Chapter 37S

Special relativity in electricity and magnetism

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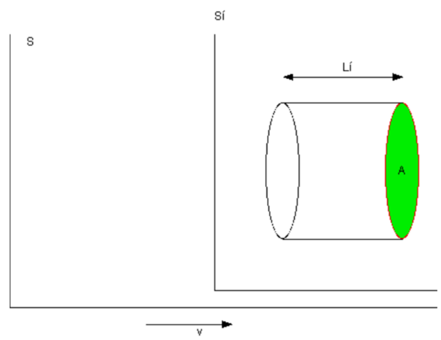
This chapter is a summary of the special relativity in the electricity and magnetism. This chapter is not taught in the class room because of the advanced topics. Nevertheless, for example, the formula for the Lorentz transformation in electric field and magnetic field may be helpful to the discussion of the origin of the Faraday's law of induction in Chapter 30.

1. Charge density and current density
1.1 Charge density



We consider the frame S' moving to the positive *x* direction with a velocity *v* relative to the frame S. Note that $\beta = v/c$. We measure the distance of the cylinder under the condition that $\Delta x_4 = 0$. Since

$$\Delta x_1' = \gamma(\Delta x_1 + i\beta\Delta x_4) = \gamma\Delta x_1$$

or

$$\Delta x_1 = \frac{1}{\gamma}\Delta x_1' = \sqrt{1-\beta^2}\,\Delta x_1' \qquad \text{(length contraction)}$$

we have

$$L = \frac{1}{\gamma}L' = \sqrt{1-\beta^2}\,L'$$

but with the same area *A* (since dimension transverse to the motion are unchangeable. If we call ρ' (= ρ_0 = the rest-charge density) the density of charges in the *S*' frame in which charges momentarily at rest, the total charge *Q* is the same in any system,

$$Q = \rho' L' A' = \rho_0 L' A' = \rho L A$$

or

$$\rho_0 L' = \rho L$$

or

$$\rho = \rho_0 \frac{L'}{L} = \gamma\rho_0 > \rho_0.$$

1.2 Four-vector current density J_μ

The current density J_μ is defined as

$$J_\mu = (\boldsymbol{J}, ic\rho) = (\rho\boldsymbol{u}, ic\rho)$$

where $\boldsymbol{u}$ is the velocity of the particle in the S frame. It is well known that the continuity equation is valid in both the frame *S* and *S*'.

$$\partial_\mu J_\mu = \frac{\partial J_\mu}{\partial x_\mu} = \frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0, \qquad \text{and} \qquad \partial_\mu ' J_\mu ' = \frac{\partial J_\mu '}{\partial x_\mu '} = 0$$

This equation is a simple consequence of Maxwell's equations.

$$\frac{\partial J_\mu '}{\partial x_\mu '} = 0 = \frac{\partial J_\mu '}{\partial J_\nu}\frac{\partial J_\nu}{\partial x_\lambda}\frac{\partial x_\lambda}{\partial x_\mu '} = a_{\mu\lambda}\frac{\partial J_\mu '}{\partial J_\nu}\frac{\partial J_\nu}{\partial x_\lambda}$$

When $\frac{\partial J_\mu '}{\partial J_\nu} = a_{\mu\nu}$, in other words,

$$J_\mu ' = a_{\mu\nu} J_\nu,$$

we have

$$\frac{\partial J_\mu '}{\partial x_\mu '} = 0 = a_{\mu\lambda}a_{\mu\nu}\frac{\partial J_\nu}{\partial x_\lambda} = \delta_{\lambda\nu}\frac{\partial J_\nu}{\partial x_\lambda} = \frac{\partial J_\nu}{\partial x_\nu} = \frac{\partial J_\mu}{\partial x_\mu}$$

Then it is concluded that J_μ is a four-vector, called the four-current density.

$$J_\mu ' = a_{\mu\nu} J_\nu$$

or

$$J_1' = \gamma(J_1 + i\beta J_4) = \gamma(J_1 - v\rho)$$
$$\rho' = \gamma(-\frac{\beta}{c}J_1 + \rho)$$

or

$$J_\mu = (a^{-1})_{\mu\nu} J_\nu' = (a^T)_{\mu\nu} J_\nu' = a_{\nu\mu} J_\nu'$$

$$J_1 = \gamma(J_1' - i\beta J_4') = \gamma(J_1' + v\rho')$$
$$\rho = \gamma(\frac{\beta}{c}J_1' + \rho')$$

Then we have

$$\rho = \gamma(\frac{\beta}{c}J_1' + \rho')$$

Note that

$$\rho = \gamma\rho' = \gamma\rho_0 = \frac{\rho_0}{\sqrt{1-\beta^2}}$$

when $J_1' = 0$. Here ρ_0 is the rest-charge density.

1.3 Invariance under the Lorentz transformation

We know that $J_\mu J_\mu$ is invariant under the Lorentz transformation

$$J'_\mu J'_\mu = a_{\mu\nu} J_\nu a_{\mu\lambda} J_\lambda = a_{\mu\nu} a_{\mu\lambda} J_\nu J_\lambda = \delta_{\mu\lambda} J_\nu J_\lambda = J_\mu J_\mu$$

or

$$J_\mu J_\mu = \boldsymbol{J}^2 - c^2\rho^2 = \boldsymbol{J}'^2 - c^2\rho'^2$$

Suppose that ***J'*** = 0 (or ***u'*** = 0) in the S' frame, where the point charge is at rest. $J = \rho\boldsymbol{u}$ (the frame S' moves at the velocity ***u*** relative to the frame S). Then we have

$$\rho^2 u^2 - c^2\rho^2 = 0 - c^2\rho'^2 = -c^2\rho_0{}^2$$

or

$$\rho\sqrt{1-\frac{u^2}{c^2}} = \rho_0, \qquad \text{or} \qquad \rho = \frac{\rho_0}{\sqrt{1-\frac{u^2}{c^2}}} = \gamma(\boldsymbol{u})\rho_0$$

The current density $\boldsymbol{J}$ is defined as

$$\boldsymbol{J} = \rho\boldsymbol{u} = \frac{\rho_0\boldsymbol{u}}{\sqrt{1-\frac{u^2}{c^2}}} = \rho_0\gamma(\boldsymbol{u})\boldsymbol{u}$$

The four-vector current density is expressed by

$$J_\mu = (\rho_0\gamma(\boldsymbol{u})\boldsymbol{u}, ic\rho_0\gamma(\boldsymbol{u}))$$

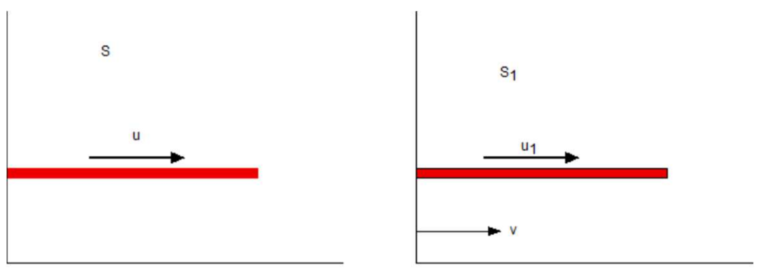
This can be expressed by

$$J_\mu = \rho_0\frac{dx_\mu}{d\tau}$$

where dτ is Lorentz-invariant and is given by

$$d\tau = \frac{dt}{\gamma(\boldsymbol{u})}$$

1.4 Simple case



Current density and charge density

$$J = (\rho u, ic\rho) = (\rho_0\gamma(u)u, ic\rho_0\gamma(u))$$
$$J' = (\rho' u', ic\rho') = (\rho_0\gamma(u')u', ic\rho_0\gamma(u'))$$

Here

$$u=\frac{u'+v}{1+\frac{v}{c^2}u'} \qquad u'=\frac{u-v}{1-\frac{v}{c^2}u}$$

$$J_1'=\gamma(v)(J_1+i\beta J_4)=\rho\gamma(v)(u-v)=\rho_0\gamma(v)\gamma(u)(u-v)$$
$$\rho'=\gamma(-\frac{\beta}{c}J_1+\rho)=\gamma(v)(1-\frac{\beta}{c}u)\rho \qquad ,$$

We also have

$$J_1=\gamma(v)(J_1'-i\beta J_4')$$
$$=\gamma(v)(u'+v)\rho'$$

$$\rho=\gamma(v)(\frac{\beta}{c}J_1'+\rho')$$
$$=\gamma(v)(1+\frac{\beta}{c}u')\rho'$$

We consider the four special cases.

(i) $u=v \quad u'=0$

$$J_1'=\rho_0\gamma(v)\gamma(u)(u-v)=0$$

$$\rho'=\gamma(v)(-\frac{\beta}{c}J_1+\rho)$$
$$=\gamma(v)(1-\frac{\beta}{c}u)\rho$$
$$=\rho_0\gamma(u)\gamma(v)(1-\frac{\beta}{c}u)$$
$$=\rho_0\gamma(v)\gamma(v)(1-\frac{\beta}{c}v)$$
$$=\rho_0\left[\gamma(v)\right]^2(1-\frac{v^2}{c^2})$$
$$=\rho_0$$

(ii) $u=-v$,

$$u'=\frac{u-v}{1-\frac{v}{c^2}u}=\frac{-2v}{1+\beta^2}$$

$$J_1' = \rho_0\gamma(v)\gamma(u)(u-v)$$
$$= -2\rho_0[\gamma(v)]^2 v$$

$$\rho' = \rho_0[\gamma(v)]^2(1+\beta^2)$$
$$= \rho_0\frac{1+\beta^2}{1-\beta^2}$$

(iii) $u' = v$,

$$u = \frac{u'+v}{1+\frac{v}{c^2}u'} = \frac{2v}{1+\frac{v^2}{c^2}}$$

$$J_1 = \gamma(v)(J_1' - i\beta J_4')$$
$$= 2v\gamma(v)\rho'$$

$$\rho = \gamma(v)(\frac{\beta}{c}J_1' + \rho')$$
$$= \gamma(v)(1+\frac{\beta}{c}u')\rho'$$
$$= \gamma(v)(1+\frac{v^2}{c^2})\rho'$$
$$= \gamma(v)(1+\frac{v^2}{c^2})\rho_0\gamma(v)$$
$$= \frac{1+\beta^2}{1-\beta^2}\rho_0$$

(iv) $u' = -v$, $\qquad u = \dfrac{u'+v}{1+\frac{v}{c^2}u'} = 0$

$$J_1 = \gamma(v)(J_1' - i\beta J_4')$$
$$= \gamma(v)(u'+v)\rho'$$
$$= 0$$

$$\rho = \gamma(v)(\frac{\beta}{c}J_1' + \rho')$$
$$= \gamma(v)(1 + \frac{\beta}{c}u')\rho'$$
$$= \gamma(v)(1 - \frac{\beta}{c}v)\rho'$$
$$= \frac{\rho'}{\gamma(v)}$$

2 Maxwell's equation field tensor

2.1 Four vectors for the vector potential and scalar potential

$$J_\mu = (\boldsymbol{J}, ic\rho), \qquad A_\mu = (\boldsymbol{A}, i\frac{1}{c}\phi), \qquad \partial_\mu = (\nabla, \frac{\partial}{\partial(ict)}).$$

The equation of continuity;

$$\partial_\mu J_\mu = \nabla\cdot\boldsymbol{J} + (-i\frac{1}{c}\frac{\partial}{\partial t})(ic\rho)$$
$$= \nabla\cdot\boldsymbol{J} + \frac{\partial}{\partial t}\rho$$
$$= 0$$

Note that

$$\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{A} - \nabla\varphi. \qquad \boldsymbol{B} = \nabla\times\boldsymbol{A}$$

2.2 Gauge transformation

$$A_\mu = (A, i\frac{1}{c}\phi).$$

Under the Gauge transformation. the new four-dimensional vector $A_\mu{}^G$ is related to the original vector A_μ through

$$\boldsymbol{A}^G = \boldsymbol{A} + \nabla\lambda,$$

$$\phi^G = \phi - \frac{\partial\lambda}{\partial t},$$

$$A^G{}_\mu = A_\mu + \partial_\mu\lambda,$$

where λ is a arbitrary function of x_u.

((**Note**))

$$i\frac{1}{c}\phi^G = i\frac{1}{c}\phi + \frac{\partial\lambda}{\partial(ict)}$$

$$A_4{}^G = A_4 + \frac{\partial\lambda}{\partial x_4}$$

((**Lorentz gauge**))
We impose the Lorentz condition given by

$$\frac{\partial A_\mu}{\partial x_\mu} = \partial_\mu A_\mu$$

$$= \nabla\cdot\boldsymbol{A} + \frac{\partial}{\partial(ict)} i\frac{\phi}{c}$$

$$= \nabla\cdot\boldsymbol{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0$$

In this gauge, A_μ is the four-vector.

$$A_\mu' = a_{\mu\nu}A_\nu\,.$$

Note that the Lorentz gauge is very convenient because it is an invariant condition.

2.3 Electromagnetic field tensor *F*

We define the field tensor as

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}\,.$$

This tensor satisfies the Jacobi identity;

$$\frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} = 0\,.$$

This equation holds automatically for the antisymmetric tensor

The magnetic field;

$$F_{12} = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}$$
$$= B_3$$

$$F_{23} = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}$$
$$= B_1$$

$$F_{31} = \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}$$
$$= B_2$$

The electric field;

$$F_{14} = \frac{\partial A_4}{\partial x_1} - \frac{\partial A_1}{\partial x_4}$$
$$= \frac{E_1}{ic}$$
$$= -\frac{i}{c} E_1$$

$$F_{24} = \frac{\partial A_4}{\partial x_2} - \frac{\partial A_2}{\partial x_4}$$
$$= \frac{E_2}{ic}$$
$$= -\frac{i}{c} E_2$$

$$F_{34} = \frac{\partial A_4}{\partial x_3} - \frac{\partial A_3}{\partial x_4}$$
$$= \frac{E_3}{ic}$$
$$= -\frac{i}{c} E_3$$

The field tensor is an anti-symmetric tensor of second rank and hence, has 6 independent components.

Electromagnetic field tensor;

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & -\frac{i}{c}E_1 \\ -B_3 & 0 & B_1 & -\frac{i}{c}E_2 \\ B_2 & -B_1 & 0 & -\frac{i}{c}E_3 \\ \frac{i}{c}E_1 & \frac{i}{c}E_2 & \frac{i}{c}E_3 & 0 \end{pmatrix}$$

We show that

$$F_{\mu\nu}{}' = \frac{\partial A_\nu{}'}{\partial x_\mu{}'} - \frac{\partial A_\mu{}'}{\partial x_\nu{}'}$$
$$= a_{\mu\sigma}a_{\nu\tau}F_{\sigma\tau}$$

$$\frac{\partial}{\partial x_\mu{}'} = a_{\mu\nu}\frac{\partial}{\partial x_\nu}, \qquad \text{and} \qquad A_\mu{}' = a_{\mu\lambda}A_\lambda$$

$$F_{\mu\nu}{}' = \frac{\partial A_\nu{}'}{\partial x_\mu{}'} - \frac{\partial A_\mu{}'}{\partial x_\nu{}'}$$
$$= a_{\mu\sigma}\frac{\partial A_\nu{}'}{\partial x_\sigma} - a_{\nu\tau}\frac{\partial A_\mu{}'}{\partial x_\tau}$$
$$= a_{\mu\sigma}\frac{\partial(a_{\nu\tau}A_\tau)}{\partial x_\sigma} - a_{\nu\tau}\frac{\partial(a_{\mu\sigma}A_\sigma)}{\partial x_\tau}$$

or

$$F_{\mu\nu}{}' = a_{\mu\sigma}a_{\nu\tau}\left(\frac{\partial A_\tau}{\partial x_\sigma} - \frac{\partial A_\sigma}{\partial x_\tau}\right)$$
$$= a_{\mu\sigma}a_{\nu\tau}F_{\sigma\tau}$$

2.4 Maxwell's equation

The Maxwell's equation can be written as

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 J_\mu$$

Note

$$\frac{\partial F_{1\mu}}{\partial x_\mu} = \frac{\partial F_{11}}{\partial x_1} + \frac{\partial F_{12}}{\partial x_2} + \frac{\partial F_{13}}{\partial x_3} + \frac{\partial F_{14}}{\partial x_4}$$

$$= \frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3} - \frac{\frac{i}{c}}{ic}\frac{\partial E_1}{\partial t}$$

$$= (\nabla \times \boldsymbol{B})_1 - \frac{1}{c^2}\frac{\partial E_1}{\partial t}$$

$$(\nabla \times \boldsymbol{B})_1 = \varepsilon_0 \mu_0 \frac{\partial E_1}{\partial t} + \mu_0 J_1$$

$$= \mu_0 (J_1 + \varepsilon_0 \frac{\partial E_1}{\partial t})$$

$$\frac{\partial F_{2\mu}}{\partial x_\mu} = \frac{\partial F_{21}}{\partial x_1} + \frac{\partial F_{22}}{\partial x_2} + \frac{\partial F_{23}}{\partial x_3} + \frac{\partial F_{24}}{\partial x_4}$$

$$= -\frac{\partial B_3}{\partial x_1} + \frac{\partial B_1}{\partial x_3} - \frac{\frac{i}{c}}{ic}\frac{\partial E_2}{\partial t}$$

$$= (\nabla \times \boldsymbol{B})_2 - \frac{1}{c^2}\frac{\partial E_2}{\partial t}$$

$$(\nabla \times \boldsymbol{B})_2 = \varepsilon_0 \mu_0 \frac{\partial E_2}{\partial t} + \mu_0 J_2$$

$$= \mu_0 (J_2 + \varepsilon_0 \frac{\partial E_2}{\partial t})$$

$$\frac{\partial F_{3\mu}}{\partial x_\mu} = \frac{\partial F_{31}}{\partial x_1} + \frac{\partial F_{32}}{\partial x_2} + \frac{\partial F_{33}}{\partial x_3} + \frac{\partial F_{34}}{\partial x_4}$$

$$= \frac{\partial B_2}{\partial x_1} - \frac{\partial B_1}{\partial x_2} - \frac{\frac{i}{c}}{ic}\frac{\partial E_3}{\partial t}$$

$$= (\nabla \times \boldsymbol{B})_3 - \frac{1}{c^2}\frac{\partial E_3}{\partial t}$$

$$(\nabla \times \boldsymbol{B})_3 = \varepsilon_0 \mu_0 \frac{\partial E_3}{\partial t} + \mu_0 J_3$$

$$= \mu_0 (J_3 + \varepsilon_0 \frac{\partial E_3}{\partial t})$$

$$\frac{\partial F_{4\mu}}{\partial x_\mu} = \frac{\partial F_{41}}{\partial x_1} + \frac{\partial F_{42}}{\partial x_2} + \frac{\partial F_{43}}{\partial x_3} + \frac{\partial F_{44}}{\partial x_4}$$
$$= \frac{i}{c}\frac{\partial E_1}{\partial x_1} + \frac{i}{c}\frac{\partial E_2}{\partial x_2} + \frac{i}{c}\frac{\partial E_3}{\partial x_3}$$
$$= \frac{i}{c}\nabla\cdot\boldsymbol{E} = \mu_0 J_4$$

$$\frac{i}{c}\nabla\cdot\boldsymbol{E} = \mu_0 ic\rho, \qquad \nabla\cdot\boldsymbol{E} = \mu_0 c^2\rho = \frac{\rho}{\varepsilon_0}$$

((**Note**))

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}$$
$$= \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ A_1 & A_2 & A_3 \end{vmatrix}$$
$$= (\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}, \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}, \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2})$$

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}$$
$$= (-\frac{\partial\phi}{\partial x_1} - ic\frac{\partial A_1}{\partial x_4}, -\frac{\partial\phi}{\partial x_2} - ic\frac{\partial A_2}{\partial x_4}, -\frac{\partial\phi}{\partial x_3} - ic\frac{\partial A_3}{\partial x_4})$$

or

$$\boldsymbol{E} = ic(\frac{\partial A_4}{\partial x_1} - \frac{\partial A_1}{\partial x_4}, \frac{\partial A_4}{\partial x_2} - \frac{\partial A_2}{\partial x_4}, \frac{\partial A_4}{\partial x_3} - \frac{\partial A_3}{\partial x_4})$$

$$\phi = \frac{c}{i}A_4$$

2.5 Invariants of the field

$F_{\mu\nu}F_{\mu\nu}$ is invariant under the Lorentz transformation

$$F_{\mu\nu}{}'F_{\mu\nu}{}' = a_{\mu\lambda}a_{\nu\rho}a_{\mu\sigma}a_{\nu\tau}F_{\lambda\rho}F_{\sigma\tau}$$
$$= \delta_{\lambda\sigma}\delta_{\rho\tau}F_{\lambda\rho}F_{\sigma\tau}$$
$$= F_{\lambda\rho}F_{\lambda\rho}$$
$$= F_{\mu\nu}F_{\mu\nu}$$

$$F_{\mu\nu}F_{\mu\nu} = 2[B_1^{\,2} + B_2^{\,2} + B_3^{\,2} - \frac{1}{c^2}(E_1^{\,2} + E_2^{\,2} + E_3^{\,2})] = \text{invariant}$$

or

$$\boldsymbol{B}^2 - \frac{1}{c^2}\boldsymbol{E}^2 = \text{invariant under the Lorentz transformation}$$

A further invariant is obtained by contraction of the field tensor with the "completely anti-symmetric unit tensor of fourth rank" defined by

$\varepsilon_{\kappa\lambda\mu\nu} =$

0 if two indices are equal,
1 if $(\kappa\lambda\mu\nu)$ is an even permutation of (1234), and
-1 if $(\kappa\lambda\mu\nu)$ is an odd permutation of (1234).

One may be convinced easily that $\varepsilon_{\kappa\lambda\mu\nu}$ is a tensor of rank 4 because

$$\varepsilon_{\kappa'\lambda'\mu'\nu'}{}' = a_{\kappa'\kappa}a_{\lambda'\lambda}a_{\mu'\mu}a_{\nu'\nu}\varepsilon_{\kappa\lambda\mu\nu}$$

Now we consider

$$\varepsilon_{\kappa\lambda\mu\nu}F_{\kappa\lambda}F_{\mu\nu} = \varepsilon_{1234}F_{12}F_{34} + \varepsilon_{1324}F_{13}F_{24} + \ldots = -\frac{8i}{c}\boldsymbol{E}\cdot\boldsymbol{B}$$

So we conclude that

$$\boldsymbol{E}\cdot\boldsymbol{B} = \text{invariant under the Lorentz transformation}$$

2.6 Equation of continuity

The equation of continuity can be derived as follows.

$$F_{\mu\nu} = -F_{\nu\mu}$$

$$F_{\mu\nu} = \frac{F_{\mu\nu} + F_{\mu\nu}}{2} = \frac{F_{\mu\nu} - F_{\nu\mu}}{2}$$

$$\frac{\partial}{\partial x_\mu}\left(\frac{\partial F_{\mu\nu}}{\partial x_\nu}\right) = \frac{1}{2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}(F_{\mu\nu} - F_{\nu\mu})$$
$$= \frac{1}{2}\left(\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}F_{\mu\nu} - \frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_\mu}F_{\nu\mu}\right)$$
$$= 0$$

Since

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 J_\mu \quad \text{(Maxwell's equation)}$$

we have

$$\frac{\partial}{\partial x_\mu} J_\mu = 0 .$$

3. Vector potential under the Lorentz transformation

$$A_\mu = (\boldsymbol{A}, i\frac{1}{c}\phi), \qquad A_\mu' = (\boldsymbol{A}', i\frac{1}{c}\phi')$$

$$A_\mu' = a_{\mu\nu}A_\nu$$

$$A_\mu = (a^{-1})_{\mu\nu}A_\nu' = (a^T)_{\mu\nu}A_\nu' = a_{\nu\mu}A_\nu'$$

$$A_1' = \frac{cA_1 - \beta\phi}{c\sqrt{1-\beta^2}} \qquad A_1' = \frac{cA_1 - \beta\phi}{c\sqrt{1-\beta^2}}$$
$$A_2' = A_2 \qquad A_2' = A_2$$
$$A_3' = A_3 \qquad A_3' = A_3$$
$$A_4' = -\frac{i}{c}\frac{(c\beta A_1 - \phi)}{\sqrt{1-\beta^2}} \qquad \phi' = \frac{\phi - c\beta A_1}{\sqrt{1-\beta^2}}$$

and

$$A_1 = \frac{cA_1'+\beta\phi'}{c\sqrt{1-\beta^2}} \qquad A_1 = \frac{cA_1'+\beta\phi'}{c\sqrt{1-\beta^2}}$$

$$A_2 = A_2' \qquad A_2 = A_2'$$

$$A_3 = A_3' \qquad A_3 = A_3'$$

$$A_4 = \frac{i}{c}\frac{(c\beta A'_1+\phi')}{\sqrt{1-\beta^2}} \qquad \phi = \frac{(c\beta A'_1+\phi')}{\sqrt{1-\beta^2}}$$

4. *E* and *B* under the Lorentz transformation

4.1 Transformation

$$F'_{\mu\nu} = a_{\mu\lambda} a_{\nu\sigma} F_{\lambda\sigma}$$

$$a_{\mu\xi} a_{\nu\eta} F'_{\mu\nu} = a_{\mu\xi} a_{\nu\eta} a_{\mu\lambda} a_{\nu\sigma} F_{\lambda\sigma} = a_{\mu\xi} a_{\mu\lambda} a_{\nu\eta} a_{\nu\sigma} F_{\lambda\sigma} = \delta_{\xi\lambda}\delta_{\eta\sigma} F_{\lambda\sigma} = F_{\xi\eta}$$

or

$$F_{\mu\nu} = a_{\lambda\mu} a_{\sigma\nu} F'_{\lambda\sigma} = (a^T)_{\mu\lambda}(a^T)_{\nu\sigma} F'_{\lambda\sigma} = (a^{-1})_{\mu\lambda}(a^{-1})_{\nu\sigma} F'_{\lambda\sigma}$$

$$E_1' = E_1 \qquad E_1' = E_1$$

$$E_2' = \gamma(E_2 - c\beta B_3) \qquad E_2' = \gamma(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})_2$$

$$E_3' = \gamma(E_3 + c\beta B_2) \qquad E_3' = \gamma(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})_3$$

$$B_1' = B_1 \qquad B_1' = B_1$$

$$B_2' = \gamma(B_2 + \frac{\beta}{c}E_3) \qquad B_2' = \gamma(\boldsymbol{B} - \frac{1}{c^2}\boldsymbol{v}\times\boldsymbol{E})_2$$

$$B_3' = \gamma(B_3 - \frac{\beta}{c}E_2) \qquad B_3' = \gamma(\boldsymbol{B} - \frac{1}{c}\boldsymbol{v}\times\boldsymbol{E})_3$$

$$E_1 = E_1' \qquad E_1 = E_1'$$

$$E_2 = \gamma(E_2' + c\beta B_3') \qquad E_2 = \gamma(\boldsymbol{E}' - \boldsymbol{v}\times\boldsymbol{B}')_2$$

$$E_3 = \gamma(E_3' - c\beta B_2') \qquad E_3 = \gamma(\boldsymbol{E}' - \boldsymbol{v}\times\boldsymbol{B}')_3$$

$$B_1 = B_1' \qquad B_1 = B_1'$$

$$B_2 = \gamma(B_2' - \frac{\beta}{c}E_3') \qquad B_2 = \gamma(\boldsymbol{B}' + \frac{1}{c^2}\boldsymbol{v}\times\boldsymbol{E}')_2$$

$$B_3 = \gamma(B_3' + \frac{\beta}{c}E_2') \qquad B_3 = \gamma(\boldsymbol{B}' + \frac{1}{c^2}\boldsymbol{v}\times\boldsymbol{E}')_3$$

((**Mathematica-1**)) See the book of Michael Trott, Springer Verlag

```
Clear["Global`*"];
```

Field tensor under the Lorentz transformatrion

```
a = {{1/Sqrt[1 - β^2], 0, 0, I β/Sqrt[1 - β^2]}, {0, 1, 0, 0},
   {0, 0, 1, 0}, {-I β/Sqrt[1 - β^2], 0, 0, 1/Sqrt[1 - β^2]}};
```

```
a // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{i\beta}{\sqrt{1-\beta^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{i\beta}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{1}{\sqrt{1-\beta^2}} \end{pmatrix}$$

Field tensor before the Lorentz transformation

$$F = \left\{\left\{0, B3, -B2, -i\frac{E1}{c}\right\}, \left\{-B3, 0, B1, -i\frac{E2}{c}\right\}, \left\{B2, -B1, 0, -i\frac{E3}{c}\right\}, \left\{i\frac{E1}{c}, i\frac{E2}{c}, i\frac{E3}{c}, 0\right\}\right\};$$

Field tensor after the Lorentz transformation

```
Fnew = {{0, B3', -B2', -ⅈ E1'/c}, {-B3', 0, B1', -ⅈ E2'/c},
   {B2', -B1', 0, -ⅈ E3'/c}, {ⅈ E1'/c, ⅈ E2'/c, ⅈ E3'/c, 0}};
```

```
Fnew // MatrixForm
```

$$\begin{pmatrix} 0 & B3' & -B2' & -\frac{i\,E1'}{c} \\ -B3' & 0 & B1' & -\frac{i\,E2'}{c} \\ B2' & -B1' & 0 & -\frac{i\,E3'}{c} \\ \frac{i\,E1'}{c} & \frac{i\,E2'}{c} & \frac{i\,E3'}{c} & 0 \end{pmatrix}$$

Electric field and magnetic field

```
ElectricFieldStrength[fieldtensor_] :=
  {fieldtensor[[4, 1]] c/ⅈ, fieldtensor[[4, 2]] c/ⅈ,
   fieldtensor[[4, 3]] c/ⅈ};
```

```
MagneticFieldStrength[fieldtensor_] :=
  {fieldtensor[[2, 3]] , -fieldtensor[[1, 3]],
   fieldtensor[[1, 2]]};
```

```
Ea = ElectricFieldStrength[F]; Ea // MatrixForm
```

$$\begin{pmatrix} E1 \\ E2 \\ E3 \end{pmatrix}$$

```
Ba = MagneticFieldStrength[F]; Ba // MatrixForm
```

$$\begin{pmatrix} B1 \\ B2 \\ B3 \end{pmatrix}$$

Field tensor after the Lorentz transfomation

```
Ftrans =
  Table[Sum[a[[μ, λ]] × a[[ν, σ]] × F[[λ, σ]], {λ, 1, 4},
     {σ, 1, 4}], {μ, 1, 4}, {ν, 1, 4}] // Simplify;
```

```
Ftrans // MatrixForm
```

$$\begin{pmatrix} 0 & \frac{B3\,c-E2\,\beta}{c\sqrt{1-\beta^2}} & \frac{-B2\,c-E3\,\beta}{c\sqrt{1-\beta^2}} & -\frac{i\,E1}{c} \\ \frac{-B3\,c+E2\,\beta}{c\sqrt{1-\beta^2}} & 0 & B1 & \frac{i\,(-E2+B3\,c\,\beta)}{c\sqrt{1-\beta^2}} \\ \frac{B2\,c+E3\,\beta}{c\sqrt{1-\beta^2}} & -B1 & 0 & -\frac{i\,(E3+B2\,c\,\beta)}{c\sqrt{1-\beta^2}} \\ \frac{i\,E1}{c} & \frac{i\,(E2-B3\,c\,\beta)}{c\sqrt{1-\beta^2}} & \frac{i\,(E3+B2\,c\,\beta)}{c\sqrt{1-\beta^2}} & 0 \end{pmatrix}$$

```
Fntrans =
  Table[Sum[a[[λ, μ]] × a[[σ, ν]] × Fnew[[λ, σ]], {λ, 1, 4},
     {σ, 1, 4}], {μ, 1, 4}, {ν, 1, 4}] // Simplify;
```

```
Fntrans // MatrixForm
```

$$\begin{pmatrix} 0 & \frac{c\,B3'+\beta\,E2'}{c\sqrt{1-\beta^2}} & \frac{-c\,B2'+\beta\,E3'}{c\sqrt{1-\beta^2}} & -\frac{i\,E1'}{c} \\ \frac{-c\,B3'-\beta\,E2'}{c\sqrt{1-\beta^2}} & 0 & B1' & -\frac{i\,(c\,\beta\,B3'+E2')}{c\sqrt{1-\beta^2}} \\ \frac{c\,B2'-\beta\,E3'}{c\sqrt{1-\beta^2}} & -B1' & 0 & \frac{i\,(c\,\beta\,B2'-E3')}{c\sqrt{1-\beta^2}} \\ \frac{i\,E1'}{c} & \frac{i\,(c\,\beta\,B3'+E2')}{c\sqrt{1-\beta^2}} & -\frac{i\,(c\,\beta\,B2'-E3')}{c\sqrt{1-\beta^2}} & 0 \end{pmatrix}$$

Eelectric field and magnetic field after the Lorentz transformation

```
Eb = ElectricFieldStrength[Ftrans]
```

$$\left\{E1, \frac{E2 - B3\, c\, \beta}{\sqrt{1-\beta^2}}, \frac{E3 + B2\, c\, \beta}{\sqrt{1-\beta^2}}\right\}$$

```
Bb = MagneticFieldStrength[Ftrans]
```

$$\left\{B1, -\frac{-B2\, c - E3\, \beta}{c\sqrt{1-\beta^2}}, \frac{B3\, c - E2\, \beta}{c\sqrt{1-\beta^2}}\right\}$$

```
Ec = ElectricFieldStrength[Fntrans]
```

$$\left\{E1', \frac{c\, \beta\, B3' + E2'}{\sqrt{1-\beta^2}}, -\frac{c\, \beta\, B2' - E3'}{\sqrt{1-\beta^2}}\right\}$$

```
Bc = MagneticFieldStrength[Fntrans]
```

$$\left\{B1', -\frac{-c\, B2' + \beta\, E3'}{c\sqrt{1-\beta^2}}, \frac{c\, B3' + \beta\, E2'}{c\sqrt{1-\beta^2}}\right\}$$

Lagrangian and Hamiltonian

```
Sum[F[[μ, ν]] × F[[μ, ν]], {μ, 1, 4}, {ν, 1, 4}] // Cancel
```

$$2\,B1^2 + 2\,B2^2 + 2\,B3^2 - \frac{2\,E1^2}{c^2} - \frac{2\,E2^2}{c^2} - \frac{2\,E3^2}{c^2}$$

```
Sum[Ftrans[[μ, ν]] × Ftrans[[μ, ν]], {μ, 1, 4}, {ν, 1, 4}] //
  Simplify // Cancel
```

$$\frac{2\left(B1^2\,c^2 + B2^2\,c^2 + B3^2\,c^2 - E1^2 - E2^2 - E3^2\right)}{c^2}$$

Dual field strength Fdual

```
Fdual = Table[Sum[Signature[{λ, μ, ν, σ}] × F[[ν, σ]] / 2,
    {ν, 1, 4}, {σ, 1, 4}], {λ, 1, 4}, {μ, 1, 4}];
```

```
Fdual // MatrixForm
```

$$\begin{pmatrix} 0 & -\frac{i\,E3}{c} & \frac{i\,E2}{c} & B1 \\ \frac{i\,E3}{c} & 0 & -\frac{i\,E1}{c} & B2 \\ -\frac{i\,E2}{c} & \frac{i\,E1}{c} & 0 & B3 \\ -B1 & -B2 & -B3 & 0 \end{pmatrix}$$

```
Sum[Fdual[[μ, ν]] × F[[μ, ν]], {μ, 1, 4}, {ν, 1, 4}] //
  Cancel
```

$$-\frac{4\,i\,B1\,E1}{c} - \frac{4\,i\,B2\,E2}{c} - \frac{4\,i\,B3\,E3}{c}$$

```
Tr[Fdual.F]
```

$$\frac{4\,i\,B1\,E1}{c} + \frac{4\,i\,B2\,E2}{c} + \frac{4\,i\,B3\,E3}{c}$$

4.2 Choice of the frame S' which has pure electric or pure magnetic fields

From the Sec.3.5, we find that

(1) $\boldsymbol{B}^2 - \frac{1}{c^2}\boldsymbol{E}^2$ = invariant under the Lorentz transformation

(2) $\boldsymbol{E}\cdot\boldsymbol{B}$ = invariant under the Lorentz transformation

Here we assume that $\boldsymbol{E}\cdot\boldsymbol{B}$=0 and $\boldsymbol{B}^2 - \frac{1}{c^2}\boldsymbol{E}^2 \neq 0$

Then one can find a frame S' in which (***E***' = 0 and ***B***' ≠ 0) [pure magnetic field], or (***B***' = 0 and ***E***' ≠ 0) [pure electric field]. The proof is given in the following.

(a) Pure magnetic field (*E*' = 0)

We assume that ***E***' = 0. From the Lorentz transformation, we have

$$E_1' = E_1 = 0 \qquad\qquad E_1 = 0$$
$$E_2' = \gamma(E_2 - c\beta B_3) = 0 \qquad \text{or} \qquad E_2 = c\beta B_3 = vB_3$$
$$E_3' = \gamma(E_3 + c\beta B_2) = 0 \qquad\qquad E_3 = -c\beta B_2 = -vB_2$$

The condition $\boldsymbol{E}\cdot\boldsymbol{B}$=0 is satisfied since

$$\boldsymbol{E}\cdot\boldsymbol{B} = E_1B_1 + E_2B_2 + E_3B_3 = vB_2B_3 - vB_2B_3 = 0$$

The condition $\boldsymbol{B}^2 - \frac{1}{c^2}\boldsymbol{E}^2 = \boldsymbol{B}'^2 - \frac{1}{c^2}\boldsymbol{E}'^2 \neq 0$ can be rewritten as

$$\boldsymbol{B}^2 - \frac{1}{c^2}\boldsymbol{E}^2 = \boldsymbol{B}'^2 > 0$$

This implies that one can find the frame where $\mathbf{B}'^2 \neq 0$ and ***E***' = 0.

((**Note**))

From the relation

$$E_1 = 0$$
$$E_2 = c\beta B_3 = vB_3$$
$$E_3 = -c\beta B_2 = -vB_2$$

we get

$$\boldsymbol{E} = -\boldsymbol{v}\times\boldsymbol{B}$$

(b) **Pure electric field (*B*' = 0)**

Next we assume that ***B***' = 0. Then we have

$$B_1' = B_1 = 0 \qquad\qquad B_1 = 0$$

$$B_2' = \gamma(B_2 + \frac{\beta}{c}E_3) = 0, \qquad \text{or} \qquad B_2 = -\frac{v}{c^2}E_3$$

$$B_3' = \gamma(B_3 - \frac{\beta}{c}E_2) = 0 \qquad\qquad B_3 = \frac{v}{c^2}E_2$$

The condition $\boldsymbol{E}\cdot\boldsymbol{B}$ =0 is satisfied since

$$\begin{aligned}\boldsymbol{E}\cdot\boldsymbol{B} &= E_1B_1 + E_2B_2 + E_3B_3 \\ &= -\frac{v}{c^2}E_2E_3 + \frac{v}{c^2}E_2E_3 \\ &= 0\end{aligned}$$

The condition $\boldsymbol{B}^2 - \frac{1}{c^2}\boldsymbol{E}^2 = \boldsymbol{B}'^2 - \frac{1}{c^2}\boldsymbol{E}'^2 \neq 0$ can be rewritten as

$$\boldsymbol{B}^2 - \frac{1}{c^2}\boldsymbol{E}^2 = -\frac{1}{c^2}\boldsymbol{E}'^2 < 0$$

This implies that one can find the frame where $\boldsymbol{E}'^2 \neq 0$ and $\boldsymbol{B}' = 0$.

((**Note**))
From the relation

$$B_1 = 0$$

$$B_2 = -\frac{v}{c^2}E_3$$

$$B_3 = \frac{v}{c^2}E_2$$

we get

$$\boldsymbol{B} = \frac{1}{c^2}(\boldsymbol{v}\times\boldsymbol{E})$$

5. Energy-momentum tensor and Maxwell's stress

5.1 Force density

We define the four-vector of the force density as f_μ

$$F_{\mu\nu}J_\nu = f_\mu$$

Here we have

$$f_i = \rho E_i + (\boldsymbol{J}\times\boldsymbol{B})_i$$

where

$$\boldsymbol{J}\times\boldsymbol{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ J_1 & J_2 & J_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$\begin{cases} f_1 = \rho E_1 + (J_2 B_3 - J_3 B_2) \\ f_2 = \rho E_2 + (J_3 B_1 - J_1 B_3) \\ f_3 = \rho E_3 + (J_1 B_2 - J_2 B_1) \end{cases}$$

$$\begin{aligned} f_1 &= F_{1\nu} J_\nu = F_{11} J_1 + F_{12} J_2 + F_{13} J_3 + F_{14} J_4 \\ &= B_3 J_2 - B_2 J_3 - \frac{i}{c} E_1 (ic\rho) \\ &= (\boldsymbol{B}\times\boldsymbol{J})_1 + \rho E_1 \end{aligned}$$

$$f_2 = (\boldsymbol{B}\times\boldsymbol{J})_2 + \rho E_2$$

$$f_3 = (\boldsymbol{B}\times\boldsymbol{J})_3 + \rho E_3$$

$$\begin{aligned} f_4 &= F_{4\nu} J_\nu = \frac{i}{c} E_1 J_1 + \frac{i}{c} E_2 J_2 + \frac{i}{c} E_3 J_3 \\ &= \frac{i}{c}(\boldsymbol{E}\cdot\boldsymbol{J}) = i\left(\frac{\boldsymbol{E}\cdot\boldsymbol{J}}{c}\right) \end{aligned}$$

5.2 Maxwell's equation

The Maxwell's equation is given by

$$\frac{\partial F_{\nu\lambda}}{\partial x_\lambda} = \mu_0 J_\nu$$

The current density is given by

$$J_\mu = (\boldsymbol{J}, ic\rho)$$

Then we have

$$f_\mu = F_{\mu\nu} J_\nu = \frac{1}{\mu_0} F_{\mu\nu} \frac{\partial F_{\nu\lambda}}{\partial x_\lambda}$$

$$\mu_0 f_\mu = F_{\mu\nu} \frac{\partial F_{\nu\lambda}}{\partial x_\lambda}$$

The left-hand side can be split into two terms,

$$\mu_0 f_\mu = \frac{\partial}{\partial x_\lambda}(F_{\mu\nu} F_{\nu\lambda}) - F_{\nu\lambda} \frac{\partial}{\partial x_\lambda} F_{\mu\nu}$$

The second term:

$$\begin{aligned} F_{\nu\lambda} \frac{\partial}{\partial x_\lambda} F_{\mu\nu} &= \frac{1}{2} F_{\nu\lambda} \frac{\partial}{\partial x_\lambda} F_{\mu\nu} + \frac{1}{2} F_{\lambda\nu} \frac{\partial}{\partial x_\nu} F_{\mu\lambda} \\ &= \frac{1}{2} F_{\nu\lambda} \frac{\partial}{\partial x_\lambda} F_{\mu\nu} + \frac{1}{2} F_{\nu\lambda} \frac{\partial}{\partial x_\nu} F_{\lambda\mu} \end{aligned}$$

or

$$\begin{aligned} F_{\nu\lambda} \frac{\partial}{\partial x_\lambda} F_{\mu\nu} &= \frac{1}{2} F_{\nu\lambda} \left(\frac{\partial}{\partial x_\lambda} F_{\mu\nu} + \frac{\partial}{\partial x_\nu} F_{\lambda\mu}\right) \\ &= -\frac{1}{2} F_{\nu\lambda} \frac{\partial}{\partial x_\mu} F_{\nu\lambda} \\ &= -\frac{1}{4} \frac{\partial}{\partial x_\mu}(F_{\nu\lambda} F_{\nu\lambda}) \end{aligned}$$

Here we use the Jacobi identity;

$$\frac{\partial}{\partial x_\lambda} F_{\mu\nu} + \frac{\partial}{\partial x_\mu} F_{\nu\lambda} + \frac{\partial}{\partial x_\nu} F_{\lambda\mu} = 0 \qquad \text{(Jacobi identity)}$$

Then we have

$$F_{\nu\lambda} \frac{\partial}{\partial x_\lambda} F_{\mu\nu} = -\frac{1}{4} \delta_{\mu\lambda} \frac{\partial}{\partial x_\lambda}(F_{\sigma\tau} F_{\sigma\tau})$$

The force density is rewritten as

$$f_\mu = \frac{1}{\mu_0}\frac{\partial}{\partial x_\lambda}(F_{\mu\nu}F_{\nu\lambda} + \frac{1}{4}\delta_{\mu\lambda}F_{\sigma\tau}F_{\sigma\tau})$$
$$= \frac{\partial T_{\mu\lambda}}{\partial x_\lambda}$$

with the symmetric energy-momentum tensor (Maxwell's stress tensor)

$$T_{\mu\nu} = \frac{1}{\mu_0}(F_{\mu\lambda}F_{\lambda\nu} + \frac{1}{4}\delta_{\mu\lambda}F_{\sigma\tau}F_{\sigma\tau})$$

$$Tr[T_{\mu\nu}] = T_{\mu\mu} = \frac{1}{\mu_0}(F_{\mu\lambda}F_{\lambda\mu} + \frac{1}{4}F_{\sigma\tau}F_{\sigma\tau}) = 0$$

5.3 Conservation law

$$\frac{\partial u}{\partial t} + \nabla\cdot\boldsymbol{S} = -\boldsymbol{E}\cdot\boldsymbol{J}$$
$$u = \frac{1}{2}(\varepsilon_0\boldsymbol{E}^2 + \frac{1}{\mu_0}\boldsymbol{B}^2)$$
$$\varepsilon_0\mu_0\frac{\partial\boldsymbol{S}}{\partial t} + \boldsymbol{f} = (\nabla\cdot\vec{\boldsymbol{T}})$$

where

$$\boldsymbol{S} = \frac{1}{\mu_0}(\boldsymbol{E}\times\boldsymbol{B}) \qquad \text{: pointing vector}$$

$$\boldsymbol{G} = \varepsilon_0\mu_0\boldsymbol{S} = \frac{1}{c^2}\boldsymbol{S} \qquad \text{: momentum of the field}$$

$$\boldsymbol{f} = \rho\boldsymbol{E} + (\boldsymbol{J}\times\boldsymbol{B}) \qquad \text{: force density}$$

$$T_{ij} = (\varepsilon_0 E_i E_j + \frac{1}{\mu_0}B_i B_j) - \frac{1}{2}\delta_{ij}(\varepsilon_0\boldsymbol{E}^2 + \frac{1}{\mu_0}\boldsymbol{B}^2)$$

or

$$\mu_0 T_{ij} = (\frac{1}{c^2}E_i E_j + B_i B_j) - \frac{1}{2}\delta_{ij}(\frac{1}{c^2}\boldsymbol{E}^2 + \boldsymbol{B}^2)$$

where $c^2 = \dfrac{1}{\varepsilon_0 \mu_0}$

$$(J_\mu) = (\boldsymbol{J}, ic\rho)$$

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & B_3 & -B_2 & -\frac{i}{c}E_1 \\ -B_3 & 0 & B_1 & -\frac{i}{c}E_2 \\ B_2 & -B_1 & 0 & -\frac{i}{c}E_3 \\ \frac{i}{c}E_1 & \frac{i}{c}E_2 & \frac{i}{c}E_3 & 0 \end{pmatrix}$$

$$\mu_0 T_{\mu\nu} = F_{\mu\alpha}F_{\alpha\nu} + \frac{1}{4}\delta_{\mu\nu}F_{\alpha\beta}F_{\alpha\beta}$$

where

$$F_{\alpha\beta}F_{\alpha\beta} = 2[(B_1^2 + B_2^2 + B_3^2) - \frac{1}{c^2}(E_1^2 + E_2^2 + E_3^2)]$$

$$F_{1\alpha}F_{\alpha 2} = \frac{1}{c^2}E_1E_2 + B_1B_2$$

$$F_{2\alpha}F_{\alpha 3} = \frac{1}{c^2}E_2E_3 + B_2B_3$$

$$F_{1\alpha}F_{\alpha 3} = \frac{1}{c^2}E_3E_1 + B_3B_1$$

$$F_{1\alpha}F_{\alpha 4} = -\frac{i}{c}(E_2B_3 - B_2E_3) = -\frac{i\mu_0}{c}S_1$$

$$F_{2\alpha}F_{\alpha 4} = -\frac{i}{c}(E_3B_1 - B_1E_3) = -\frac{i\mu_0}{c}S_2$$

$$F_{3\alpha}F_{\alpha 4} = -\frac{i}{c}(E_1B_2 - B_2E_1) = -\frac{i\mu_0}{c}S_3$$

$$F_{1\alpha}F_{\alpha 1} = \frac{1}{c^2}E_1^2 - (B_2{}^2 + B_3{}^2)$$

$$F_{2\alpha}F_{\alpha 2} = \frac{1}{c^2}E_2^2 - (B_3{}^2 + B_1{}^2)$$

$$F_{3\alpha}F_{\alpha 3} = \frac{1}{c^2}E_3^2 - (B_1{}^2 + B_2{}^2)$$

$$F_{4\alpha}F_{\alpha 4} = \frac{1}{c^2}(E_1^2 + E_2^2 + E_3^2)$$

The Maxwell's stress tensor is given by

$$\mu_0 T_{11} = \frac{1}{2c^2}(E_1{}^2 - E_2{}^2 - E_3{}^2) + \frac{1}{2}(B_1{}^2 - B_2{}^2 - B_3{}^2)$$

$$\mu_0 T_{12} = \frac{1}{c^2}E_1E_2 + B_1B_2$$

$$\mu_0 T_{13} = \frac{1}{c^2}E_3E_1 + B_3B_1$$

$$\mu_0 T_{14} = -\frac{i}{c}(E_2B_3 - B_2E_3) = -\frac{i\mu_0}{c}S_1 = -i\mu_0 c G_1$$

$$\mu_0 T_{22} = \frac{1}{2c^2}(-E_1{}^2 + E_2{}^2 - E_3{}^2) + \frac{1}{2}(-B_1{}^2 + B_2{}^2 - B_3{}^2)$$

$$\mu_0 T_{23} = \frac{1}{c^2}E_2E_3 + B_2B_3$$

$$\mu_0 T_{24} = -\frac{i}{c}(E_3B_1 - B_1E_3) = -\frac{i\mu_0}{c}S_2 = -i\mu_0 c G_2$$

$$\mu_0 T_{33} = \frac{1}{2c^2}(-E_1{}^2 - E_2{}^2 + E_3{}^2) + \frac{1}{2}(-B_1{}^2 - B_2{}^2 + B_3{}^2)$$

$$\mu_0 T_{34} = -\frac{i}{c}(E_1B_2 - B_2E_1) = -\frac{i\mu_0}{c}S_3 = -i\mu_0 c G_3$$

$$\mu_0 T_{44} = \frac{1}{2}(B_1{}^2 + B_2{}^2 + B_3{}^2) + \frac{1}{2c^2}(E_1{}^2 + E_2{}^2 + E_3{}^2) = \mu_0 u$$

Explicitly, the elements of T are

$$(T_{\mu\nu}) = \begin{pmatrix} T_{11} & T_{12} & T_{13} & -icG_1 \\ T_{21} & T_{22} & T_{23} & -icG_2 \\ T_{31} & T_{32} & T_{33} & -icG_3 \\ -icG_1 & -icG_2 & -icG_3 & u \end{pmatrix}$$

$$T_{11} + T_{22} + T_{33} = -u$$

6. Lorentz force
6.1 Origin of the Lorentz force

Consider a particle of charge q moving with velocity $\boldsymbol{v}$ (along the x axis) with respect to the reference frame S in a region with electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$.

In the frame S, the Lorentz force on this charge is given by

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) = (qE_1, q(E_2 - vB_3), q(E_3 + vB_2))$$

In the frame S', the Lorentz force is given by

$$\boldsymbol{F}' = q\boldsymbol{E}' = (qE_1', qE_2', qE_3')$$

where q is a relativistic invariant and is at rest in the S' frame. Note that the particle is at rest in the S' frame.

((**Note**))
In general case, we have

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})$$
$$\boldsymbol{F}' = q(\boldsymbol{E}' + \boldsymbol{u}' \times \boldsymbol{B}')$$

where $\boldsymbol{u}$ and $\boldsymbol{u}$' are the velocity of the particle in the S-frame and S'-frame (moving with the velocity $\boldsymbol{v}$). When $\boldsymbol{u} = \boldsymbol{v}$, it follows that $\mathbf{u}' = 0$. Then we have

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$
$$\boldsymbol{F}' = q\boldsymbol{E}'$$

The fields in S and S' are related by

$$E_1' = E_1$$
$$E_2' = \gamma(E_2 - vB_3)$$
$$E_3' = \gamma(E_3 + vB_2)$$

Then we have

$$F_1' = qE_1' = qE_1$$
$$F_2' = qE_2' = q\gamma(E_2 - vB_3)$$
$$F_3' = qE_3' = q\gamma(E_3 + vB_2)$$

What is the relation between ***F*** and ***F***'?

$$F_1' = qE_1' = qE_1 = F_1$$
$$F_2' = qE_2' = q\gamma(E_2 - vB_3) = \gamma F_2$$
$$F_3' = qE_3' = q\gamma(E_3 + vB_2) = \gamma F_3$$

or

$$F_1 = F_1'$$
$$\gamma F_2 = F_2'$$
$$\gamma F_3 = F_3'$$

6.2 force density and charge density

$$\boldsymbol{f} = \rho \boldsymbol{E} + (\boldsymbol{J} \times \boldsymbol{B})$$

We choose the frame *S*' in which the system with the charge density is at rest. We now calculate the force density vector

$$\boldsymbol{f}' = \rho' \boldsymbol{E}'$$

since $\boldsymbol{J}' = 0$ (the system is at rest).

We note the Lorentz transformation of 4-dimensional vector, current density and charge density

$$J_\mu = (\boldsymbol{J}, ic\rho)$$

$$J_1 = \gamma(J_1' - i\beta J_4') = \gamma(J_1' + v\rho')$$
$$\rho = \gamma(\frac{\beta}{c} J_1' + \rho')$$

Then we have

$$\rho = \gamma\rho'$$
$$J_1 = \gamma v\rho' = \rho v$$

The Lorentz transformation of ***E*** and ***B***,

$$E_1' = E_1$$
$$E_2' = \gamma(E_2 - vB_3)$$
$$E_3' = \gamma(E_3 + vB_2)$$

Then we have

$$\gamma \boldsymbol{f}' = (\rho' \gamma E_1, \rho' \gamma^2 (E_2 - vB_3), \rho' \gamma^2 (E_3 + vB_2))$$

or

$$\gamma \boldsymbol{f}' = (\rho E_1, \rho\gamma (E_2 - vB_3), \rho\gamma (E_3 + vB_2))$$

In the frame *S*, the Lorentz force is given by

$$\boldsymbol{f} = \rho[\boldsymbol{E} + (\boldsymbol{v} \times \boldsymbol{B})] = (\rho E_1, \rho(E_2 - vB_3), \rho(E_3 + vB_2))$$

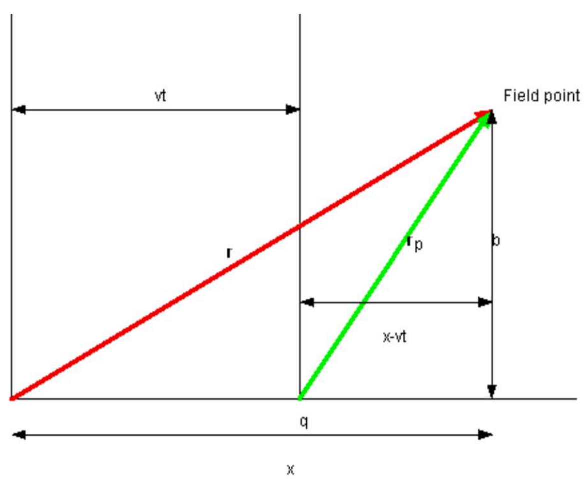
Thus we have

$$\gamma f_1' = f_1$$
$$f_2' = f_2$$
$$f_3' = f_3$$

7. Lienard-Wiechert potential

7.1 Lienard-Wiechert potential

What are the scalar potential and vector potential of a charge *q* at th the velocity *v* moving along the *x* direction. The problem is easy in a coordinate system moving with the charge, since in this frame the charge is standing still.



In the *S'* frame:

$$\phi' = \frac{q}{4\pi\varepsilon_0} \frac{1}{r'}, \qquad \boldsymbol{A}' = 0 .$$

Using the Lorentz transformation given by

$$A_1 = \frac{A_1' + \frac{v}{c^2}\phi'}{\sqrt{1-\beta^2}} \qquad \text{or} \qquad A_1 = \frac{\frac{v}{c^2}\phi'}{\sqrt{1-\beta^2}}$$

$$A_2 = A_2' \qquad \text{or} \qquad A_2 = 0$$

$$A_3 = A_3' \qquad \text{or} \qquad A_3 = 0$$

$$\phi = \frac{vA'_1 + \phi'}{\sqrt{1-\beta^2}} \qquad \text{or} \qquad \phi = \frac{\phi'}{\sqrt{1-\beta^2}} = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{1-\beta^2}}\frac{1}{r'}$$

we get

$$\phi = \frac{\phi'}{\sqrt{1-\beta^2}} = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{1-\beta^2}}\frac{1}{\sqrt{x_1'^2 + x_2'^2 + x_3'^2}}$$

where

$$x_1' = \gamma(x_1 - vt)$$
$$x_2' = x_2$$
$$x_3' = x_3$$

The scalar potential ϕ is given by

$$\varphi = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{1-\beta^2}}\frac{1}{\sqrt{\gamma^2(x_1 - vt)^2 + x_2^2 + x_3^2}}$$
$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{(x_1 - vt)^2 + (1-\beta^2)(x_2^2 + x_3^2)}}$$

or

$$\phi = \frac{q}{4\pi\varepsilon_0}\frac{1}{R^*}$$

with

$$R^* = \sqrt{(x_1 - vt)^2 + (1-\beta^2)(x_2^2 + x_3^2)}$$

Similarly we have for the vector potential

$$\boldsymbol{A} = (A_1, 0, 0)$$

with

$$A_1 = \frac{\frac{v}{c^2}\phi'}{\sqrt{1-\beta^2}} = \frac{v}{c^2}\frac{q}{4\pi\varepsilon_0}\frac{1}{R^*} = \frac{qv\mu_0}{4\pi}\frac{1}{R^*}$$

Then the electric field ***E*** and the magnetic field ***B*** are given by

$$\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{A} - \nabla\varphi \qquad \text{and} \qquad \boldsymbol{B} = \nabla\times\boldsymbol{A}$$

Using the Mathematica, we get

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0}(1-\beta^2)\frac{\boldsymbol{R}}{R^{*3}}$$

$$\boldsymbol{B} = \frac{\boldsymbol{v}}{c^2}\times\boldsymbol{E}$$

where

$$\boldsymbol{R} = (x - vt, y, z)$$

For a slow moving charge ($v<<c$), we can take for ***E*** the Coulomb field. Then w have

$$\boldsymbol{B} = \frac{\boldsymbol{v}}{c^2}\times\boldsymbol{E} = \frac{q\boldsymbol{v}\times\boldsymbol{r}}{4\pi\varepsilon_0 c^2 r^2} = \frac{\mu_0}{4\pi}\frac{q\boldsymbol{v}\times\boldsymbol{r}}{r^2} \qquad \text{(Bio-Savart law)}$$

((**Mathematica-II**))

Lienard-Wiechert potential

Electric field in the frame S

```
E1 = -Grad[ϕ, {x, y, z}] - D[A, t] /. {μ0 → 1/(ϵ0 c^2)} //
  FullSimplify
```

$$\left\{\frac{q\left(-c^2+v^2\right)(t\,v-x)}{4\,c^2\,\pi\left((-t\,v+x)^2-\left(y^2+z^2\right)\left(-1+\beta^2\right)\right)^{3/2}\epsilon 0},\; -\frac{q\,y\left(-1+\beta^2\right)}{4\,\pi\left((-t\,v+x)^2-\left(y^2+z^2\right)\left(-1+\beta^2\right)\right)^{3/2}\epsilon 0},\; -\frac{q\,z\left(-1+\beta^2\right)}{4\,\pi\left((-t\,v+x)^2-\left(y^2+z^2\right)\left(-1+\beta^2\right)\right)^{3/2}\epsilon 0}\right\}$$

```
V1 = {v, 0, 0}; eq1 = 1/c^2 Cross[V1, E1] // Simplify
```

$$\left\{0,\; \frac{q\,v\,z\left(-1+\beta^2\right)}{4\,c^2\,\pi\left((-t\,v+x)^2-\left(y^2+z^2\right)\left(-1+\beta^2\right)\right)^{3/2}\epsilon 0},\; -\frac{q\,v\,y\left(-1+\beta^2\right)}{4\,c^2\,\pi\left((-t\,v+x)^2-\left(y^2+z^2\right)\left(-1+\beta^2\right)\right)^{3/2}\epsilon 0}\right\}$$

```
eq1 - B1 /. {μ0 → 1/(c^2 ϵ0)} // Simplify
```

{0, 0, 0}

Lienard-Wiechert potential

```
Clear["Global`*"];
R = Sqrt[(x - v t)^2 + (1 - β^2) (y^2 + z^2)];
ϕ = q/(4 π ϵ0) 1/R;
A1 = (q v μ0)/(4 π) 1/R;
A = {A1, 0, 0};
B1 = Curl[A, {x, y, z}] // FullSimplify
```

$$\left\{0, \frac{q\, v\, z\, (-1+\beta^2)\, \mu 0}{4\pi\left((-t\, v+x)^2-(y^2+z^2)(-1+\beta^2)\right)^{3/2}}, -\frac{q\, v\, y\, (-1+\beta^2)\, \mu 0}{4\pi\left((-t\, v+x)^2-(y^2+z^2)(-1+\beta^2)\right)^{3/2}}\right\}$$

7.2 Distribution of the electric field

$$\boldsymbol{E}=\frac{q}{4\pi\varepsilon_0}(1-\beta^2)\frac{\boldsymbol{R}_p}{R^{*3}}$$

where

$$R^*=\sqrt{(x-vt)^2+(1-\beta^2)(y^2+z^2)}$$

$$\boldsymbol{R}^*=(x-vt,\sqrt{1-\beta^2}\,y,\sqrt{1-\beta^2}\,z)$$

$$\boldsymbol{R}_p=(x-vt,y,z)$$

$\boldsymbol{R}_p$ is the relative coordinate of the field point and the charge point. The electric field is along the position vector $\boldsymbol{R}_p$. $\boldsymbol{R}_p$ is a vector from the instantaneous location of the charge in S to the point where $\boldsymbol{E}$ is measured in *S*.

((**Mathematica-3**))
The electric field of a charge moving with the constant speed with v ($\beta = v/c$) on the unit circle of the real space.

Lienard-Wiechert problem; field for a uniformly moving charge

```
Clear["Global`*"];
```

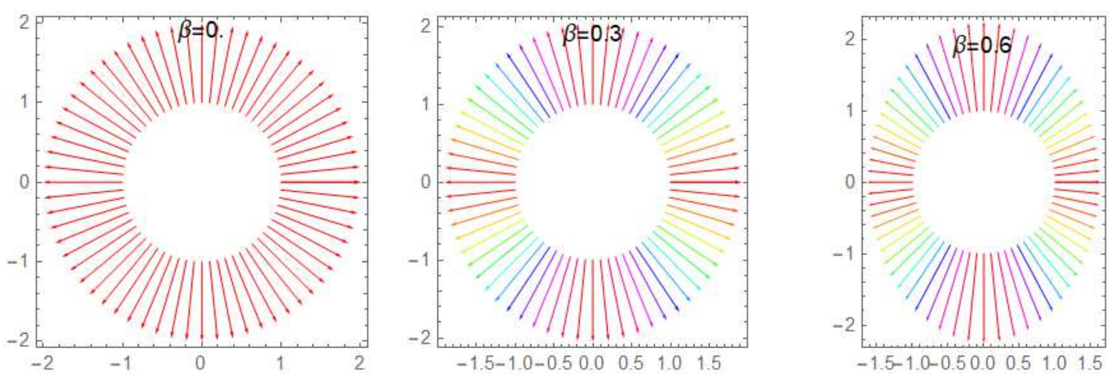
$$E1X[\theta\_, \beta\_] := \frac{x\left(1-\beta^2\right)}{\left(\sqrt{x^2+\left(1-\beta^2\right)y^2}\right)^3} \;/.\; \{x \to Cos[\theta],\; y \to Sin[\theta]\} \;//\; Simplify;$$

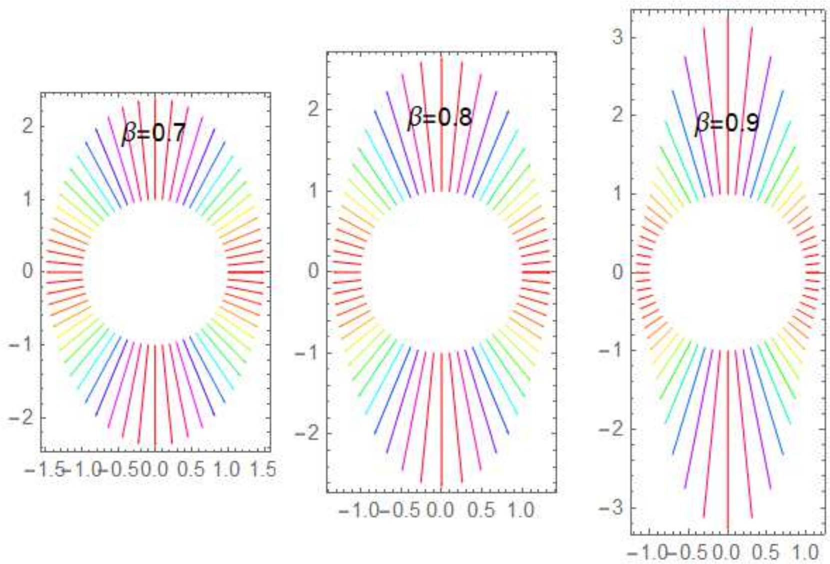
$$E1Y[\theta\_, \beta\_] := \frac{y\left(1-\beta^2\right)}{\left(\sqrt{x^2+\left(1-\beta^2\right)y^2}\right)^3} \;/.\; \{x \to Cos[\theta],\; y \to Sin[\theta]\} \;//\; Simplify;$$

```
s1[β_] := Table[{{Cos[θ], Sin[θ]}, {E1X[θ, β], E1Y[θ, β]}}, {θ, 0, 2π, π/32}];
Needs["VectorFieldPlots`"];
s2[β_] := ListVectorFieldPlot[Evaluate[s1[β]], ColorFunction → Hue,
  AspectRatio → Automatic, Frame → True, AxesOrigin → {0, 0}];
ps1[β_] := Module[{h1, h2, h3}, h1 = Evaluate[s2[β]];
  h2 = Graphics[{
     Text[Style["β=" <> ToString[β], Black, 12], {0, 1.9}]}];
  h3 = Show[h1, h2, PlotRange → All]];
```

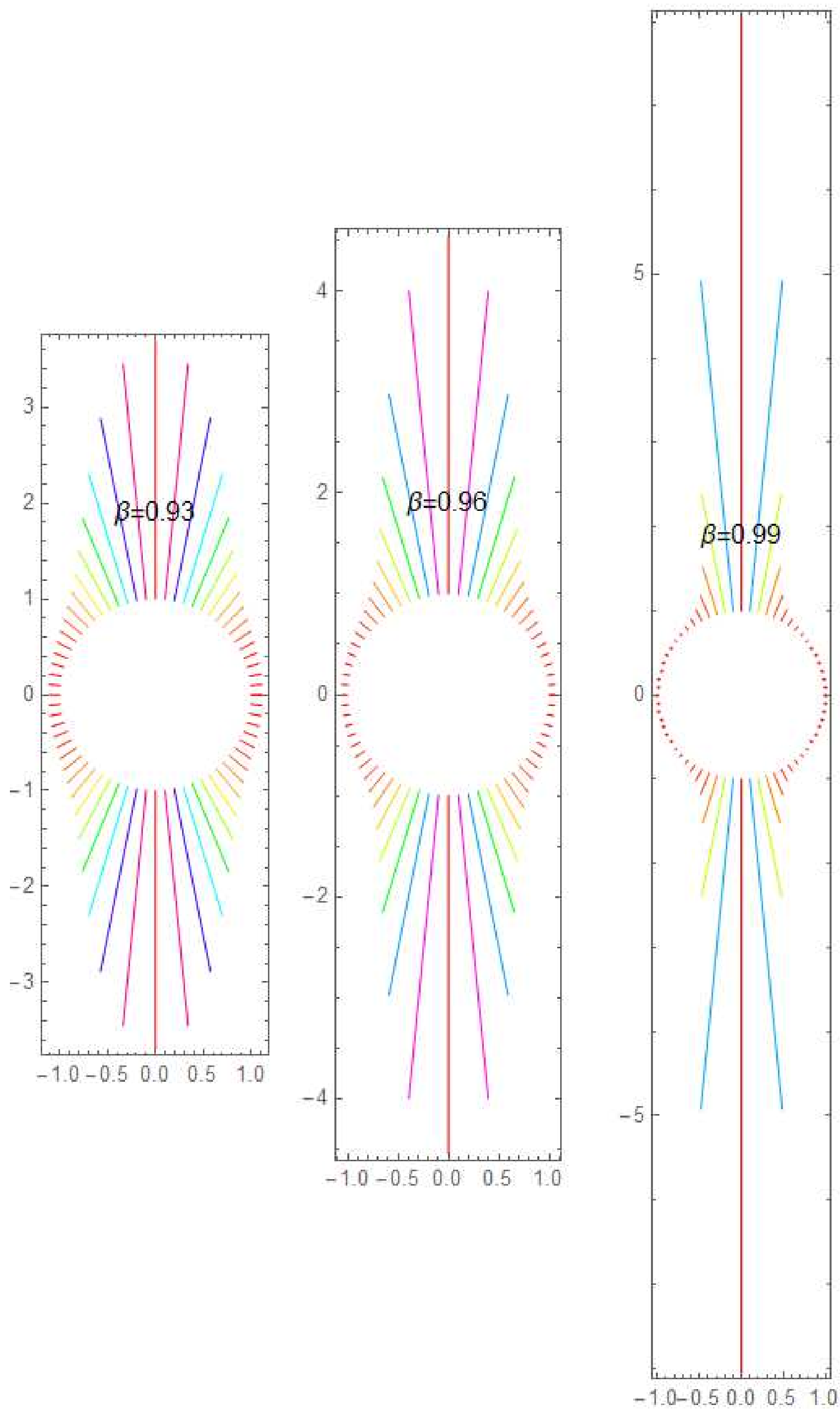
```
qs1 = Table[ps1[β], {β, 0, 0.6, 0.3}]; GraphicsGrid[Partition[qs1, 3]]
```


```
qs1 = Table[ps1[β], {β, 0, 0.6, 0.3}]; GraphicsGrid[Partition[qs1, 3]]
```

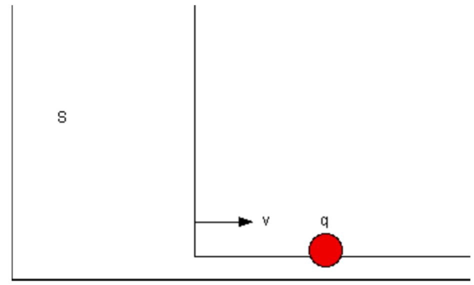






```
qs3 = Table[ps1[β], {β, 0.93, 0.99, 0.03}];
GraphicsGrid[Partition[qs3, 3]]
```



8. Relativity of Electric field and magnetic field



$E = 0$ and $B \neq 0$

We consider the charge q moving along the x axis in the presence of the magnetic field $\boldsymbol{B}$ (the frame S). In *the frame S, there is only an external magnetic field* $\boldsymbol{B}$. Thus the magnetic force on the charge is given by

$$\boldsymbol{F}_\perp = q(\boldsymbol{v} \times \boldsymbol{B})$$

Suppose that there is no electric field ($\boldsymbol{E} = 0$) in the frame S $(\mathbf{B} \neq 0)$. The $\boldsymbol{E}$' and $\boldsymbol{B}$' in the frame S' are related to those in the frame S as

$$E_1' = E_1 = 0$$
$$E_2' = \gamma(E_2 - c\beta B_3) = -\gamma v B_3$$
$$E_3' = \gamma(E_3 + c\beta B_2) = \gamma v B_2$$

$$B_1' = B_1$$
$$B_2' = \gamma(B_2 + \frac{\beta}{c} E_3) = \gamma B_2$$
$$B_3' = \gamma(B_3 - \frac{\beta}{c} E_2) = \gamma B_3$$

or

$$\boldsymbol{E}' = \gamma(\boldsymbol{v} \times \boldsymbol{B}) = \boldsymbol{v} \times \boldsymbol{B}' \qquad (1)$$

Then the force (electric force) on the charge q in the frame S' is

$$\boldsymbol{F}'_\perp = q\boldsymbol{E}' = q\gamma(\boldsymbol{v} \times \boldsymbol{B})$$

since the charge q is the same for any frame and the particle is at rest in the frame S'. There is no force due to $\boldsymbol{B}$' since the particle is at rest in the frame S'. $\mathbf{F}'_\perp$ is the force of $\boldsymbol{F}$' in a direction perpendicular to the velocity $\boldsymbol{v}$. Thus we have

$$\boldsymbol{F}_\perp = \frac{1}{\gamma} \boldsymbol{F}_\perp'$$

9. Derivation of the Biot Savart law

$\boldsymbol{B}$'= 0 and $\boldsymbol{E}$'≠ 0.

We consider that the magnetic field $\boldsymbol{B}'=0$ *in the frame S'.* In the frame *S'*, there is only an external electric field $\boldsymbol{E}'$ (the point charge is at rest). The $\boldsymbol{E}$ and $\boldsymbol{B}$ in the frame *S* are related to those in the frame *S'* as

$$E_1 = E_1' \qquad B_1 = 0$$

$$E_2 = \gamma E_2' \qquad B_2 = -\frac{\gamma}{c^2} v E_3'$$

$$E_3 = \gamma E_3' \qquad B_3 = \frac{\gamma}{c}\beta E_2' = \frac{\gamma}{c^2} v E_2'$$

or

$$\boldsymbol{B} = \frac{\gamma}{c^2}(\boldsymbol{v} \times \boldsymbol{E}') = \frac{1}{c^2}(\boldsymbol{v} \times \boldsymbol{E}), \tag{2}$$

Using the result from the Lienard-Wiechert potential (β<<1) (see Sec.8)

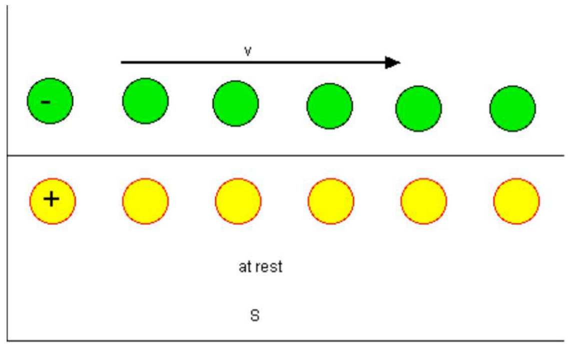
$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0}(1-\beta^2)\frac{\boldsymbol{R}}{R^{*3}} \approx \frac{q}{4\pi\varepsilon_0}\frac{\boldsymbol{r}}{r^3}$$

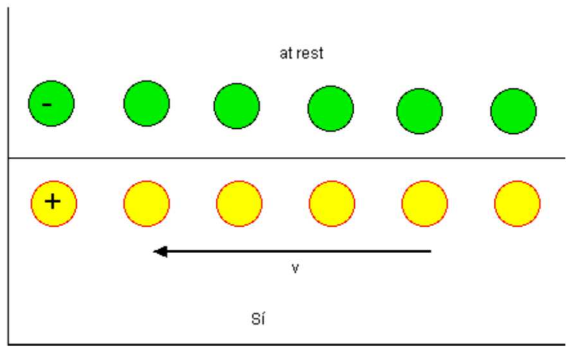
$$\boldsymbol{B} = \frac{1}{c^2}(\boldsymbol{v} \times \boldsymbol{E}) = \frac{1}{c^2}\frac{q}{4\pi\varepsilon_0}\frac{\boldsymbol{v} \times \boldsymbol{r}}{r^3} = \frac{\mu_0}{4\pi}\frac{q\boldsymbol{v} \times \boldsymbol{r}}{r^3}$$

which is the application of the Biot-Savart law to a point charge.

10. Ampere's law (Feynman 13-9)

We consider that the electrons located on the linear chain (the line density $-\lambda_0$) moves at the velocity $\boldsymbol{v}$. At the same time there are positive ions located on the same chain (the line density λ_0). We now consider the frame S' which moves at the velocity $\boldsymbol{v}$.





((**Formula**))

$$\rho = \gamma \rho'$$

where ρ for the frame where the particle moves at the velocity *v* along the *x* axis, and ρ' for the frame where the particle is at rest.

We assume that
(1) The line densities of electrons and positive ions are given by $-\lambda_0$ and λ_0 in the frame S.
(2) The line densities of electrons and positive ions are given by $-\lambda_-$ and λ_+ in the frame *S*'

$$(-\lambda_0) = \gamma(-\lambda_-) \qquad \text{or} \qquad \lambda_- = \frac{1}{\gamma}\lambda_0 = \sqrt{1-\frac{v^2}{c^2}}\lambda_0 \qquad \text{for electrons}$$

$$\lambda_+ = \gamma\lambda_0 \qquad \text{or} \qquad \lambda_+ = \gamma\lambda_0 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\lambda_0 \qquad \text{for ions}$$

The net line charge density in the frame S' is

$$\lambda' = \lambda_+ - \lambda_- = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\lambda_0 - \sqrt{1-\frac{v^2}{c^2}}\lambda_0 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{v^2}{c^2}\lambda_0 = \gamma\frac{v^2}{c^2}\lambda_0$$

((**Note**))

This relation can be also derived from the Lorentz transformation of the 4-dimensional current density

$$J_\mu = (\boldsymbol{J}, ic\rho) = (\rho\boldsymbol{u}, ic\rho)$$

$$J_\mu' = a_{\mu\nu}J_\nu$$

$$J_1' = \gamma(J_1 + i\beta J_4) = \gamma(J_1 - v\rho)$$

$$\rho' = \gamma(-\frac{\beta}{c}J_1 + \rho)$$

$$J_\mu = a_{\nu\mu}J_\nu'$$

$$J_1 = \gamma(J_1' - i\beta J_4') = \gamma(J_1' + v\rho')$$

$$\rho = \gamma(\frac{\beta}{c}J_1' + \rho')$$

Here we define

$$\lambda = A\rho$$

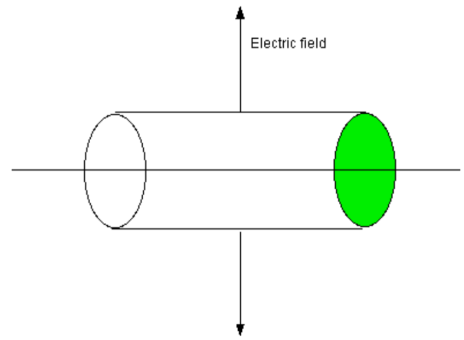
$$\lambda' = A\rho'$$

A is the same for the S and S', since the plane of A is perpendicular to $\boldsymbol{v}$.

$$\lambda' = \gamma(-\frac{\beta}{c}J_1 + \lambda) = \gamma\frac{\beta}{c}\lambda_0 v = \gamma\frac{v^2}{c^2}\lambda_0$$

where $I_1 = AJ_1 = (-\lambda_0)v$ and $\lambda = 0$.

So the positive line density produces an electric field ***E***'. We use the Gauss's law.



The electric field ***E***' at the distance *s* from the axis of the cylinder,

$$E'(2\pi s h) = \frac{1}{\varepsilon_0}(h\lambda')$$

where *s* is the radius of the Gaussian surface (cylinder).

or

$$E' = \frac{\lambda'}{2\pi\varepsilon_0 s} = \frac{1}{2\pi\varepsilon_0 s}\gamma\frac{v^2}{c^2}\lambda_0$$

So there is an electrical force on *q* in *S*';

$$F_\perp' = qE' = \frac{q}{2\pi\varepsilon_0 s}\gamma\frac{v^2}{c^2}\lambda_0 .$$

But if there is a force on the test charge *q* in *S*', there must be one in *S*. In fact, one can calculate it by using the transformation rules for forces. Since *q* is at rest *S*' and $F_\perp$ is perpendicular to the *x* axis, we have

$$F_\perp' = \gamma F_\perp \qquad \text{or} \qquad F_\perp = \frac{1}{\gamma}F_\perp'$$

Using this result we have

$$F_\perp = \frac{1}{\gamma}F_\perp' = \frac{q}{2\pi\varepsilon_0 s}\frac{v^2}{c^2}\lambda_0 = \frac{q\mu_0}{2\pi s}v(v\lambda_0) = q\frac{\mu_0(v\lambda_0)}{2\pi s}v$$

where $B = \dfrac{\mu_0(v\lambda_0)}{2\pi s}$ is a magnetic field due to the line current density $v\lambda_0$ (Ampere's law). The force has a form as $F = qvB$.

11. Derivation of the Ampere's law

We analyze the fields and currents as viewed from two frames; *S* where the ions are at rest. *S*' where the electrons are, on the average, at rest.

$$J_\mu = (\boldsymbol{J}, ic\rho)$$

$$J_\mu = (a^{-1})_{\mu\nu} J_\nu' = a_{\nu\mu} J_\nu'$$

Multiplying the cross-sectional area (*A*) of the wires, we obtain the following transformation for currents and linear charge densities.

$$I_\mu = AJ_\mu = (A\boldsymbol{J}, icA\rho) = (\boldsymbol{I}, ic\lambda)$$

$$I_\mu = a_{\nu\mu} I_\nu'$$

$$I_1 = \gamma(I_1' - i\beta I_4') = \gamma(I_1' + v\lambda')$$

$$I_4 = ic\lambda = \gamma(i\beta I_1' + I_4') = \gamma(i\frac{v}{c} I_1' + ic\lambda')$$

or

$$I_\pm = \gamma(I_\pm' + v\lambda_\pm')$$

$$\lambda_\pm = \gamma(\frac{v}{c^2} I_\pm' + \lambda_\pm')$$

where $\lambda = A\rho$, the subscript 1 (*x* axis) is neglected and the plus and minus subscript refer to the ions and the electrons, respectively.

In *S*' we know that $I_-' = 0$ since the electrons are at rest.

$$\lambda_- = \gamma(\frac{v}{c^2} I_-' + \lambda_-') = \gamma\lambda_-'$$

In *S* the net charge per unit length must vanish.

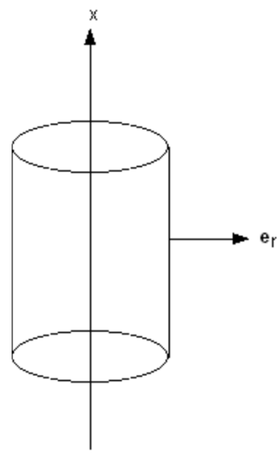
$$0 = \lambda_+ + \lambda_- = \lambda_+ + \gamma\lambda_-'$$

or

$$\lambda_-' = -\frac{\lambda_+}{\gamma}$$

The fields in *S*' due to λ_-' are

$$\boldsymbol{E}_-' = \frac{\lambda_-'}{2\pi\varepsilon_0 r'}\boldsymbol{e}_r$$
$$\boldsymbol{B}_-' = 0$$



The fields in *S* due to λ_+ are

$$\boldsymbol{E}_+ = \frac{\lambda_+}{2\pi\varepsilon_0 r}\boldsymbol{e}_r$$
$$\boldsymbol{B}_+ = 0$$

We now consider the field transformation from

$$\boldsymbol{E}_-' = \frac{\lambda_-'}{2\pi\varepsilon_0 r'}\boldsymbol{e}_r$$
$$\boldsymbol{B}_-' = 0$$

to $\boldsymbol{E}_+$ and $\boldsymbol{B}_-$. Noting that $\hat{r} = \hat{r}'$ (perpendicular to the *x* axis), we find that the fields in *S* are

$$E_1 = E_1' = 0$$
$$E_2 = \gamma(E_2' + c\beta B_3') = \gamma E_2'$$
$$E_3 = \gamma E_3'$$

$$B_1 = B_1' = 0$$
$$B_2 = \frac{\gamma}{c}(cB_2' - \beta E_3') = -\frac{\gamma v}{c^2}E_3'$$
$$B_3 = \frac{\gamma}{c}(\beta E_2' + cB_3') = \frac{\gamma v}{c^2}E_2'$$

or

$$E_- = \gamma \frac{e_r}{2\pi\varepsilon_0 r}\lambda_-',$$

$$B_- = \frac{\gamma}{c^2}(v \times E_-')$$

Then the total fields in the frame *S* are

$$E = E_+ + E_- = \frac{e_r}{2\pi\varepsilon_0 r}(\lambda_+ + \gamma\lambda_-') = 0$$

$$\begin{aligned}
B &= B_+ + B_- \\
&= B_- \\
&= \frac{\gamma}{c^2}(v \times E_-') \\
&= \frac{\gamma}{c^2}(v \times e_r)\frac{\lambda_-'}{2\pi\varepsilon_0 r'} \\
&= \frac{\gamma\lambda_-' v}{2\pi\varepsilon_0 c^2}\frac{(e_x \times e_r)}{r}
\end{aligned}$$

Since $I_- = \gamma(I_-' + v\lambda_-')$ and $I_-' = 0$, we have

$$I_- = \gamma v \lambda_-'$$

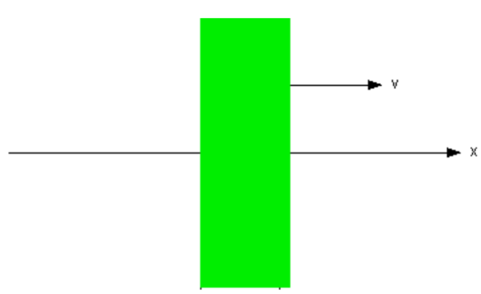
Using $\mathbf{e}_x \times \mathbf{e}_r = \mathbf{e}_\phi$, we obtain

$$B = \frac{\mu_0 I_-}{2\pi r}e_\varphi$$
$$E = 0$$

We see that a magnetic field due to current flow is a relativistic effect.

12. Capacitance moving along the *x* axis with a uniform velocity

12.1 The capacitance moves along the *x* direction which is parallel to the electric field of the capacitance.



In the frame S' where the charges are at rest.

$$E_1' = \frac{\sigma'}{\varepsilon_0}$$
$$E_2' = 0$$
$$E_3' = 0$$

and

$$B_1' = 0$$
$$B_2' = 0$$
$$B_3' = 0$$

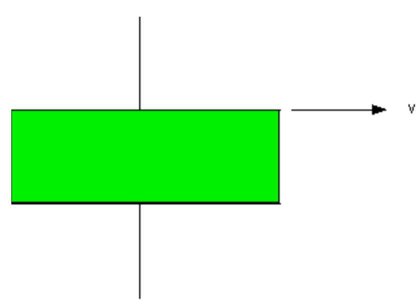
$$E_1 = E_1' \qquad B_1 = 0$$
$$E_2 = 0 \qquad B_2 = 0$$
$$E_3 = 0 \qquad B_3 = 0$$

Thus we have

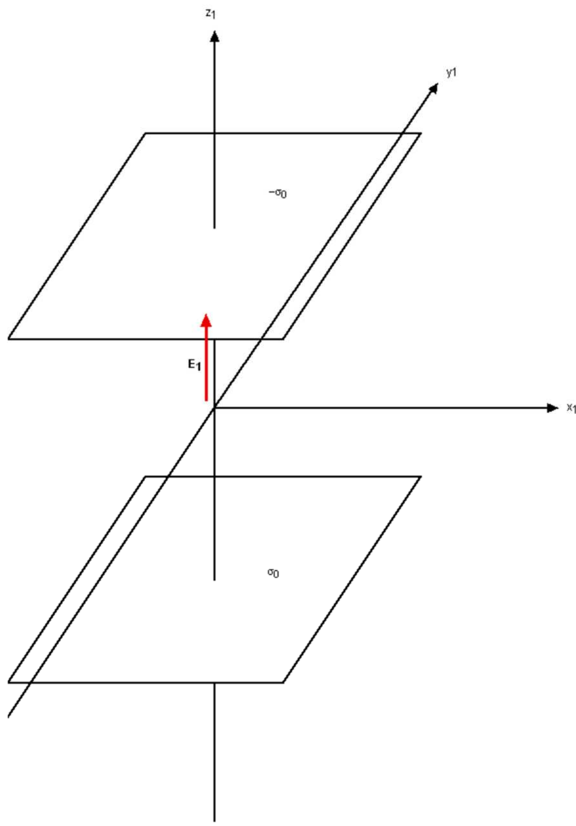
$$E_1 = E_1'$$

where $\sigma' = \sigma$

12.2 The capacitance moves along the *x* direction which is perpendicular to the electric field of the capacitance.



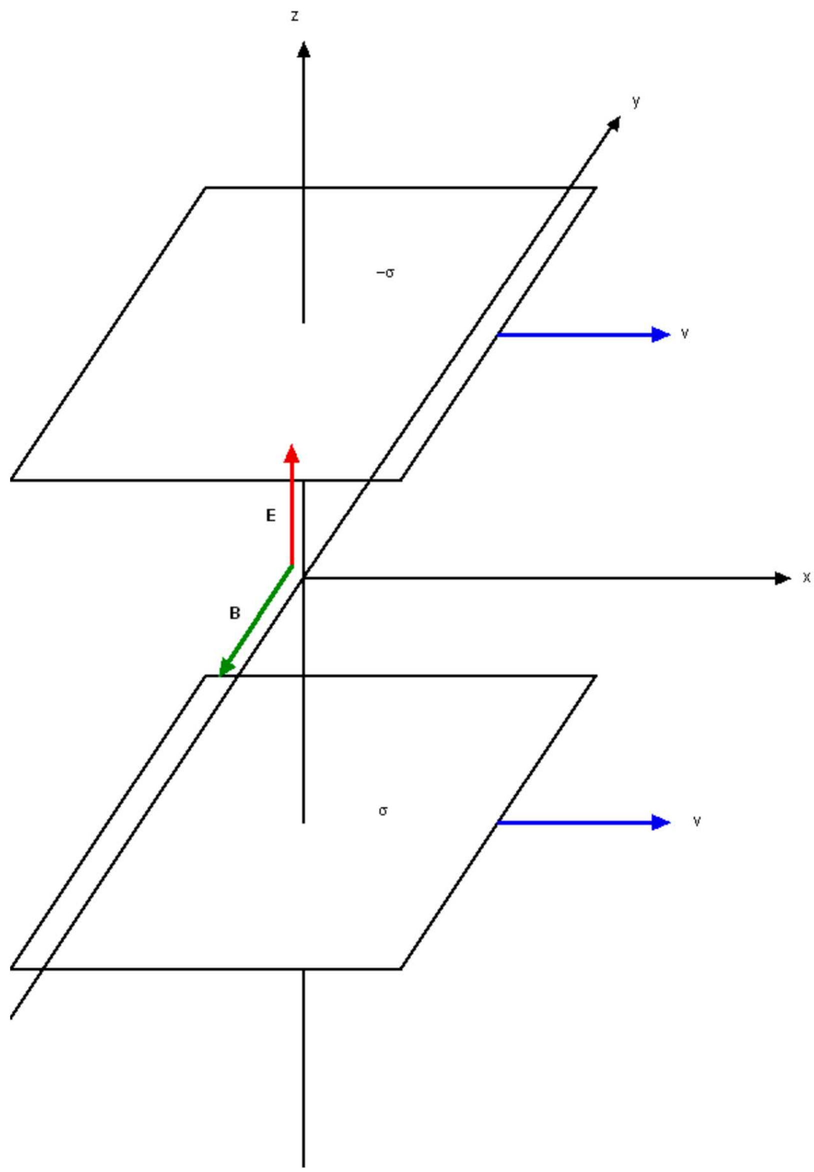
In the frame S' where the charges are at rest.



$$E_1' = 0 \qquad B_1' = 0$$
$$E_2' = 0 \qquad B_2' = 0$$
$$E_3' = \frac{\sigma'}{\varepsilon_0} \qquad B_3' = 0$$

where

$$\sigma = \gamma\sigma' = \gamma\sigma_0$$



Lorentz transformation

$$E_1 = E_1' = 0$$
$$E_2 = \gamma(E_2' + c\beta B_3') = 0$$
$$E_3 = \gamma(-c\beta B_2' + E_3') = \gamma E_3' = \gamma \frac{\sigma'}{\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

$$B_1 = B_1' = 0$$

$$B_2 = \gamma(B_2' - \frac{\beta}{c}E_3') = \frac{-\beta E_3'}{c\sqrt{1-\beta^2}} = -\gamma\frac{v}{c^2}\frac{\sigma'}{\varepsilon_0} = -\frac{v}{c^2}\frac{\sigma}{\varepsilon_0} = -\mu_0 v\sigma$$

$$B_3 = \gamma(\frac{\beta}{c}E_2' + B_3') = 0$$

13. Lagrangian and Hamiltonian in terms of the field tensor $F_{\mu\nu}$

$$F_{\mu\nu}F_{\mu\nu} = 2(B_1^2 + B_2^2 + B_3^2) - \frac{2}{c^2}(E_1^2 + E_2^2 + E_3^2)$$

This is invariant under the Lorentz transformation.

We may try the Lagrangian density

$$L = -\frac{1}{4\mu_0}F_{\mu\nu}F_{\mu\nu} + J_\mu A_\mu$$

By regarding each component of A_μ as an independent field, we find that the Lagrange equation

$$\frac{\partial L}{\partial A_\mu} = \frac{\partial}{\partial x_\nu}[\frac{\partial L}{\partial(\frac{\partial A_\mu}{\partial x_\nu})}]$$

is equivalent to

$$\frac{\partial F_{\mu\nu}}{\partial x_\mu} = \mu_0 J_\mu .$$

The Hamiltonian density H_{em} for the free Maxwell field can be evaluated as follows.

$$L_{em} = -\frac{1}{4\mu_0}F_{\mu\nu}F_{\mu\nu}$$

$$H_{em} = \frac{\partial L_{em}}{\partial\left(\frac{\partial A_\mu}{\partial x_4}\right)}\frac{\partial A_\mu}{\partial x_4} - L_{em} = -\frac{F_{4\mu}}{\mu_0}(F_{4\mu} + \frac{\partial A_4}{\partial x_\mu}) - \frac{1}{2\mu_0}(\boldsymbol{B}^2 - \frac{1}{c^2}\boldsymbol{E}^2)$$

or

$$H_{em} = \frac{1}{2}\varepsilon_0 \boldsymbol{E}^2 + \frac{1}{2\mu_0}\boldsymbol{B}^2 - \varepsilon_0 \boldsymbol{E}\cdot\nabla\phi$$

$$\int H_{em} d\boldsymbol{r} = \frac{1}{2}\int (\varepsilon_0 \boldsymbol{E}^2 + \frac{1}{2\mu_0}\boldsymbol{B}^2) d\boldsymbol{r} - \int \varepsilon_0 (\boldsymbol{E}\cdot\nabla\phi) d\boldsymbol{r}$$
$$= \frac{1}{2}\int (\varepsilon_0 \boldsymbol{E}^2 + \frac{1}{2\mu_0}\boldsymbol{B}^2) dr$$

((**Note**))

$$\int (\boldsymbol{E}\cdot\nabla\phi) d\boldsymbol{r} = \int [\nabla\cdot(\boldsymbol{E}\phi) - \phi\nabla\cdot\boldsymbol{E}] d\boldsymbol{r}$$
$$= \int \nabla\cdot(\boldsymbol{E}\phi) d\boldsymbol{r}$$
$$= \int (\boldsymbol{E}\phi)\cdot d\boldsymbol{a} = 0$$

where $\boldsymbol{E}\phi$ vanishes sufficiently rapidly at infinity.

$$\nabla\cdot\boldsymbol{E} = \frac{\rho}{\varepsilon_0} = 0 \text{ (in this case).}$$

E is the energy of a free particle

14 Lorentz force in the relativistic mechanics

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} = q[\boldsymbol{E} + (\boldsymbol{u}\times\boldsymbol{B})]$$

holds in an arbitrary frame *S*, where u is the velocity of the system. This expression is the correct relativistic form for Newton's second law. The momentum form is more fundamental.

The four-dimensional momentum is given by

$$\boldsymbol{p} = m_0 \frac{\boldsymbol{u}}{\sqrt{1 - \frac{\boldsymbol{u}^2}{c^2}}} = m_0 \gamma(\boldsymbol{u})\boldsymbol{u}$$

$$E_{kin} = \frac{m_0 c^2}{\sqrt{1 - \frac{\boldsymbol{u}^2}{c^2}}} = m_0 c^2 \gamma(\boldsymbol{u})$$

$$\frac{E_{kin}{}^2}{c^2} = \frac{m_0{}^2 c^2}{1 - \frac{\boldsymbol{u}^2}{c^2}} = \frac{m_0{}^2 c^2 (1 - \frac{\boldsymbol{u}^2}{c^2}) + m_0{}^2 \boldsymbol{u}^2}{1 - \frac{\boldsymbol{u}^2}{c^2}} = m_0{}^2 c^2 + \boldsymbol{p}^2$$

or

$$E_{kin} = c(m_0{}^2 c^2 + \boldsymbol{p}^2)^{1/2}$$

where we use E_{kin} instead of E in order to avoid confusion between the kinetic energy and the electric field.

The final form of the equation of motion is given by

$$\frac{d}{dt}\boldsymbol{p} = \boldsymbol{F} = q[\boldsymbol{E} + (\boldsymbol{u} \times \boldsymbol{B})] \qquad (1)$$

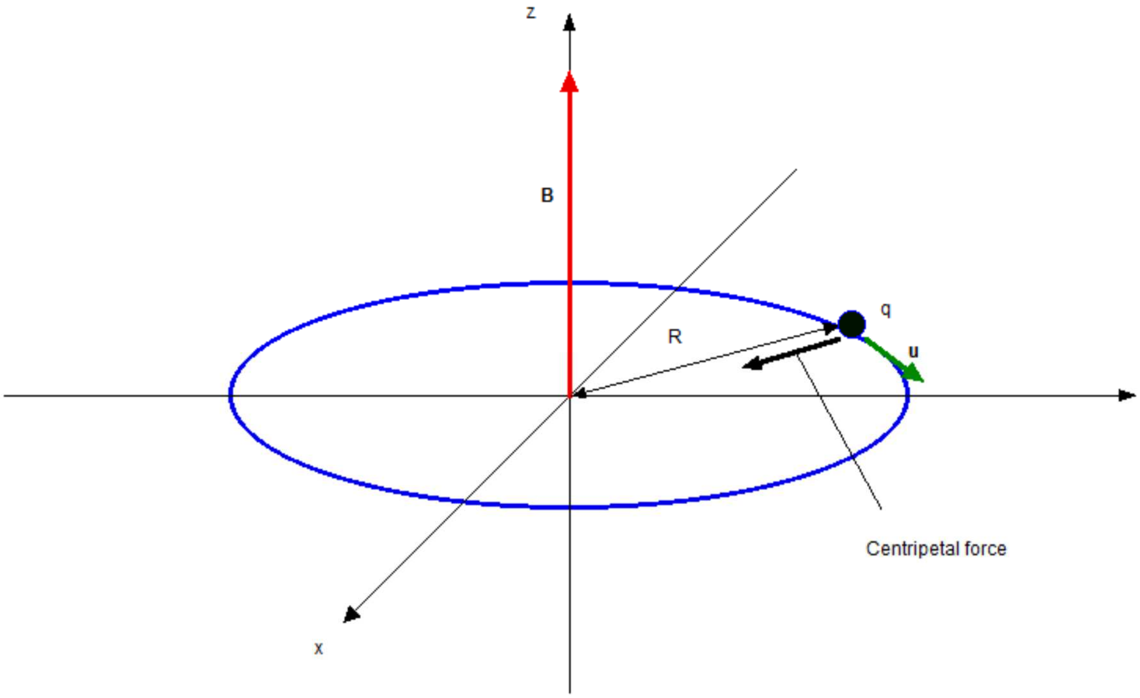
$$\boldsymbol{p} = \frac{m_0 \boldsymbol{u}}{\sqrt{1 - \frac{\boldsymbol{u}^2}{c^2}}},$$

$$\frac{dE_{kin}}{dt} = \boldsymbol{F} \cdot \boldsymbol{u} = q(\boldsymbol{u} \cdot \boldsymbol{E})$$

where

$$E_{kin} = \frac{m_0 c^2}{\sqrt{1 - \frac{\boldsymbol{u}^2}{c^2}}} = c\sqrt{m_0{}^2 c^2 + \boldsymbol{p}^2} \qquad (2)$$

15. Cyclotron motion: a particle in a uniform magnetic field along the *z* axis.



We now consider the case of $\boldsymbol{E} = 0$.

$$\frac{d}{dt}E_{kin} = \boldsymbol{F}\cdot\boldsymbol{u}$$
$$= q(\boldsymbol{u}\times\boldsymbol{B})\cdot\boldsymbol{u}$$
$$= 0$$

Thus we have $\gamma(\boldsymbol{u}) = \dfrac{1}{\sqrt{1-\dfrac{\boldsymbol{u}^2}{c^2}}} = \dfrac{E_{kin}}{m_0c^2}$ = constant. This means that the magnitude of the velocity remains unchanged.

The momentum:

$$\boldsymbol{p} = \frac{E_{kin}\boldsymbol{u}}{c^2}$$

From the equation of motion

$$\frac{d}{dt}\boldsymbol{p} = \frac{d}{dt}\frac{E_{kin}\boldsymbol{u}}{c^2}$$
$$= \boldsymbol{F}$$
$$= q(\boldsymbol{u}\times\boldsymbol{B})$$

we have

$$\frac{d}{dt}\boldsymbol{u} = \frac{c^2}{E_{kin}} q(\boldsymbol{u} \times \boldsymbol{B})$$

or

$$\dot{u}_x = \frac{c^2 qB}{E_{kin}} u_y$$

$$\dot{u}_y = -\frac{c^2 qB}{E_{kin}} u_x$$

$$\dot{u}_z = 0$$

We use the complex plane for the solution.

$$\frac{d}{dt}(u_x + iu_y) = -\frac{ic^2 qB}{E_{kin}}(u_x + iu_y)$$

or

$$(u_x + iu_y) = (u_x{}^0 + iu_y{}^0)\exp[-\frac{ic^2 qBt}{E_{kin}}] = v\exp[-i(\omega t + \alpha)]$$

where

$$\omega = \frac{c^2 qB}{E_{kin}} = \frac{c^2 qB}{m_0 c^2 \gamma} = \frac{qB}{m_0 \gamma}$$

$$u_x{}^0 + iu_y{}^0 = ue^{-i\alpha}$$

Then we have

$$v_x = \frac{dx}{dt} = v\cos(\omega t + \alpha)$$

$$v_x = \frac{dy}{dt} = -v\sin(\omega t + \alpha)$$

or

$$u_x{}^2 + u_y{}^2 = u^2 = constant$$

$$x = \frac{u}{\omega}\sin(\omega t + \alpha) + x_1$$

$$y = \frac{u}{\omega}\cos(\omega t + \alpha) + y_1$$

This equation describes a cyclotron motion (circular motion with radius *R*).

$$R = \frac{v}{\omega} = \frac{vE_{kin}}{c^2 qB} = \frac{p}{qB}$$

where ω is the angular frequency,

$$\omega = \frac{c^2 qB}{E_{kin}}$$

or

$$p = qBR$$

The radius has a maximum when $\frac{v}{c} = \frac{1}{\sqrt{2}}$

In summary

$$x = \frac{\sqrt{p_{0x}{}^2 + p_{0y}{}^2}}{qB}\sin(\frac{c^2 qB}{E_{kin}}t + \alpha)$$

$$y = \frac{\sqrt{p_{0x}{}^2 + p_{0y}{}^2}}{qB}\cos(\frac{c^2 qB}{E_{kin}}t + \alpha)$$

16. The motion of the particle under an electric field ($\boldsymbol{E} = -\nabla\varphi$)

$$\frac{d}{dt}E_{kin} = q(\boldsymbol{u}\cdot\boldsymbol{E}) = -qv\cdot\nabla\varphi = -q\frac{d}{dt}\varphi$$

or

$$\frac{d}{dt}(E_{kin} + q\phi) = 0$$

or

$$E_{kin} + q\phi = \text{constant}$$

We now consider the capacitance consisting of two parallel planes. Suppose that the particle with charge q on the one plate moves to the other plate. The initial velocity is equal to zero. What is the velocity of the particle arriving at the other plate?

$$E_{kin} + q\phi_2 = m_0 c^2 + q\phi_1$$

When $\phi = \phi_1 - \phi_2$,

$$\frac{1}{1 - \frac{\boldsymbol{u}^2}{c^2}} = \left(1 + \frac{q\varphi}{m_0 c^2}\right)^2$$

or

$$u = c\left[1 - \frac{1}{\left(1 + \frac{q\phi}{m_0 c^2}\right)^2}\right]^{1/2}$$

17. Equation of motion under a constant electric field

We assume that $\boldsymbol{E}$ is along the y axis. The initial momentum $\boldsymbol{p}_0$ is in the (x, y) plane. The particle is at the origin at $t = 0$.

$$\frac{d}{dt}\boldsymbol{p} = q\boldsymbol{E} \qquad (1)$$

$$\boldsymbol{p} = \boldsymbol{p}_0 + q\boldsymbol{E}t$$

or

$$\boldsymbol{p} = (p_{0x}, qEt + p_{0y}, 0)$$

$$\begin{aligned} E_{kin} &= c(m_0{}^2 c^2 + \boldsymbol{p}^2)^{1/2} \\ &= [m_0{}^2 c^4 + c^2(p_{0x}{}^2 + p_{0y}{}^2) + c^2(q^2 E^2 t^2 + 2p_{0y} qEt)]^{1/2} \end{aligned}$$

or

$$E_{kin} = [(E_{kin}{}^0)^2 + c^2(q^2 E^2 t^2 + 2p_{0y} qEt)]^{1/2}$$

where $E_{kin}{}^0$ is the kinetic energy at the beginning of the motion ($t = 0$).

$$E_{kin}{}^0 = \sqrt{m_0{}^2 c^4 + c^2 (p_{0x}{}^2 + p_{0y}{}^2)}$$

$$\begin{aligned}\boldsymbol{p} &= \frac{m_0 \boldsymbol{u}}{\sqrt{1-\frac{\boldsymbol{v}^2}{c^2}}} \\ &= m_0 \boldsymbol{u} \frac{E_{kin}}{m_0 c^2} \\ &= \frac{E_{kin} \boldsymbol{u}}{c^2}\end{aligned}$$

Thus we have

$$\begin{aligned}\boldsymbol{u} &= \frac{c^2}{E_{kin}} \boldsymbol{p} \\ &= \frac{c^2}{E_{kin}} (p_{0x}, qEt + p_{0y}, 0) \\ &= \frac{c^2}{[(E_{kin}{}^0)^2 + c^2 (q^2 E^2 t^2 + 2 p_{0y} qEt)]^{1/2}} (p_{0x}, qEt + p_{0y}, 0)\end{aligned}$$

or

$$\frac{dx}{dt} = \frac{c^2 p_{0x}}{[(E_{kin}{}^0)^2 + c^2 (q^2 E^2 t^2 + 2 p_{0y} qEt)]^{1/2}}$$

$$\frac{dy}{dt} = \frac{c^2 (p_{0y} + qEt)}{[(E_{kin}{}^0)^2 + c^2 (q^2 E^2 t^2 + 2 p_{0y} qEt)]^{1/2}}$$

$$\frac{dz}{dt} = 0$$

Solving these differential equations (we use the Mathematica),

$$y = \frac{1}{qE} [\sqrt{(E_{kin}{}^0)^2 + c^2 q^2 E^2 t^2 + 2 p_{0y} c^2 Eqt} - E_{kin}{}^0]$$

or

$$y = \frac{c}{qE}[\sqrt{(p_{0y} + qEt)^2 + m_0{}^2c^2 + p_{0x}{}^2} - \frac{E_{kin}{}^0}{c}]$$

$$x = \frac{cp_{0x}}{qE}\ln\frac{p_{0y} + qEt + \sqrt{(p_{0y} + qEt)^2 + m_0{}^2c^2 + p_{0x}{}^2}}{p_{0y} + \frac{E_{kin}{}^0}{c}}$$

$$z = 0$$

We now consider the special case when $p_{0y} = 0$.

$$\frac{qEy}{c} = \sqrt{(qEt)^2 + m_0{}^2c^2 + p_{0x}{}^2} - \sqrt{m_0{}^2c^2 + p_{0x}{}^2}$$

$$x = \frac{cp_{0x}}{qE}\ln\frac{qEt + \sqrt{(qEt)^2 + m_0{}^2c^2 + p_{0x}{}^2}}{\sqrt{m_0{}^2c^2 + p_{0x}{}^2}}$$

or

$$\exp(\frac{qEx}{cp_{0x}}) = \frac{qEt + \sqrt{(qEt)^2 + m_0{}^2c^2 + p_{0x}{}^2}}{\sqrt{m_0{}^2c^2 + p_{0x}{}^2}}$$

$$\exp(-\frac{qEx}{cp_{0y}}) = \frac{\sqrt{m_0{}^2c^2 + p_{0x}{}^2}}{qEt + \sqrt{(qEt)^2 + m_0{}^2c^2 + p_{0x}{}^2}}$$
$$= \frac{-qEt + \sqrt{(qEt)^2 + m_0{}^2c^2 + p_{0x}{}^2}}{\sqrt{m_0{}^2c^2 + p_{0x}{}^2}}$$

$$\cosh(\frac{qEx}{cp_{0x}}) = \frac{\sqrt{(qEt)^2 + m_0{}^2c^2 + p_{0x}{}^2}}{\sqrt{m_0{}^2c^2 + p_{0x}{}^2}}$$

$$\frac{qEy}{c} + \sqrt{m_0{}^2c^2 + p_{0x}{}^2} = \sqrt{(qEt)^2 + m_0{}^2c^2 + p_{0x}{}^2}$$
$$= \sqrt{m_0{}^2c^2 + p_{0x}{}^2}\cosh(\frac{qEx}{cp_{0x}})$$

or

$$y = \frac{E_{kin}{}^0}{qE}[\cosh(\frac{qEx}{cp_{0x}}) - 1]$$

$$= \frac{E_{kin}{}^0}{qE}[\cosh(\frac{qEx}{cp_{0x}}) - 1]$$

$$= \frac{c}{qE}\sqrt{m_0{}^2c^2 + p_{0x}{}^2}[\cosh(\frac{qEx}{cp_{0x}}) - 1]$$

Thus in a uniform electric field, a charge q moves along a catenary curve.

((**Mathematica-4**))

Zimmerman

2D motion of a relativistic particle in a uniform electric field

Y1 = $\{\frac{px0}{m}, \frac{q\,E0\,t}{m}, 0\}$

$\{\frac{px0}{m}, \frac{E0\,q\,t}{m}, 0\}$

eq1 = $\frac{\xi^2}{1-\frac{\xi^2}{c^2}}$ == **Y1.Y1**

$\frac{\xi^2}{1-\frac{\xi^2}{c^2}} == \frac{px0^2}{m^2} + \frac{E0^2\,q^2\,t^2}{m^2}$

```
eq2=Solve[eq1,ξ]//Simplify
```

$\{\{\xi \to -\frac{\sqrt{-c^2\,(px0^2 + E0^2\,q^2\,t^2)}}{\sqrt{-c^2\,m^2 - px0^2 - E0^2\,q^2\,t^2}}\}, \{\xi \to \frac{\sqrt{-c^2\,(px0^2 + E0^2\,q^2\,t^2)}}{\sqrt{-c^2\,m^2 - px0^2 - E0^2\,q^2\,t^2}}\}\}$

eq3 = $\sqrt{1-\frac{\xi^2}{c^2}}$ **/. eq2[[2]] // Simplify**

$\sqrt{\frac{c^2\,m^2}{c^2\,m^2 + px0^2 + E0^2\,q^2\,t^2}}$

```
V={x'[t],y'[t],z'[t]}
```

{x′[t], y′[t], z′[t]}

```
eq4=eq3 Y1
```

$\{\frac{px0\sqrt{\frac{c^2\,m^2}{c^2\,m^2+px0^2+E0^2\,q^2\,t^2}}}{m}, \frac{E0\,q\,t\sqrt{\frac{c^2\,m^2}{c^2\,m^2+px0^2+E0^2\,q^2\,t^2}}}{m}, 0\}$

```
eq5=Table[V[[i]]==eq4[[i]],{i,1,3}]
```

$\{x'[t] == \frac{px0\sqrt{\frac{c^2\,m^2}{c^2\,m^2+px0^2+E0^2\,q^2\,t^2}}}{m},$

$y'[t] == \frac{E0\,q\,t\sqrt{\frac{c^2\,m^2}{c^2\,m^2+px0^2+E0^2\,q^2\,t^2}}}{m}, z'[t] == 0\}$

```
eq6=DSolve[{eq5,{x[0]==0,y[0]==0,z[0]==0}},{x[t],y[t],z[t
]},t]//Simplify
```

$$\left\{\left\{x[t] \to \frac{1}{E0\, m\, q}\left(-px0\sqrt{\frac{c^2 m^2}{c^2 m^2 + px0^2}}\sqrt{c^2 m^2 + px0^2}\,\mathrm{Log}\left[2\sqrt{c^2 m^2 + px0^2}\right] + px0\sqrt{\frac{c^2 m^2}{c^2 m^2 + px0^2 + E0^2 q^2 t^2}}\sqrt{c^2 m^2 + px0^2 + E0^2 q^2 t^2}\,\mathrm{Log}\left[2\left(E0\, q\, t + \sqrt{c^2 m^2 + px0^2 + E0^2 q^2 t^2}\right)\right]\right),\right.\right.$$

$$\left.\left. y[t] \to \frac{c^2 m\left(\sqrt{\frac{c^2 m^2}{c^2 m^2+px0^2}} - \sqrt{\frac{c^2 m^2}{c^2 m^2+px0^2+E0^2 q^2 t^2}}\right)}{E0\sqrt{\frac{c^2 m^2}{c^2 m^2+px0^2}}\, q\sqrt{\frac{c^2 m^2}{c^2 m^2+px0^2+E0^2 q^2 t^2}}},\ z[t] \to 0\right\}\right\}$$

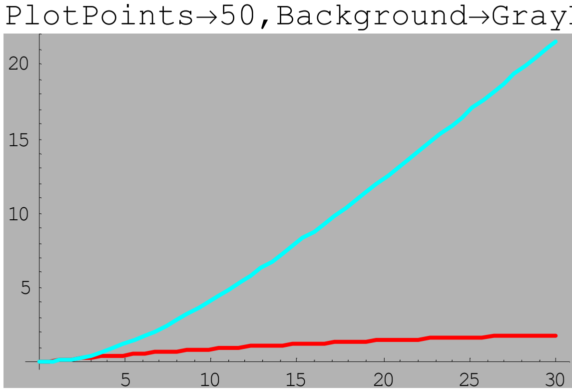
```
rule1={m→1,q→1,E0→0.1,px0→0.1,c→1}
```
{m→1,q→1,E0→0.1,px0→0.1,c→1}

```
x[t_]=x[t]/.eq6[[1,1]]/.rule1//Simplify
```
$$-0.698122 + 10.\sqrt{1.01 + 0.01 t^2}\sqrt{\frac{1}{101. + 1. t^2}}\,\mathrm{Log}\left[2\left(0.1 t + \sqrt{1.01 + 0.01 t^2}\right)\right]$$

```
y[t_]=y[t]/.eq6[[1,2]]/.rule1//Simplify
```
$$\frac{1. - 10.0499\sqrt{\frac{1}{101.+1. t^2}}}{\sqrt{\frac{1}{101.+1. t^2}}}$$

```
z[t_]=z[t]/.eq6[[1,3]]/.rule1//Simplify
```
0

```
pl1=Plot[{x[t],y[t]},{t,0,30},PlotStyle→Table[Hue[0.5 i],{i,0,1}],           Prolog→AbsoluteThickness[2], PlotPoints→50,Background→GrayLevel[0.7]]
```



-Graphics-

Nonrelativistic motion

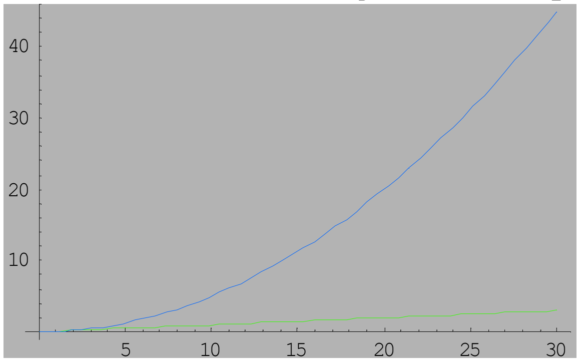
xnon[t_] = $\frac{px0\, t}{m}$ /. rule1

0.1 t

ynon[t_] = $\frac{q\, E0\, t^2}{2\, m}$ /. rule1

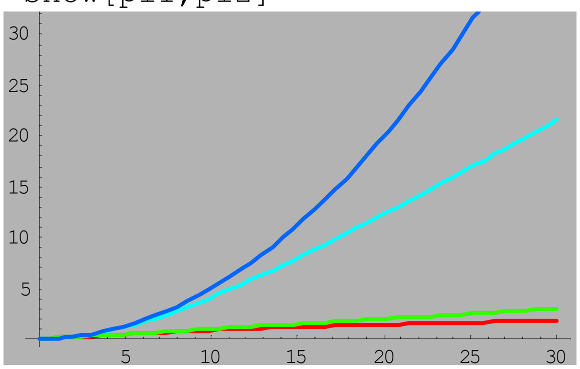
$0.05\, t^2$

```
pl2=Plot[{xnon[t],ynon[t]},{t,0,30},PlotStyle→Table[Hue[0.3    i],{i,1,2}],    Prolog→AbsoluteThickness[1.6],
PlotPoints→50,Background→GrayLevel[0.7]]
```



-Graphics-

```
Show[pl1,pl2]
```



-Graphics-

18. A particle in a uniform electric field and a magnetic field

Let the electric field $\boldsymbol{E}$ be parallel to the y axis and the magnetic field $\boldsymbol{B}$ parallel to the z axis. At $t = 0$ the particle is at the point (0,0,0) and has a momentum $\boldsymbol{p}_0$.

Lorentz invariant:

$$\frac{d}{dt}\boldsymbol{p} = \boldsymbol{F} = q[\boldsymbol{E} + (\boldsymbol{v} \times \boldsymbol{B})] \tag{1}$$

According to the Lorentz invariance, we have

$$\boldsymbol{B}^2 - \frac{1}{c^2}\boldsymbol{E}^2 = \boldsymbol{B}'^2 - \frac{1}{c^2}\boldsymbol{E}'^2$$

$$\boldsymbol{E} \cdot \boldsymbol{B} = \boldsymbol{E}' \cdot \boldsymbol{B}'$$

Since $\boldsymbol{E} \cdot \boldsymbol{B} = 0$, we have $\boldsymbol{E}' \cdot \boldsymbol{B}' = 0$

(1)

We assume a frame that $\boldsymbol{B}'=0$.

In this case, we have

$$\boldsymbol{B}^2-\frac{1}{c^2}\boldsymbol{E}^2=-\frac{1}{c^2}\boldsymbol{E}'^2<0$$

or

$$\boldsymbol{B}^2<\frac{1}{c^2}\boldsymbol{E}^2$$

or

$$B_3<\frac{1}{c}E_2$$

This is a condition for ***E*** and ***B***. Using the Lorentz transformation, we have

$$\begin{aligned} E_1' &= E_1=0 \\ E_2' &= \gamma(E_2-c\beta B_3) \\ E_3' &= \gamma(E_3+c\beta B_2)=0 \end{aligned}$$

$$\begin{aligned} B_1' &= B_1=0 \\ B_2' &= \gamma(B_2+\frac{\beta}{c}E_3)=0 \\ B_3' &= \gamma(B_3-\frac{\beta}{c}E_2)=0 \end{aligned}$$

We choose $B_3'=0$

$$B_3'=B_3-\frac{\beta}{c}E_2=0$$

or

$$v=c^2\frac{B_3}{E_2}<c$$

In this case,

$$\boldsymbol{B}'=0$$

$$E_2'=\gamma(E_2-c\beta B_3)=\frac{1}{\gamma}E_2=\frac{E}{\gamma}=E'$$

$$E_1'=0$$

$$E_3'=0$$

The frame *S*' move relative to the frame *S* with a velocity *v* along the *x* axis. We know the equation of motion for the particle in a uniform electric field $\boldsymbol{E}'$ along the *y* axis.

$$x' = \frac{cp_{0x}'}{qE'} \ln \frac{p_{0y}' + qE't' + \sqrt{(p_{0y}' + qE't')^2 + m_0^2 c^2 + p_{0x}'^2}}{p_{0y}' + \frac{E_{kin}^{0}{}'}{c}}$$

$$y' = \frac{c}{qE'}[\sqrt{(p_{0y}' + qE't')^2 + m_0^2 c^2 + p_{0x}'^2} - \frac{E_{kin}^{0}{}'}{c}]$$

$$E_{kin}'^0 = c(m_0^2 c^2 + p_{0x}'^2 + p_{0y}'^2)^{1/2}$$

with $v = c^2 \frac{B_3}{E_2} < c$

The Lorentz transformation between $p_\mu^0 = (\boldsymbol{p}_0, i\frac{E_{kin}^0}{c})$ and $p_\mu^0{}' = (\boldsymbol{p}_0', i\frac{E_{kin}^0{}'}{c})$ is given by

$$p_{01}' = \gamma(p_{01} - \frac{\beta}{c} E_{kin}^0)$$
$$p_{02}' = p_{02}$$
$$p_{03}' = p_{03}$$
$$E_{kin}^0{}' = \gamma(E_{kin}^0 - \beta c p_{01})$$

The required equations of motion for the particle in the frame S is obtained using the Lorentz transformation.

$$x_1 = \gamma(x_1' - i\beta x_4') \qquad x = \gamma(x' + vt')$$
$$x_2 = x_2' \qquad y = y'$$
$$x_3 = x_3' \qquad z = z'$$
$$x_4 = \gamma(i\beta x_1' + x_4') \qquad t = \gamma(\frac{\beta}{c} x' + t')$$

(2)

We assume a frame *S*' that $\boldsymbol{E}' = 0$.

In this case, we have

$$\boldsymbol{B}^2 - \frac{1}{c^2}\boldsymbol{E}^2 = \boldsymbol{B}'^2 > 0$$

or

$$\boldsymbol{B}^2 > \frac{1}{c^2}\boldsymbol{E}^2$$

or

$$B_3 > \frac{1}{c}E_2$$

This is a condition for ***E*** and ***B***. Using the Lorentz transformation, we have

$$E_1' = E_1 = 0$$
$$E_2' = \gamma(E_2 - c\beta B_3)$$
$$E_3' = \gamma(E_3 + c\beta B_2) = 0$$

$$B_1' = B_1 = 0$$
$$B_2' = \gamma(B_2 + \frac{\beta}{c}E_3) = 0$$
$$B_3' = \gamma(B_3 - \frac{\beta}{c}E_2)$$

We choose $E_2' = 0$

$$E_2' = \gamma(E_2 - c\beta B_3) = 0$$

or

$$v = \frac{E_2}{B_3}$$

In this case,

$$\boldsymbol{E}' = 0$$
$$B_3' = \gamma(B_3 - \frac{\beta}{c}vB_3) = \frac{1}{\gamma}B_3$$
$$B_1' = 0$$
$$B_2' = 0$$

The frame *S*' move relative to the frame *S* with a velocity v ($=E_2/B_3<c$) along the x axis. We know the equation of motion for the particle in a uniform electric field $\mathbf{B}'$ along the z' axis.

$$x' = \frac{\sqrt{p_{0x}'^2 + p_{0y}'^2}}{qB'} \sin\left(\frac{c^2 qB'}{E_{kin}'} t' + \alpha'\right)$$

$$y' = \frac{\sqrt{p_{0x}'^2 + p_{0y}'^2}}{qB'} \cos\left(\frac{c^2 qB'}{E_{kin}'} t' + \alpha'\right)$$

with $v = E_2/B_3 < c$.

The Lorentz transformation between $p_\mu{}^0 = (\boldsymbol{p}_0, i\frac{E_{kin}{}^0}{c})$ and $p_\mu{}^0{}' = (\boldsymbol{p}_0{}', i\frac{E_{kin}{}^0{}'}{c})$ is given by

$$p_{01}' = \gamma\left(p_{01} - \frac{\beta}{c} E_{kin}{}^0\right)$$
$$p_{02}' = p_{02}$$
$$p_{03}' = p_{03}$$
$$E_{kin}{}^0{}' = \gamma(E_{kin}{}^0 - \beta c p_{01})$$

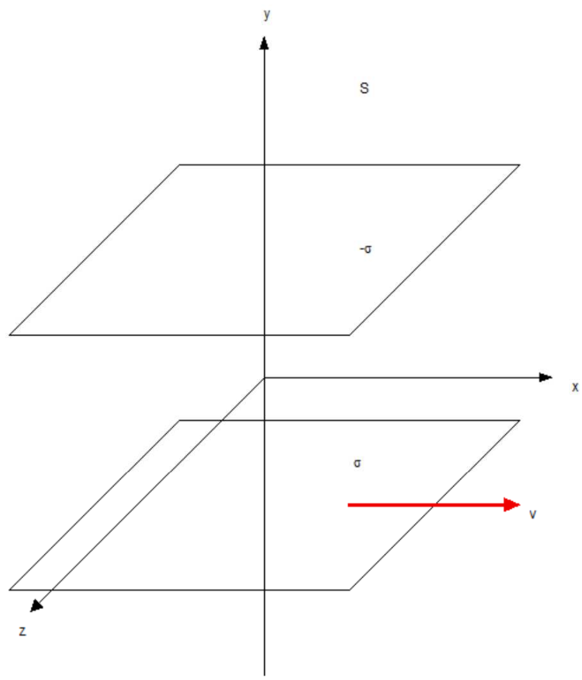
The required equations of motion for the particle in the frame S is obtained using the Lorentz transformation.

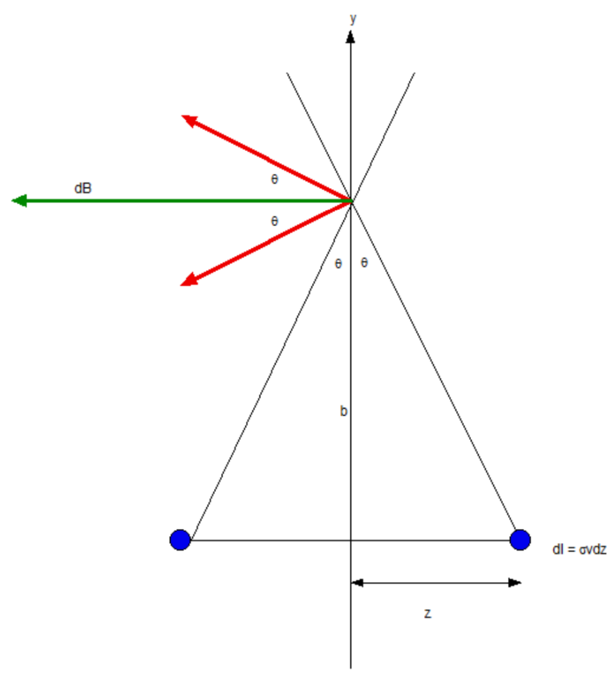
$$x_1 = \gamma(x_1' - i\beta x_4') \qquad x = \gamma(x' + vt')$$
$$x_2 = x_2' \qquad y = y'$$
$$x_3 = x_3' \qquad z = z'$$
$$x_4 = \gamma(i\beta x_1' + x_4') \qquad t = \gamma\left(\frac{\beta}{c} x' + t'\right)$$

19. Examples

19.1 Problem

There are two infinitely large planes Π_1 and Π_2 separated by a distance a. In the frame S, the plane Π_1 with a uniform surface charge density $-\sigma$, is at rest, while the plane Π_2 wth a uniform surface charge density σ, moves to the x direction at the constant velocity v. (a) Find the electric field and magnetic field in the frame S. (b) Find the electric field and magnetic field in the frame S' which moves to the x axis at the velocity v together with the Π_2 plane. (c) Suppose that a positive charge e is put in the outside of the two planes. The charge is at rest in the frame S at that moment. Find the force exerted on the charge both in the frame S and S' frames.





((***S*-frame**))

From the Ampere's law, we have

$$dB = 2\frac{\mu_0(\sigma v dz)}{2\pi\sqrt{z^2+b^2}}\cos\theta = 2\frac{\mu_0(\sigma v dz)}{2\pi\sqrt{z^2+b^2}}\frac{b}{\sqrt{z^2+b^2}}$$

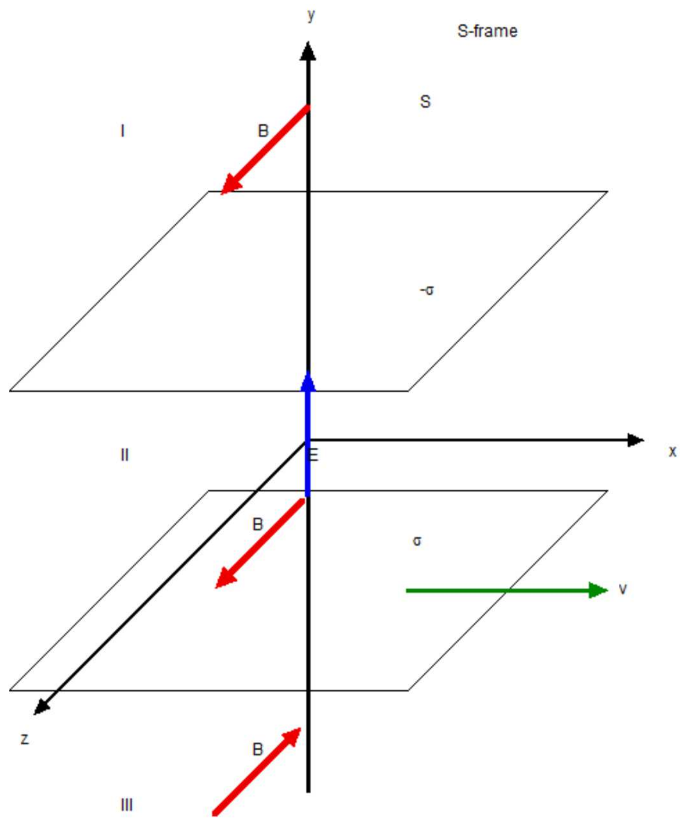
$$= \frac{\mu_0 b\sigma v}{\pi}\frac{dz}{z^2+b^2}$$

Then we have

$$B_z = \frac{\mu_0 b \sigma v}{\pi} \int_0^\infty \frac{dz}{z^2 + b^2} = \frac{\mu_0 b \sigma v}{\pi} \frac{\pi}{2b} = \frac{\mu_0 \sigma v}{2}$$

The electric field:

$$E_y = \frac{\sigma}{\varepsilon_0}$$



((*S'* frame))

<u>Region I</u>

$$E_1' = E_1 = 0$$

$$E_2' = \gamma(E_2 - c\beta B_3) = -\gamma c\beta \frac{\mu_0 \sigma v}{2} = -\gamma \frac{\sigma \beta^2}{2\varepsilon_0}$$

$$E_3' = \gamma(E_3 + c\beta B_2) = 0$$

$$B_1' = B_1 = 0$$

$$B_2' = \gamma(B_2 + \frac{\beta}{c}E_3) = 0$$

$$B_3' = \gamma(B_3 - \frac{\beta}{c}E_2) = \gamma\frac{\mu_0\sigma v}{2}$$

$$\boldsymbol{E}' = (0, -\gamma\frac{\sigma\beta^2}{2\varepsilon_0}, 0), \quad \boldsymbol{B}' = (0, 0, \gamma\frac{\mu_0\sigma v}{2})$$

$$\boldsymbol{E} = (0,0,0), \quad \boldsymbol{B} = (0,0,\frac{\mu_0\sigma v}{2})$$

Region II

$$E_1' = E_1 = 0$$

$$\begin{aligned} E_2' &= \gamma(E_2 - c\beta B_3) \\ &= \gamma(\frac{\sigma}{\varepsilon_0} - c\beta\frac{\mu_0\sigma v}{2}) \\ &= \gamma\sigma(\frac{1}{\varepsilon_0} - c\beta\frac{\mu_0 v}{2}) \\ &= \gamma\sigma(\frac{1}{\varepsilon_0} - c^2\beta^2\frac{\mu_0}{2}) \\ &= \frac{\gamma\sigma}{2\varepsilon_0}(2 - \beta^2) \end{aligned}$$

$$E_3' = \gamma(E_3 + c\beta B_2) = 0$$

\

$$B_1' = B_1 = 0$$

$$B_2' = \gamma(B_2 + \frac{\beta}{c}E_3) = 0$$

$$B_3' = \gamma(B_3 - \frac{\beta}{c}E_2)$$

$$= \gamma(\frac{\mu_0 \sigma v}{2} - \frac{\beta}{c}\frac{\sigma}{\varepsilon_0})$$

$$= \mu_0 \gamma \sigma(\frac{v}{2} - \frac{v}{c^2}\frac{1}{\varepsilon_0 \mu_0})$$

$$= -\gamma \frac{\mu_0 \sigma v}{2}$$

$$\boldsymbol{E}' = (0, \frac{\gamma\sigma}{2\varepsilon_0}(2-\beta^2), 0), \qquad \boldsymbol{B}' = (0, 0, -\gamma\frac{\mu_0 \sigma v}{2})$$

$$\boldsymbol{E} = (0, \frac{\sigma}{\varepsilon_0}, 0), \qquad \boldsymbol{B} = (0, 0, \frac{\mu_0 \sigma v}{2})$$

Region III

$$E_1' = E_1 = 0$$

$$E_2' = \gamma(E_2 - c\beta B_3) = \sigma\gamma\frac{\mu_0 v^2}{2} = \gamma\frac{\sigma\beta^2}{2\varepsilon_0}$$

$$E_3' = \gamma(E_3 + c\beta B_2) = 0$$

$$B_1' = B_1 = 0$$

$$B_2' = \gamma(B_2 + \frac{\beta}{c}E_3) = 0$$

$$B_3' = \gamma(B_3 - \frac{\beta}{c}E_2) = -\gamma\frac{\mu_0 \sigma v}{2}$$

$$\boldsymbol{E}' = (0, \frac{\gamma\sigma\beta^2}{2\varepsilon_0}, 0), \qquad \boldsymbol{B}' = (0, 0, -\gamma\frac{\mu_0 \sigma v}{2})$$

$$\boldsymbol{E} = (0, 0, 0), \qquad \boldsymbol{B} = (0, 0, \frac{\mu_0 \sigma v}{2})$$

Suppose that a charge q is put at rest in the S-frame (the region I). What is the force applied on the charge (q>0)?

First we consider the velocity of the particle in the frame S'.

$$u_1' = \frac{u_1 - v}{1 - \frac{\beta}{c} u_1} = -v$$

$$u_2' = \frac{1}{\gamma} \frac{u_2}{1 - \frac{\beta}{c} u_1} = 0$$

$$u_3' = \frac{1}{\gamma} \frac{u_3}{1 - \frac{\beta}{c} u_1} = 0$$

or

$$\boldsymbol{u}' = (-v, 0, 0) \qquad \text{and} \qquad \boldsymbol{u} = (0, 0, 0)$$

Region I

$$\boldsymbol{F} = q\boldsymbol{E} = (0, 0, 0)$$

$$\boldsymbol{F}' = q(\boldsymbol{E}' + \boldsymbol{u}' \times \boldsymbol{B}') = (0, 0, 0)$$

where

$$\boldsymbol{E}' = (0, -\gamma \frac{\sigma \beta^2}{2\varepsilon_0}, 0), \qquad \boldsymbol{B}' = (0, 0, \gamma \frac{\mu_0 \sigma v}{2})$$

$$\boldsymbol{E} = (0, 0, 0), \qquad \boldsymbol{B} = (0, 0, \frac{\mu_0 \sigma v}{2})$$

Region II

$$\boldsymbol{F} = q\boldsymbol{E} = (0, \frac{q\sigma}{\varepsilon_0}, 0)$$

$$\begin{aligned}\boldsymbol{F}' &= q(\boldsymbol{E}' + \boldsymbol{u}' \times \boldsymbol{B}') \\ &= (0, \frac{q\gamma\sigma}{\varepsilon_0}, 0) \\ &= \gamma \boldsymbol{F}\end{aligned}$$

with

$$E' = (0, \frac{\gamma\sigma}{2\varepsilon_0}(2-\beta^2), 0), \qquad B' = (0, 0, -\gamma\frac{\mu_0\sigma v}{2})$$

$$E = (0, \frac{\sigma}{\varepsilon_0}, 0), \qquad B = (0, 0, \frac{\mu_0\sigma v}{2})$$

Region III

$$F = qE = (0,0,0)$$

$$F' = q(E' + u' \times B') = 0$$

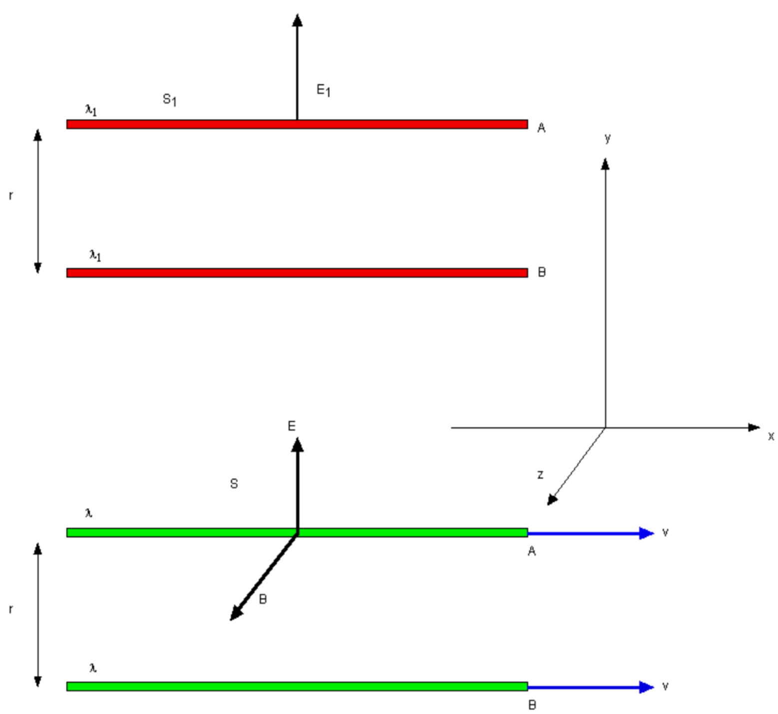
with

$$E' = (0, \frac{\gamma\sigma\beta^2}{2\varepsilon_0}, 0), \qquad B' = (0, 0, -\gamma\frac{\mu_0\sigma v}{2})$$

$$E = (0,0,0), \qquad B = (0, 0, \frac{\mu_0\sigma v}{2})$$

19.2 Problem-2

Suppose that two parallel infinite wires A and B (line charge density λ) move along the x axis with the velocity v. The separation distance between two wires is r. What is the force between two wires?



We now consider the electric field in the *S*' frame were the line wires are at rest. Using the Gauss's law we have an electric field at the line A (the line charge density $\lambda' = \frac{\lambda}{\gamma}$) due to the line charge of B.

$$E_2' = \frac{\lambda'}{2\pi r}$$
$$E_1' = E_3' = 0$$

No magnetic field is generated since the two lines are at rest in the *S*' frame:

$$B_1' = B_2' = B_3' = 0$$

The repulsive force (per unit length) between two wires is obtained as

$$F' = \lambda' E_2' = \frac{\lambda'^2}{2\pi\varepsilon_0 r}.$$

We now consider that Lorentz transformation for the electric and magnetic fields,

$$E_1 = E_1' = 0$$

$$E_2 = \gamma(E_2' + c\beta B_3')$$
$$= \frac{\gamma\lambda'}{2\pi\varepsilon_0 r}$$
$$= \frac{\lambda}{2\pi\varepsilon_0 r}$$

$$E_3 = \gamma(E_3' - c\beta B_2') = 0$$

and

$$B_1 = B_1' = 0$$

$$B_2 = \gamma(B_2' - \frac{\beta}{c} E_3') = 0$$

$$B_3 = \gamma(B_3' + \frac{\beta}{c} E_2')$$
$$= \frac{\gamma\beta}{c} \frac{\lambda'}{2\pi\varepsilon_0 r}$$
$$= \frac{\gamma\lambda' v}{2\pi\varepsilon_0 c^2 r}$$
$$= \frac{\mu_0 \gamma\lambda' v}{2\pi r}$$
$$= \frac{\mu_0 \lambda v}{2\pi r}$$

The repulsive force between two wires is

$$F = \lambda E_2 - \lambda v B_3$$
$$= \frac{\lambda^2}{2\pi\varepsilon_0 r} - \frac{\mu_0 \lambda^2 v^2}{2\pi r}$$
$$= \frac{\lambda^2}{2\pi\varepsilon_0 r}(1 - \varepsilon_0 \mu_0 v^2)$$
$$= \frac{\lambda^2}{2\pi\varepsilon_0 r}(1 - \frac{v^2}{c^2})$$
$$= \frac{1}{2\pi\varepsilon_0 r} \frac{\lambda^2}{\gamma^2}$$
$$= \frac{\lambda'^2}{2\pi\varepsilon_0 r}$$

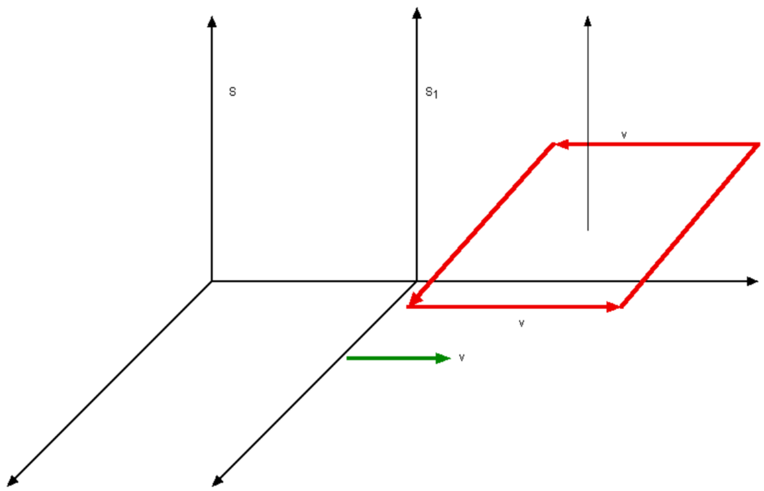
19.3 Problem

From the magnetic dipole moment to the electric dipole moment
((**Griffiths Problem 12-64**))

The magnetic moment consists of a uniform charge density ρ_0 (the line charge density $\lambda_0 = A\rho_0$ and A is the cross section of the wire) circulating at speed v around a square loop of side l_0, so that the magnetic moment is equal to

$$m = A\rho_0 v l_0^2 = \lambda_0 v l_0^2$$

Suppose that this magnetic moment moves in the x direction at speed v. The frame S' moves toward the x axis with the velocity v, relative to the frame S. Show that in the frame S the current loop carries an electric dipole moment.



Charge density of the front side in the *S* frame

$$u' = v,$$

$$u = \frac{u' + v}{1 + \frac{v}{c^2}u'}$$

$$= \frac{2v}{1 + \frac{v^2}{c^2}}$$

$$J_1 = \gamma(v)(J_1' - i\beta J_4')$$
$$= 2v\gamma(v)\rho'$$

$$\rho = \rho_f$$
$$= \gamma(v)(\frac{\beta}{c}J_1' + \rho')$$
$$= \gamma(v)(1 + \frac{\beta}{c}u')\rho'$$
$$= \gamma(v)(1 + \frac{v^2}{c^2})\rho'$$
$$= \gamma(v)(1 + \frac{v^2}{c^2})\rho_0\gamma(v)$$
$$= \frac{1+\beta^2}{1-\beta^2}\rho_0$$

Charge density of the back side in the *S* frame

$$u' = -v, \qquad u = \frac{u'+v}{1+\frac{v}{c^2}u'} = 0$$

$$\begin{aligned} J_1 &= \gamma(v)(J_1' - i\beta J_4') \\ &= \gamma(v)(u'+v)\rho' \\ &= 0 \end{aligned}$$

$$\begin{aligned} \rho &= \rho_b \\ &= \gamma(v)(\frac{\beta}{c}J_1' + \rho') \\ &= \gamma(v)(1+\frac{\beta}{c}u')\rho' \\ &= \gamma(v)(1-\frac{\beta}{c}v)\rho' \\ &= \frac{\rho'}{\gamma(v)} \\ &= \rho_0 \end{aligned}$$

Here we note that the length l in the S-frame is related to the length l' by

$$l = \frac{l'}{\gamma(v)} = \frac{l_0}{\gamma(v)} \qquad \text{(length contraction)}$$

No length contraction occurs along the y direction. The total charge in the front side is

$$\begin{aligned} Q_f &= \rho_f lA \\ &= \rho_0 A \frac{1+\beta^2}{1-\beta^2}\frac{l_0}{\gamma(v)} \\ &= \lambda_0 \frac{1+\beta^2}{1-\beta^2}\frac{l_0}{\gamma(v)} \end{aligned}$$

The total charge in the back side is

$$\begin{aligned} Q_b &= \rho_b lA \\ &= \rho_0 A \frac{l_0}{\gamma(v)} \\ &= \lambda_0 \frac{l_0}{\gamma(v)} \end{aligned}$$

The electric dipole moment is given by

$$\begin{aligned}
\boldsymbol{p} &= Q_b(\frac{l_0}{2}\boldsymbol{e}_y) + Q_f(-\frac{l_0}{2}\boldsymbol{e}_y) \\
&= [\rho_0 A \frac{l_0^{\ 2}}{2\gamma(v)} - \rho_0 A \frac{1+\beta^2}{1-\beta^2}\frac{l_0^{\ 2}}{2\gamma(v)}]\boldsymbol{e}_y \\
&= \rho_0 A \frac{l_0^{\ 2}}{2\gamma(v)}(1 - \frac{1+\beta^2}{1-\beta^2})\boldsymbol{e}_y \\
&= -\rho_0 A \frac{l_0^{\ 2}}{\gamma(v)}\frac{\beta^2}{1-\beta^2}\boldsymbol{e}_y \\
&= -(\lambda_0 l_0^{\ 2})\beta^2\gamma(v)\boldsymbol{e}_y \\
&= -(\lambda_0 l_0^{\ 2})\frac{v^2}{c^2}\gamma(v)\boldsymbol{e}_y
\end{aligned}$$

19.4 Problem

((Griffiths Problem 12-67))

A charge q is released from rest at the origin in the presence of a uniform electric field $\boldsymbol{E} = (0, 0, E_0)$ and a uniform magnetic field $\boldsymbol{B} = (B_0, 0, 0)$. Determine the trajectory of the particle by transforming to a system in which $\boldsymbol{E} = 0$, finding the path in that system and the transforming back to the original system. Assuming $E_0 < cB_0$.

We know that

(1) $\boldsymbol{B}^2 - \frac{1}{c^2}\boldsymbol{E}^2$ = invariant under the Lorentz transformation

(2) $\boldsymbol{E}\cdot\boldsymbol{B}$ = invariant under the Lorentz transformation

This means that

$$\begin{aligned}
\boldsymbol{B}^2 - \frac{1}{c^2}\boldsymbol{E}^2 &= B_0^{\ 2} - \frac{1}{c^2}E_0^{\ 2} \\
&= \boldsymbol{B}'^2 - \frac{1}{c^2}\boldsymbol{E}'^2 \\
&= \boldsymbol{B}'^2 > 0
\end{aligned}$$

and

$$\boldsymbol{E}\cdot\boldsymbol{B} = \boldsymbol{E}'\cdot\boldsymbol{B}' = 0$$

Suppose that $\boldsymbol{E}' = 0$, satisfying the second condition. The first condition is also satisfied since $E_0 < cB_0$.

$$E_2' = E_2 = 0$$

$$\begin{aligned} E_3' &= \gamma(E_3 - c\beta B_1) \\ &= \gamma(E_3 - vB_1) \\ &= \gamma(E_0 - vB_0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} E_1' &= \gamma(E_1 + c\beta B_3) \\ &= 0 \end{aligned}$$

Since $E_3' = 0$, we have $v = \dfrac{E_0}{B_0}$.

$$B_2' = B_2 = 0$$

$$B_3' = \gamma(B_3 + \frac{\beta}{c} E_1) = 0$$

$$\begin{aligned} B_1' &= \gamma(B_1 - \frac{\beta}{c} E_3) \\ &= \gamma(B_0 - \frac{\beta}{c} E_0) \\ &= \gamma B_0 (1 - \frac{v^2}{c^2}) \\ &= \frac{B_0}{\gamma} \end{aligned}$$

The magnetic field is along the *x*' direction in the *S*' frame.

The trajectory in *S*' system.

The particle started out at rest and at the origin in the *S* frame.

$$u_2' = \frac{u_2 - v}{1 - \frac{\beta}{c} u_2} = -v$$

$$u_3' = \frac{1}{\gamma} \frac{u_3}{1 - \frac{\beta}{c} u_2} = 0$$

$$u_1' = \frac{1}{\gamma} \frac{u_1}{1 - \frac{\beta}{c} u_2} = 0$$

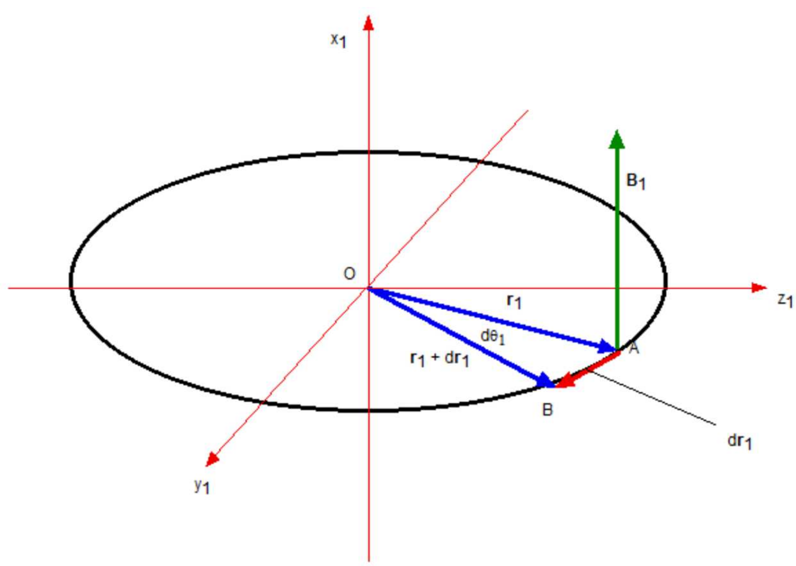
So it started out the velocity $\boldsymbol{u}_2' = -\boldsymbol{v}$ (in the x'-y' plane) in the S' frame. The magnitude of $\boldsymbol{u}'$ remains unchanged since no work is done in the presence of the magnetic field. We now consider the equation of motion of the particle in the S' frame.

$$\frac{d}{dt'}[m_0 \gamma_{u'} \boldsymbol{u}'] = q(\boldsymbol{u}' \times \boldsymbol{B}')$$

$$= q(\frac{d\boldsymbol{r}'}{dt'} \times \boldsymbol{B}')$$

or

$$\boldsymbol{u}' = \frac{q}{m_0 \gamma_{u'}} (\boldsymbol{r}' \times \boldsymbol{B}'), \quad \text{or} \qquad d\boldsymbol{r}' = \frac{q}{m_0 \gamma_{u'}} (\boldsymbol{r}' \times \boldsymbol{B}') dt'$$



In this figure, we have

$$dr' = r'd\theta' = \frac{q}{m_0 \gamma_{u'}} r'B'dt'$$

or

$$\omega' = \frac{d\theta'}{dt'} = \frac{qB'}{m_0 \gamma_{u'}} = \frac{qB_0}{m_0 {\gamma_{u'}}^2} .$$

Since $v = \omega' R$, the particle moves in a circle of radius

$$R = \frac{v}{\omega'} = \frac{m_0 \gamma^2 v}{qB_0}$$

with

$$\gamma_{u'} = \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The actual trajectory is given by

$$\begin{aligned} x' &= 0 \\ y' &= -R\sin(\omega' t') \\ z' &= R(1 - \cos\omega' t') \end{aligned}$$

The trajectory in *S*:

$$\begin{aligned} x' &= x & x &= x' \\ y' &= \gamma(y - vt) & y &= \gamma(y' + vt') \\ z' &= z & z &= z' \\ t' &= \gamma(t - \frac{v}{c^2} y) & t &= \gamma(t' + \frac{v}{c^2} y) \end{aligned}$$

So the trajectory in *S* is given by

$$x = 0$$

$$y = -\gamma R \sin[\gamma\omega'(t - \frac{v}{c^2}y)] + \gamma^2 v(t - \frac{v}{c^2}y)]$$

or

$$y = vt - \frac{R}{\gamma}\sin[\gamma\omega'(t - \frac{v}{c^2}y)]$$

$$z = R\{1 - \cos[\gamma\omega'(t - \frac{v}{c^2}y)]\}$$

Finally, we have

$$\gamma^2(y - vt)^2 + (z - R)^2 = R^2$$

with

$$R = \frac{m_0\gamma^2 v}{qB_0} = R_0\gamma^2$$

19.5 Problem

((Griffiths Problem 12-64))

In a certain inertial fame *S*, the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$ are neither parallel nor perpendicular, at a particular space-time point. Show that in a different inertial system $\bar{S}$, moving relative to *S* with velocity $\boldsymbol{v}$ given by

$$\frac{\boldsymbol{v}}{1 + \frac{v^2}{c^2}} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2 + E^2 / c^2}$$

the fields $\bar{\boldsymbol{E}}$ and $\bar{\boldsymbol{B}}$ are parallel at that point. Is there a frame in which the two are perpendicular.

We know that

(1) $\boldsymbol{B}^2 - \frac{1}{c^2}\boldsymbol{E}^2$ = invariant under the Lorentz transformation

(2) $\boldsymbol{E} \cdot \boldsymbol{B}$ = invariant under the Lorentz transformation

This means that

$$\boldsymbol{B}^2 - \frac{1}{c^2}\boldsymbol{E}^2 = \boldsymbol{B}'^2 - \frac{1}{c^2}\boldsymbol{E}'^2$$

$$\boldsymbol{E} \cdot \boldsymbol{B} = \boldsymbol{E}' \cdot \boldsymbol{B}' = \text{constant}$$

We choose axes so that ***E*** points in the *z* direction and ***B*** in the *y*-*z* plane. The *S*' frame moves at the velocity *v* along the *x* axis relative to the *S* frame.

The Lorentz transformation:

$$E_1' = E_1 = 0$$

$$E_2' = \gamma(E_2 - c\beta B_3)$$
$$= -c\gamma\beta B_3$$

$$E_3' = \gamma(E_3 + c\beta B_2)$$

$$B_1' = B_1 = 0,$$

$$B_2' = \gamma(B_2 + \frac{\beta}{c} E_3)$$

$$B_3' = \gamma(B_3 - \frac{\beta}{c} E_2)$$
$$= \gamma B_3$$

The vectors ***E***' and ***B***' are parallel. So we have

$$\boldsymbol{E}'\times\boldsymbol{B}' = 0 = \begin{vmatrix} \boldsymbol{e}_1' & \boldsymbol{e}_2' & \boldsymbol{e}_3' \\ 0 & -c\gamma\beta B_3 & \gamma(E_3 + c\beta B_2) \\ 0 & \gamma(B_2 + \frac{\beta}{c}E_3) & \gamma B_3 \end{vmatrix} =$$

From this we have

$$[B_2{}^2 c^2 + (B_3{}^2 c^2 + E_3{}^2)]v + B_2 E_3 (c^2 + v^2) = 0$$

or

$$\frac{v}{1+\frac{v^2}{c^2}} = \frac{(\boldsymbol{E}\times\boldsymbol{B})_1}{\boldsymbol{B}^2 + \frac{\boldsymbol{E}^2}{c^2}}$$

APPENDIX

A. Relativistic-covariant Lagrangian formalism

A.1 Lagrangian *L* (simple case)

Proper time

$$\begin{aligned}\left(dx_\mu '\right)^2 &= a_{\mu\lambda}a_{\mu\sigma}dx_\lambda dx_\sigma \\ &= \delta_{\lambda\sigma}dx_\lambda dx_\sigma \\ &= \left(dx_\mu\right)^2\end{aligned}$$

We define the proper time as

$$\begin{aligned}\left(ds\right)^2 &= c^2(dt)^2-(dx_1)^2-(dx_2)^2-(dx_3)^2 \\ &= c^2(dt')^2-(dx_1')^2-(dx_2')^2-(dx_3')^2\end{aligned}$$

$$\begin{aligned}\left(ds\right)^2 &= c^2(dt)^2\{1-\frac{1}{c^2}[(\frac{dx_1}{dt})^2+(\frac{dx_2}{dt})^2+(\frac{dx_3}{dt})^2]\} \\ &= c^2(dt)^2(1-\frac{\boldsymbol{u}^2}{c^2})\end{aligned}$$

or

$$d\tau=\frac{ds}{c}=dt\sqrt{1-\frac{\boldsymbol{u}^2}{c^2}}$$

where τ is a proper time and $\boldsymbol{u}$ is the velocity of the particle in the frame *S*.

The integral $\int_a^b ds$ taken between a given pair of world points has its maximum value if it is taken along the straight line joining two points.

$$S=-\alpha\int_a^b ds=-\alpha c\int_{t_a}^{t_b} dt\sqrt{1-\frac{\boldsymbol{u}^2}{c^2}}=\int_{t_a}^{t_b} Ldt$$

where

$$L=-\alpha c\sqrt{1-\frac{\boldsymbol{u}^2}{c^2}}$$

Nonrelativistic case

$$L = -\alpha c(1 - \frac{\boldsymbol{u}^2}{c^2})^{1/2} = -\alpha c(1 - \frac{\boldsymbol{u}^2}{2c^2}) = \frac{\alpha}{2c}\boldsymbol{u}^2 - \alpha c$$

In the classical mechanics,

$$\frac{\alpha}{2c} = \frac{m_0}{2} \qquad \text{or} \qquad \alpha = m_0 c$$

Therefore, the Lagrangian L is given by

$$L = -m_0 c^2 (1 - \frac{\boldsymbol{u}^2}{c^2})^{1/2}.$$

The momentum $\boldsymbol{p}$ is defined by

$$\begin{aligned} \boldsymbol{p} &= \frac{\partial L}{\partial \boldsymbol{u}} \\ &= \frac{m_0 \boldsymbol{u}}{\sqrt{1 - \frac{\boldsymbol{u}^2}{c^2}}} \\ &= m_0 \boldsymbol{u} \gamma(\boldsymbol{u}) \\ &= m_0 \frac{d\boldsymbol{r}}{d\tau} \\ &= m_0 \frac{d\boldsymbol{r}}{dt}\frac{dt}{d\tau} \end{aligned}$$

((**Note**))
This momentum coincides with the components of four-vector momentum p_μ defined by

$$p_\mu = m_0 \frac{dx_\mu}{d\tau}$$

A.2 Hamiltonian

The Hamiltonian H is defined by

$$H = \boldsymbol{p}\cdot\boldsymbol{u} - L = \gamma(\boldsymbol{u})m_0\boldsymbol{u}^2 + m_0c^2\frac{1}{\gamma(\boldsymbol{u})}$$
$$= \frac{\gamma(\boldsymbol{u})^2 m_0\boldsymbol{u}^2 + m_0c^2}{\gamma(\boldsymbol{u})}$$
$$= \gamma(\boldsymbol{u})m_0c^2$$
$$= \frac{m_0c^2}{\sqrt{1-\dfrac{\boldsymbol{u}^2}{c^2}}}$$
$$= E$$

or

$$E = \frac{m_0c^2}{\sqrt{1-\dfrac{\boldsymbol{u}^2}{c^2}}}$$

We have

$$\frac{E^2}{c^2} = \frac{m_0{}^2c^2}{1-\dfrac{\boldsymbol{u}^2}{c^2}} = \frac{m_0{}^2c^2(1-\dfrac{\boldsymbol{u}^2}{c^2}) + m_0{}^2\boldsymbol{u}^2}{1-\dfrac{\boldsymbol{u}^2}{c^2}} = m_0{}^2c^2 + \boldsymbol{p}^2$$

A.3 Lagrangian form in the presence of an electromagnetic field

The action function for a charge in an electromagnetic field

$$S = \int_a^b (-m_0cds + qA_\mu dx_\mu)$$

where the second term is invariant under the Lorentz transformation.

$$A_\mu = (\boldsymbol{A}, i\frac{1}{c}\phi), \qquad \text{and} \qquad dx_\mu = (dx_1, dx_2, dx_3, icdt)$$

Then we have

$$S = \int_a^b (-mcds + qA_\mu dx_\mu)$$
$$= \int_a^b [-m_0c^2\sqrt{1-\frac{\boldsymbol{u}^2}{c^2}} + q(\boldsymbol{A}\cdot\boldsymbol{u} - \phi)]dt$$

The integrand in the Lagrangian function of a charge (q) in the electromagnetic field,

$$L = -m_0 c^2 \sqrt{1 - \frac{\boldsymbol{u}^2}{c^2}} + q(\boldsymbol{A} \cdot \boldsymbol{u} - \phi)$$

$$\boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{u}} = \frac{m_0 \boldsymbol{u}}{\sqrt{1 - \frac{\boldsymbol{u}^2}{c^2}}} + q\boldsymbol{A}$$

where

$$A_\mu = (\boldsymbol{A}, i\frac{1}{c}\phi)$$

The Hamiltonian H is given by

$$\begin{aligned} H &= \boldsymbol{p} \cdot \boldsymbol{u} - L \\ &= \frac{m_0 \boldsymbol{u}^2}{\sqrt{1 - \frac{\boldsymbol{u}^2}{c^2}}} + e\boldsymbol{A} \cdot \boldsymbol{u} - (-m_0 c^2 \sqrt{1 - \frac{\boldsymbol{u}^2}{c^2}} + q\boldsymbol{A} \cdot \boldsymbol{u} - q\phi) \end{aligned}$$

or

$$H = \frac{m_0 c^2}{\sqrt{1 - \frac{\mathbf{u}^2}{c^2}}} + q\phi$$

or

$$\begin{aligned} \left(\frac{H - q\phi}{c} \right)^2 &= \frac{m_0{}^2 c^2 (1 - \frac{\boldsymbol{u}^2}{c^2}) + m_0{}^2 \boldsymbol{u}^2}{1 - \frac{\boldsymbol{u}^2}{c^2}} \\ &= m_0{}^2 c^2 + (\boldsymbol{p} - q\boldsymbol{A})^2 \end{aligned}$$

A.4 Expression for the Lagrangian in terms of 4-dimensional velocity

Here we use dτ instead of dt in the expression of Lagrangian.

$$ds = c d\tau$$

η_μ is a four-dimensional velocity defined by

$$\begin{aligned}\eta_\mu &= \frac{dx_\mu}{d\tau} \\ &= \frac{dt}{d\tau}\frac{dx_\mu}{dt} \\ &= (\gamma(\boldsymbol{u})u_1, \gamma(\boldsymbol{u})u_2, \gamma(\boldsymbol{u})u_3, ic\gamma(\boldsymbol{u}))\end{aligned}$$

$$\begin{aligned}A_\mu\eta_\mu &= A_1\eta_1 + A_2\eta_2 + A_3\eta_3 + A_4\eta_4 \\ &= \gamma(\boldsymbol{u})(\boldsymbol{u}\cdot\boldsymbol{A} - \phi)\end{aligned}$$

since

$$A_\mu = (\boldsymbol{A}, i\frac{1}{c}\phi), \qquad \eta_4 = \frac{dt}{d\tau}\frac{dx_4}{dt} = ic\frac{dt}{d\tau}$$

$$\begin{aligned}S &= \int_a^b (-m_0 c ds + qA_\mu dx_\mu) \\ &= \int_a^b (-m_0 c^2 + qA_\mu \cdot \eta_\mu) d\tau\end{aligned}$$

$$L = -m_0 c^2 + qA_\mu \eta_\mu$$

A.5 Lagrangian and Hamiltonian in terms of the field tensor $F_{\mu\nu}$

$$F_{\mu\nu}F_{\mu\nu} = 2(B_1^2 + B_2^2 + B_3^2) - \frac{2}{c^2}(E_1^2 + E_2^2 + E_3^2)$$

This is invariant under the Lorentz transformation.

We may try the Lagrangian density

$$L = -\frac{1}{4\mu_0}F_{\mu\nu}F_{\mu\nu} + J_\mu A_\mu$$

By regarding each component of A_μ as an independent field, we find that the Lagrange equation

$$\frac{\partial L}{\partial A_\mu} = \frac{\partial}{\partial x_\nu}[\frac{\partial L}{\partial(\frac{\partial A_\mu}{\partial x_\nu})}]$$

is equivalent to

$$\frac{\partial F_{\mu\nu}}{\partial x_\mu} = \mu_0 J_\mu .$$

The Hamiltonian density H_{em} for the free Maxwell field can be evaluated as follows.

$$L_{em} = -\frac{1}{4\mu_0} F_{\mu\nu} F_{\mu\nu}$$

$$H_{em} = \frac{\partial L_{em}}{\partial\left(\frac{\partial A_\mu}{\partial x_4}\right)} \frac{\partial A_\mu}{\partial x_4} - L_{em}$$

$$= -\frac{F_{4\mu}}{\mu_0}(F_{4\mu} + \frac{\partial A_4}{\partial x_\mu}) - \frac{1}{2\mu_0}(\boldsymbol{B}^2 - \frac{1}{c^2}\boldsymbol{E}^2)$$

or

$$H_{em} = \frac{1}{2}\varepsilon_0 \boldsymbol{E}^2 + \frac{1}{2\mu_0}\boldsymbol{B}^2 - \varepsilon_0 \boldsymbol{E}\cdot\nabla\phi$$

$$\int H_{em} d\boldsymbol{r} = \frac{1}{2}\int (\varepsilon_0 \boldsymbol{E}^2 + \frac{1}{2\mu_0}\boldsymbol{B}^2) d\boldsymbol{r} - \int \varepsilon_0 (\boldsymbol{E}\cdot\nabla\phi) d\boldsymbol{r}$$
$$= \frac{1}{2}\int (\varepsilon_0 \boldsymbol{E}^2 + \frac{1}{2\mu_0}\boldsymbol{B}^2) dr$$

((**Note**))

$$\int (\boldsymbol{E}\cdot\nabla\phi) d\boldsymbol{r} = \int [\nabla\cdot(\boldsymbol{E}\phi) - \phi\nabla\cdot\boldsymbol{E}] d\boldsymbol{r}$$
$$= \int \nabla\cdot(\boldsymbol{E}\phi) d\boldsymbol{r}$$
$$= \int (\boldsymbol{E}\phi)\cdot d\boldsymbol{a}$$
$$= 0$$

where $\boldsymbol{E}\phi$ vanishes sufficiently rapidly at infinity.

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} = 0 \text{ (in this case).}$$