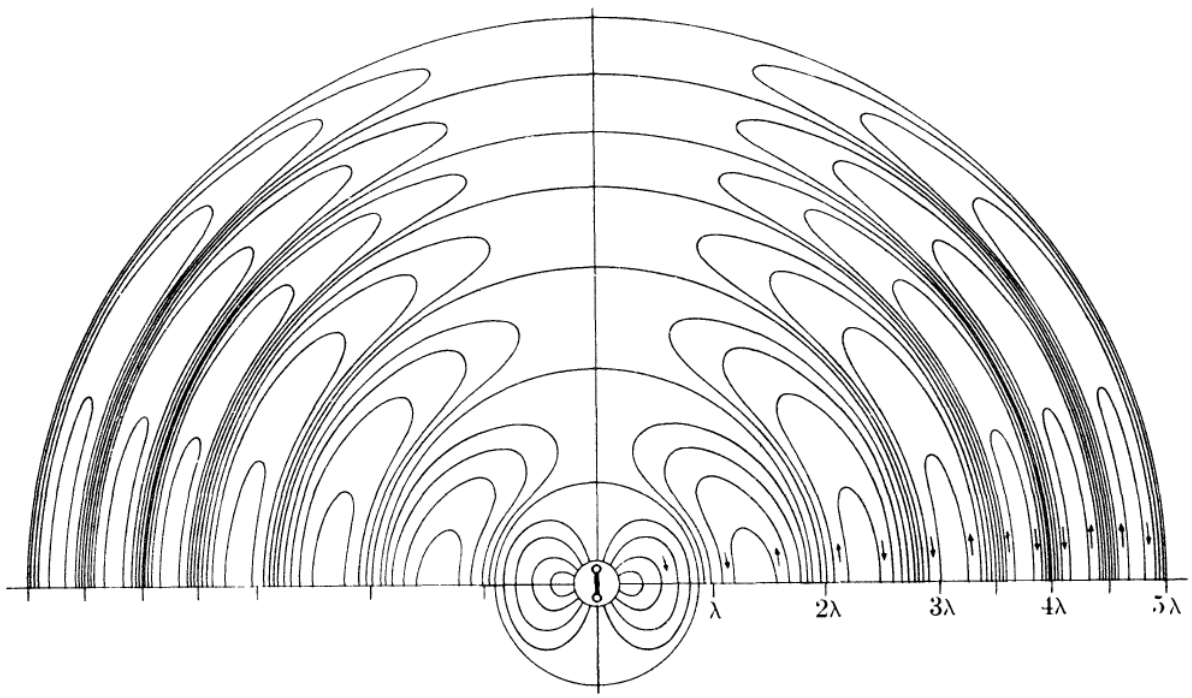


Fig. Electric field lines produced by an oscillating dipole.[W.K.H. Panofsky and M. Phillips, Classical Electricity and Magnetism (Dover. 1990)].

(2) ContourPlot of $E_x^{\ 2} + E_z^{\ 2} = \beta$

The ContourPlot (Mathematica) can be used to make a plot of the contour lines where the magnitude of $E_x^{\ 2} + E_z^{\ 2}$ is kept constant (= β). β is changed as a parameter. This figure is made by using the ContourPlot of the Mathematica.

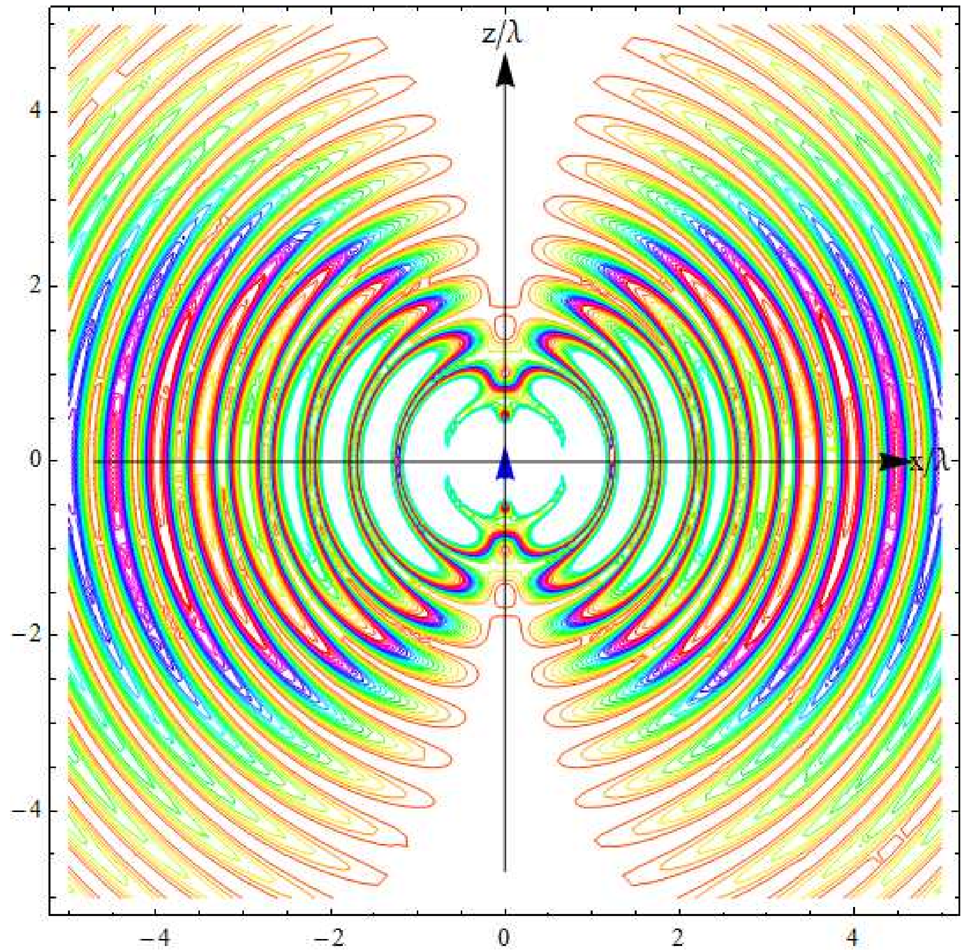


Fig. Contour plot of $E_x^{\ 2} + E_z^{\ 2} = \beta$ in the $(x/\lambda, z/\lambda)$ plane, where $y = 0$ and β is changed as a parameter. The direction of the oscillating electric dipole (located at the origin) is the z axis. This figure is made usimg the ContourPlot of the Mathematica. During one period, the loop of $\boldsymbol{E}$ shown closest to the source moves out and expands to become the loop shown farthest from the source.

(3) Propagation of the electric field lines

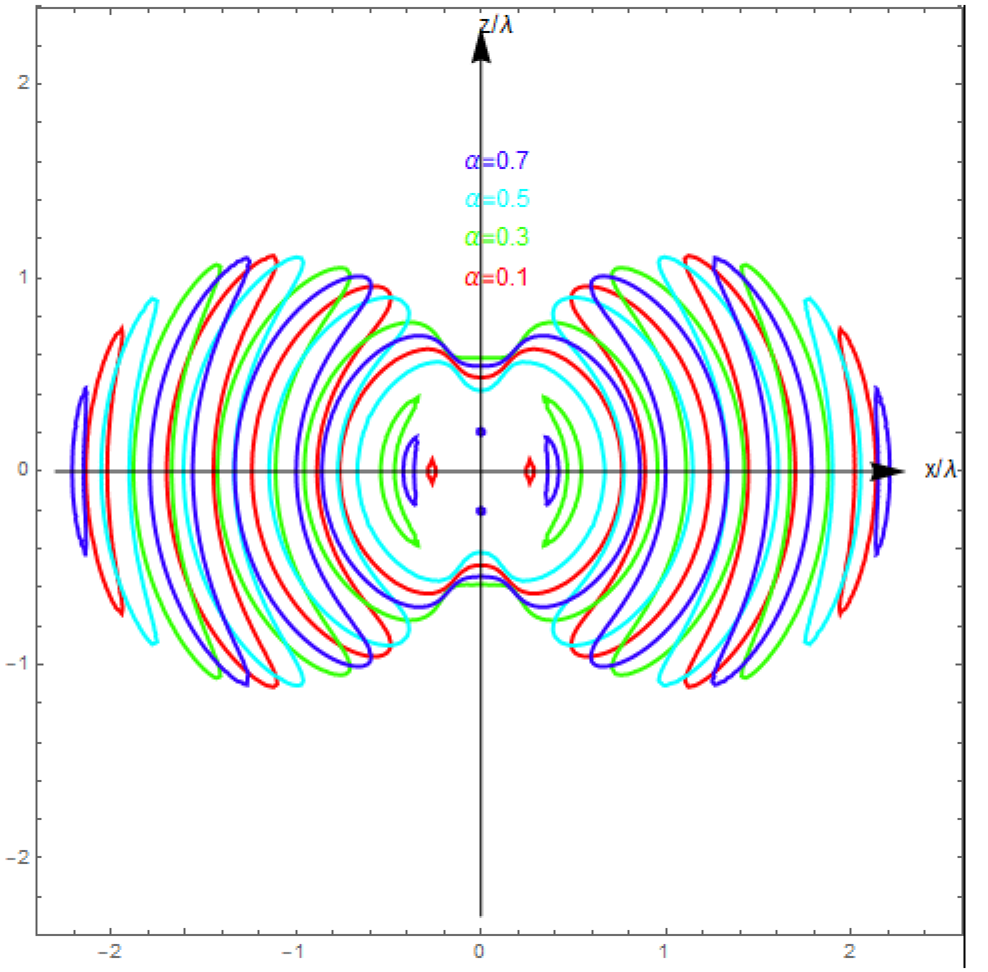


Fig. Propagation of the electric field lines, whose time dependence is periodic with a period of T (= $2\pi/\omega$). It takes time for ***E*** and ***B*** fields to spread outward from oscillating charges on two conductors (the antenna) connected to an ac source, to distant points. $A = 1$. $\lambda = 1$. $y = 0$. $\beta = 8$. The parameter a is changed as a parameter; $\alpha = \omega t = 0.1$. 0.3, 0.5, and 0.7.

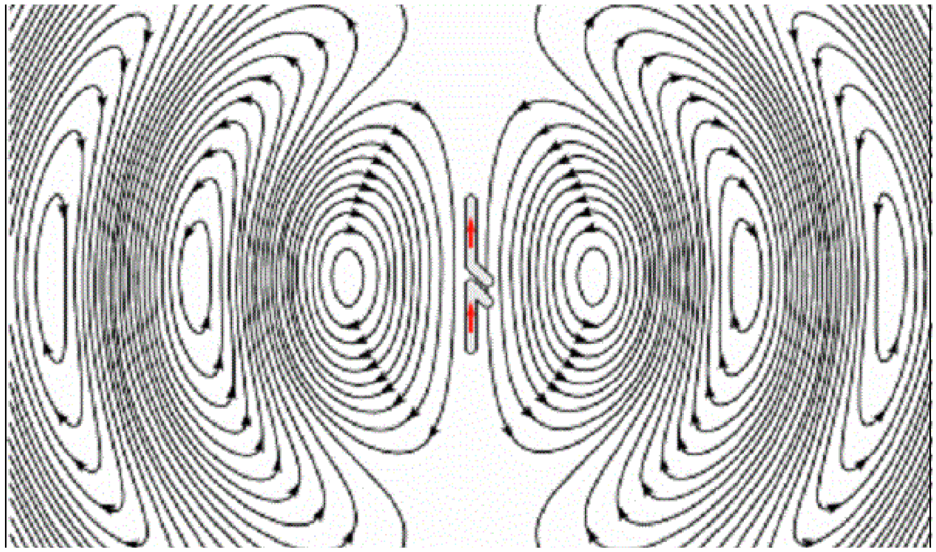


Fig. Electric field lines and magnetic field lines produced by an oscillating electric dipole. Each magnetic field line is a circle with the long axis of the dipole as its axis of revolution. The cross product $\boldsymbol{E} \times \boldsymbol{B}$ is directed away from the dipole at all points.

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