Lecture Note on
Oscillations and waves

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Abstract

This note is presented to the undergraduate students who are interested in the oscillations and waves. One of the simplest models in the classical mechanics is a simple harmonics. A more general oscillation is described by a superposition of the so-called modes. This mode is quantized into elementary excitation in quantum mechanics. In this sense, the concept of the oscillations and waves is fundamental but is essential to understanding the physics from the classical mechanics to the quantum mechanics. The duality of waves and particles plays a central role in quantum mechanics.

This note is written on the basis of a book (Oscillations and waves) [in Japanese][1] written by Prof. M. Ogata of the University of Tokyo. This summer (July, 2009), we visited Japan. We stopped by Kanda Book Stores near the University of Tokyo, in order to buy used books on physics (mainly written in Japanese), which are usually much cheaper than the new books. Fortunately we found a very interesting book on oscillations and waves, which was written by Prof. Masao Ogata. According to the preface of the book, this book was written for undergraduate students in Japan who major in physics, as one of the text-book series published from Syokabo. We read some part of this book, standing at the book store. We were very impressed by the contents of the book; the physics of the oscillation and wave for both the longitudinal waves and the transverse waves. After we came back from Japan, we have written this lecture notes using the Mathematica (variational method) along the content of the book (written by Prof. Ogata). It took several weeks for us to finish writing this note. We think that this note will be useful for the undergraduate students in U.S.A. who are interested in the physics of oscillations. This lecture note will be helpful for understanding the fundamental concept of the

oscillations and waves. We note that the lecture notes of Phys.131 and 132 (general physics course), which is given in the Chapters 16 and 17 of our home page

http://bingweb.binghamton.edu/~suzuki/

will be also useful in understanding the fundamental concept of the longitudinal and transverse waves. Note that we use the Lagrangian and the Lagrange's equation, which simplify the derivation of the equations of motion for the *N* (= 2, 3, 4, and 5) systems. The spread of the wave packet will be discussed with the concept of duality principle (wave and particle). We strongly suggest the undergraduate students to use Mathematica since there are some complicated calculations including integrals. Through the use of the Mathematica , students may understand the concept of physics with minimum effort in mathematics.

There are many excellent textbooks on the waves and oscillations (see References). We refer to 6 references for the writing this note.[1-6]. There is an excellent description on the wave packet in the Quantum Theory written by D. Bohm.[7]

CONTENT

1. Longitudinal waves in chains

Longitudinal waves are waves that have same direction of oscillations or vibrations along or parallel to their direction of travel, which means that the oscillations of the medium (particle) is in the same direction or opposite direction as the motion of the wave.

1.1. *N* = 1 longitudinal wave

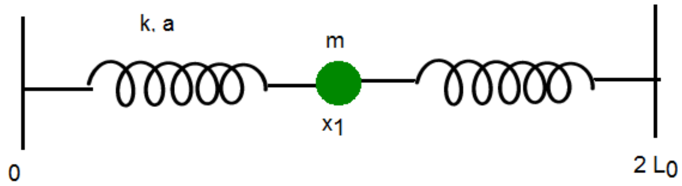

Fig.1 A system with a mass m (located at $x = x_1$) and two springs with a spring constant k. a is the length of un-stretched spring. x_1 is the position of the mass along the chain. The total distance is 2 L_0. Both sides are fixed.

For simplicity, we use the Lagrange's method for the motion of the spring systems with one point mass (mass m). The spring constant is k. We assume that both sides of the chain are fixed. The Lagrangian is defined as the difference of the kinetic energy and the potential energy and is given by

$$L = \frac{1}{2}m[\dot{x}_1(t)]^2 - \frac{1}{2}k[x_1(t) - a]^2 - \frac{1}{2}k[2L_0 - x_1(t) - a]^2, \quad (1.1)$$

where a is the length of un-stretched spring and $2L_0$ is the total length of the system. The Lagrange's equations are obtained as

$$m\ddot{x}_1(t) = kL_0 - 2kx_1(t) + kx_1(t). \quad (1.2)$$

In equilibrium ($\ddot{x}_1 = 0$), we have

$$0 = -2kx_1^0 + 2kL_0,$$

or

$$x_1^0 = L_0.$$

Here a new variables as the deviation of the displacement from the position in thermal equilibrium, is introduced,

$$y = x - x^0. \quad (1.3)$$

Note that y is a just variable and the displacement along the chain. This notation (y) has nothing to do with the displacement along the direction perpendicular to the chain. Then we have

$$m\ddot{y} = -2ky. \quad (1.4)$$

We assume that

$$y = \mathrm{Re}[Ye^{i\omega t}], \quad (1.5)$$

where Y is the complex amplitude and ω is the angular frequency. Y satisfies the equation given by

$$\frac{m\omega^2}{k}Y = 2Y.$$

Then we have

$$\omega = \sqrt{2}\sqrt{\frac{k}{m}}, \quad Y = 1, \tag{1.6}$$

or

u_1=1 for $n = 1$.

We make a plot of $\boldsymbol{u}_1$ as a function of the position n. (n = 0, 1, 2). Here we assume that the 0-th and 2–th components of $\boldsymbol{u}_1$ is equal to zero.

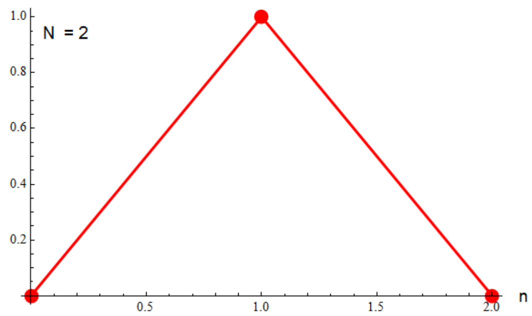

Fig.2 Plot of $\boldsymbol{u}_1$ as a function of n (n = 0, 1, and 2).

1.2 N = 2 longitudinal wave

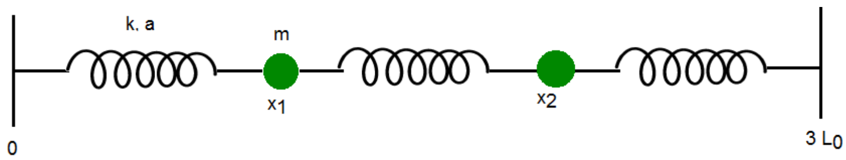

Fig.3 A system with two masses (m) (located at $x = x_1$ and x_2) and three springs with a spring constant k. a is the length of un-stretched spring. The total distance is $3L_0$. Both sides are fixed.

For simplicity, we use the Lagrange's method for the motion of the spring systems with two point masses with mass m. The spring constant is k. The Lagrangian is given by

$$L = \frac{1}{2}m[\dot{x}_1(t)]^2 + \frac{1}{2}m[\dot{x}_2(t)]^2 - \frac{1}{2}k[x_1(t) - a]^2 - \frac{1}{2}k[x_2(t) - x_1(t) - a]^2$$
$$- \frac{1}{2}k[3L_0 - x_2(t) - a]^2 \quad , \qquad (1.7)$$

where a is the length of un-stretched spring and $3L_0$ is the total length of the system. The Lagrange's equations are

$$m\ddot{x}_1(t) = -2kx_1(t) + kx_2(t)$$
$$m\ddot{x}_2(t) = 3kL_0 + kx_1(t) - 2kx_2(t) \, . \qquad (1.8)$$

In equilibrium ($\ddot{x}_1 = \ddot{x}_2 = 0$), we have

$$0 = -2kx_1^0 + kx_2^0$$
$$0 = kx_1^0 - 2kx_2^0 + 3kL_0 \, ,$$

or

$$x_1^0 = L_0$$
$$x_2^0 = 2L_0 \, .$$

For convenience, new variables are introduced as the deviation of the displacements from the positions in thermal equilibrium,

$$y_1 = x_1 - x_1^0$$
$$y_2 = x_2 - x_2^0 \, . \qquad (1.9)$$

Then we have

$$m\ddot{y}_1 = -2ky_1 + ky_2$$
$$m\ddot{y}_2 = ky_1 - 2ky_2 \quad , \qquad (1.10)$$

$$y_1 = \mathrm{Re}[Y_1 e^{i\omega t}]$$
$$y_2 = \mathrm{Re}[Y_2 e^{i\omega t}] \, , \qquad (1.11)$$

where Y_1 and Y_2 are the complex amplitudes, and ω is the angular frequency. Y_1 and Y_2 satisfy the equations defined by

$$-m\omega^2 Y_1 = -2kY_1 + kY_2$$
$$-m\omega^2 Y_2 = kY_1 - 2kY_2 \quad , \qquad (1.12)$$

or

$$-m\omega^2\begin{pmatrix}Y_1\\Y_2\end{pmatrix}=\begin{pmatrix}-2k & k\\ k & -2k\end{pmatrix}\begin{pmatrix}Y_1\\Y_2\end{pmatrix},$$

or

$$\frac{m\omega^2}{k}\begin{pmatrix}Y_1\\Y_3\end{pmatrix}=\begin{pmatrix}2k & -k\\ -k & 2k\end{pmatrix}\begin{pmatrix}Y_1\\Y_3\end{pmatrix}$$

$$\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}\begin{pmatrix}Y_1\\Y_3\end{pmatrix}=\lambda^2\begin{pmatrix}Y_1\\Y_3\end{pmatrix} \qquad (1.13).$$

This is the eigenvalue problem with the eigenvalue λ^2 ($\lambda=\sqrt{\frac{m\omega^2}{k}}=\omega\sqrt{\frac{m}{k}}$). We use the Mathematica to solve this problem for simplicity.

(a) The out-of-phase (mode-1)

$$\begin{pmatrix}Y_1\\Y_2\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}, \qquad (1.14)$$

for $\lambda_1 = m\omega_1^2 = 3k$; $\omega_1=\sqrt{\frac{3k}{m}}=\sqrt{3}\sqrt{\frac{k}{m}}$. Since $Y_1 = -Y_2 = 1/\sqrt{2}$, the movement of the mass m_1 (= m) is opposite to that of mass m_2 (= m) (the out-of-phase). The angular frequency of the mode 2 is the simple oscillation where the middle spring contributes to the restoring force.

(b) In-phase mode (mode 2)

$$\begin{pmatrix}Y_1\\Y_2\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}, \qquad (1.15)$$

for $\lambda_2 = m\omega_2^2 = k$; $\omega_2=\sqrt{\frac{k}{m}}$. Since $Y_1 = Y_2 = 1/\sqrt{2}$, the masses m_1 (= m) and m_2 (= m) undergoes the same displacement in the same direction (in-phase). This means that the spring between m_1 and m_2 remains un-stretched and un-shrunk. The mode 2 is the simple oscillation where the middle spring remains unchanged. The angular frequency ω_2 is smaller than ω_1.

(i) $\lambda_1 = 1.73205$

$$\mathbf{u}_1 = \begin{pmatrix} 0.707107 \\ -0.707107 \end{pmatrix}, \tag{1.16}$$

where $\boldsymbol{u}_1$ is the eigenvector and is normalized so that

$$\|\mathbf{u}_1\| = 1.$$

(ii) $\lambda_2 = 1.0$

$$\mathbf{u}_2 = \begin{pmatrix} 0.707107 \\ 0.707107 \end{pmatrix}, \tag{1.17}$$

where $\boldsymbol{u}_2$ is the eigenvector and is normalized so that

$$\|\mathbf{u}_2\| = 1.$$

We make a plot of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ as a function of the position n. (n = 0, 1, 2, 3). Here we assume that the 0-th and 3–th components of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are equal to zero.

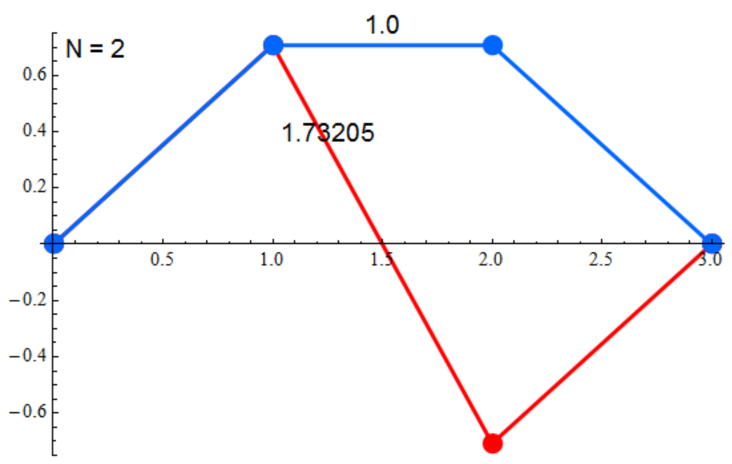

Fig.4 Plot of $\boldsymbol{u}_1$ (λ_1 = 1.73205) and $\boldsymbol{u}_2$ (λ_2 = 1.0) as a function of n (n = 0, 1, 2, 3).

Here we make a plot of the sine function define by

$$a_k(n, N=2) = \frac{\sin(\frac{\pi}{3}kn)}{\sqrt{\sum_{m=0}^{3}\sin^2(\frac{\pi}{3}km)}}, \tag{1.18}$$

as a function of n, where k is the mode number (k = 1, 2). We also make the plot of $\boldsymbol{u}_k$, (k = 1, 2) at integer n. We find that these points well fall on the curve of $a_k(n, N = 2)$ vs n for each mode k.

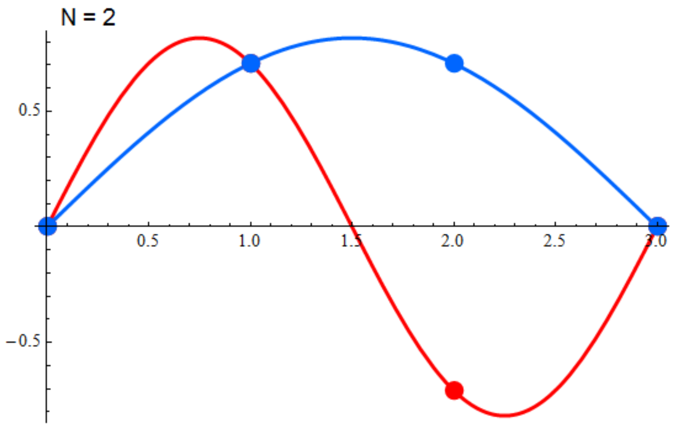

Fig.5 Plot of $a_k(n, N = 2)$ as a function of n (n = 0, 1, 2, 3). The solid circles are denoted by the two modes $\boldsymbol{u}_1$ (λ_1 = 1.73205) and $\boldsymbol{u}_2$ (λ_2 = 1.0).

1.3. N = 3 longitudinal wave

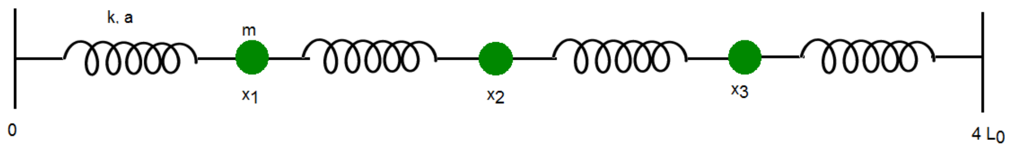

Fig.6 A system with three masses (m) (located at the positions $x = x_1$, x_2, and x_3) and four springs with a spring constant k. a is the length of un-stretched spring. The total distance is $4L_0$. Both sides are fixed.

For simplicity, we use the Lagrange's method for the motion of the spring systems with three point masses with mass m. The spring constant is k. Both sides are fixed. The Lagrangian of this system is given by

$$L = \frac{1}{2}m[\dot{x}_1(t)]^2 + \frac{1}{2}m[\dot{x}_2(t)]^2 + \frac{1}{2}m[\dot{x}_3(t)]^2 - \frac{1}{2}k[x_1(t) - a]^2 - \frac{1}{2}k[x_2(t) - x_1(t) - a]^2$$
$$-\frac{1}{2}k[x_3(t) - x_2(t) - a]^2 - \frac{1}{2}k[4L_0 - x_3(t) - a]^2$$

(1.19),

where a is the length of un-stretched spring and $4L_0$ is the total length of the system. The Lagrange's equations:

$$m\ddot{x}_1(t) = -2kx_1(t) + kx_2(t)$$
$$m\ddot{x}_2(t) = kx_1(t) - 2kx_2(t) + kx_3(t)$$
$$m\ddot{x}_3(t) = kx_2(t) - 2kx_3(t) + 4kL_0 \quad . \qquad (1.20)$$

In equilibrium ($\ddot{x}_1 = \ddot{x}_2 = \ddot{x}_3 = 0$), we have

$$0 = -2kx_1^0 + kx_2^0$$
$$0 = kx_1^0 - 2kx_2^0 + kx_3^0$$
$$0 = kx_2^0 - 2kx_3^0 + 4kL_0 \, ,$$

or

$$x_1^0 = L_0$$
$$x_2^0 = 2L_0$$
$$x_2^0 = 3L_0 \, .$$

Here new variables are introduced as the deviation of the displacements from the positions in thermal equilibrium,

$$y_1 = x_1 - x_1^0$$
$$y_2 = x_2 - x_2^0$$
$$y_3 = x_3 - x_3^0 \, . \qquad (1.21)$$

Then we have

$$m\ddot{y}_1 = -2ky_1 + ky_2$$
$$m\ddot{y}_2 = ky_1 - 2ky_2 + ky_3$$
$$m\ddot{y}_3 = ky_2 - 2ky_3 \quad . \qquad (1.22)$$

We assume that

$$y_1 = \mathrm{Re}[Y_1 e^{i\omega t}]$$
$$y_2 = \mathrm{Re}[Y_2 e^{i\omega t}]$$
$$y_3 = \mathrm{Re}[Y_3 e^{i\omega t}] \, , \qquad (1.23)$$

where Y_1, Y_2, and Y_3 are the complex amplitudes, and ω is the angular frequency, Y_1, Y_2, and Y_3 satisfy the equations defined by

$$-m\omega^2 Y_1 = -2kY_1 + kY_2$$
$$-m\omega^2 Y_2 = kY_1 - 2kY_2 + kY_3$$
$$-m\omega^2 Y_3 = kY_2 - 2kY_3 \qquad (1.24),$$

$$\frac{m\omega^2}{k}\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \lambda^2 \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}. \qquad (1.25)$$

This is the eigenvalue problem with the eigenvalue λ^2 ($\lambda = \sqrt{\frac{m\omega^2}{k}} = \omega\sqrt{\frac{m}{k}}$). We use the Mathematica to solve the problem. There are three eigenvalues

(1) $\lambda_1 = 1.84776$

$$\mathbf{u}_1 = \begin{pmatrix} 0.5 \\ -0.707107 \\ 0.5 \end{pmatrix}, \qquad (1.26)$$

where $\boldsymbol{u}_1$ is the eigenvector and is normalized so that

$$\|\mathbf{u}_1\| = 1.$$

(ii) $\lambda_2 = 1.41421$

$$\mathbf{u}_2 = \begin{pmatrix} 0.707107 \\ 0 \\ -0.707107 \end{pmatrix}, \qquad (1.27)$$

where $\boldsymbol{u}_2$ is the eigenvector and is normalized so that

$$\|\mathbf{u}_2\| = 1.$$

(iii) $\lambda_3 = 0.765367$

$$\mathbf{u}_3 = \begin{pmatrix} 0.5 \\ 0.707107 \\ 0.5 \end{pmatrix}, \tag{1.28}$$

where $\boldsymbol{u}_3$ is the eigenvector and is normalized so that

$$\|\mathbf{u}_3\| = 1.$$

We make a plot of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, and $\boldsymbol{u}_3$ as a function of the position n. (n = 0, 1, 2, 3, 4). Here we assume that the 0-th and 4–th components of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, and $\boldsymbol{u}_3$ are equal to zero.

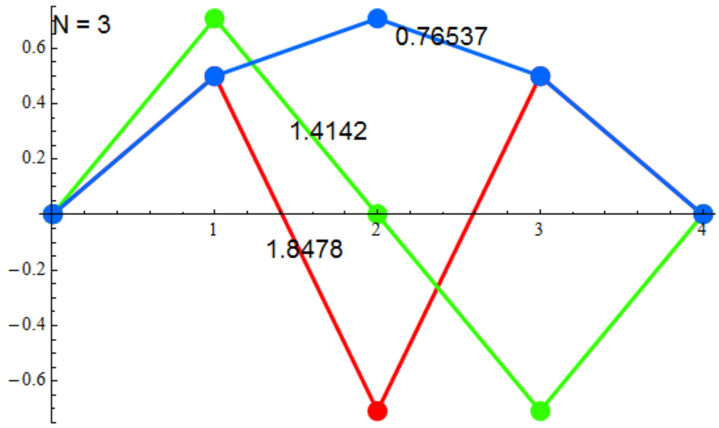

Fig.7 Plot of $\boldsymbol{u}_1$ (λ_1 = 1.8478), $\boldsymbol{u}_2$ (λ_2 = 1.4142), and $\boldsymbol{u}_3$ (λ_3 = 0.76537), as a function of n (n = 0, 1, 2, 3, 4).

Here we make a plot of the sine function defined by

$$a_k(n, N=3) = \frac{\sin(\frac{\pi}{4}kn)}{\sqrt{\sum_{m=0}^{4}\sin^2(\frac{\pi}{4}km)}}, \tag{1.29}$$

as a function of n, where k is the mode number (k = 1, 2, 3). We also make the plot of $\boldsymbol{u}_k$, (k = 1, 2, 3) at integer n. We find that these points well fall on the curve of $a_k(n, N = 3)$ vs n for each mode k.

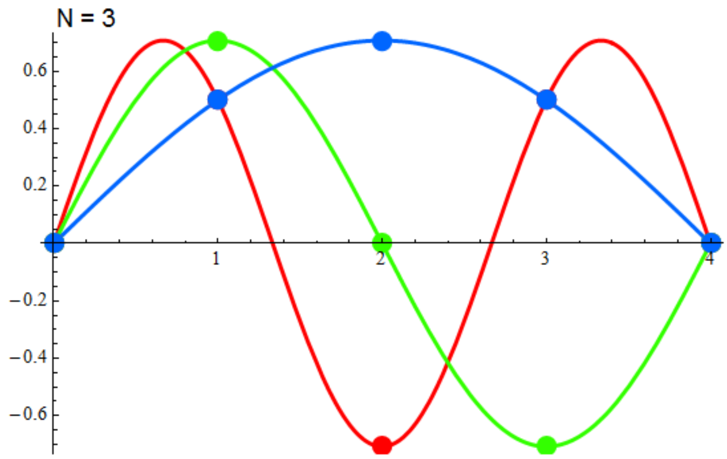

Fig.8 Plot of $a_k(n, N = 3)$ as a function of n ($n = 0, 1, 2, 3, 4$). The solid circles are denoted by the three modes, $\boldsymbol{u}_1$ ($\lambda_1 = 1.8478$), $\boldsymbol{u}_2$ ($\lambda_2 = 1.4142$), and $\boldsymbol{u}_3$ ($\lambda_2 = 0.76537$).

1.4. $N = 4$ longitudinal wave

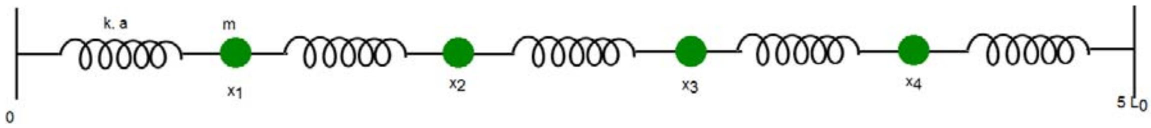

Fig.9A system with four masses (m) (located at the positions $x = x_1$, x_2, x_3, and x_4) and five springs with a spring constant k. a is the length of un-stretched spring. The total distance is $5L_0$. Both sides are fixed.

For the system with 4 masses, the eigenvalue problem to be solved is as follows.

$$\frac{m\omega^2}{k}Y_1 = 2Y_1 - Y_2$$
$$\frac{m\omega^2}{k}Y_2 = -Y_1 + 2Y_2 - Y_3$$
$$\frac{m\omega^2}{k}Y_3 = -Y_2 + 2Y_3 - Y_4$$
$$\frac{m\omega^2}{k}Y_4 = -Y_3 + 2Y_4 \quad , \qquad (1.30)$$

or

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} = \lambda^2 \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix}. \qquad (1.31)$$

((**Mathematica**)) We use the Mathematica to solve the eigenvalue problem.

```
A1 = -{{-2, 1, 0, 0}, {1, -2, 1, 0}, {0, 1, -2, 1}, {0, 0, 1, -2}}
{{2, -1, 0, 0}, {-1, 2, -1, 0}, {0, -1, 2, -1}, {0, 0, -1, 2}}

A1 // MatrixForm
( 2  -1  0   0 )
( -1  2  -1  0 )
( 0  -1  2  -1 )
( 0   0  -1  2 )

s1 = Eigensystem[A1] // N
{{3.61803, 2.61803, 1.38197, 0.381966},
 {{-1., 1.61803, -1.61803, 1.}, {1., -0.618034, -0.618034, 1.},
  {-1., -0.618034, 0.618034, 1.}, {1., 1.61803, 1.61803, 1.}}}
```

(1) $\lambda_1 = 1.90211$

$$\mathbf{u}_1 = \begin{pmatrix} 0.371748 \\ -0.601501 \\ 0.601501 \\ -0.371748 \end{pmatrix}, \qquad (1.32)$$

where $\boldsymbol{u}_1$ is the eigenvector and is normalized so that

$$\|\mathbf{u}_1\| = 1.$$

(ii) $\lambda_2 = 1.61803$

$$\mathbf{u}_2 = \begin{pmatrix} 0.601501 \\ -0.371748 \\ -0.371748 \\ 0.601501 \end{pmatrix} \qquad (1.33),$$

where $\boldsymbol{u}_2$ is the eigenvector and is normalized so that

$$\|\mathbf{u}_2\|=1 .$$

(iii) $\lambda_3 = 1.17557$

$$\mathbf{u}_3 = \begin{pmatrix} 0.601501 \\ 0.371748 \\ -0.371748 \\ -0.601501 \end{pmatrix}, \tag{1.34}$$

where $\boldsymbol{u}_3$ is the eigenvector and is normalized so that

$$\|\mathbf{u}_3\|=1 .$$

(iv) $\lambda_4 = 0.618043$

$$\mathbf{u}_4 = \begin{pmatrix} 0.371748 \\ 0.601501 \\ 0.601501 \\ 0.371748 \end{pmatrix}, \tag{1.35}$$

where $\boldsymbol{u}_4$ is the eigenvector and is normalized so that

$$\|\mathbf{u}_4\|=1 .$$

We make a plot of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$, and $\boldsymbol{u}_4$ as a function of the position n. (n = 0, 1, 2, 3, 4, 5). Here we assume that the 0-th and 5–th components of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$ and $\boldsymbol{u}_4$ are equal to zero.

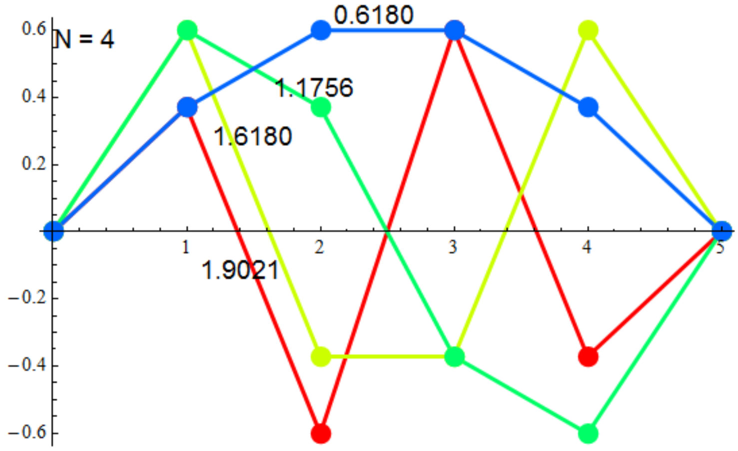

Fig.10 Plot of $\boldsymbol{u}_1$ (λ_1 = 1.9021), $\boldsymbol{u}_2$ (λ_2 = 1.6180), $\boldsymbol{u}_3$ (λ_3 = 1.1756), and $\boldsymbol{u}_4$ (λ_4 = 0.6180), as a function of n (n = 0, 1, 2, 3, 4, 5).

Here we make a plot of the sine function define by

$$a_k(n, N=4) = \frac{\sin(\frac{\pi}{5}kn)}{\sqrt{\sum_{m=0}^{5}\sin^2(\frac{\pi}{5}km)}}, \tag{1.36}$$

as a function of n, where k is the mode number (k = 1, 2, …, 4). We also make the plot of $\boldsymbol{u}_k$, (k = 1, 2, …, 4) at integer n. We find that these points well fall on the curve of $a_k(n, N = 4)$ vs n for each mode k.

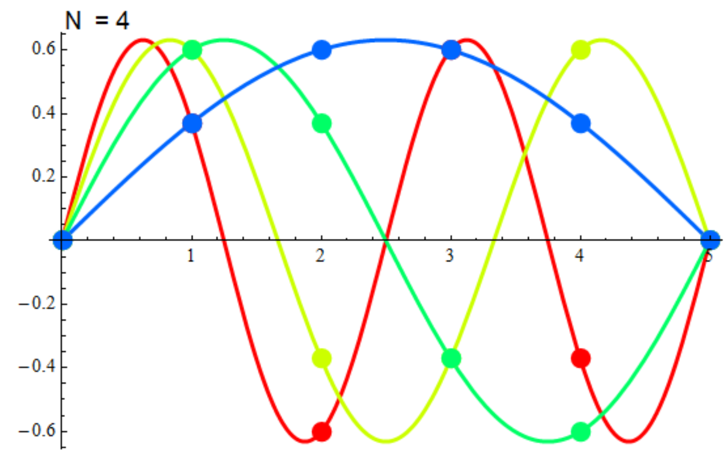

Fig.11 Plot of $a_k(n, N = 4)$ as a function of n (n = 0, 1, 2, 3, 4, 5). The solid circles are denoted by the four modes, $\boldsymbol{u}_1$ (λ_1 = 1.9021), $\boldsymbol{u}_2$ (λ_2 = 1.6180), $\boldsymbol{u}_3$ (λ_3 = 1.1756), and $\boldsymbol{u}_4$ (λ_4 = 0.6180).

1.5. N = 5 longitudinal wave

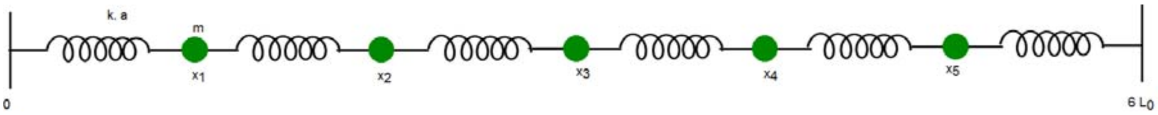

Fig.12 A system with five masses (m) (located at the positions $x = x_1$, x_2, x_3, x_4, and x_5) and six springs with a spring constant k. a is the length of un-stretched spring. The total distance is $6L_0$. Both sides are fixed.

For the system with 5 masses, the eigenvalue problem to be solve is as follows.

$$\frac{m\omega^2}{k}Y_1 = 2Y_1 - Y_2$$
$$\frac{m\omega^2}{k}Y_2 = -Y_1 + 2Y_2 - Y_3$$
$$\frac{m\omega^2}{k}Y_3 = -Y_2 + 2Y_3 - Y_4$$
$$\frac{m\omega^2}{k}Y_4 = -Y_3 + 2Y_4 - Y_5$$
$$\frac{m\omega^2}{k}Y_5 = -Y_4 + 2Y_5 , \qquad (1.37)$$

or

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{pmatrix} = \lambda^2 \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{pmatrix} . \qquad (1.38)$$

((**Mathematica**)) We use the Mathematica to solve the eigenvalue-problem.

```
A1 = {{2, -1, 0, 0, 0}, {-1, 2, -1, 0, 0}, {0, -1, 2, -1, 0},
   {0, 0, -1, 2, -1}, {0, 0, 0, -1, 2}}

{{2, -1, 0, 0, 0}, {-1, 2, -1, 0, 0},
 {0, -1, 2, -1, 0}, {0, 0, -1, 2, -1}, {0, 0, 0, -1, 2}}

A1 // MatrixForm
```

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

```
s1 = Eigensystem[A1] // N

{{3.73205, 3., 2., 1., 0.267949}, {{1., -1.73205, 2., -1.73205, 1.},
   {-1., 1., 0., -1., 1.}, {1., 0., -1., 0., 1.},
   {-1., -1., 0., 1., 1.}, {1., 1.73205, 2., 1.73205, 1.}}}
```

For the system with 5 masses

(1) $\lambda_1 = 1.93185$

$$\mathbf{u}_1 = \begin{pmatrix} 0.288675 \\ -0.50 \\ 0.57735 \\ -0.5 \\ 0.288675 \end{pmatrix}, \tag{1.39}$$

where $\boldsymbol{u}_1$ is the eigenvector and is normalized so that

$$\| \mathbf{u}_1 \| = 1$$

(ii) $\lambda_2 = 1.73205$

$$\mathbf{u}_2 = \begin{pmatrix} 0.5 \\ -0.5 \\ 0 \\ 0.5 \\ -0.5 \end{pmatrix}, \tag{1.40}$$

where $\boldsymbol{u}_2$ is the eigenvector and is normalized so that

$$\| \mathbf{u}_2 \| = 1 .$$

(iii) $\lambda_3 = 1.41421$

$$\mathbf{u}_3 = \begin{pmatrix} 0.57735 \\ 0 \\ -0.57735 \\ 0 \\ 0.57735 \end{pmatrix}, \tag{1.41}$$

where $\boldsymbol{u}_3$ is the eigenvector and is normalized so that

$$\| \mathbf{u}_3 \| = 1 .$$

(iv) $\lambda_4 = 1$

$$\mathbf{u}_4 = \begin{pmatrix} 0.5 \\ 0.5 \\ 0 \\ -0.5 \\ -0.5 \end{pmatrix}, \tag{1.42}$$

where $\boldsymbol{u}_4$ is the eigenvector and is normalized so that

$$\| \mathbf{u}_4 \| = 1 .$$

(iv) $\lambda_5 = 0.517638$

$$\mathbf{u}_5 = \begin{pmatrix} 0.288675 \\ 0.5 \\ 0.57735 \\ 0.5 \\ 0.288675 \end{pmatrix}, \tag{1.43}$$

where $\boldsymbol{u}_5$ is the eigenvector and is normalized such that

$$\| \mathbf{u}_5 \| = 1 .$$

We make a plot of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$, $\boldsymbol{u}_4$, and $\boldsymbol{u}_5$ as a function of the position n. (n = 0, 1, 2, 3, 4, 5, 6). Here we assume that the 0-th and 6–th components of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$ $\boldsymbol{u}_4$, u_5, and $\boldsymbol{u}_6$ are equal to zero.

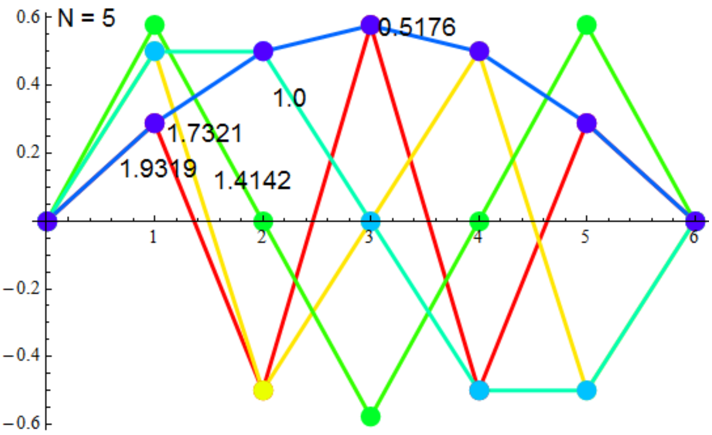

Fig.13 Plot of $\boldsymbol{u}_1$ (λ_1 = 1.9319), $\boldsymbol{u}_2$ (λ_2 = 1.7310), $\boldsymbol{u}_3$ (λ_3 = 1.4142), $\boldsymbol{u}_4$ (λ_4 = 1.0), and $\boldsymbol{u}_5$ (λ_5 = 0.5176), as a function of n (n = 0, 1, 2, 3, 4, 5,6).

Here we make a plot of the sine function define by

$$a_k(n, N=5) = \frac{\sin(\frac{\pi}{6}kn)}{\sqrt{\sum_{m=0}^{6}\sin^2(\frac{\pi}{6}km)}}, \qquad (1.44)$$

as a function of n, where k is the mode number (k = 1, 2, …, 5). We also make the plot of $\boldsymbol{u}_k$, (k = 1, 2, …, 5) at integer n. We find that these points well fall on the curve of $a_k(n, N = 5)$ vs n for each mode k.

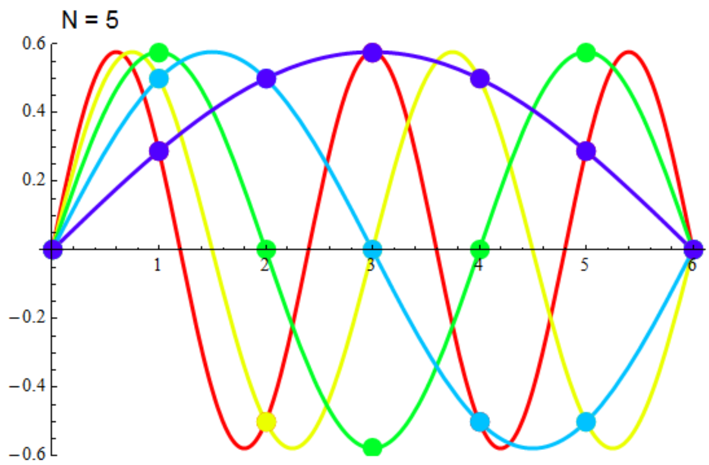

Fig.14 Plot of $a_k(n, N = 5)$ as a function of n (n = 0, 1, 2, 3, 4, 5, 6). The solid circles are denoted by the five modes, $\boldsymbol{u}_1$ (λ_1 = 1.9319), $\boldsymbol{u}_2$ (λ_2 = 1.7310), $\boldsymbol{u}_3$ (λ_3 = 1.4142), $\boldsymbol{u}_4$ (λ_4 = 1.0), and $\boldsymbol{u}_5$ (λ_5 = 0.5176).

This means that

$$\mathbf{u}_k \approx \mathbf{a}_k(n, N=5),$$

where

$$\mathbf{a}_k = \begin{pmatrix} a_k(1) \\ a_k(2) \\ a_k(3) \\ a_k(4) \\ a_k(5) \end{pmatrix}$$

1.6. Longitudinal wave for *N* masses

We now consider the modes for the *N* masses on the chain. In equilibrium, the masses are located at

$x = L_0/(N+1)$, $2L_0/(N+1)$, $3\,L_0/(N+1)$,

where L_0 is the length of the system and $\Delta x = L_0/(N+1)$ is the separation distance between the nearest masses in thermal equilibrium. The mass is the same (= *m*) and the spring constant is the same (= *k*). y_i, (i = 1, 2, …, *N*) is the displacement of the *i*-th mass from the equilibrium position ($x_i = i\,L_0/(N+1)$. Using the Lagrange's method, we have the equation s of motion

$$\begin{aligned}
m\ddot{y}_1 &= -k(y_1 - y_0) + k(y_2 - y_1) \\
m\ddot{y}_2 &= -k(y_2 - y_1) + k(y_3 - y_2) \\
m\ddot{y}_3 &= -k(y_3 - y_2) + k(y_4 - y_3) \\
m\ddot{y}_4 &= -k(y_4 - y_3) + k(y_5 - y_4) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
m\ddot{y}_n &= -k(y_n - y_{n-1}) + k(y_{n+1} - y_n) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
m\ddot{y}_N &= -k(y_N - y_{N-1}) + k(y_{N+1} - y_N),
\end{aligned} \tag{1.45}$$

where $y_0(t) = y_{N+1}(t) = 0$.

$$y_i = \mathrm{Re}[Y_i e^{i\omega t}], \tag{1.46}$$

$$-m\omega^2 Y_n = -k(Y_n - Y_{n-1}) + k(Y_{n+1} - Y_n) \tag{1.47}$$

where ω is the angular frequency, or

$$\frac{m\omega^2}{k}Y_n = (Y_n - Y_{n-1}) - (Y_{n+1} - Y_n) = 2Y_n - Y_{n+1} - Y_{n-1} \quad . \tag{1.48}$$

From the case of N = 2, 3, 4, and 5, it is reasonable to assume that Y_n can be expressed in the form of

$$Y_n = \sin(pn) \quad . \tag{1.49}$$

The displacements of the system are described by

$$\mathbf{u} = \begin{pmatrix} Y_0 = 0 \\ Y_1 \\ Y_2 \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ Y_N \\ Y_{N+1} = 0 \end{pmatrix} \quad . \tag{1.50}$$

Here p is constant and is determined from the condition $Y_{N+1} = 0$.

$$Y_{N+1} = \sin[p(N+1)] = 0 \quad , \tag{1.51}$$

$$p_\kappa = \frac{k\pi}{N+1} \qquad (\kappa = 1, 2, \quad , N). \tag{1.52}$$

This means that there are N modes in this system.

For $p = p_{N+1}$,

$$Y_n = \sin(p_{N+1}n) = \sin(\frac{(N+1)}{N+1}\pi n) = \sin(\pi n) = 0 \quad . \tag{1.53}$$

For $p = p_{N+j}$, (j = 2, 3, 4,   ),

$$\begin{aligned} Y_n &= \sin(p_{N+j}n) \\ &= \sin(\frac{(N+j)}{N+1}\pi n) \\ &= \sin(2\pi n - \frac{N-j+2}{N+1}\pi n) \\ &= -\sin(\frac{N-j+2}{N+1}\pi n) \quad . \end{aligned} \tag{1.54}$$

Then the solution of Y_n with $p = p_{N+j}$ is the same as that with $p = p_{N-j+2}$. The substitution of the form of Y_n into Eq.(1) yields to the dispersion relation (relation between ω and p_k)

$$\frac{\omega_\kappa^{\ 2}}{\omega_0^{\ 2}} = 2[1-\cos(p_\kappa)] = 4\sin^2(\frac{p_\kappa}{2})$$

$$\omega_\kappa = 2\omega_0 \left|\sin(\frac{p_\kappa}{2})\right| = 2\omega_0 \left|\sin[\frac{\kappa\pi}{2(N+1)}]\right|, \qquad (1.55)$$

where

$$\omega_0 = \sqrt{k/m}.$$

Note that k is a spring constant and that κ is the mode number. The dispersion relation of for N = 50 is shown in Fig.15, where p_κ is the wave number and the angular frequency is normarized by $2\omega_0$.

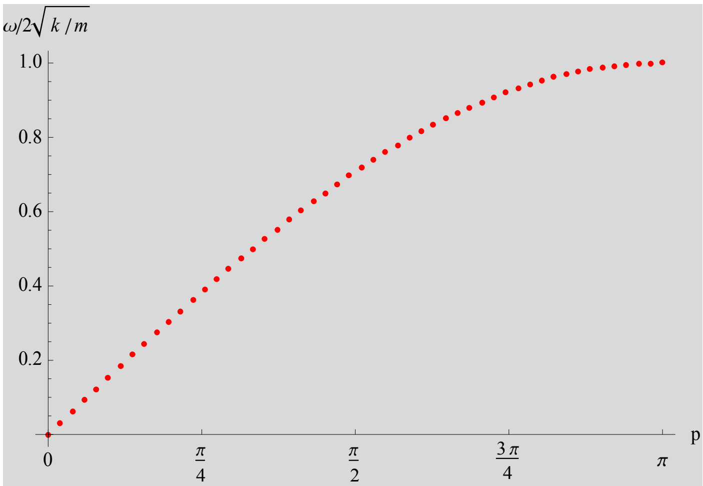

Fig.15 Dispersion relation for N = 50. The angular frequency ω_k vs the wave number p_κ.

The chosen modes with κ = 1, 2, 3, …, 10 for N = 50 are plotted in **Fig.16**.

$$a_\kappa(n, N=50) = \frac{\sin(\frac{\pi}{51}\kappa n)}{\sqrt{\sum_{m=0}^{50}\sin^2(\frac{\pi}{51}\kappa m)}}. \qquad (1.56)$$

Fig. 16 Plot of $a_k(n, N=50)$ with k = 1, 2, ..., 10, as a function of *n*.

2. Double pendulum (another example)

Here we consider the motion of the double pendulum. This motion is similar to that of the longitudinal wave in the chain system with $N = 2$.

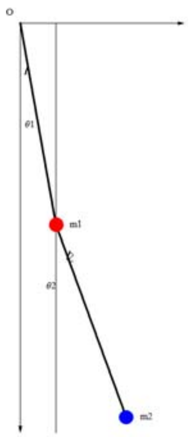

Fig.17 Schematic diagram of the double pendulum. $l_1 = l_2$. $m_1 = m_2$.

θ_1 and θ_2 are angled (normal coordinates)

$$\mathbf{r}_1 = (l_1 \sin\theta_1, l_1 \cos\theta_1)$$
$$\mathbf{r}_2 = (l_1 \sin\theta_1 + l_2 \sin\theta_2, l_1 \cos\theta_1 + l_2 \cos\theta_2) . \qquad (2.1)$$

The kinetic energy *K* is given by

$$K = \frac{1}{2}m_1(l_1\dot{\theta}_1)^2 + \frac{1}{2}m_2[(l_1\dot{\theta}_1)^2 + (l_2\dot{\theta}_2)^2 + 2l_1l_2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2] \quad . \qquad (2.2)$$

The potential energy U is given by

$$U = -m_1gl_1\cos\theta_1 - m_2g(l_1\cos\theta_1 + l_2\cos\theta_2) \quad . \qquad (2.3)$$

From the definition, the Lagrangian L is

$$L = K - U \quad . \qquad (2.4)$$

The Lagrange's equation is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) = \frac{\partial L}{\partial\theta_1}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right) = \frac{\partial L}{\partial\theta_2} \quad . \qquad (2.5)$$

The equations of motion are obtained as

$$(m_1 + m_2)l_1\ddot{\theta}_1 + m_2l_2\cos(\theta_1 - \theta_2)\ddot{\theta}_1 + m_2l_2\sin(\theta_1 - \theta_2)\dot{\theta}_2^{\,2} + (m_1 + m_2)g\sin\theta_1 = 0$$

$$l_2\ddot{\theta}_2 + l_1\cos(\theta_1 - \theta_2)\ddot{\theta}_1 - l_1\sin(\theta_1 - \theta_2)\dot{\theta}_1^{\,2} + g\sin\theta_2 = 0$$

(2.6).

((**Mathematica**)) Variational method

```
(*Lagrange equation*)

Clear["Global`*"]

<< "VariationalMethods`"

L = 1/2 m1 ℓ1^2 θ1'[t]^2 + 1/2 m2 (ℓ1^2 θ1'[t]^2 + ℓ2^2 θ2'[t]^2) +
  m2 ℓ1 ℓ2 Cos[θ1[t] - θ2[t]] θ1'[t] θ2'[t] + m1 g ℓ1 Cos[θ1[t]] +
  m2 g (ℓ1 Cos[θ1[t]] + ℓ2 Cos[θ2[t]])

g m1 ℓ1 Cos[θ1[t]] + g m2 (ℓ1 Cos[θ1[t]] + ℓ2 Cos[θ2[t]]) + 1/2 m1 ℓ1^2 θ1′[t]^2 +
 m2 ℓ1 ℓ2 Cos[θ1[t] - θ2[t]] θ1′[t] θ2′[t] + 1/2 m2 (ℓ1^2 θ1′[t]^2 + ℓ2^2 θ2′[t]^2)

eq11 = VariationalD[L, θ1[t], t]

-ℓ1 (g m1 Sin[θ1[t]] + g m2 Sin[θ1[t]] + m2 ℓ2 Sin[θ1[t] - θ2[t]] θ2′[t]^2 +
    (m1 + m2) ℓ1 θ1″[t] + m2 ℓ2 Cos[θ1[t] - θ2[t]] θ2″[t])

eq12 = VariationalD[L, θ2[t], t]

-m2 ℓ2 (g Sin[θ2[t]] - ℓ1 Sin[θ1[t] - θ2[t]] θ1′[t]^2 +
    ℓ1 Cos[θ1[t] - θ2[t]] θ1″[t] + ℓ2 θ2″[t])

eq21 = EulerEquations[L, θ1[t], t]

-ℓ1 (g m1 Sin[θ1[t]] + g m2 Sin[θ1[t]] + m2 ℓ2 Sin[θ1[t] - θ2[t]] θ2′[t]^2 +
     (m1 + m2) ℓ1 θ1″[t] + m2 ℓ2 Cos[θ1[t] - θ2[t]] θ2″[t]) == 0

eq22 = EulerEquations[L, θ2[t], t]

-m2 ℓ2 (g Sin[θ2[t]] - ℓ1 Sin[θ1[t] - θ2[t]] θ1′[t]^2 +
     ℓ1 Cos[θ1[t] - θ2[t]] θ1″[t] + ℓ2 θ2″[t]) == 0

eq31 = FirstIntegrals[L, θ1[t], t] // Simplify

{FirstIntegral[t] → 1/2 (-2 g ((m1 + m2) ℓ1 Cos[θ1[t]] + m2 ℓ2 Cos[θ2[t]]) +
      (m1 + m2) ℓ1^2 θ1′[t]^2 - m2 ℓ2^2 θ2′[t]^2)}

eq31 = FirstIntegrals[L, θ2[t], t] // Simplify

{FirstIntegral[t] → 1/2 (-2 g ((m1 + m2) ℓ1 Cos[θ1[t]] + m2 ℓ2 Cos[θ2[t]]) -
      (m1 + m2) ℓ1^2 θ1′[t]^2 + m2 ℓ2^2 θ2′[t]^2)}
```

In the limit of small angles, the above equations are simplified as

$$\begin{aligned}&(m_1+m_2)l_1\ddot{\theta}_1+m_2 l_2\ddot{\theta}_1+(m_1+m_2)g\theta_1=0\\&l_2\ddot{\theta}_2+l_1\ddot{\theta}_1+g\sin\theta_2=0\end{aligned}. \quad (2.7)$$

Here we use the following approximations,

$$\begin{aligned}&\sin(\theta_1-\theta_2)=0\\&\cos(\theta_1-\theta_2)=0\\&\sin\theta_1\approx\theta_1\\&\sin\theta_2\approx\theta_2\end{aligned}, \quad (2.8)$$

For simplicity, hereafter we assume that $l_1 = l_2 = l$ and $m_1 = m_2 = m$.

$$\begin{aligned}&2\ddot{\theta}_1+\ddot{\theta}_1=-2\gamma\theta_1\\&\ddot{\theta}_1+\ddot{\theta}_2=-\gamma\theta_2\end{aligned}, \quad (2.9)$$

where

$$\gamma=\frac{g}{l}.$$

We assume that

$$\begin{aligned}&\theta_1=\mathrm{Re}[\Theta_1 e^{i\omega t}]\\&\theta_2=\mathrm{Re}[\Theta_2 e^{i\omega t}]\end{aligned}, \quad (2.10)$$

where Θ_1 and Θ_2 are the complex amplitudes and ω is the angular frequency.

$$\begin{aligned}&-\omega^2(2\Theta_1+\Theta_2)=-2\gamma\Theta_1\\&-\omega^2(\Theta_1+\Theta_2)=-\gamma\theta_2\end{aligned}. \quad (2.11)$$

This is reduced to the eigenvalue problem after simple procedures,

$$\frac{\omega^2}{\gamma}\begin{pmatrix}2&1\\1&1\end{pmatrix}\begin{pmatrix}\Theta_1\\\Theta_2\end{pmatrix}=\begin{pmatrix}2&0\\0&1\end{pmatrix}\begin{pmatrix}\Theta_1\\\Theta_2\end{pmatrix},$$

or

$$\frac{\omega^2}{\gamma}\begin{pmatrix}\Theta_1\\\Theta_2\end{pmatrix}=\begin{pmatrix}2&1\\1&1\end{pmatrix}^{-1}\begin{pmatrix}2&0\\0&1\end{pmatrix}\begin{pmatrix}\Theta_1\\\Theta_2\end{pmatrix}=\begin{pmatrix}2&-1\\-2&2\end{pmatrix}\begin{pmatrix}\Theta_1\\\Theta_2\end{pmatrix},$$

or

$$\begin{pmatrix}2&-1\\-2&2\end{pmatrix}\begin{pmatrix}\Theta_1\\\Theta_2\end{pmatrix}=\lambda\begin{pmatrix}\Theta_1\\\Theta_2\end{pmatrix}, \quad (2.12)$$

where $\lambda = \dfrac{\omega^2}{\gamma}$.

((**Mathematica**)) Eigenvalue problem

```
A1 = (2 1
      1 1); B1 = (2 0
                  0 1)
{{2, 0}, {0, 1}}

C1 = Inverse[A1].B1
{{2, -1}, {-2, 2}}

Eigensystem[C1] // Simplify
```

$$\left\{\left\{2+\sqrt{2},\ 2-\sqrt{2}\right\},\ \left\{\left\{-\frac{1}{\sqrt{2}},\ 1\right\},\ \left\{\frac{1}{\sqrt{2}},\ 1\right\}\right\}\right\}$$

(a) The in-phase mode (normal mode)

The eigenvalue is

$$\lambda = \lambda_1 = \frac{\omega_1{}^2}{\gamma} = 2-\sqrt{2}, \qquad \omega_1 = \sqrt{2-\sqrt{2}}\sqrt{\gamma} = 0.76537\sqrt{\gamma}. \tag{2.13}$$

The eigenvector belonging to the eigenvalue λ_1 is

$$\mathbf{u}^{(1)} = \begin{pmatrix}\Theta_1 \\ \Theta_2\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}} \\ 1\end{pmatrix}. \tag{2.14}$$

(b) The out-of-phase mode (normal mode)

The eigenvalue of the out-of-phase mode is

$$\lambda = \lambda_2 = \frac{\omega_2{}^2}{\gamma} = 2+\sqrt{2}, \qquad \omega_2 = \sqrt{2+\sqrt{2}}\sqrt{\gamma} = 1.84776\sqrt{\gamma}.$$

The eigenvector belonging to the eigenvalue λ_2 is

$$\mathbf{u}^{(2)} = \begin{pmatrix}\Theta_1 \\ \Theta_2\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}} \\ -1\end{pmatrix}. \tag{2.15}$$

Since the differential equations are linear, we have the general form given by any superposition of $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$,

$$\begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix} = C_1\mathbf{u}^{(1)} + C_2\mathbf{u}^{(2)} = C_1\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \end{pmatrix} + C_2\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{C_1 + C_2}{\sqrt{2}} \\ C_1 - C_2 \end{pmatrix}. \tag{2.16}$$

where C_1 and C_2 are complex constants. From Eqs.(2.10) and (2.16), we have

$$\theta_1(t) = \frac{1}{\sqrt{2}}\mathrm{Re}[(C_1 + C_2)e^{i\omega t}] \qquad \dot{\theta}_1(t) = \frac{1}{\sqrt{2}}\mathrm{Re}[i\omega(C_1 + C_2)e^{i\omega t}]$$
$$\theta_2(t) = \mathrm{Re}[(C_1 - C_2)e^{i\omega t}] \qquad \dot{\theta}_2(t) = \mathrm{Re}[i\omega(C_1 - C_2)e^{i\omega t}] \tag{2.17}$$

and the initial conditions (at $t = 0$)

$$\theta_1(t=0) = \frac{1}{\sqrt{2}}\mathrm{Re}[(C_1 + C_2)] \qquad \dot{\theta}_1(t=0) = \frac{1}{\sqrt{2}}\mathrm{Re}[i\omega(C_1 + C_2)]$$
$$\theta_2(t=0) = \mathrm{Re}[(C_1 - C_2)] \qquad \dot{\theta}_2(t=0) = \mathrm{Re}[i\omega(C_1 - C_2)] \tag{2.18}$$

(i) Initial condition for the observation of only the in-phase mode

To this end, it is necessary to have $C_2 = 0$. Then we get

$$\theta_1(t) = \frac{1}{\sqrt{2}}\mathrm{Re}[C_1)e^{i\omega t}]$$
$$\theta_2(t) = \mathrm{Re}[C_1e^{i\omega t}] \quad , \tag{2.19}$$

which leads to $\sqrt{2}\theta_1(t) = \theta_2(t)$ In other words, if we have the initial condition such that

$$\sqrt{2}\theta_1(t=0) = \theta_2(t=0) = \mathrm{Re}[C_1], \qquad \sqrt{2}\dot{\theta}_1(t=0) = \dot{\theta}_2(t=0) = \mathrm{Re}[i\omega C_1],$$

then the in-phase mode can be realized experimentally.

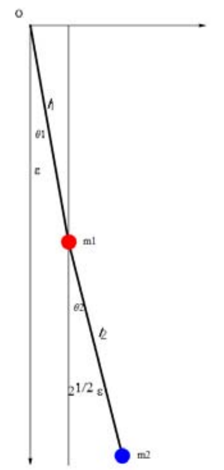

Fig.18(a) The initial conditions for the In-phase mode of the double pendulum. We choose the initial conditions such that $\theta_1(t=0) = \varepsilon$. $\theta_2(t=0) = \sqrt{2}\varepsilon$. $\dot{\theta}_1(t=0) = 0$. $\dot{\theta}_2(t=0) = 0$. ε is a very small angle.

(ii) Initial condition for the observation of the out-of-phase mode

To this end, it is necessary to have $C_1 = 0$. Then we get

$$\theta_1(t) = \frac{1}{\sqrt{2}} \mathrm{Re}[C_2 e^{i\omega t}]$$

$$\theta_2(t) = -\mathrm{Re}[C_2 e^{i\omega t}] \quad , \qquad (2.20)$$

which leads to $\sqrt{2}\theta_1(t) = -\theta_2(t)$. In other words, if we have the initial condition such that

$$\sqrt{2}\theta_1(t=0) = -\theta_2(t=0) = \mathrm{Re}[C_2], \qquad \sqrt{2}\dot{\theta}_1(t=0) = -\dot{\theta}_2(t=0) = \mathrm{Re}[i\omega C_2],$$

then the out-of-phase mode can be realized experimentally.

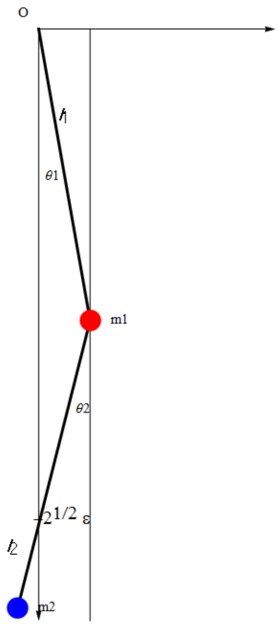

Fig.18(b) The initial conditions for the out-of-phase mode of the double pendulum. We choose the initial conditions such that $\theta_1(t=0) = \varepsilon$. $\theta_2(t=0) = -\sqrt{2}\varepsilon$. $\dot{\theta}_1(t=0) = 0$. $\dot{\theta}_2(t=0) = 0$. ε is a very small angle.

3 Transverse waves

We now consider the transverse wave in the system. In equilibrium, the positions of masses and the springs are aligned along the straight line. The direction of the displacement is perpendicular to the straight line in equilibrium. In other words, the masses can move along the *y* direction. We assume that the distance between masses in equilibrium is longer than the un-stretched length of springs. Otherwise, the system becomes nonlinear.

A transverse wave is a moving wave that consists of oscillations occurring perpendicular to the direction of energy transfer. If a transverse wave is moving in the positive x direction, its oscillations are in up and down directions that lie in the *y direction*.

3.1 $N = 1$ transverse wave

We consider the case of the transverse wave. The mass is located at the fixed position ($x = L_0$). The displacement of the mass occurs along the direction perpendicular to the x axis (transverse direction). When the displacement y_1 is very small, we find that the mass m undergoes a simple harmonic oscillation (the transverse mode).

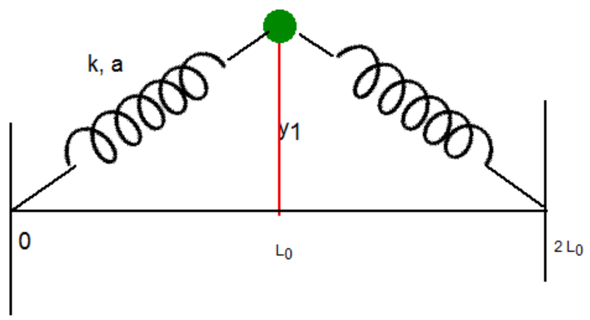

Fig.19 Transverse oscillation for $N = 1$ system. The displacement of the mass (m) is restricted to the direction perpendicular to the chain in equilibrium (the lenght $2L_0$). The spring constant is k. The length L_0 is assumed to be larger than the length of un-stretched spring.

The Lagramgian is given by

$$L = \frac{1}{2}m\dot{y}_1^{\ 2} - 2\frac{1}{2}k(\sqrt{y_1^{\ 2} + L_0^{\ 2}} - a)^2 \quad . \tag{3.1}$$

The Lagrange equation is

$$\ddot{y}_1 = \frac{k\xi}{m}(-2y_1) \quad , \tag{3.2}$$

with

$\xi = (L_0 - a)/L_0$ (>0),

where a is the unstretched length of spring and y1 is much shorter than L_0 and a. When

$$y_1 = \mathrm{Re}[Y_1 e^{i\omega t}] \quad . \tag{3.3}$$

we have

$$\omega = \sqrt{2}\sqrt{\frac{k}{m}}\sqrt{\xi} \quad . \tag{3.4}$$

Note that the angular frequency of the transverse wave is different from that of the longitudinal wave by the factor ($\sqrt{\xi}$).

((Mathematica))

```
(*Lagrange equation*)

Clear["Global`*"]

<< "VariationalMethods`"

L = 1/2 m y'[t]^2 - 2 × 1/2 k (Sqrt[y[t]^2 + L0^2] - a)^2

-k (-a + Sqrt[L0^2 + y[t]^2])^2 + 1/2 m y'[t]^2

eq11 =
 VariationalD[L, y[t], t] // Series[#, {y[t], 0, 1}] & //
   Simplify[#, L0 > 0] & // Normal

(2 k (a - L0) y[t])/L0 - m y''[t]
```

3.2 N = 2 transverse wave

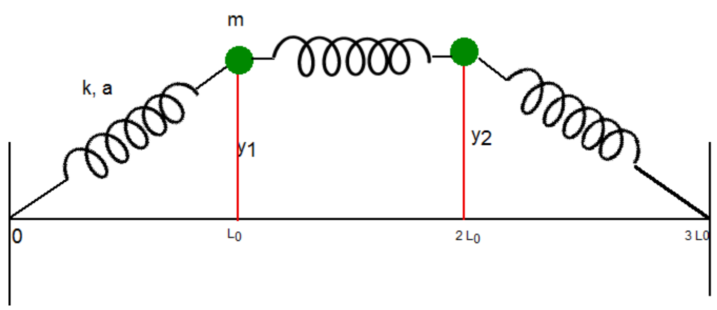

Fig.20 Transverse oscillation for $N = 2$ system. The displacement of each mass (m) is restricted to the direction perpendicular to the chain in equilibrium (the length $3L_0$). The spring constant is k. The length L_0 is assumed to be larger than the length of un-stretched spring.

The differential equations for the transverse displacements y_1 and y_2 are given by

$$\ddot{y}_1 = \frac{k\xi}{m}(-2y_1 + y_2)$$
$$\ddot{y}_2 = \frac{k\xi}{m}(y_1 - 2y_2) \quad . \tag{3.4}$$

Then we have the eigenvalue problem given by

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \lambda^2 \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \tag{3.5}$$

where λ^2 is the eigenvalue and $\lambda = \omega\sqrt{m/(k\xi)}$. This eigenvalue problem is almost the same as that for the longitudinal waves for $N = 2$, except for the factor $1/\sqrt{\xi}$ in the value of λ.

((**Mathematica**))

```
(*Lagrange equation*)

Clear["Global`*"]

<< "VariationalMethods`"

L = 1/2 m y1'[t]^2 + 1/2 m y2'[t]^2 - 1/2 k (Sqrt[y1[t]^2 + L0^2] - a)^2 -
   1/2 k (Sqrt[(y2[t] - y1[t])^2 + L0^2] - a)^2 - 1/2 k (Sqrt[y2[t]^2 + L0^2] - a)^2;

eq11 = VariationalD[L, y1[t], t] // Series[#, {y1[t], 0, 1}, {y2[t], 0, 1}] & //
    Simplify[#, L0 > 0] & // Normal

2 k (a - L0) y1[t]/L0 + (k - a k/L0) y2[t] - m y1''[t]

eq11 = VariationalD[L, y2[t], t] // Series[#, {y2[t], 0, 1}, {y1[t], 0, 1}] & //
    Simplify[#, L0 > 0] & // Normal

(k - a k/L0) y1[t] + 2 k (a - L0) y2[t]/L0 - m y2''[t]
```

3.3 $N = 3$ Transverse wave

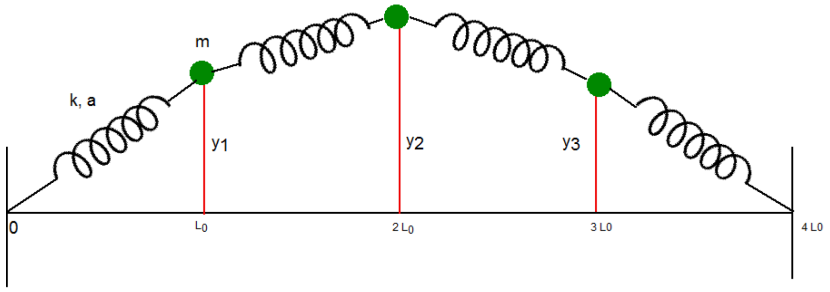

Fig.21 Transverse oscillation for $N = 3$ system. The displacement of each mass (m) is restricted to the direction perpendicular to the chain in equilibrium (the length $4L_0$). The spring constant is k. The length L_0 is assumed to be larger than the length of un-stretched spring.

The differential equations for the transverse displacements y_1, y_2, and y_3 are given by

$$\ddot{y}_1 = \frac{k\xi}{m}(-2y_1 + y_2)$$
$$\ddot{y}_2 = \frac{k\xi}{m}(y_1 - 2y_2 + y_3)$$
$$\ddot{y}_3 = \frac{k\xi}{m}(y_2 - 2y_3) \quad , \qquad (3.6)$$

or

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \lambda^2 \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} , \qquad (3.7)$$

where λ^2 is the eigenvalue and $\lambda = \omega\sqrt{m/(k\xi)}$. This eigenvalue problem is almost the same as that for the longitudinal waves for $N = 3$, except for the factor $1/\sqrt{\xi}$ in the value of λ.

((Mathematica))

```
(*Lagrange equation*)

Clear["Global`*"]

<< "VariationalMethods`"

L = 1/2 m y1'[t]^2 + 1/2 m y2'[t]^2 + 1/2 m y3'[t]^2 - 1/2 k (Sqrt[y1[t]^2 + L0^2] - a)^2 -
  1/2 k (Sqrt[(y2[t] - y1[t])^2 + L0^2] - a)^2 - 1/2 k (Sqrt[(y3[t] - y2[t])^2 + L0^2] - a)^2 -
  1/2 k (Sqrt[(y3[t])^2 + L0^2] - a)^2;
eq11 =
 VariationalD[L, y1[t], t] //
    Series[#, {y1[t], 0, 1}, {y2[t], 0, 1}, {y3[t], 0, 1}] & //
   Simplify[#, L0 > 0] & // Normal
```

$$\frac{2\,k\,(a - L0)\,y1[t]}{L0} + \left(k - \frac{a\,k}{L0}\right) y2[t] - m\,y1''[t]$$

```
eq12 =
 VariationalD[L, y2[t], t] //
    Series[#, {y1[t], 0, 1}, {y2[t], 0, 1}, {y3[t], 0, 1}] & //
   Simplify[#, L0 > 0] & // Normal
```

$$\left(k - \frac{a\,k}{L0}\right) y1[t] + \frac{2\,k\,(a - L0)\,y2[t]}{L0} + \left(k - \frac{a\,k}{L0}\right) y3[t] - m\,y2''[t]$$

```
eq13 =
 VariationalD[L, y3[t], t] //
    Series[#, {y1[t], 0, 1}, {y2[t], 0, 1}, {y3[t], 0, 1}] & //
   Simplify[#, L0 > 0] & // Normal
```

$$\left(k - \frac{a\,k}{L0}\right) y2[t] + \frac{2\,k\,(a - L0)\,y3[t]}{L0} - m\,y3''[t]$$

3.4 $N = 4$ Transverse wave

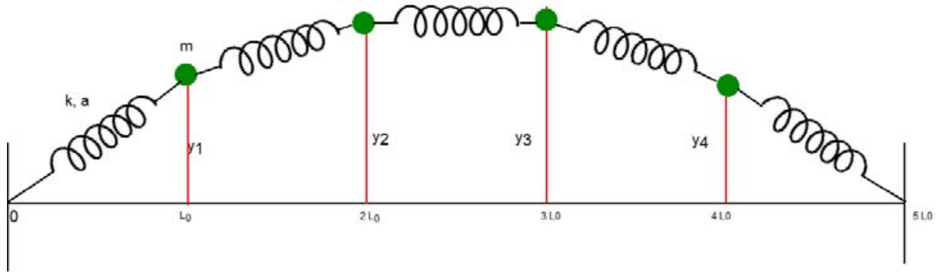

Fig.22 Transverse oscillation for $N = 4$ system. The displacement of each mass (m) is restricted to the direction perpendicular to the chain in equilibrium (the length $5L_0$). The spring constant is k. The length L_0 is assumed to be larger than the length of un-stretched spring.

The Lagramgian is given by

$$L = \frac{1}{2}my_1'^2 + \frac{1}{2}my_2'^2 + \frac{1}{2}my_3'^2 + \frac{1}{2}my_4'^2 - \frac{1}{2}k(\sqrt{y_1^2 + L_0^2} - a)^2 - \frac{1}{2}k(\sqrt{(y_2 - y_1)^2 + L_0^2} - a)^2$$
$$- \frac{1}{2}k(\sqrt{(y_3 - y_2)^2 + L_0^2} - a)^2 - \frac{1}{2}k(\sqrt{(y_4 - y_3)^2 + L_0^2} - a)^2 - \frac{1}{2}k(\sqrt{y_4^2 + L_0^2} - a)^2 \quad . \tag{3.8}$$

The Lagrange's equations up to the first order of y_1, y_2, y_3, and y_4, are given by

$$\ddot{y}_1 = \frac{k\xi}{m}(-2y_1 + y_2)$$
$$\ddot{y}_2 = \frac{k\xi}{m}(y_1 - 2y_2 + y_3)$$
$$\ddot{y}_3 = \frac{k\xi}{m}(y_2 - 2y_3 + y_4)$$
$$\ddot{y}_4 = \frac{k\xi}{m}(y_3 - 2y_4) \quad , \tag{3.9}$$

where $\xi = (L_0 - a)/L_0 > 0$.

The eigenvalue problem for the transverse wave is the same as that for the longitudinal wave, when k is replaced by $k\xi$ for the longitudinal wave.

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} = \lambda^2 \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} , \tag{3.10}$$

where λ^2 is the eigenvalue and $\lambda = \omega\sqrt{m/(k\xi)}$. This eigenvalue problem is almost the same as that for the longitudinal waves for $N = 4$, except for the factor $1/\sqrt{\xi}$ in the value of λ.

((**Mathematica**)) We use the Mathematica to derive the Lagrange's equations.

```
(*Lagrange equation*)

Clear["Global`*"]

<< "VariationalMethods`"

L = 1/2 m y1'[t]^2 + 1/2 m y2'[t]^2 + 1/2 m y3'[t]^2 + 1/2 m y4'[t]^2 -
   1/2 k (Sqrt[y1[t]^2 + L0^2] - a)^2 - 1/2 k (Sqrt[(y2[t] - y1[t])^2 + L0^2] - a)^2 -
   1/2 k (Sqrt[(y3[t] - y2[t])^2 + L0^2] - a)^2 - 1/2 k (Sqrt[(y4[t] - y3[t])^2 + L0^2] - a)^2 -
   1/2 k (Sqrt[y4[t]^2 + L0^2] - a)^2;
eq11 =
 VariationalD[L, y1[t], t] //
    Series[#, {y1[t], 0, 1}, {y2[t], 0, 1}, {y3[t], 0, 1}, {y4[t], 0, 1}] & //
   Simplify[#, L0 > 0] & // Normal
```

$$\frac{2\,k\,(a-L0)\,y1[t]}{L0}+\left(k-\frac{a\,k}{L0}\right)y2[t]-m\,y1''[t]$$

```
eq12 =
 VariationalD[L, y2[t], t] //
    Series[#, {y1[t], 0, 1}, {y2[t], 0, 1}, {y3[t], 0, 1}, {y4[t], 0, 1}] & //
   Simplify[#, L0 > 0] & // Normal
```

$$\left(k-\frac{a\,k}{L0}\right)y1[t]+\frac{2\,k\,(a-L0)\,y2[t]}{L0}+\left(k-\frac{a\,k}{L0}\right)y3[t]-m\,y2''[t]$$

```
eq13 =
 VariationalD[L, y3[t], t] //
    Series[#, {y1[t], 0, 1}, {y2[t], 0, 1}, {y3[t], 0, 1}, {y4[t], 0, 1}] & //
   Simplify[#, L0 > 0] & // Normal
```

$$\left(k-\frac{a\,k}{L0}\right)y2[t]+\frac{2\,k\,(a-L0)\,y3[t]}{L0}+\left(k-\frac{a\,k}{L0}\right)y4[t]-m\,y3''[t]$$

```
eq14 =
 VariationalD[L, y4[t], t] //
    Series[#, {y1[t], 0, 1}, {y2[t], 0, 1}, {y3[t], 0, 1}, {y4[t], 0, 1}] & //
   Simplify[#, L0 > 0] & // Normal
```

$$\left(k-\frac{a\,k}{L0}\right)y3[t]+\frac{2\,k\,(a-L0)\,y4[t]}{L0}-m\,y4''[t]$$

3.5 N = 5 Transverse wave

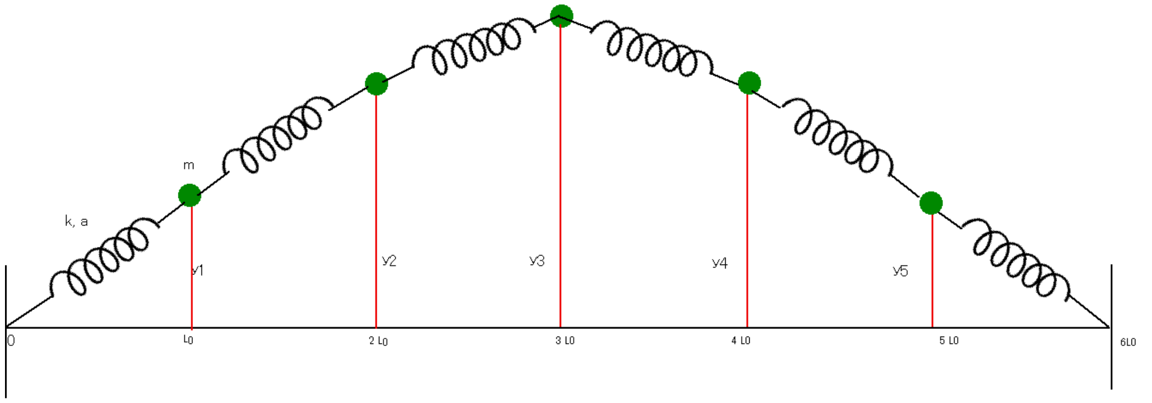

Fig.21 Transverse oscillation for $N = 5$ system. The displacement of each mass (m) is restricted to the direction perpendicular to the chain in equilibrium (the length $6L_0$). The spring constant is k. The length L_0 is assumed to be larger than the length of un-stretched spring.

The eigenvalue problem is given by

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{pmatrix} = \lambda^2 \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{pmatrix} . \tag{3.11}$$

where λ^2 is the eigenvalue and $\lambda = \omega\sqrt{m/(k\xi)}$. This eigenvalue problem is almost the same as that for the longitudinal waves for $N = 5$, except for the factor $1/\sqrt{\xi}$ in the value of λ.

4. Continuous chain for the longitudinal and transverse waves

4.1 Wave equation

From the above discussion, we find that the wave equations of the longitudinal and transverse waves are given by

$$m\frac{d^2u_s}{dt^2} = k(u_{s+1} - 2u_s + u_{s-1}) . \tag{4.1}$$

$$m\frac{d^2u_s}{dt^2} = \xi k(u_{s+1} - 2u_s + u_{s-1}) , \tag{4.2}$$

respectively. We use the same form $u_s(x) = u(x, t)$ for both waves. For the transverse wave, $u(x, t)$ is the displacement along the direction (y axis) perpendicular to the chain axis. For the longitudinal wave, $u(x, t)$ is the displacement along the chain direction (x axis). Only difference between Eqs.(1) and (2) is that the effective spring constant in the transverse wave is different from that for the longitudinal wave by the factor ξ. For convenience, hereafter we use one wave equation given by Eq.(4.1) for both the longitudinal and transverse waves. In the limit of Δx ($\rightarrow 0$), $s\Delta x = x$ and s is continuous variable. Under this assumption, $u(x, t)$ can be expanded using a Taylor expansion,

$$u_{s+1}(t) = u(x+\Delta x, t) = u(x,t) + \frac{\Delta x}{1!}\frac{\partial u(x,t)}{\partial x} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 u(x,t)}{\partial x^2} + O(\Delta x)^3 , \qquad (4.3)$$

Then the original wave equation can be rewritten as

$$m\frac{\partial^2 u(x,t)}{\partial t^2}$$
$$= k[u(x,t) + \frac{\Delta x}{1!}\frac{\partial u(x,t)}{\partial x} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 u(x,t)}{\partial x^2} - 2u(x,t) + u(x,t) - \frac{\Delta x}{1!}\frac{\partial u(x,t)}{\partial x} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 u(x,t)}{\partial x^2}] \qquad (4.4)$$

or

$$m\frac{\partial^2 u(x,t)}{\partial t^2} = k(\Delta x)^2\frac{\partial^2 u(x,t)}{\partial x^2} , \qquad (4.5)$$

or

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{k}{m}(\Delta x)^2\frac{\partial^2 u(x,t)}{\partial x^2} = v^2\frac{\partial^2 u(x,t)}{\partial x^2} , \qquad (4.6)$$

which is the continuum elastic wave equation with the velocity of sound given by

$$v = \sqrt{\frac{k}{m}}\Delta x . \qquad (4.7)$$

This applies in general to various types of traveling waves. *u(x, t)* represents various positions. For a string, it is the vertical displacement of the elements of the string. For a sound wave, it is the longitudinal position of the elements from the equilibrium position. For electromagnetic waves, it is the electric or magnetic field components.

4.2 Solution of the wave function (1)

The wave function can be solved as follows. For convenience, we put

$$u(x,t) = \psi(x,t) \quad , \tag{4.8}$$

which satisfies the wave equation given by

$$\frac{\partial^2}{\partial t^2}\psi = v^2 \frac{\partial^2}{\partial x^2}\psi \quad , \tag{4.9}$$

$$(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x})(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x})f = 0 \quad ,$$

We introduce new variables

$$\xi = t - \frac{x}{v}$$

$$\eta = t + \frac{x}{v} \, .$$

So that the equation for ψ becomes

$$\frac{\partial^2 \psi}{\partial \xi \partial \eta} = 0 \, .$$

The solution obviously has the form

$$\psi = f_1(\xi) + f_2(\eta) \, ,$$

where f_1 and f_2 are arbitrary function.
Or

$$\psi = f_1(t - \frac{x}{v}) + f_2(t + \frac{x}{v}) \, . \tag{4.10}$$

The function f_1 represents a plane wave moving in the positive x-direction, while the function f_2 represents a plane wave moving in the negative x-direction.

4.3 General solution using the Fourier transform:

We introduce a Fourier transform defined by

$$\Psi(x,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t}\psi(x,t)dt \, . \tag{4.11}$$

The inverse Fourier transform is given by

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t}\Psi(x,\omega)d\omega \, . \tag{4.12}$$

Then the wave equation can be rewritten as

$$\frac{\partial^2}{\partial x^2}\psi(x,t) - \frac{1}{v^2}\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t}[\frac{\partial^2}{\partial x^2}\Psi(x,\omega) + \frac{\omega^2}{v^2}\Psi(x,\omega)]d\omega = 0 \, .$$

Then we have

$$\frac{\partial^2}{\partial x^2}\Psi(x,\omega)+\kappa^2\Psi(x,\omega)=0 \quad , \qquad (4.13)$$

where κ is the wave number and is defined by $\kappa=\omega/v$. Here we use κ instead of k since k is used as a spring constant. The solution of this equation is

$$\Psi(x,\omega)=\phi(\omega)\frac{e^{\pm ikx}}{\sqrt{2\pi}}=\phi(\omega)\frac{e^{\pm i\frac{\omega x}{v}}}{\sqrt{2\pi}} \quad , \qquad (4.14)$$

where $\phi(\omega)$ is an arbitrary function of ω. Finally we get

$$\psi(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(\omega)e^{i\omega(t\pm\frac{x}{v})}d\omega \quad . \qquad (4.15)$$

This means that $u(x, t)$ is an arbitrary function of $(t\pm\frac{x}{v})$.

5 Modes of the oscillation in the string

5.1 Wave function of the mode

We consider the oscillation in the string using the wave equation. In this case both sides of the string are fixed.

$$\frac{\partial^2 u(x,t)}{\partial t^2}=v^2\frac{\partial^2 u(x,t)}{\partial x^2} \quad . \qquad (5.1)$$

We assume that

$$u(x,t)=\mathrm{Re}[U(x,\omega)e^{i\omega t}] \quad , \qquad (5.2)$$

where $U(x, w)$ is the complex amplitude and w is the angular frequency. Then we have

$$\mathrm{Re}[-\omega^2U(x,\omega)e^{i\omega t}]=\mathrm{Re}[v^2\frac{\partial^2 U(x,\omega)}{\partial x^2}e^{i\omega t}] \quad ,$$

or

$$\frac{\partial^2 U(x,\omega)}{\partial x^2}+\kappa^2U(x,\omega)=0 \quad , \qquad (5.3)$$

with the wave number κ given by

$$\kappa=\frac{\omega}{v} \quad . \qquad (5.4)$$

under the boundary condition

$$U(x=0,\omega)=0$$
$$U(x=L,\omega)=0 \quad , \tag{5.5}$$

where L is the total distance of the string. The solution of the second order differential equation is simply obtained as

$$U(x,\omega)\approx \sin(\kappa_m x) \quad , \tag{5.6}$$

with

$$\kappa_m=\frac{m\pi}{L} \qquad (m = 1, 2, 3, ...), \tag{5.7}$$

where m is an positive integer and denotes the number of mode m. The angular frequency ω_m of the mode m is related to the wave number κ_m of the mode m through a relation

$$\omega_m=v\kappa_m=v\frac{m\pi}{L}=m\frac{v\pi}{L}=m\omega_1 \quad , \tag{5.8}$$

where ω_1 is the fundamental angular frequency and ω_2, ω_3, ω_4, ... are the angular frequency of the second, the third, the fourth, ..., harmonics. The period (T_1) and wavelength (λ_1) of the wave with the mode m = 1 is

$$T_1=\frac{2\pi}{\omega_1}$$
$$\lambda_1=\frac{2\pi}{\kappa_1}=\frac{2\pi}{\frac{\pi}{L}}=2L \quad . \tag{5.9}$$

The wave function of the mode m is

$$\begin{aligned} u_m(x,t) &= \mathrm{Re}[A_m\sin(\kappa_m x)e^{i\omega_m t}] \\ &= \mathrm{Re}[A_m\sin(\kappa_m x)e^{i(\omega_m t+\phi_m)}] \\ &= A_m\sin(\kappa_m x)\cos(\omega_m t+\phi_m) \\ &= \frac{A_m}{2}[\sin(\kappa_m x+\omega_m t+\phi_m)+\sin(\kappa_m x-\omega_m t-\phi_m)] \\ &= \frac{A_m}{2}\{\sin[\kappa_m(x+vt+\phi_m)]+\sin[\kappa_m(x-vt-\phi_m)\} \quad , \end{aligned} \tag{5.10}$$

where $A_m e^{i\phi_m}$ is a constant complex amplitude. Here we use the dispersion relation, $\omega_m = v\kappa_m$. The phase velocity v_p and the group velocity v_g are defined as

$$v_p = \frac{\omega}{\kappa}$$

$$v_g = \frac{\partial \omega}{\partial \kappa}, \qquad (5.11)$$

respectively. In the present case we have

$$v_p = v_g = v. \qquad (5.12)$$

It is found that the wave function of the mode m is a sum of the travelling wave propagating along the (+x) direction [$\frac{A_m}{2}\sin[\kappa_m(x - vt - \phi_m)]$ and the travelling wave propagating along the (-x) direction [$\frac{A_m}{2}\sin[\kappa_m(x + vt + \phi_m)]$, leading to the standing wave.

5.2 General solution for the oscillation in the continuous string

The general solution is a superposition of the wave function for the mode m (m =1, 2, 3, ...).

$$u(x,t) = \sum_m A_m \sin(\kappa_m x)\cos(\omega_m t + \phi_m). \qquad (5.13)$$

where A_m and ϕ_m (m = 1, 2, ...) can be determined from the initial condition,

$$u(x,0) = f(x)$$

$$\frac{\partial u(x,t)}{\partial t}\Big|_{t=0} = 0, \qquad (5.14)$$

where the form of function, $f(x)$ is given. The wave function is periodic function of t with a period of T_1 (= $2\pi/\omega_1$) at the fixed x. From the second initial condition, we obtain

$$\phi_1 = \phi_2 = ... = 0. \qquad (5.15)$$

From the first initial condition, we have

$$u(x,0) = f(x) = \sum_m A_m \sin(\kappa_m x). \qquad (5.16)$$

The coefficient A_m of the Fourier series can be calculated as

$$A_m = \frac{2}{L}\int_0^L f(x)\sin(\kappa_m x)dx. \qquad (5.17)$$

5.3 Example

For simplicity, we assume that $f(x)$ is given by a triangle function

$f(x) = (2a/L)\,x \qquad (0 \le x \le L/2). \qquad (5.18)$

$f(x) = (2a/L)\,(L - x) \qquad (0 \le x \le L/2). \qquad (5.19)$

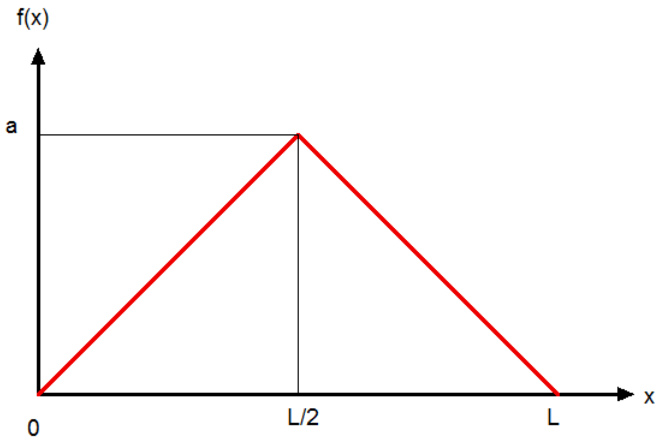

Fig.22 Initial state of the oscillation of the string

The Fourier coefficient A_m is calculated as

$$A_m = \frac{2}{L}[\int_0^{L/2} \frac{2a}{L} x \sin(\kappa_m x) dx + \int_{L/2}^{L} \frac{2a}{L}(L - x)\sin(\kappa_m x) dx$$
$$= \frac{32a}{m^2\pi^2}\cos(\frac{m\pi}{4})\sin^3(\frac{m\pi}{4}) \qquad (5.20)$$

using the Mathematica. Note that

$A_1 = \frac{8a}{\pi^2}$, $A_2 = 0$, $A_3 = -\frac{8a}{9\pi^2}$, $A_4 = 0$, $A_5 = \frac{8a}{25\pi^2}$,

$A_6 = 0$, $A_7 = -\frac{8a}{49\pi^2}$, $A_8 = 0$, $A_9 = \frac{8a}{81\pi^2}$, $A_{10} = 0$

(5.21)

Then $u(x, \mathrm{t})$ can be expressed by

$$u(x,t) = \sum_{m=1}^{N_0} \frac{32a}{m^2\pi^2}\cos(\frac{m\pi}{4})\sin^3(\frac{m\pi}{4})\sin(\frac{m\pi x}{L})\cos(\frac{m\pi}{L}vt) . \qquad (5.22)$$

In Fig.23, we show a plot of $u(x, t)$ as a function of x, where t is changed as a parameter. We choose $N_0 = 50$ as the highest term of the summation. When N_0 is small (for example, $N_0 = 3$), the form of $u(x, t = 0)$ is slightly deviated from a triangle function near at $x = L/2$. The period is $2L/v$. We use $L = 2$, $a = 1$, and $v = 1$ for the numerical calculation.

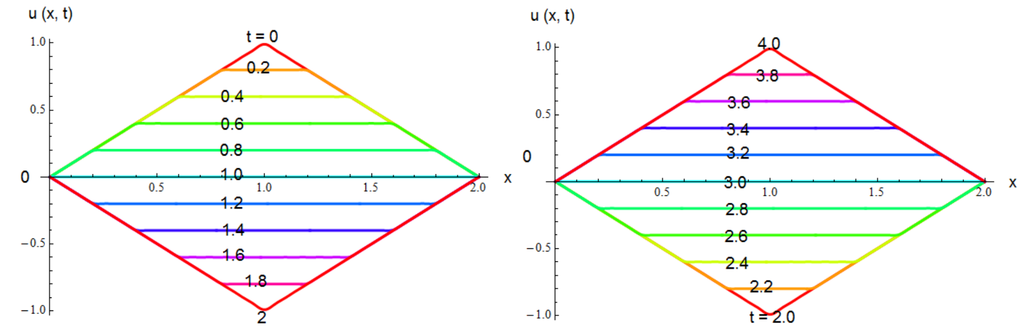

Fig.23 Plot of $u(x, t)$ as a function of x ($x = 0 – 2$), where t is changed as a parameter ($t = 0 – 4$). $L = 2$. $a = 1$. $v = 1$. The period $T = 2L/v = 4$. $N_0 = 50$.

In Fig.24, we show a plot of $u(x, t)$ as a function of t, where x is changed as a parameter. The function $u(x, t)$ at any fixed x is a periodic function of t with the period T (=$2L/v$ which is equal to 2 in the present case).

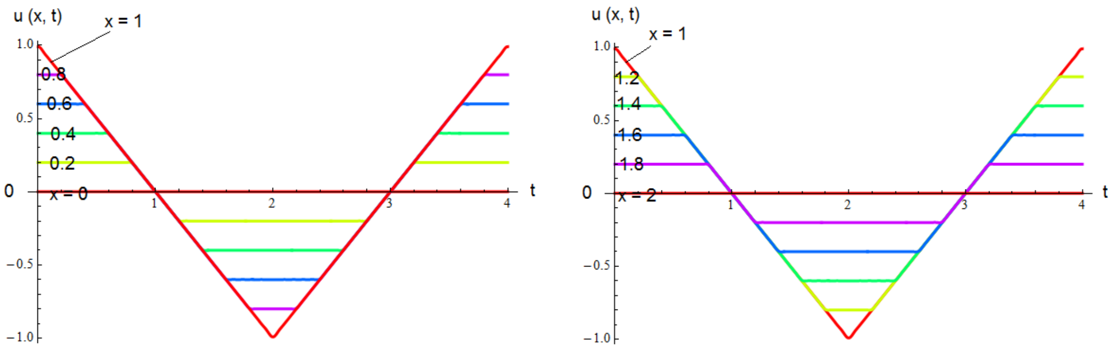

Fig.24 Plot of $u(x, t)$ as a function of t ($t = 0$ - 4) where x is changed as a parameter ($x = 0$ - 2). $L = 2$. $a = 1$. $v = 1$. The period $T = 2L/v = 4$. $N_0 = 50$.

((Mathematica))

```
u[x_, t_] :=
 Sum[(32 a)/(m^2 π^2) Cos[(m π)/4] Sin[(m π)/4]^3 Sin[(m π)/L x] Cos[m (π v)/L t], {m, 1, 50}];
rule1 = {a → 1, L → 2, v → 1}; H[x_, t_] := u[x, t] /. rule1;
Plot[Evaluate[Table[H[x, t], {t, 0, 2, 0.2}]],
 {x, 0, 2},
 PlotStyle → Table[{Thick, Hue[0.1 i]}, {i, 0, 9}]]

Plot[Evaluate[Table[H[x, t], {t, 2, 4, 0.2}]],
 {x, 0, 2},
 PlotStyle → Table[{Thick, Hue[0.1 i]}, {i, 0, 9}]]

Plot[Evaluate[Table[H[x, t], {x, 0, 1, 0.2}]],
 {t, 0, 4},
 PlotStyle → Table[{Thick, Hue[0.2 i]}, {i, 0, 4}]]

Plot[Evaluate[Table[H[x, t], {x, 1, 2, 0.2}]],
 {t, 0, 4},
 PlotStyle → Table[{Thick, Hue[0.2 i]}, {i, 0, 4}]]
```

6. Wave packet

In this section we discuss the propagation of the wave packet. Here we use k as the wave number (but not κ).

6.1 Traveling wave

We consider the solution the wave function for the infinite chain,

$$\frac{\partial^2 u(x,t)}{\partial t^2} = v^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad (6.1)$$

for x = -∞ to ∞. There is no limit on the value of x. The solution with the mode k is given by a travelling wave

$$u(x,t) = u_0 e^{i(kx-\omega t)}, \qquad (6.2)$$

where A is the complex amplitude . We choose only the positive value of ω, given by

$$\omega = \omega_k = v|k|. \qquad (6.3)$$

The k is the wave number (positive and negative values this time). The wave with the positive k denotes the traveling wave propagating along the positive x axis, while the wave with the negative of k denotes the traveling wave propagating along the negative x axis. Then the general solution is expressed by a sum of these modes,

$$u(x,t) = \mathrm{Re}[\psi(x,t)], \tag{6.4}$$

with

$$\psi(x,t) = \int_{-\infty}^{\infty} f(k) e^{i(kx-\omega_k t)}, \tag{6.5}$$

where $f(k)$ is the complex amplitude.

6.2 Wave packet for the dispersion relation $\omega = \omega_k = v|k|$

Suppose that $f(k)$ is described by a Gaussian distribution function,

$$f(k) = A_0 \frac{1}{\sqrt{2\pi}\Delta k} \exp[-\frac{(k-k_0)^2}{2(\Delta k)^2}], \tag{6.6}$$

where A_0 is the complex amplitude(constant), k_0 is the mean wave number, and Δk is the width of the Gaussian distribution (standard deviation of k).

$$\psi(x,t) = A_0 \frac{1}{\sqrt{2\pi}\Delta k} \int_{-\infty}^{\infty} dk \exp[-\frac{(k-k_0)^2}{2(\Delta k)^2}] e^{i(kx-\omega_k t)}, \tag{6.7}$$

with $\omega_k = v|k|$. We use the Mathematica for the calculation of integrals. At $t = 0$, we have

$$\psi(x,t=0) = A_0 e^{ik_0 x - \frac{1}{2}x^2(\Delta k)^2}. \tag{6.8}$$

For any t, we obtain

$$\psi(x,t) = \frac{1}{2} A_0 e^{ik_0(x-vt) - \frac{1}{2}(\Delta k)^2 (x-vt)^2)} \{(1 + erf[\frac{k_0 + i(x-vt)(\Delta k)^2}{\sqrt{2}\Delta k}])\} + \frac{1}{2} A_0 e^{ik_0(x+vt) - \frac{1}{2}(\Delta k)^2 (x+vt)^2} erfc[\frac{k_0 + i(x+vt)(\Delta k)^2}{\sqrt{2}\Delta k}], \tag{6.9}$$

where erf(z) and erfc(z) are the error functions of z.

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt, \qquad \mathrm{erfc(z)} = 1 - \mathrm{erf(z)}. \tag{6.10}$$

The first term of $u(x, t)$ is a function of $(x - vt)$, indicating the traveling wave along the positive x axis. The second term of $u(x, t)$ is function of $(x + vt)$, indicating the traveling wave along the negative x axis. When $k_0/(\sqrt{2}\Delta k) >> 1$ and $x \approx vt$, $\psi(x,t)$ can be approximated by

$$\psi(x,t) = A_0 e^{ik_0(x-vt)-\frac{1}{2}(\Delta k)^2(x-vt)^2)} . \quad (6.11)$$

since erf =1 and erfc = 0 in Eq.(6.11). Then $u(x, t)$ can be expressed by

$$u(x,t) = \mathrm{Re}[\psi(x,t)] = A_0 e^{-\frac{1}{2}(\Delta k)^2(x-vt)^2)} \cos[k_0(x - vt)] . \quad (6.12)$$

This indicates that the wave packet propagates with the group velocity v. Note that the group velocity (v_g) is equal to the phase velocity (v_p) for the dispersion ($\omega = vk$).

$$v_g = \frac{\partial\omega}{\partial k} = v$$

$$v_p = \frac{\omega}{k} = v . \quad (6.13)$$

The shape of $u(x, t)$ given by Eq.(6.13) is described by a traveling wave along the positive x axis, whose amplitude is modulated by the Gaussian function centered at $x = vt$. The width of the Gaussian function (Δx) is equal to $1/\Delta k$;

$$\Delta x \cdot \Delta k \approx 1 . \quad (6.14)$$

This means that a wave packet with a narrow range in k space must be very broad in x space and vice versa.

(a) Numerical calculation: A_0 = 1, k_0 = 20, v = 1, and Δk = 3.

In this case, $u(x, t)$ is the traveling wave along the positive x axis. since $k_0>0$.

The angular frequency: $\omega_0 = vk_0 = 20$.

The period: $T_0 = 2\pi/\omega_0 = 2\pi/20 = \pi/10 = 0.314$

The wave length: $\lambda_0 = 2\pi/k_0 = \pi/10 = 0.314$

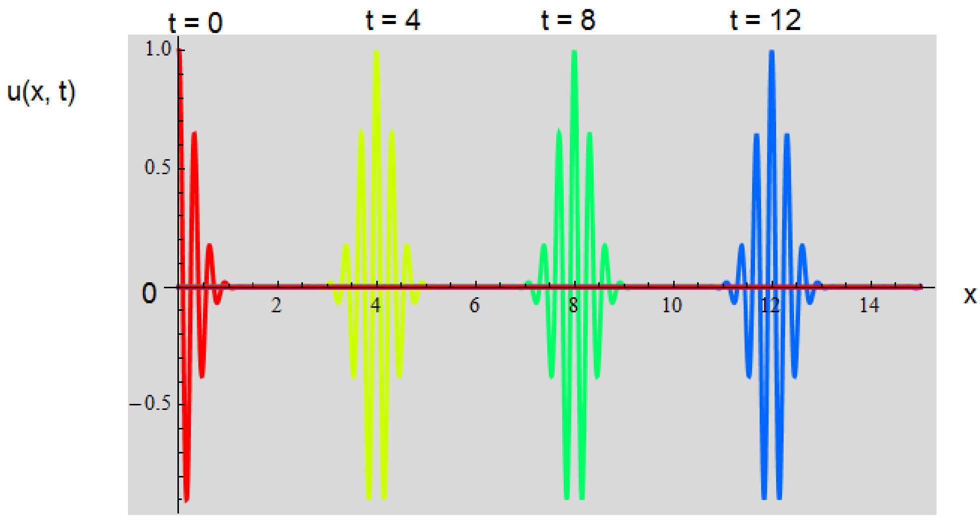

Fig.25 Propagation of the wave packet along the positive x axis. $A_0 = 1$, $k_0 = 20$, $v = 1$, and $\Delta k = 3$.

(b) Numerical calculation: A_0 = 1, k_0 = -20, v = 1, and Δk = 3.

In this case, $u(x, t)$ is the traveling wave along the negative x axis. since $k_0<0$.

The angular frequency: $\omega_0 = vk_0 = 20$.

The period: $T_0 = 2\pi/\omega_0 = 2\pi/20 = \pi/10 = 0.314$

The wave length: $\lambda_0 = 2\pi/k_0 = \pi/10 = 0.314$

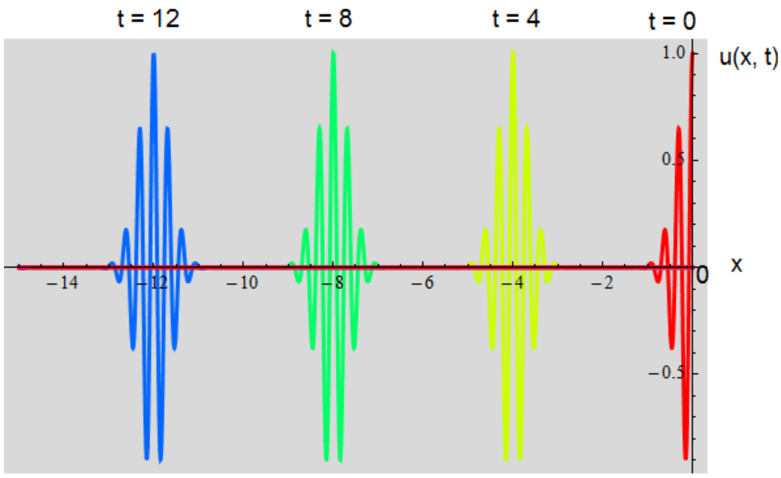

Fig.26 Propagation of the wave packet along the negative x axis. $A_0 = 1$, $k_0 = -20$, $v = 1$, and $\Delta k = 3$.

6.3 Spread of the Wave packet for the dispersion relation $\omega = \omega_k = E_k / \hbar = \hbar k^2/(2m)$

We now consider the case when the angular frequency ω_k is given by

$$\omega_k = \frac{E_k}{\hbar} = \frac{\hbar k^2}{2m}, \qquad (6.15)$$

which corresponds to the case of free electron, where $\hbar$ is the Planck's constant, m is the mass of electron, and k is the wave number. The group velocity and the phase velocity are defined as

$$v_g = \frac{\partial \omega_k}{\partial k} = \frac{\hbar k}{m}, \qquad v_p = \frac{\omega_k}{k} = \frac{\hbar k}{2m}, \qquad (6.16)$$

respectively. The wave function $\psi(x, t)$ is given by

$$\psi(x,t) = A_0 \frac{1}{\sqrt{2\pi}\Delta k} \int_{-\infty}^{\infty} dk \exp[-\frac{(k-k_0)^2}{2(\Delta k)^2}] \exp[i(kx - \frac{\hbar k^2}{2m} t) \qquad (6.17),$$

Using the Mathematica, Eq.(6.17) an be calculated as

$$\psi(x,t) = \frac{1}{\sqrt{1+\frac{i\hbar t(\Delta k)^2}{m}}} \exp[\frac{ik_0 x - \frac{1}{2}(\Delta k)^2 x^2 - i\frac{k_0{}^2 \hbar t}{2m}}{1+\frac{i\hbar t(\Delta k)^2}{m}}]$$

$$= \frac{1}{\sqrt{1+\frac{i\hbar t(\Delta k)^2}{m}}} \exp[\frac{(ik_0 x - \frac{1}{2}(\Delta k)^2 x^2 - i\frac{k_0{}^2 \hbar t}{2m})(1-\frac{i\hbar t(\Delta k)^2}{m})}{1+\frac{\hbar^2 t^2 (\Delta k)^4}{m^2}}]$$

$$= \frac{1}{\sqrt{1+\frac{i\hbar t(\Delta k)^2}{m}}} \exp[\frac{(ik_0 x - \frac{1}{2}(\Delta k)^2 x^2 - i\frac{k_0{}^2 \hbar t}{2m})(1-\frac{i\hbar t(\Delta k)^2}{m})}{1+\frac{\hbar^2 t^2 (\Delta k)^4}{m^2}}]$$

$$= \frac{1}{\sqrt{1+\frac{i\hbar t(\Delta k)^2}{m}}} \exp[\frac{i(k_0 x - \frac{k_0{}^2 \hbar t}{2m} + \frac{t x^2 \hbar (\Delta k)^4}{2m}) - \frac{1}{2}(\Delta k)^2 (x - \frac{\hbar k_0}{m} t)^2}{1+\frac{\hbar^2 t^2 (\Delta k)^4}{m^2}}],$$

$$(6.18)$$

From Eq.(6.18), the amplitude of $\psi(x,t)$ is obtained as a Gaussian function

$$A(x,t) = \frac{1}{\left(1+\frac{\hbar^2 t^2 (\Delta k)^2}{m^2}\right)^{1/4}} \exp\left[\frac{-\frac{1}{2}(\Delta k)^2 (x - \frac{\hbar k_0}{m} t)^2}{1+\frac{\hbar^2 t^2 (\Delta k)^4}{m^2}}\right] . \tag{6.19}$$

The peak of the Gaussian distribution (x_p) appears at

$$x_p = \frac{\hbar k_0}{m} t , \tag{6.20}$$

which indicates that the center of the Gaussian distribution propagates with the group velocity, but not with the phase velocity. The mean width of the Gaussian distribution (Δx) is related to Δk as

$$\Delta x = \frac{1}{\Delta k} \sqrt{1+\frac{\hbar^2 t^2 (\Delta k)^4}{m^2}} . \tag{6.21}$$

For times so short that $\hbar^2 t^2 (\Delta k)^4 / m^2 << 1$, we have $\Delta x = 1/\Delta k$, which is the spread at $t = 0$. The wave packet begins to spread only when $\hbar^2 t^2 (\Delta k)^4 / m^2 > 1$. Equation (6.21) can be rewritten as

$$\Delta x \Delta k = \sqrt{1+\frac{\hbar^2 t^2 (\Delta k)^4}{m^2}} > 1 . \tag{6.22}$$

Therefore, after a long time, the product $\Delta x \Delta k$ becomes very large. Equation (6.21) corresponds to the Heisenberg's principle of uncertainty.

We also note the oscillatory part in Eq.(6.21) given by

$$\exp\left[\frac{i(k_0 x - \frac{k_0{}^2 \hbar t}{2m}) + i\frac{t x^2 \hbar (\Delta k)^4}{2m})}{1+\frac{\hbar^2 t^2 (\Delta k)^4}{m^2}}\right] = \exp\left[\frac{i(k_0 x - \frac{k_0{}^2 \hbar t}{2m})}{1+\frac{\hbar^2 t^2 (\Delta k)^4}{m^2}}\right] \exp\left[\frac{i\frac{t x^2 \hbar (\Delta k)^4}{2m}}{1+\frac{\hbar^2 t^2 (\Delta k)^4}{m^2}}\right] . \tag{6.23}$$

For times so short that $\hbar^2 t^2 (\Delta k)^4 / m^2 << 1$, the oscillatory part is approximated by

$$\exp[i(k_0 (x - \frac{k_0 \hbar t}{2m})] \exp[i\frac{t x^2 \hbar (\Delta k)^4}{2m}] , \tag{6.24}$$

where $k_0 \hbar / (2m)$ in the first exponential term is the phase velocity ω_k / k at $k = k_0$. The second exponential term indicates the rapid oscillation of the wave packet with increasing t and x.

((**Mathematica**))

We make a plot of $|u(x, t)|$ for the Gaussian wave packet (instead of $u(x, t)$). Note that $u(x, t)$ is an oscillatory function of x at the fixed t. In order to show the semi-log plot (y axis is in a logarithmic scale), we need to make such a plot of $|u(x, t)|$.

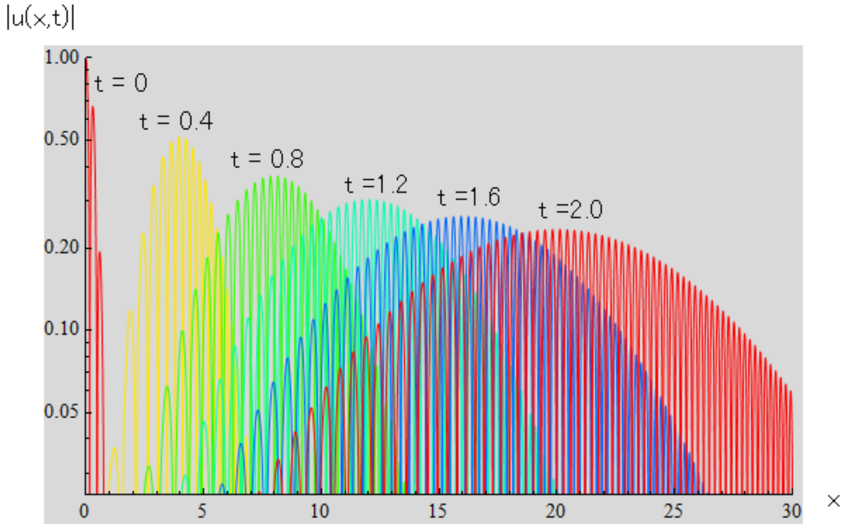

Fig.27 Propagation pattern of the wave packet along the positive x axis at the fixed times. t = 0, 0,4, 0.8, 1.2, 1.6, and 2.0. $A_0 = 1$, $k_0 = 10$, $m = 1$, $\hbar = 1$, and $\Delta k = 3$.

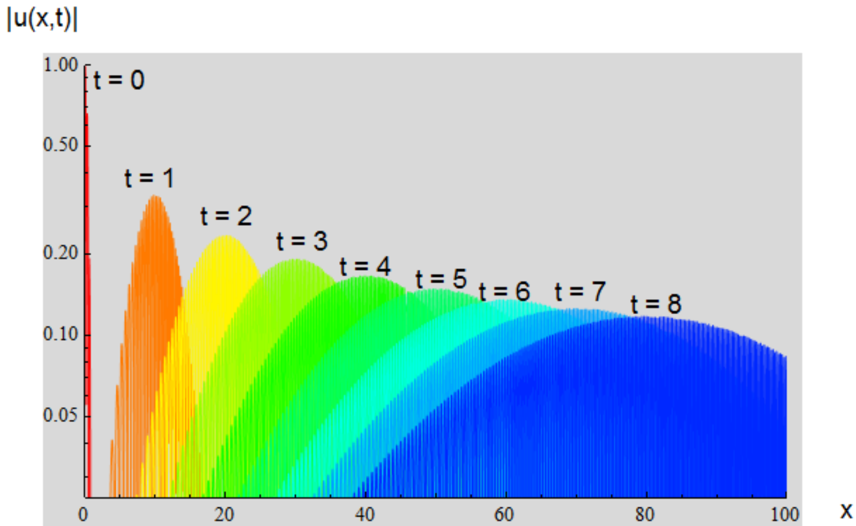

Fig.28 Propagation pattern of the wave packet along the positive x axis at fixed times. t = 0, 1, 2, 3, 4, 5, 6, 7, and 8. $A_0 = 1$, $k_0 = 10$, $m = 1$, $\hbar = 1$, and $\Delta k = 3$.

CONCLUSION

The oscillations and waves are universal phenomena which appears in a variety of physical systems. The wave packet is a superposition of the traveling waves as normal modes with various wave numbers. The wave packet consisting of many waves behaves like a particle. The peak of the wave packet propagates with the velocity as the nature of the particle. The discussion on the propagation of the Gaussian wave packet leads to the concept of the Heisenberg's principle of uncertainty.

In quantum mechanics, the wave–particle duality principle (de Broglie hypothesis) is established. All matters exhibits the behaviors of both waves and particles. The French physicist de Broglie proposed that particles behave as if they possessed a wave length that was inversely proportional to their momentum p; $\lambda = h/\lambda$ where h is the Planck's constant and λ is the wave length. The de Broglie hypothesis was experimentally confirmed from the experiment by the Americans C. Davisson and L. Germer. They demonstrated diffraction effect when they scattered electrons off a polished nickel crystal.

In these senses, we think that the concept on the oscillations and waves is closely related to the fundamental principle in quantum mechanics. We hope that this note is useful for undergraduate students in physics who will study the quantum mechanics and advanced electromagnetic theory.

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APPENDIX Variational method using Mathematica

A.1 Definition of VariationalD (Mathematica program)

We suppose that the functional is given by $f(y, y', x)$. Using the VariationalD [Mathematica], one can calculate

$$\frac{\partial f}{\partial y} - \frac{d}{dx}(\frac{\partial f}{\partial y'})$$

where the variation of the integral J is defined as

$$\delta J = \int_1^2 [\frac{\partial f}{\partial y} - \frac{d}{dx}(\frac{\partial f}{\partial y'})]\delta y dx$$

((**Mathematica**)) description of the programs was copied from the Mathematica 7.

VariationalD

`VariationalD[f, u[x], x]`
returns the variational derivative of the integral $\int f \, dx$ with respect to $u[x]$, where the integrand f is a function of u, its derivatives, and x.

`VariationalD[f, u[x, y, ...], {x, y, ...}]`
returns the variational derivative of the multiple integral $\int f \, dx \, dy \ldots$ with respect to $u[x, y, \ldots]$, where f is a function of u, its derivatives and the coordinates $x, y, \ldots$.

`VariationalD[f, {u[x, y, ...], v[x, y, ...], ...}, {x, y, ...}]`
gives a list of variational derivatives with respect to $u, v, \ldots$.

A.2 Definition of EulerEquations (Mathematica program)

Using the EulerEquations [Mathematica], one can derive the Euler (Lagrange, in physics) equation given by

$$\frac{\partial f}{\partial y} - \frac{d}{dx}(\frac{\partial f}{\partial y'}) = 0 \quad \text{(A.1)}$$

((**Mathematica**)) The description of the programs was copied from the Mathematica 7.

EulerEquations

`EulerEquations[f, u[x], x]`
returns the Euler–Lagrange differential equation obeyed by `u[x]` derived from the functional *f*, where *f* depends on the function `u[x]` and its derivatives as well as the independent variable *x*.

`EulerEquations[f, u[x, y, ...], {x, y, ...}]`
returns the Euler–Lagrange differential equation obeyed by `u[x, y, ...]`.

`EulerEquations[f, {u[x, y, ...], v[x, y, ...], ...}, {x, y, ...}]`
returns a list of Euler–Lagrange differential equations obeyed by `u[x, y, ...]`, `v[x, y, ...]`, ...

A.3 Definition of FirstIntegral (Mathematica program)

Here we note that the Euler (Lagrange) equation can be rewritten as

$$\frac{\partial f}{\partial x}-\frac{d}{dx}(f-y'\frac{\partial f}{\partial y'})=0 , \qquad \text{(A.2)}$$

since

$$\frac{\partial f}{\partial x}-\frac{d}{dx}(f-y'\frac{\partial f}{\partial y'})=\frac{\partial f}{\partial x}-\frac{df}{dx}+\frac{d}{dx}(y'\frac{\partial f}{\partial y'})$$

or

$$\frac{\partial f}{\partial x}-\frac{d}{dx}(f-y'\frac{\partial f}{\partial y'})=\frac{\partial f}{\partial x}-(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}y'+\frac{\partial f}{\partial y'}y'')+y''\frac{\partial f}{\partial y'}+y'\frac{d}{dx}(\frac{\partial f}{\partial y'})$$

$$=-\frac{\partial f}{\partial y}y'+y'\frac{d}{dx}(\frac{\partial f}{\partial y'})=y'[-\frac{\partial f}{\partial y}+\frac{d}{dx}(\frac{\partial f}{\partial y'})]=0$$

(a) The case when *f* is independent of *x*.

Since $\frac{\partial f}{\partial x}=0$ in Eq.(2), we have

$$\frac{d}{dx}(f-y'\frac{\partial f}{\partial y'})=0$$

or

$$f-y'\frac{\partial f}{\partial y'}=constant \qquad \text{(A.3)}$$

When f is independent of x, FirstIntegrals[f, $y(x)$, x] leads to the calculation of $\frac{\partial f}{\partial y'}$

(b) The case when f is independent of y.

Since $\frac{\partial f}{\partial y} = 0$ in Eq.(1), we have

$$\frac{\partial f}{\partial y'} = constant \quad \text{(A.4)}$$

When f is independent of y, FirstIntegrals[f, $y(x)$, x] leads to the calculation of $f - y'\frac{\partial f}{\partial y'}$

((**Mathematica**)) The description of the programs was copied from the Mathematica 7.

FirstIntegrals

`FirstIntegrals[f, x[t], t]`
returns a list of first integrals corresponding to the coordinate $x[t]$ and independent variable t of the integrand f.

`FirstIntegrals[f, {x[t], y[t], …}, t]`
returns a list of first integrals corresponding to the coordinates x, y, … and independent variable t.

B Wave traveling in the string (transverse wave)

B.1 Simple model

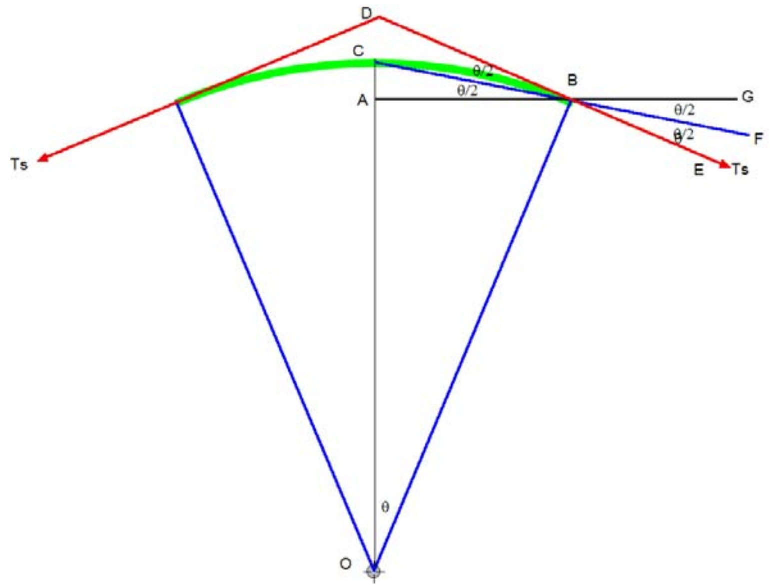

Fig.B1 The element of string (Δs) under the tension $T = T_s$.

We consider one small string element of length Δs. The net force acting in the y direction is

$$F = 2T_s \sin\theta \approx 2T_s\theta \quad \text{(B.1)}$$

Note that $\mu\Delta s$ is the mass of the element and that Δs is equal to $2R\theta$. We apply the Newton's second law to this element (centripetal force).

$$\mu\Delta s \frac{v^2}{R} = F = 2T_s\theta$$

$$\mu 2R\theta \frac{v^2}{R} = F = 2T_s\theta \quad \text{(B.2)}$$

The velocity is obtained as

$$v = \sqrt{\frac{T_s}{\mu}} \quad \text{(B.3)}$$

B.2 General case

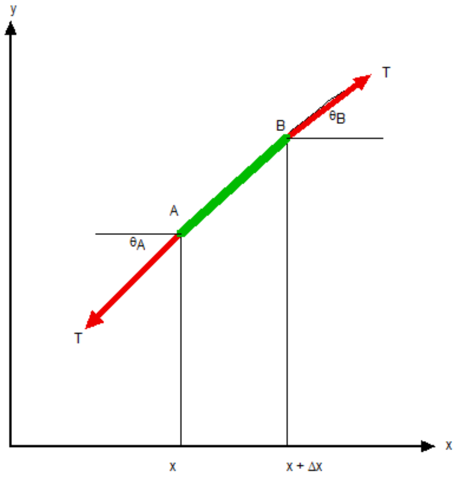

Fig.B2 Tension T_s on the string

Suppose that a traveling wave is propagating along a string that is under a tension T_s. Let us consider one small element of length Δx. The ends of the element make small angle θ_A and θ_B with the x axis. The net force acting on the element along the y-axis is

$$\sum F_y = T_s \sin\theta_B - T_s \sin\theta_A$$
$$= T_s(\sin\theta_B - \sin\theta_A) \approx T_s(\tan\theta_B - \tan\theta_A), \qquad \text{(B.4)}$$

or

$$F_y = T_s[\left(\frac{\partial y}{\partial x}\right)_B - \left(\frac{\partial y}{\partial x}\right)_A]$$
$$= T_s[\left(\frac{\partial y}{\partial x}\right)_{x+dx} - \left(\frac{\partial y}{\partial x}\right)_x] = \Delta x T_s \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \Delta x T_s \frac{\partial^2 y}{\partial x^2}, \qquad \text{(B.5)}$$

where we use the Taylor expansion.

We now apply the Newton's second law to the element, with the mass of the element given by

$$m = \mu\Delta x\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2} \approx \mu\Delta x, \qquad \text{(B.6)}$$

$$F_y = ma_y = \mu\Delta x\frac{\partial^2 y}{\partial t^2}. \qquad \text{(B.7)}$$

Then we have

$$\mu \Delta x \frac{\partial^2 y}{\partial t^2} = T_s \Delta x \frac{\partial^2 y}{\partial x^2}$$

$$\frac{\mu}{T_s} \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} , \qquad \text{(B.8)}$$

which leads to a wave equation given by

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T_s} \frac{\partial^2 y}{\partial t^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} , \qquad \text{(B.9)}$$

where

$$v = \sqrt{\frac{T_s}{\mu}} . \qquad \text{(B.10)}$$