Hodographic solutions to the Kepler problems."

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Johannes Kepler, working with data collected by Tycho Brahe, developed the three laws which described the orbital motion of the planets such as the Earth.

Kepler's laws

(i) The orbit of a planet is an ellipse with the Sun at one of the two foci.
(ii) A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time. (angular momentum conservation)
(iii) The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

One of the triumph of classical mechanics is that the observed elliptical orbits of the planets around the Sun can be explained directly from the Newton's laws of motion and the inverse square law of gravity. Such a demonstration starts with the differential equation for $u = 1/r$ as a function of angle θ and solve the Kepler problem analytically. There is another approach to the solution of the Kepler problem, the *hodograph* proposed by Sir William Rowan Hamilton and August Möbius. This approach has been originated mainly from the Newton Principia Mathematica. Because of the use of geometry instead of analysis, *the physics* has not always been clear to physicists. It has been often forgotten and independently rediscovered several times. In my view, one of the best known rediscoveries is the lecture given by Richard P. Feynman at March 13, 1964. Thanks to a book edited by D. and J. Goodstein, it is available for one to read this lecture note of Feynman. We note that Sir William Rowan Hamilton called the graph of the tip of the velocity vector as a function of time as the *hodograph*, from the Greek ‘‘to draw’’ *graphein*, and ‘‘path’’ *hodos*.

Here we present a hodorographics solution of Kepler's law of ellipses using both the analytical method and the Euclidean geometry. Most part of the approach was already discussed by Feynman and James C. Maxwell. The visualization of the hodograph diagram by using Mathematica makes much easier for one to understand the physics of orbital motion, in particular, the vectors of the velocity (tangential direction) and the acceleration (centripetal, toward the Sun).

((**Feynman's comment**))

"It is not easy to use the geometrical method to discover things. It is very difficult, but the elegance of the demonstrations after the discoveries are made, is really very great. The power of the analytical method is that it is much easier to discover things than to prove things. But not in any degree of elegance, it is a lot of dirty paper, with x's and y's and crossed out, cancellations and so on."

D.L. Goodstein and J.R. Goodstein, Feynman's Lost Lecture: The Motion of Planets Around the Sun (W.W. Norton, New York, 1996). Lecture by R.P. Feynman at March 13, 1964.

1. Geometry of ellipse orbit

(a) Definition of ellipse

An ellipse is the curve that can be made, by taking one string and two tracks and putting a pencil here and going around. Or mathematically, it is the locus, such that the sum of the distance SQ and the distance FQ remains constant (see Fig.1) , where S (the Sun) and F are the two fixed points. One may have heard another definition of an ellipse: these two points are called the foci, and this focus means that the light emitted from S will bounce to F from any point on the ellipse (ellipse optic theorem).

Suppose that the Earth undergoes an orbital motion of ellipse where the Sun (S) is one of the focus of the ellipse, and F is another focus. We consider the point Q on the ellipse orbit.

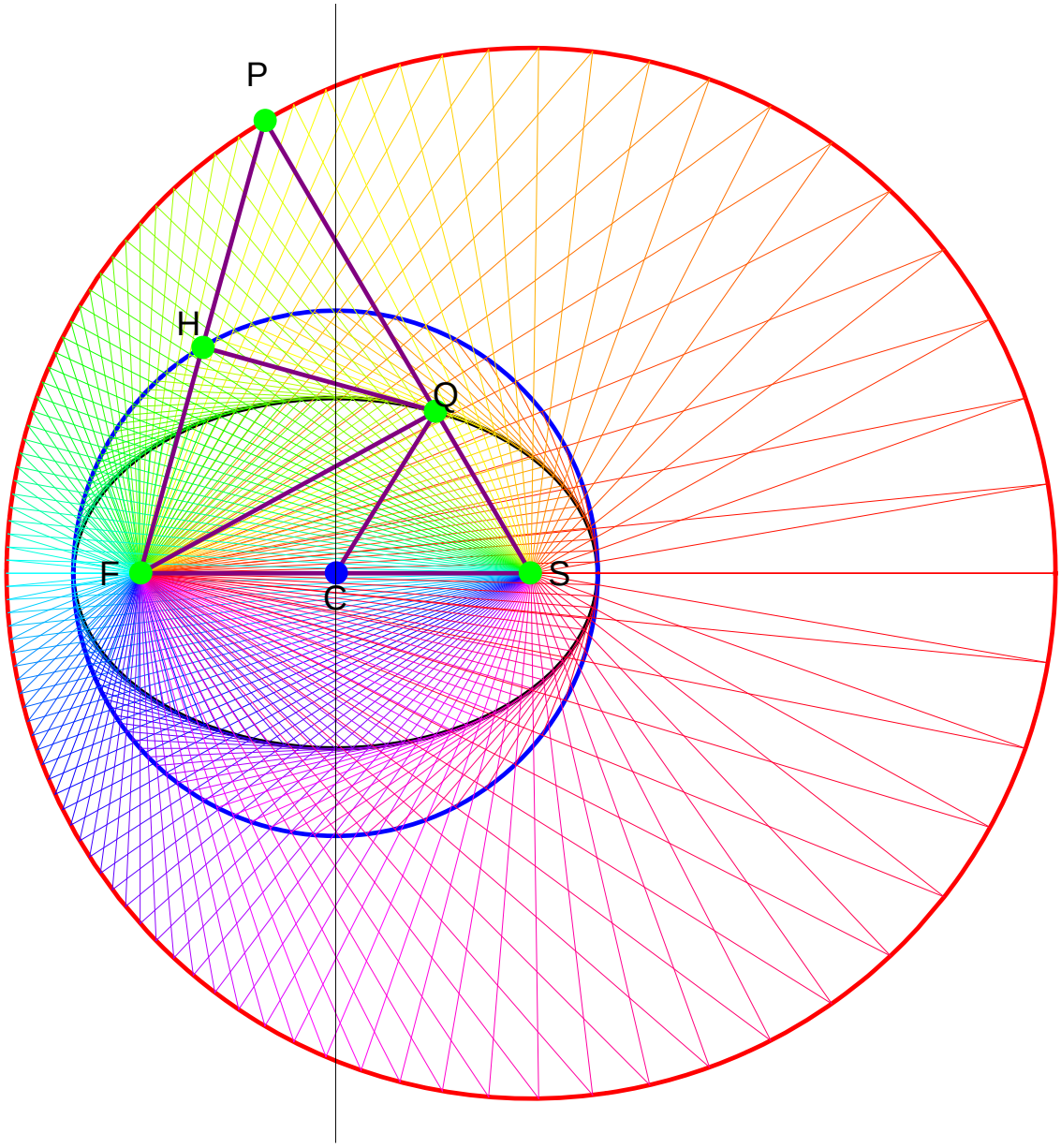


Fig.1 Hodograph diagram. Q (the Earth) is on the ellipse. F and S (the Sun) are foci. FQ + QS = 2 *a*. $\angle FQC = \angle SQC$ (ellipse optic theorem). The point P is on the circle (radius 2a) centered at S. FP is proportional to the velocity at the point Q. The direction of the velocity is parallel to the tangential line at Q.

From the property of the ellipse, we have

$FQ + SQ = 2a$,

$OF = OS = ae$,

where a is the semi-major axis and e is the eccentricity; 0<e<1. When $FQ = QP$, we have

$SP = 2a$.

The point P is located at the circle with radius 2a centered at the focal point S.

(b) Ellipse optic theorem

First we demonstrate the equivalence of these two definitions for ellipse. The light is reflected as though the surface were a plane tangent to the actual curve. We know that the law of reflection for the light from a plane is that the angle of incidence and reflection are the same. In other words, the angles made with the two lines FQ and SQ are equal, that that line is then tangent to the ellipse.

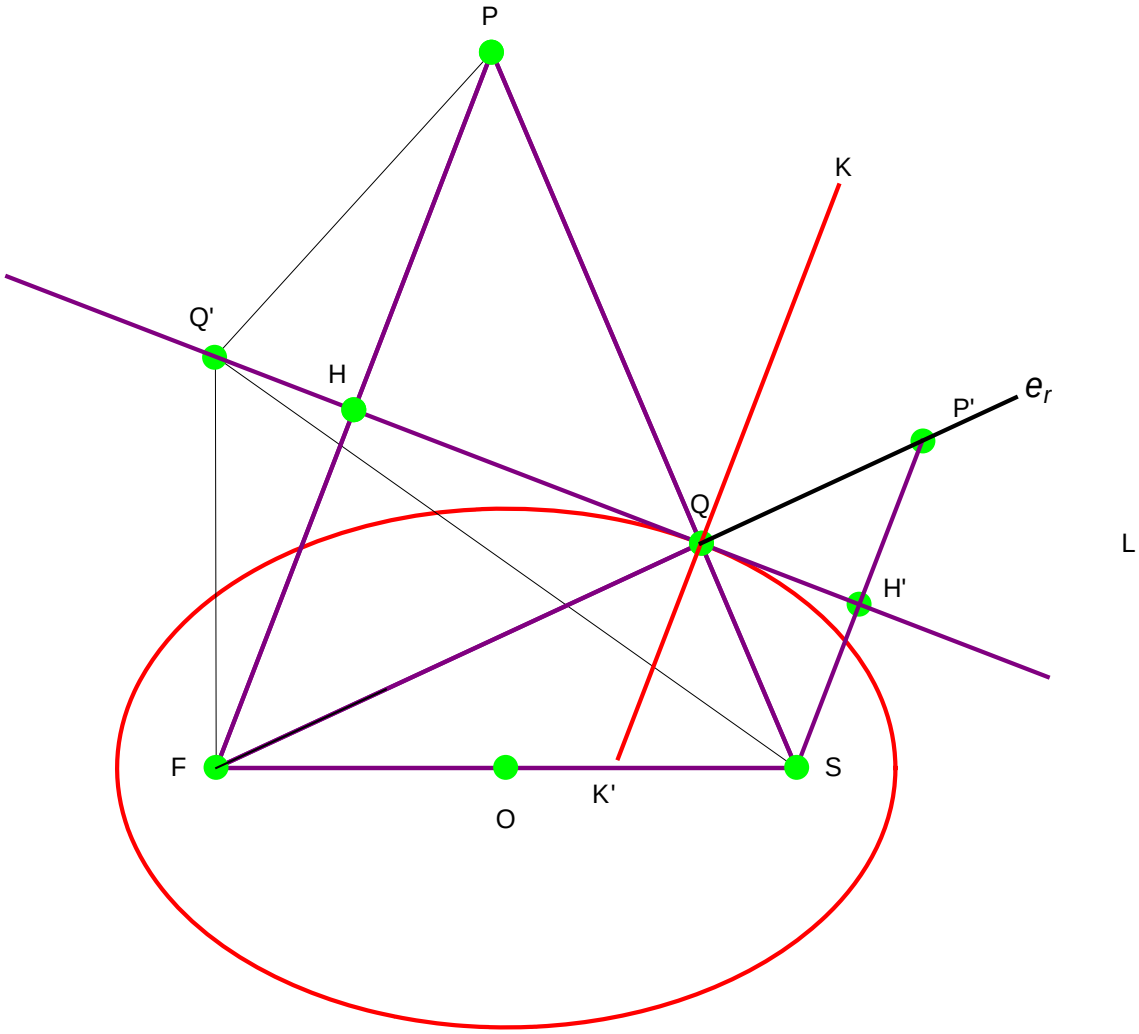


Fig.2 Q (the Earth) on the ellipse with foci S (the Sun) and F. The green circle (radius a) centered at the origin O. FS = 2*ae*.

((**Proof**))

First we extend the perpendicular from F to the tangential line at the point Q, the same distance on the other side, to obtain P, the image of F; Now connect the point Q to P. Two right triangles are exactly the same (see Fig.2). Thus we have

$$\angle FQH = \angle PQH\,, \qquad PQ = FQ\,. \qquad \text{(ellipse optic theorem)}$$

So we get

$$FQ + QS = PQ + QS = SP = 2a\,,$$

Suppose that we takes any other point on the tangent, Q'. We take the sum of distances,

$$FQ'+Q'S = PQ'+Q'S\,,$$

where

$$FQ' = PQ'\,.$$

It is clear that the inequality

$$PQ'+Q'S > SP = 2a\,,$$

in the triangle $\Delta PQ'S$ .In other words, for any point on the tangential line, the sum of the distances from Q' to F and from Q' to F is greater than it is for a point Q on the ellipse.

2. Geometrical theorems

Using the Mathematica, we prove analytically that

(i) $\left|\overrightarrow{FH}\right| \cdot \left|\overrightarrow{SH}\right| = b^2$.

(ii) $\left|\overrightarrow{SP}\right| = 2a$. (P is on the circle (radius 2a) centered at the point S)

(iii) $\left|\overrightarrow{OH}\right| = \left|\overrightarrow{OH'}\right| = a$. [H and H' are on the same circle (radius a) centered at O]

where b is the semi-minor axis and is given by $b = a\sqrt{1-e^2}$ (see Fig.2). We consider the point Q on the ellipse. The position vector of Q is given by

$$\overrightarrow{OQ} = (x_1, y_1) = (a\cos u, b\sin u) = a(\cos u, \sqrt{1-e^2}\sin u)\,,$$

where u is a angle such that $\cos u = \frac{x_1}{a}$. u is a little larger than $\angle QOS$. The position vector of the focus F and the Sun S is given by

$$\overrightarrow{OF} = (-ae, 0)\,, \qquad \overrightarrow{OS} = (ae, 0)\,.$$

The line QQ' which is tangential line at Q on the ellipse is expressed by

$$y - y_1 = -\frac{(1-e^2)x_1}{y_1}(x - x_1), \tag{1}$$

The line SQP is expressed by

$$y = \frac{y_1}{x_1 - ae}(x - ae). \tag{2}$$

The line FHP, which is perpendicular to the tangential line H'QH, is expressed by

$$y = \frac{y_1}{(1-e^2)x_1}(x + ae). \tag{3}$$

The line SH'P', which is perpendicular to the tangential line H'QH,

$$y = \frac{y_1}{(1-e^2)x_1}(x - ae). \tag{4}$$

From Eqs.(2) and (3), the position vector of P can be obtained as

$$\overrightarrow{OP} = (x_P, y_P),$$

with

$$x_P = -\frac{a[e + (e^2 - 2)\cos u]}{1 - e\cos u}, \qquad y_P = \frac{2a\sqrt{1-e^2}\sin u}{1 - e\cos u}.$$

The point H is the middle point between the points F and P,

$$\overrightarrow{OH} = \frac{1}{2}(\overrightarrow{OF} + \overrightarrow{OP}) = (x_H, y_H)$$

with

$$x_H = -\frac{a(e-\cos u)}{1-e\cos u}, \qquad y_H = -\frac{a\sqrt{1-e^2}\sin u}{1-e\cos u}.$$

where

$$\left|\overrightarrow{OH}\right| = a\,.$$

The position vector $\overrightarrow{SP}$ is given by

$$\overrightarrow{SP} = \overrightarrow{OP} - \overrightarrow{OS} = (-\frac{2a(e-\cos u)}{1-e\cos u}, \frac{2a\sqrt{1-e^2}\sin u}{1-e\cos u}),$$

where

$$\left|\overrightarrow{SP}\right| = 2a.$$

The position vector $\overrightarrow{SH}$ is obtained as

$$\overrightarrow{SH} = \overrightarrow{OH} - \overrightarrow{OS} = (\frac{a(1-e^2)(1+e\cos u)}{1-e\cos u}, \frac{a\sqrt{1-e^2}\sin u}{1-e\cos u}).$$

From Eqs.(1) and (4), we get the position vector of H' as

$$\overrightarrow{OH'} = (x_{H'}, y_{H'}),$$

with

$$x_{H'} = \frac{a[e+\cos u]}{1+e\cos u}, \qquad y_{H'} = \frac{a\sqrt{1-e^2}\sin u}{1+e\cos u}.$$

Note that

$$\left|\overrightarrow{OH'}\right| = a\,.$$

The position vector $\overrightarrow{SH'}$ is obtained as

$$\overrightarrow{SH'} = \overrightarrow{OH'} - \overrightarrow{OS} = \left(\frac{a(1-e^2)\cos u)}{1+e\cos u}, \frac{a\sqrt{1-e^2}\sin u}{1+e\cos u}\right),$$

then we have

$$\left|\overrightarrow{FH}\right|\left|\overrightarrow{SH'}\right| = b^2.$$

((**Mathematica**))

```
Clear["Global`*"]; F1 = {a e1, 0};
F2 = {- a e1, 0}; O1 = {0, 0};
rule1 = {x1 → a Cos[u],
   y1 → a √(1 - e1^2) Sin[u]};
eq1 = y - y1 == ((-(1 - e1^2) x1) / y1) (x - x1);
eq2 = y == ( y1 / (x1 - a e1)) (x - a e1);
eq3 = y == ( y1 / ((1 - e1^2) x1)) (x + a e1);
eq23 = Solve[{eq2, eq3}, {x, y}] // Simplify;
eq4 = y == ( y1 / ((1 - e1^2) x1)) (x - a e1);
eq14 = Solve[{eq1, eq4}, {x, y}];
P1 = {x, y} /. eq23[[1]] // Simplify;
M1 = {x, y} /. eq14[[1]] // Simplify;
H1 = (1 / 2) (P1 + F2); Q1 = {x1, y1};
P2 = P1 /. rule1 // Simplify;
H2 = H1 /. rule1 // Simplify; Q2 = Q1 /. rule1;
M2 = M1 /. rule1 // Simplify;
```

```
P2
```

$$\left\{\frac{a\left(e1+\left(-2+e1^2\right)\text{Cos}[u]\right)}{-1+e1\,\text{Cos}[u]},\ \frac{2\,a\sqrt{1-e1^2}\,\text{Sin}[u]}{1-e1\,\text{Cos}[u]}\right\}$$

```
Q2
```

$$\left\{a\,\text{Cos}[u],\ a\sqrt{1-e1^2}\,\text{Sin}[u]\right\}$$

```
H2.H2 // Simplify
```

$$a^2$$

```
H2 // FullSimplify[#, {a > 0, 0 < e1 < 1}] &
```

$$\left\{\frac{a\,(e1-\text{Cos}[u])}{-1+e1\,\text{Cos}[u]},\ \frac{a\sqrt{1-e1^2}\,\text{Sin}[u]}{1-e1\,\text{Cos}[u]}\right\}$$

```
F2P2 = P2 - {a e1, 0}; F2P2.F2P2 // FullSimplify
```

$$4\,a^2$$

```
F2P2 // Simplify
```

$$\left\{\frac{2\,a\,(e1-\text{Cos}[u])}{-1+e1\,\text{Cos}[u]},\ \frac{2\,a\sqrt{1-e1^2}\,\text{Sin}[u]}{1-e1\,\text{Cos}[u]}\right\}$$

```
F1H2 = H2 + {a e1, 0}; f1 = F1H2.F1H2 // FullSimplify
```

$$\frac{a^2\left(-1+e1^2\right)(1+e1\,\text{Cos}[u])}{-1+e1\,\text{Cos}[u]}$$

```
F1H2 // Simplify
```

$$\left\{\frac{a\left(-1+e1^2\right)\text{Cos}[u]}{-1+e1\,\text{Cos}[u]},\ \frac{a\sqrt{1-e1^2}\,\text{Sin}[u]}{1-e1\,\text{Cos}[u]}\right\}$$

```
M2
```

$$\left\{\frac{a\,(e1+\text{Cos}[u])}{1+e1\,\text{Cos}[u]},\ \frac{a\sqrt{1-e1^2}\,\text{Sin}[u]}{1+e1\,\text{Cos}[u]}\right\}$$

```
M2.M2 // Simplify
```

$$a^2$$

```
M2F2 = M2 - {a e1, 0}; f2 = M2F2.M2F2 // FullSimplify
```

$$\frac{a^2\left(-1+e1^2\right)(-1+e1\,\text{Cos}[u])}{1+e1\,\text{Cos}[u]}$$

```
M2F2 // Simplify
```

$$\left\{-\frac{a\left(-1+e1^2\right)\text{Cos}[u]}{1+e1\,\text{Cos}[u]},\ \frac{a\sqrt{1-e1^2}\,\text{Sin}[u]}{1+e1\,\text{Cos}[u]}\right\}$$

```
f1 f2 // Simplify
```

$$a^4\left(-1+e1^2\right)^2$$

3. The velocity on the ellipse orbit ((Maxwell)

Here we show the discussion on the velocity, which was given by J.C. Maxwell (see Fig.3). The physics given by Maxwell is very clear for me. We consider the ellipse A_0QP_0 with foci F and S (S standing for the Sun, A_0 the aphelion, and P_0 the perihelion). Let Q be any point on the ellipse, and draw SP through Q, such that SP = A_0P_0 = 2*a*. In order to avoid the confusion, we use A_0 and P_0 for the aphelion and perihelion. Draw a line from F to P. It remains to be shown that PF is perpendicular to, and proportional to, the velocity at point Q, and that the locus of P is a circle.

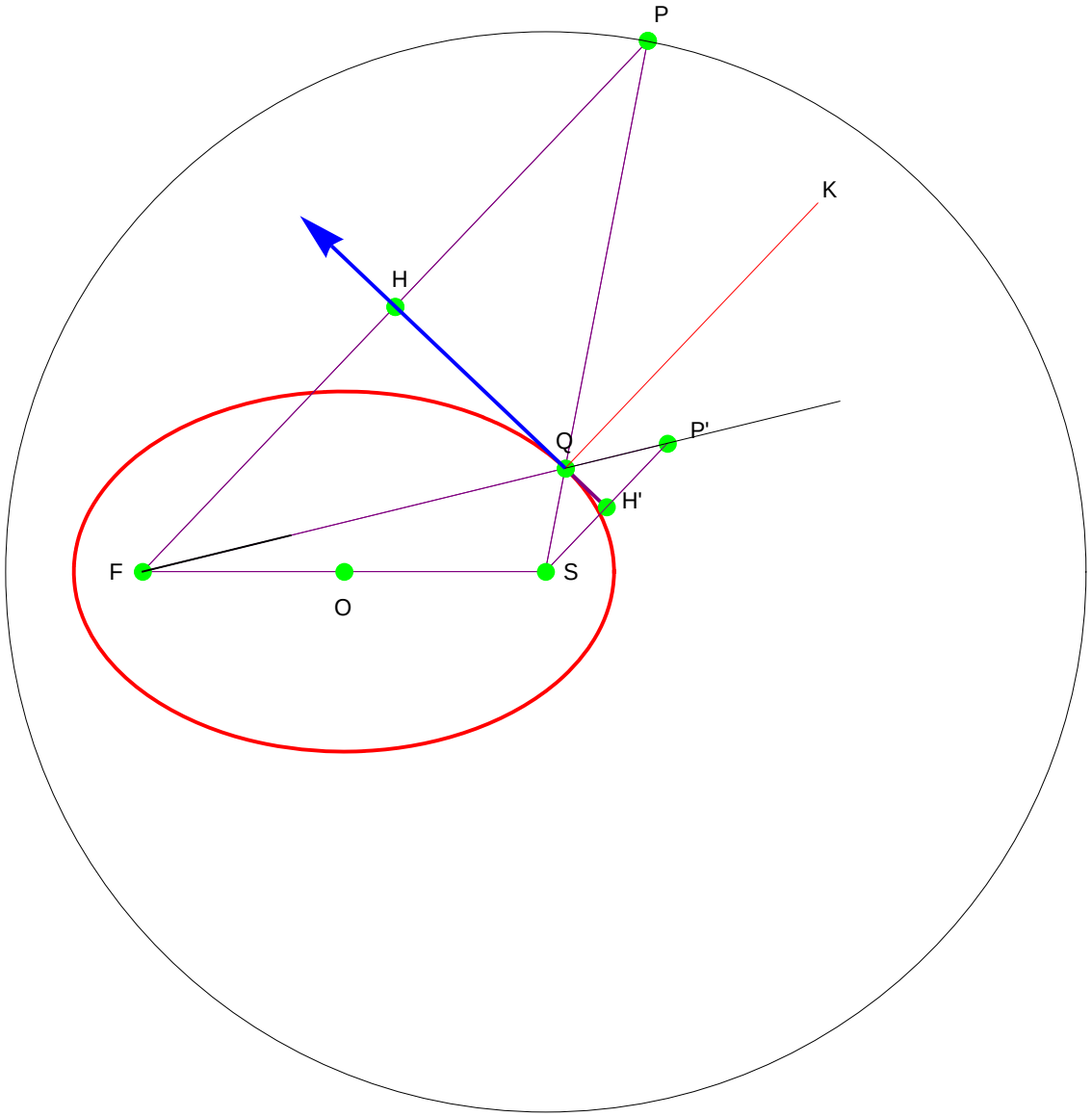


Fig.3 The direction of the velocity vector. The magnitude of the velocity is proportional to the distance FP.

((**Proof**))

In the ellipse, PF is perpendicular to the velocity at the point Q. Draw a tangent from Q to intersect PF at H. Then by the ellipse optical theorem,

$$\angle SQH' = \angle FQH ,$$

and

$$\angle FQH = \angle PQH\,.$$

We also have

$$PQ = PS - QS = 2a - QS = FQ\,.$$

where

$$PS = 2a\,.$$ (2a: the distance between the perihelion and aphelion)

So HQ is perpendicular to PF. Then the direction of PF is perpendicular to the tangent, and hence the velocity at Q.

In the ellipse, PF is proportional to the velocity at Q. Draw a perpendicular line from S to the tangent to intersect the tangent at H'. Let $\boldsymbol{v}$ be the velocity at Q, of the magnitude v. By the conservation of angular momentum, we get

$$mvH'S = l\,,$$

where l is a constant. Using the geometrical theorem

$$HF \cdot H'S = b^2$$

we get

$$\frac{1}{H'S} = \frac{mv}{l} = \frac{HF}{b^2}$$

or

$$HF = \frac{mb^2}{l} v = \frac{1}{2} PF$$

so PF is proportional to v.

Since SP is always equal to the major axis, it follows that the locus of P is a circle, with common origin of the velocity vectors at F; this circle is the hodograph turned through 90° because PF is perpendicular to $\boldsymbol{v}$. Then we have

$$v = \frac{l}{mb^2} HF\,, \qquad \text{or} \qquad v = \frac{l}{2mb^2} PF$$

$\frac{l}{mb^2}$ is the scaling factor and the unit is [1/s].

Note that this scaling factor. We consider the aphelion and the perihelion

(i) At the aphelion

$$l = mv_A r_A,$$

$$v_A = \alpha a(1-e), \qquad r_A = a(1+e)$$

Then we have

$$l = m\alpha a^2(1+e)(1-e) = m\alpha a^2(1-e^2) = \alpha m b^2.$$

The scaling factor is obtained as

$$\alpha = \frac{l}{mb^2}.$$

(ii) At the perihelion,

$$l = mv_P r_P,$$

$$v_p = \alpha a(1+e), \qquad r_p = a(1-e).$$

Then we have

$$l = m\alpha a^2(1-e)(1+e) = m\alpha a^2(1-e^2) = m\alpha b^2,$$

or

$$\alpha = \frac{l}{mb^2} = \frac{l}{ma^2(1-e^2)} = \frac{l}{map_0},$$

where b is the minor axis distance,

$$b = a\sqrt{1-e^2},$$

and p is the semi-latus rectum;

$$p_0 = a(1-e^2).$$

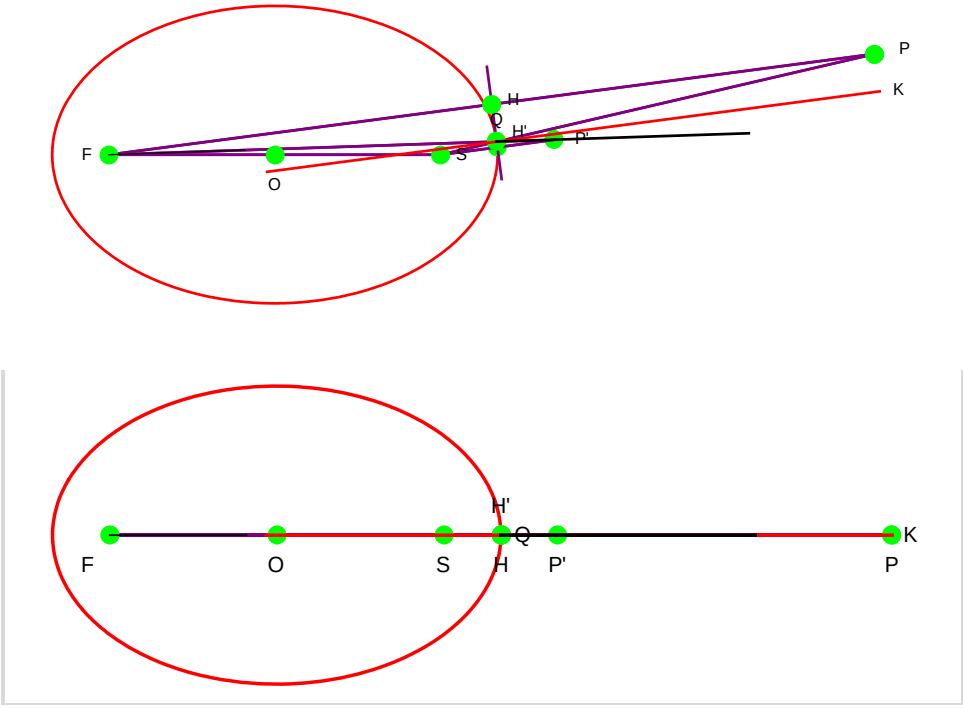


Fig.4 (a) Q is near the perihelion and (b) Q is on the perihelion.

4. Centripetal acceleration

First we discuss the velocity vectors at the point Q_1 and Q_2 on the same ellipse, where Q_1 and Q_2 are very close. The velocity at the point Q_1 is proportional to the length FP_1.

$$v_1 = \frac{l}{2map_0}\left|\overrightarrow{FP_1}\right|.$$

The velocity is directed along the tangential line at the point Q_1. The rotation of the vector $\overrightarrow{FP_1}$ around the point F by $\pi/2$ in a counterclockwise leads to the direction of the velocity. For this rotation we use the geometrical rotation operator $\Re(z,\frac{\pi}{2})$.

$$\boldsymbol{v}_1 = \frac{l}{2map_0}\Re(z,\frac{\pi}{2})\overrightarrow{FP_1}.$$

When the particle rotates from Q_1 to $Q_{2'}$ on the ellipse during the time Δt, the instantaneous acceleration $\boldsymbol{a}$ is

$$\boldsymbol{a} = \frac{\Delta \boldsymbol{v}}{\Delta t} = \frac{\boldsymbol{v}_2 - \boldsymbol{v}_1}{\Delta t} = \frac{l}{2map_0 \Delta t} \Re(z, \frac{\pi}{2})[\overrightarrow{FP_2} - \overrightarrow{FP_1}] = \frac{l}{2map_0 \Delta t} \Re(z, \frac{\pi}{2})[\overrightarrow{P_1P_2}].$$

where

$$\boldsymbol{v}_2 = \frac{l}{2map_0} \Re(z, \frac{\pi}{2}) \overrightarrow{FP_2}.$$

In the limit where the point P_2 is very close to P_1. the vector $\overrightarrow{P_1P_2}$ is perpendicular to the vector $\overrightarrow{SP_1}$. Then the acceraltion is directed toward the Sun (one of the focus in the ellipsoid).

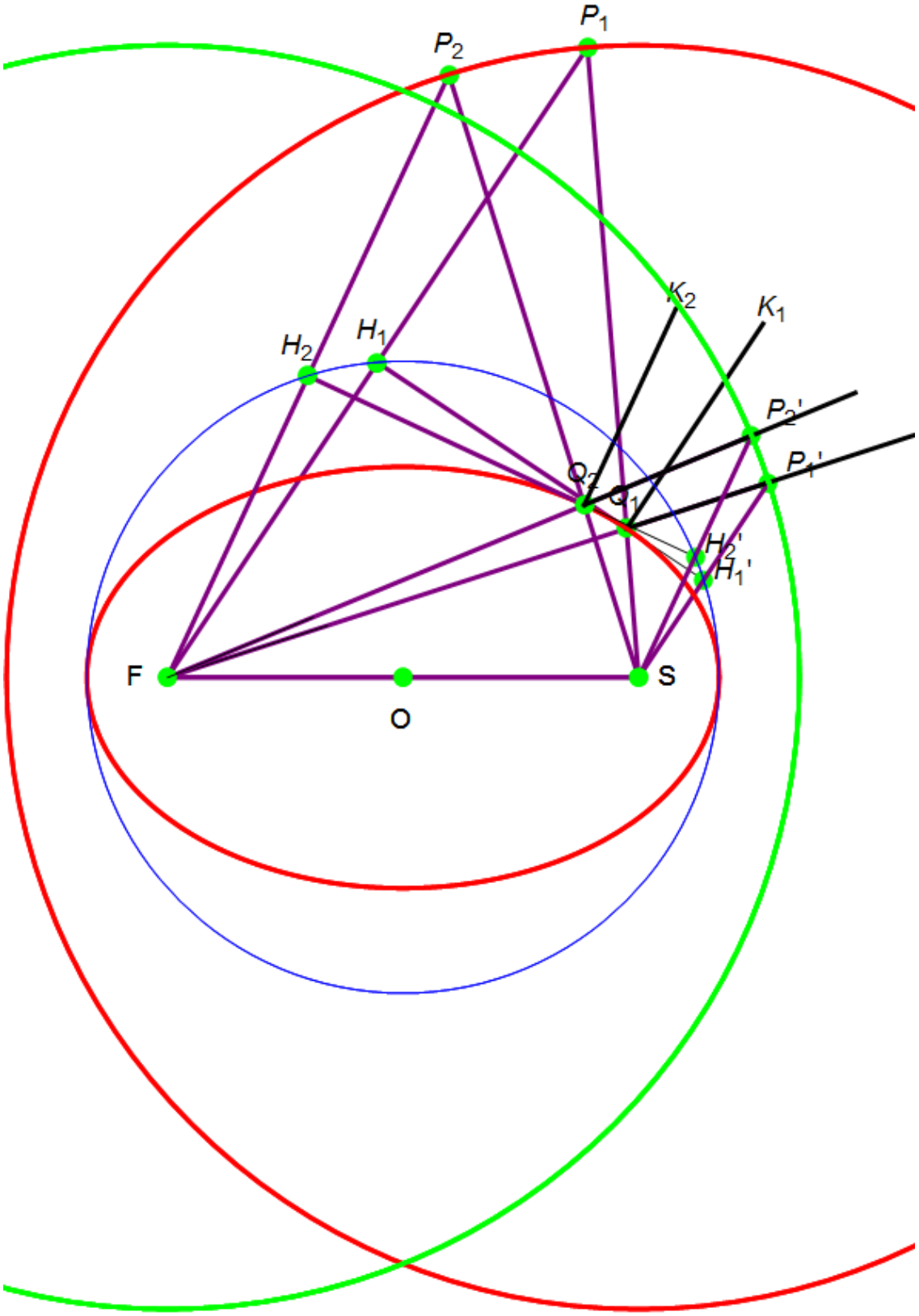
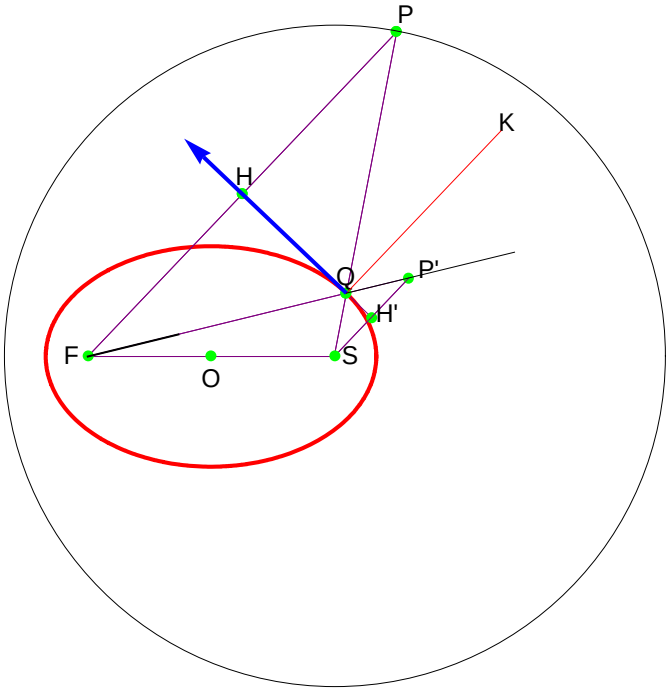
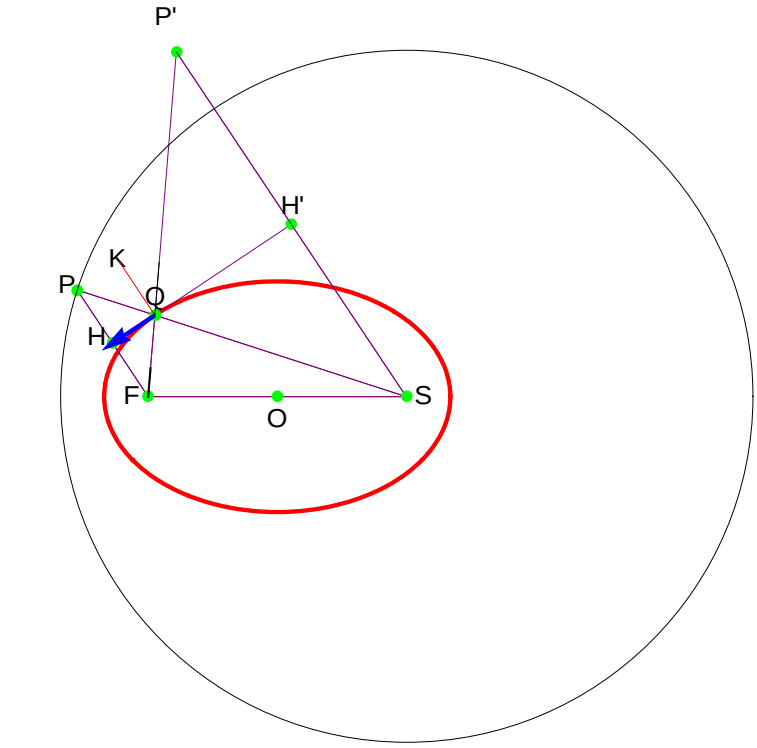

P_1
P_2
K_2
K_1
H_2
H_1
P_2'
P_1'
Q_2
Q_1
H_2'
H_1'
F
O
S

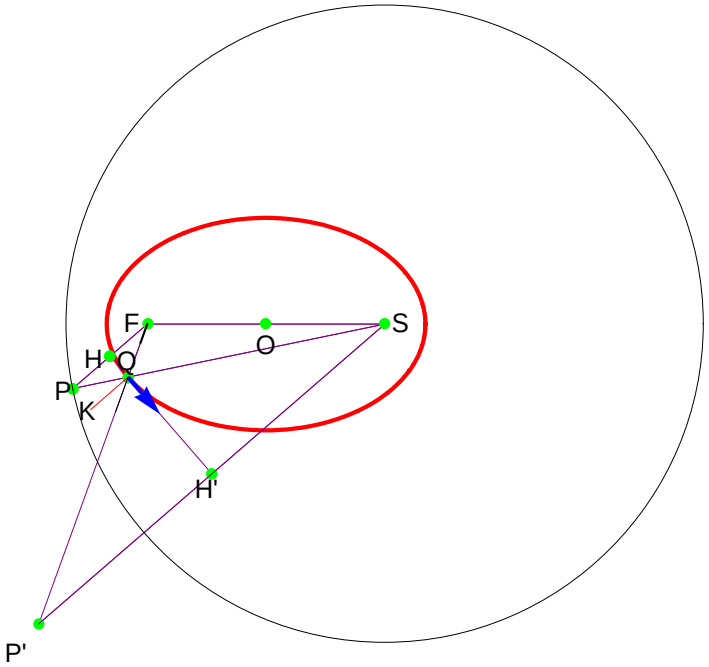
Fig.5 The points P_1 and P_2 are on the circle (radius 2a) centered at S. The point P_1’ and P_2' are on the circle (radius 2a) centered at F. The points H_1, H_2, H_1', and H_2' are on the circle (radius a) centered at the origin O. $K_1Q_1 \perp H_1Q_1$. $K_2Q_2 \perp H_2Q_2$.

5. Hodograph diagram

Using the Mathematica we draw the hodograph for typical position of Q on the ellipse.




P'
H'
K
P
Q
H
F
O
S


F
S
O
H
Q
P
K
H'
P'

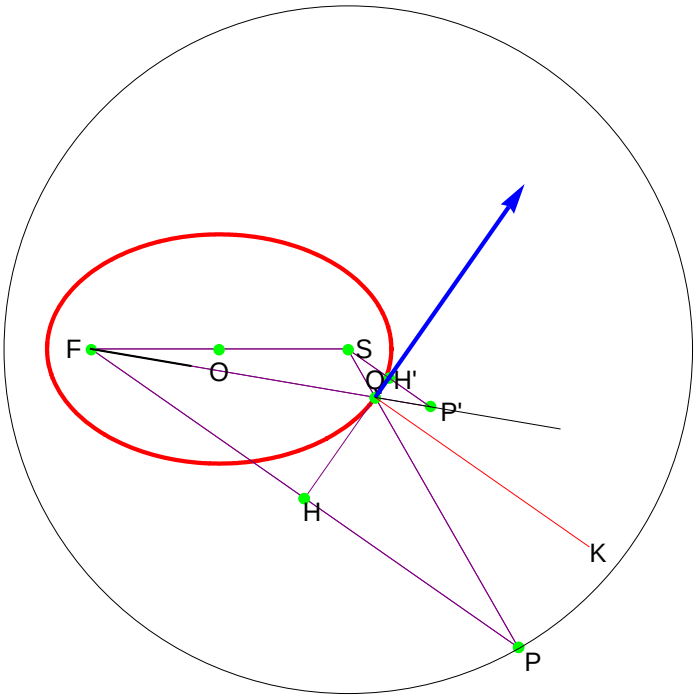


Fig.6 The velocity vector where the point Q is located at different positions on the ellipse orbit.

6. The velocity distribution

We make a plot of the velocity at various points on the ellipse. The circle denoted by green line is the hodograph. It is clear that the velocity is the largest at the perihelion, and that the velocity is the smallest at the aphelion, indicating the Kepler's second law (the conservation of the angular momentum.

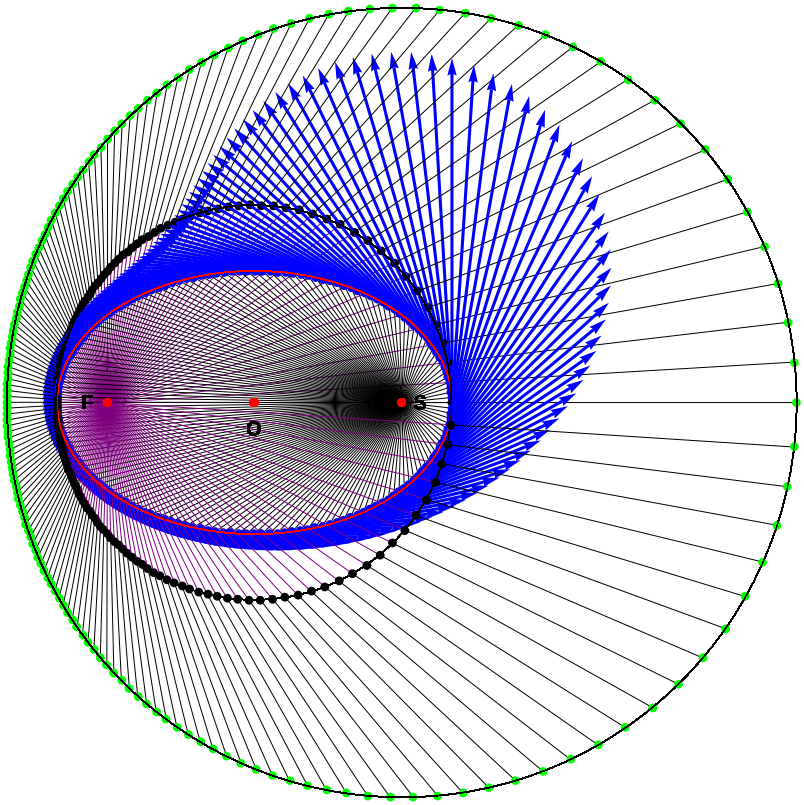


Fig.7 Velocity of various points on the ellipse. The direction of the velocity is parallel to the tangential line at the points. For convenience, the magnitude of the velocity is assumed to be equal to the length of FH.

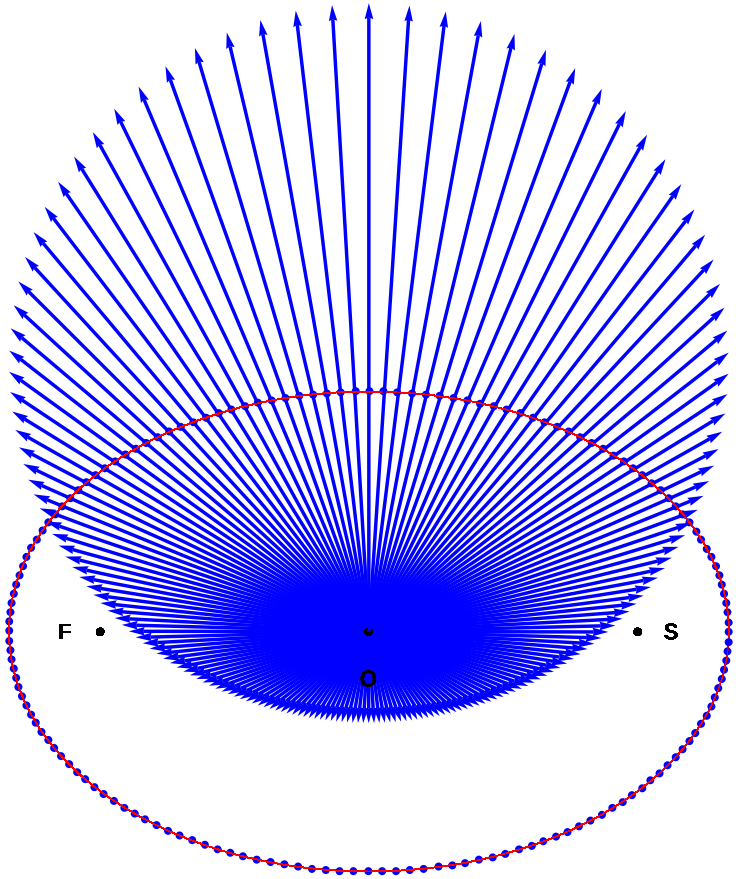


Fig.8 The velocity distribution. The starting point of the velocity vector coincides with the origin, while the magnitude of the velocity is assumed to be equal to the length of FH.

7. The hodographical solution ((Maxwell))

The hodographical solution was discussed by Maxwell. The detail is as follows.

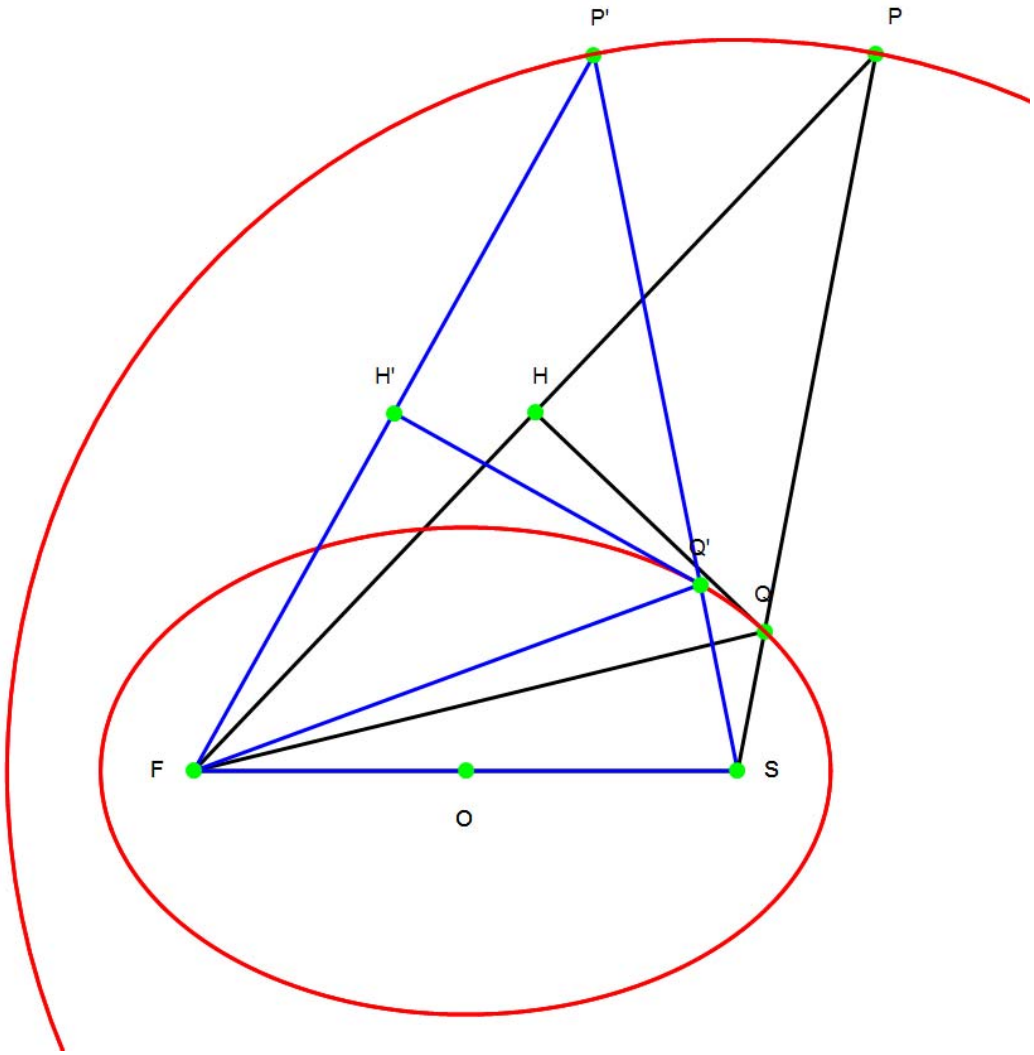


Fig.9 The hodograph. The points P and P’ are located on the circle (radius 2*a*) centered at S. When the planet moves from Q to Q' during the time Δt, the velocity (which is proportional to FP) changes from FP to FP'. $\angle QSQ' = \Delta\theta$. SQ = *r*. SQ' = $r + dr$. QQ'= $r\Delta\theta$. QQ'= $rd\Delta = r\dfrac{\Delta\theta}{\Delta t}\Delta t = r\omega\Delta t$. $\omega = \dfrac{\Delta\theta}{\Delta t}$ is the angular velocity at the point Q. PP'= $2a\Delta\theta = 2a\omega\Delta t$

On the diagram of velocity the line traced out moving point is called the hodograph of the body which it corresponds.

The study of the hodograph was introduced by Sir W.R. Hamilton. The hodograph may be defined as the path traced out by the extremity of a vector which continually represents, in direction and magnitude, the velocity of a moving body. In applying the method of the hodograph to a planet, the orbit of which is in one plane, we shall find it convenient to suppose

the hodograph turned round its origin through a right angle, so that the vector of the hodograph is perpendicular instead of parallel to the velocity it represents.

Hence FP is always proportional to the velocity, and it is perpendicular to its direction. Now SP is always equal to 2*a* (the distance between the aphelion and the perihelion). Hence the circle whose center is S (Sun) and radius 2*a* is the hodograph of the planet, F being the origin of the hodograph.

The corresponding points of the orbit (Q) and the hodograph (P) are those which lie in the same straight line through S (in other words, P, Q and S are on the same straight line).

Thus Q corresponds to P and Q' corresponds to P'. The velocity communicated to the body during its passage from Q to Q' is represented by the geometrical difference between the vector FP and FP', that is, by the line PP', and it is perpendicular to this arc of the circle, and is therefore directed toward the Sun S.

8. Centripetal acceleration: central-force problem

If PP' is the arc described in unit of time, then PP' represents the acceleration, and since P P' is on a circle whose center is S, the distance of arc PP' will be a measure of angular velocity,

$$\text{PP'} = 2a\Delta\theta = 2a\omega\Delta t = \frac{2al}{mr^2}\Delta t\,,$$

where ω is the angular velocity at the point Q,

$$\omega = \frac{l}{mr^2}\,.$$

Note that the angular momentum l is conserved and is given by

$$l = mr^2\frac{d\theta}{dt} = mr^2\omega\,.$$

The acceleration $\boldsymbol{a}$ is obtained as

$$\begin{aligned}\boldsymbol{a} &= \frac{\Delta \boldsymbol{v}}{\Delta t} \\ &= \frac{l}{2map_0 \Delta t}\Re(z,\frac{\pi}{2})[\overrightarrow{PP'}] \\ &= \frac{l}{2map_0 \Delta t}\Re(z,\frac{\pi}{2})[\frac{2al}{mr^2}\Delta t\boldsymbol{e}_\theta] \\ &= \frac{l}{map_0}\frac{al}{mr^2}\Re(z,\frac{\pi}{2})[\boldsymbol{e}_\theta]\end{aligned}$$

or

$$\boldsymbol{F} = m\boldsymbol{a} = -\frac{l^2}{mp_0 r^2}\boldsymbol{e}_r,$$

where $\Re(z,\frac{\pi}{2})$ is the geometrical rotation operator (counter-clockwise rotation of the system around the *z* axis by .

$$\Re(z,\frac{\pi}{2})[\boldsymbol{e}_\theta] = -\boldsymbol{e}_r,$$

and

$$p_0 = a(1-e^2).$$

The acceleration is inversely as the square of the distance SQ. Hence the acceleration of the planet is in the direction of the Sun, and is inversely as the square of the distance from the Sun.

This, therefore, is the law according to which the attraction of the Sun on a planet varies as the planet moves in the orbit and alters its distance from the Sun.

9. Derivation of the gravitational constant

From the Kepler's second law

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\omega = \frac{l}{2m}.$$

T is the time of going completely around the orbit and is obtained by

$$T = \frac{\pi ab}{\frac{dA}{dt}} = \frac{\pi ab}{\frac{l}{2m}} = \frac{2m\pi ab}{l},$$

or

$$l = \frac{2\pi mab}{T}.$$

Using this we get

$$\frac{l^2}{mp_0{}^2} = \frac{4\pi^2 a^3 m}{T^2}.$$

We note that the attractive central force for the Earth, due to the Sun, is given by

$$\boldsymbol{F} = -\frac{GmM}{r^2}\boldsymbol{e}_r$$

where G is the gravitational constant. Comparing two equations for the force $\boldsymbol{F}$, we have

$$\frac{l^2}{mp_0{}^2} = GmM = \frac{4\pi^2 a^3 m}{T^2}.$$

Then the gravitational constant G is derived as

$$G = \frac{4\pi^2 a^3}{MT^2}.$$

where M is the mass of the Sun, T is the orbital period of the Earth, and a is the major axis distance of the ellipse orbit. From this equation we can get

$$T^2 = \frac{4\pi^2 a^3}{GM} \qquad \text{or} \qquad T = \left(\frac{4\pi^2}{GM}\right)^{1/2} a^{3/2},$$

which is the Kepler's third law.

((**Note**))

The Earth's orbit around the Sun is not a circle. The Earth's orbit around the Sun is slightly elliptical. Therefore, the distance between the Earth and the Sun varies throughout the year. At its nearest point on the ellipse that is the Earth's orbit around the Sun, the Earth is 91,445,000 miles (147,166,462 km) from the Sun. This point in the Earth's orbit is known as perihelion and it occurs around January 3. The Earth is farthest away from the Sun around July 4 when it is 94,555,000 miles (152,171,522 km) from the Sun. This point in the Earth's orbit is called aphelion.

The slight ellipse in the Earth's orbit does have a slight impact on the amount of solar energy being received by the Earth. This 3.3% difference in distance does not impact the Earth as much as the seasonal variations, however. Scientists utilize the average distance from the Earth to the Sun as the standard for one astronomical unit (1 AU). This average distance from the Earth to the Sun is 92,955,807 miles (149,597,870.691 km). It takes light from the Sun about 8.317 minutes to reach the Earth.

The Earth takes 365 days, 5 hours, 48 minutes, and 46 seconds (365.242199 days) to make a full revolution around the Sun.

http://geography.about.com/od/physicalgeography/a/orbitSun.htm

Here we use

T = 365.242199 days. (the orbital period of the Earth)

$r_p = a(1-e)$ = 1.47166462 x 10^{11} m, (distance between the Sun and the perihelion)

$r_a = a(1+e)$ = 1.52171522 x 10^{11} m, (distance between the Sun and the aphelion)

a = 1.49669 x 1011 m, (semi-major axis

e = 0.0167204. (eccentricity)

Then we get

$G = 6.68428 \times 10^{-11}$ $m^3/(kg\ s^2)$,

which is close to the 2010 CODATA-recommended value of the gravitational constant

$G = 6.67384(80)$ x 10^{-11} $m^3/(kg\ s^2)$.

((**Mathematica**))

```
Clear["Global`*"];
rule1 = { M → 1.988435 × 10^30, a → 1.49669 10^11,
   T → 365.242199 × 24 × 3600};
G1 = 4 π^2 a^3 / (M T^2) //. rule1
```

$$6.68428 \times 10^{-11}$$

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APPENDIX The Hodograph by using Mathematica

Here we show the locus of the point P, H, H' and P' as Q moves on the ellipse orbit.

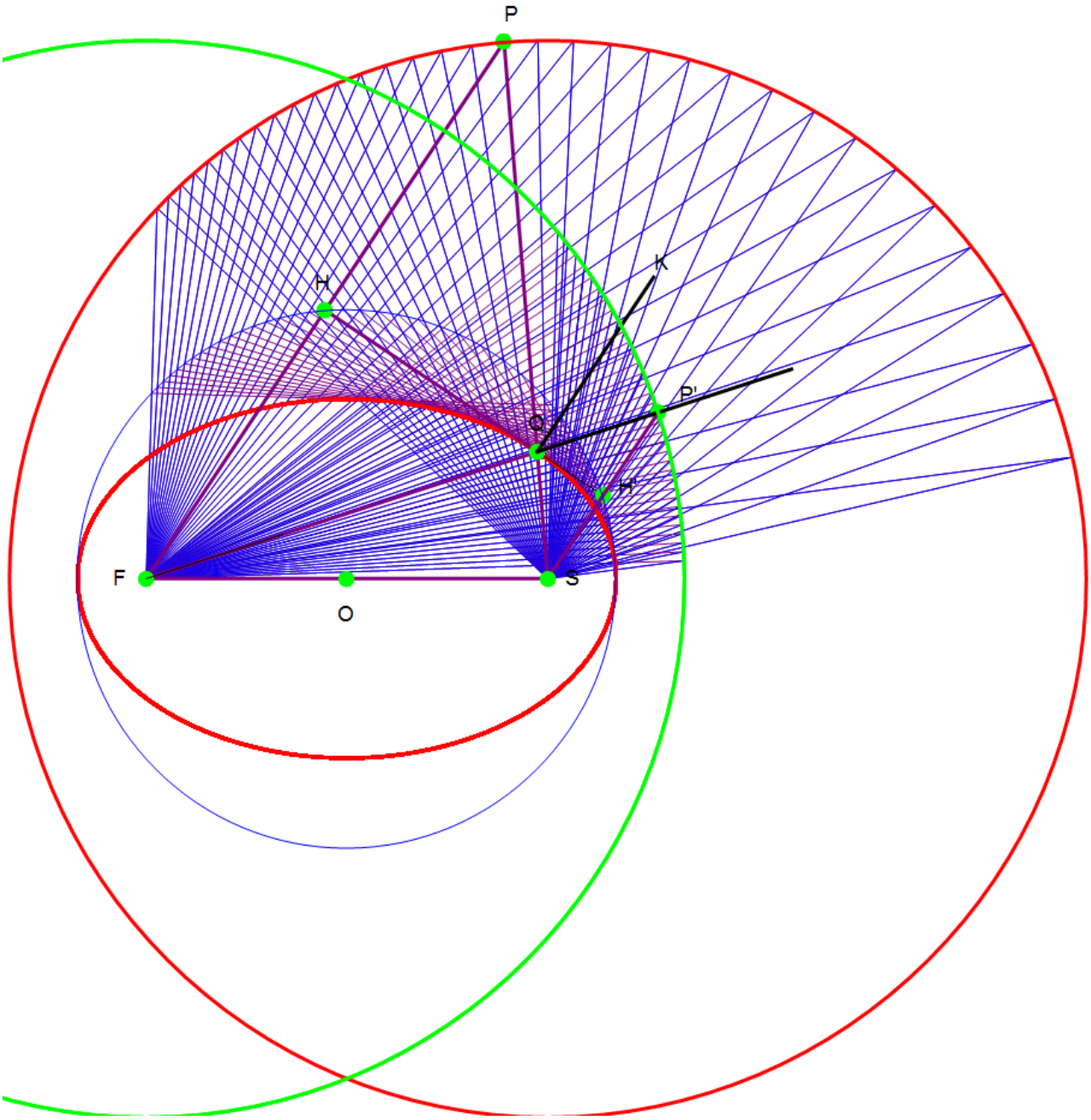

Fig.10 Hodograph for the ellipse orbit.