Physics on Magnetic Moment in Classical Physics and Quantum Mechanics

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Overview

After a series of experiments (**Oersted**, **Arago**, and **Ampère**) around 1820, it became clear that the magnetic moment is equivalent to a loop current. The macro-scale loop current consists of collective atomic-scale loop currents, since the atomic-scale currents inside the large loop current cancel out each other. The atomic-scale current is equivalent to the orbital angular momentum (spin angular momentum is also included). Quantum mechanically, this angular momentum is quantized in the units of the Planck constant $\hbar$. It follows that the origin of the bar magnet (or compass needle) is based on the quantum mechanics; magnetic moment = loop current, and atomic-scale loop current = orbital angular momentum.

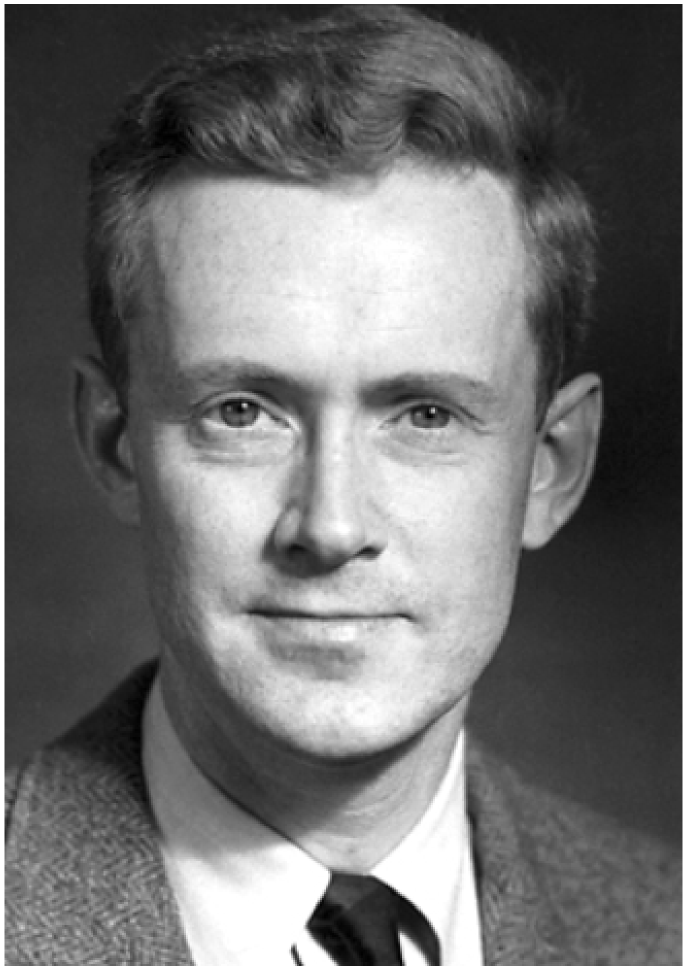
Here we discuss the origin of magnetic (dipole) moment from classical physics as well as quantum mechanics. Through the concept of the magnetic moment, as **Faraday** originally predicted, the electricity and magnetism are found to be closely related.

The present article does not include any topics which are something new. We just collect interesting topics related to the magnetic moment. This article is helpful to undergraduate students who want to understand the physics of magnetic moment extensively from both classical physics and quantum mechanics..

((**Note**))

We use two kinds of notations for the magnetic moment; $\boldsymbol{m}$ (classical physics) and $\boldsymbol{\mu}$ (quantum mechanics).

Edward Mills Purcell (August 30, 1912 – March 7, 1997) was an American physicist who shared the 1952 Nobel Prize for Physics for his independent discovery (published 1946) of nuclear magnetic resonance in liquids and in solids. Nuclear magnetic resonance (NMR) has become widely used to study the molecular structure of pure materials and the composition of mixtures. Friends and colleagues knew him as Ed Purcell.



https://en.wikipedia.org/wiki/Edward_Mills_Purcell

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1. Maxwell's equations

Maxwell's theory of electromagnetism can be summarized in beautiful four equations (SI units)

(I) $$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\varepsilon_0}.$$ (Gauss' law)

(II) $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$ (Faraday's law of induction)

(1II) $$\nabla \cdot \mathbf{B} = 0.$$ (Absence of magnetic monopole)

(IV) $$\nabla \times \mathbf{B} = \mu_0 (\mathbf{J} + \varepsilon_0 \frac{\partial}{\partial t} \mathbf{E}).$$ (Ampère-Maxwell's law)

where ρ_e is the electric charge density and $\boldsymbol{J}$ is the current density.

In electricity, the ***E***-field simply radiates away from positive charges or converge towards negative charges, so they do not circulate. Although the magnetic monopole (single magnetic charge) was theoretically predicted by Dirac (1931), no isolated magnetic poles have been found in nature. So, the ***B***-filed lines have neither beginning nor end. The relation $\nabla \cdot \mathbf{B} = 0$ (III) expresses this fact. In an ideal situation, the lines of ***B*** are closed curves, in contrast to the lines of ***E***, which must originate and terminate on charges. The magnetic field **B** appear" *in the presence of* currents from the Ampère's law. Wherever there are currents, there are lines of magnetic field making loops around the currents. The Ampère's law indicates that magnetism in matter is to be accounted for by a multitude of tiny rings of electric current distributed through the substance. The magnetic moment arises from loop current (**Biot-Savart** law).

The magnetic moment $\boldsymbol{m}$ is proportional to the orbital angular momentum. In quantum mechanics, the z component of orbital angular momentum is quantized as $L_z = n\hbar$, where $\hbar$ is the Planck constant and n is integer. We note that the angular momentum plays a significant role in quantum mechanics. There is a commutation relation in angular momentum.

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z, \qquad [\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x, \qquad [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$$

Note that there is also a spin magnetic moment, which is proportional to the spin angular momentum. The origin of the spin magnetic moment is rather different from that of orbital magnetic moment.

((Note)) **Chow** (Electromagnetic theory)

If the magnetic monopole exists in nature, the Maxwell's equations (II) and (III) change as follows. while the Maxwell's equations (I) and (II) remain unchanged.

(II'))

$$\nabla \cdot \mathbf{B} = \mu_0 \rho_m .$$

(III')

$$\nabla \times \mathbf{E} = -(\frac{\partial}{\partial t}\mathbf{B} + \mu_0 \mathbf{J}_m) .$$

where ρ_m is the magnetic charge density if it exists and $\mathbf{J}_m$ is the magnetic current density. Since $\nabla \cdot \nabla \times \mathbf{E} = 0$ (mathematically), we get the equation of continuity,

$$\frac{\partial}{\partial t}\rho_m + \nabla \cdot \mathbf{J}_m = 0 .$$

Dirac showed that the existence of magnetic monopole would explain why electric charge is quantized.

2. *B*-field (magnetic moment) and *E*-field (electric dipole moment)

Here, we consider (i) the electric field ***E*** (arising from the electric dipole moment), and (ii) the magnetic field ***B*** (arising from the magnetic (dipole) moment ***m***). The electric field **E** and the magnetic field ***B*** are described simply in spherical polar coordinates:

$$\mathbf{E} = \frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\mathbf{e}_r + \sin\theta\mathbf{e}_\theta),$$

and

$$\mathbf{B} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\mathbf{e}_r + \sin\theta\mathbf{e}_\theta),$$

Figures 1-4 show the field distribution of ***B*** and ***E*** in the *y-z* plane. The magnetic field ***B*** *close* to a current loop is entirely different from the electric field close to a pair of separated positive and negative charges. Note that between the charges the electric field points down, while inside the current ring the magnetic field points up, although the far fields are alike. This reflects the fact that the magnetic field satisfies

$$\nabla \cdot \mathbf{B} = 0,$$

everywhere, *even inside the source.* The magnetic field lines do not end. By *near* and *far* we mean, of course, relative to the size of the current loop or the separation of the charges. If we imagine the current ring shrinking in size, the current meanwhile increasing, so that the magnetic moment remains constant. We approach the infinitesimal magnetic (dipole) moment, the counterpart of the infinitesimal electric dipole moment.

We expressed the components in the *y-z* plane of the field ***E*** of an electric dipole moment ***p***, which was situated exactly like our magnetic (dipole) moment ***m***. The expressions are essentially identical, the only changes being

$$\mathbf{p} \rightarrow \mathbf{m} \qquad \textit{and} \qquad \frac{1}{\varepsilon_0} \rightarrow \mu_0 .$$

So, it is found that the magnetic field of a small current loop, has at remote points, the same form as the electric field of two separated charges.

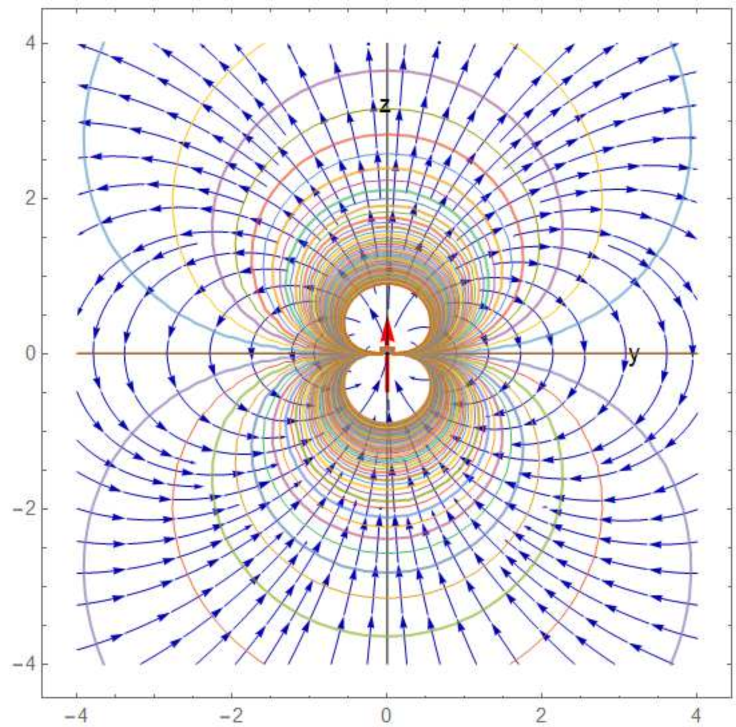


Fig.1 StreamPlot (Mathematica Version 13). The magnetic field of a magnetic (dipole) moment in the *y*-*z* plane. The *y*-*z* plane. Far away the ***B***-field becomes similar to the ***E***-field of an electric dipole moment. The magnetic (dipole) moment along the *z* axis is located at the origin. Note that ***B*** is continuous around the origin, unlike the electric field from the electric dipole moment. The lines of constant magnetic scalar potential ϕ_m^* are also shown. We use the magnetic scalar potential $\phi_m^*(\mathbf{r}) = \mu_0 \mathbf{m} \cdot \mathbf{r} / (4\pi r^3)$, from the analogy of the electric potential due to the electric dipole moment. We use the Mathematica program (ContourPlot and Stream Plot).

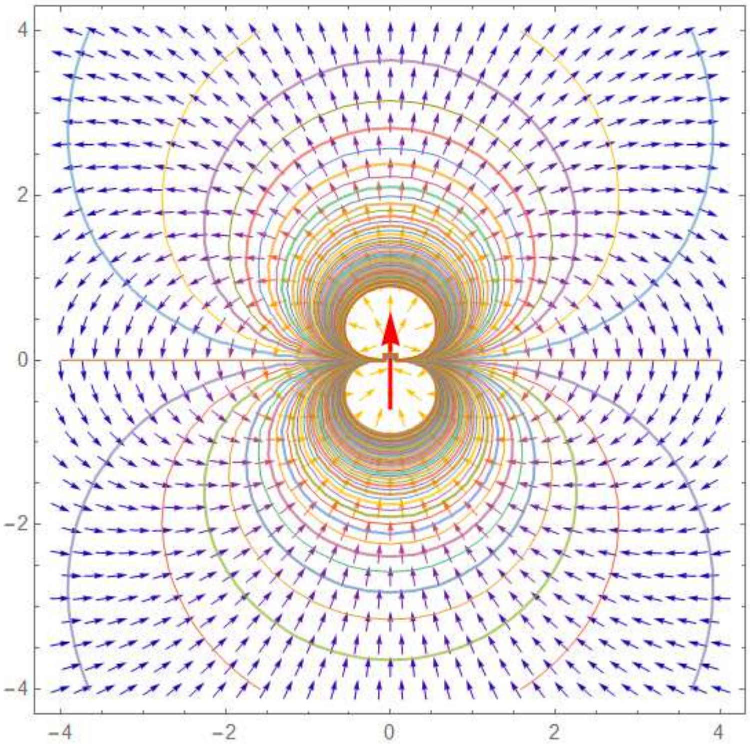


Fig.2 VectorPlot of magnetic field ***B*** in the *y*-*z* plane (Mathematica version 13). The lines of constant magnetic scalar potential ϕ_m^* are also shown. Note that ***B*** is continuous around the origin.

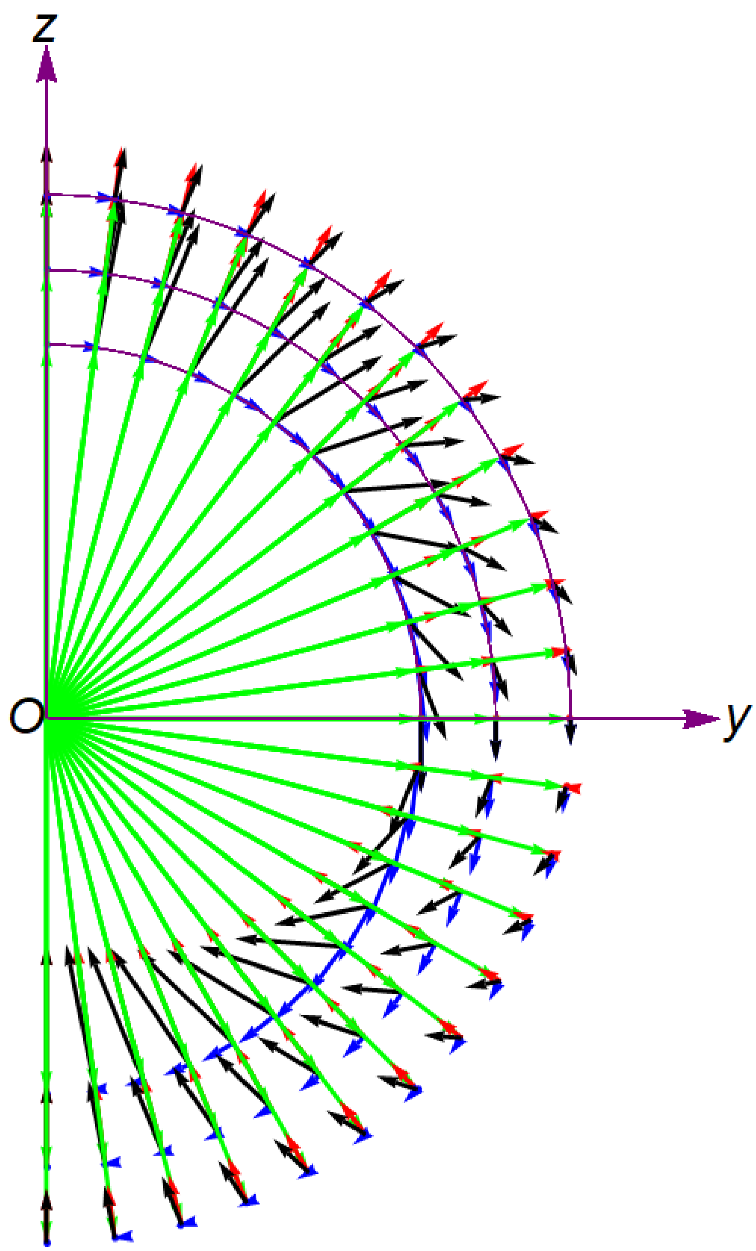


Fig.3 Magnetic field distribution from the magnetic moment at the origin, directed along the *z* axis, in the *y*-*z* plane. The magnetic field $\boldsymbol{B}$ is separated into the $\boldsymbol{e}_r$-component and $\boldsymbol{e}_\theta$-component in the *y*-*z* plane.

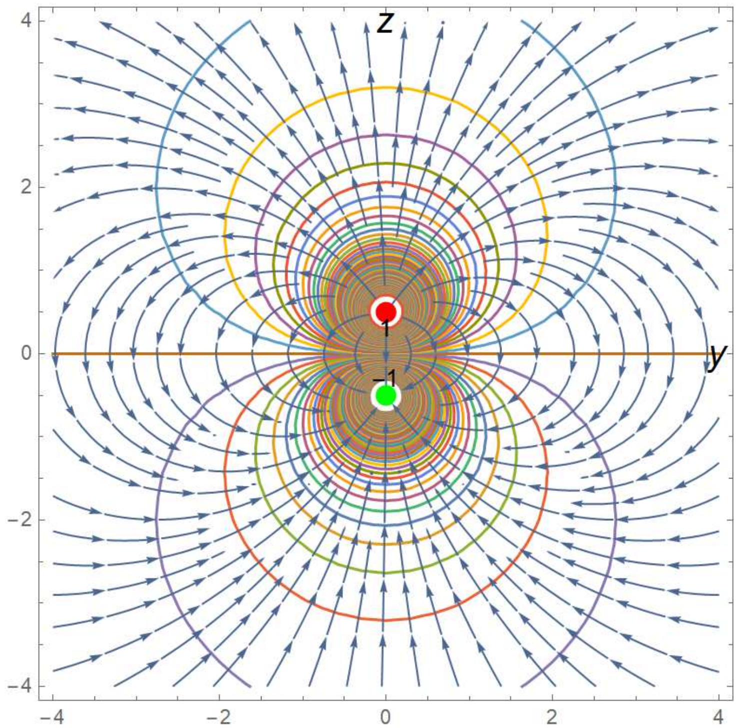


Fig.4 StreamPlot (Mathematica Version 13). The electric field ***E*** of a pair of equal and opposite charges. Far away the ***E***-field becomes similar to the ***B***-field of magnetic (dipole) moment. The electric dipole moment along the *z* axis is located around the origin. The electric field is directed from the positive charge to the negative charge (green) in the vicinity of the origin. In other words, the **E**-field is discontinuous there. The *y*-*z* plane.

3. Historical views on magnetic moment of current loop (Isaac Asimov)

((From the article written by **Isaac Asimov**))

In 1820, possibly during a lecture, **Hans Christian Oersted** (1777 – 1851). happened to move a compass near a wire that carried a current. He noticed that the compass's needle jumped. People knew that compasses worked via magnetism and at the same time realized that current was flowing electricity. He had been using a strong battery in his lecture, and he closed by placing a current-carrying wire over a compass in such a way as to have the wire parallel to the north-south alignment of the compass needle. (It is not certain now what point he was trying to make in doing this).

However, when he put the wire over the needle, the needle turned violently, as though, thanks to the presence of the current, it now wanted to orient itself east-west. **Oersted**, surprised, carried the matter further by inverting the flow of current – that is, he connected the wire to electrodes in reverse manner. Now the compass needle turned violently again, but in the opposite sense.

As soon as **Oersted** announced this, physicists all over Europe began to carry out further experimentation, and it quickly became plain that electricity and magnetism were intimately related, and that one might well speak of electromagnetism in referring to the manner in which one of the two forces gave rise to the other.

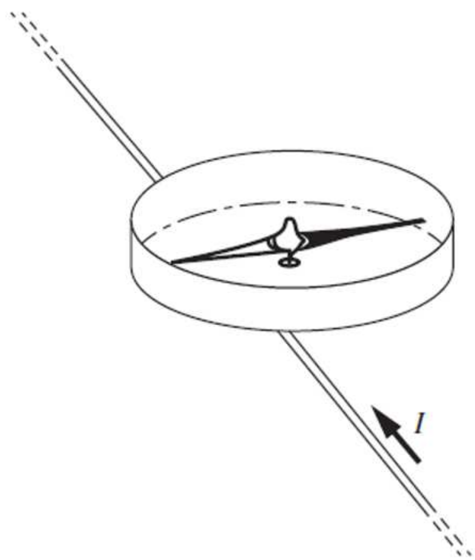


Fig.5 Experiment of Oersted's experiment. When the constant current flows through an electric wire, the axis of a compass needle (a kind of bar magnet) directed toward a direction normal to the direction of current.

The French physicist **Dominique François Jean Arago** (1786-1853) showed almost at once that a wire carrying an electric current attracted not only magnetized needles but ordinary unmagnetized iron filings, just as a straightforward magnet would. A magnetic force, indistinguishable from that of ordinary magnets, originated in the electric current. Indeed, a flow of electric current was a magnet.

To show this more dramatically, it was possible to do away with iron, either magnetized or unmagnetized, altogether. If two magnets attracted each other or repelled each other (depending on how their poles were oriented), then the same should be true of two wires, each carrying an electric current. This was indeed demonstrated in 1820 by the French physicist **André-Marie Ampère**, after whom the unit of current intensity was named. Ampère began with two parallel wires, each connected to a separate battery. One wire was fixed, while the other was capable of sliding toward its neighbor or away from it. When the current was travelling in the same direction in both wires, the movable wire slid toward the other, indicating an attraction between the wires. If the current traveled in opposite directions in the two wires, the movable wire slid away, indicating repulsion between the two wires. Furthermore, if Ampère arranged matters so that the movable wire was free to rotate, it did so when the current was in opposite directions, turning through 180° until the two wires were parallel again with the current in each now flowing in the same direction. (This is analogous to the manner in which, if the north pole of one small magnet is brought near the north pole of another, the second magnet is brought near the north pole of another, the second magnet will flip so as to present its south pole end to the approaching north pole).

Again, if a flowing current is a magnet, it should exhibit magnetic lines of force as an ordinary magnet does, and these lines of force should be followed by a compass needle. Since the compass needle tends to align itself in a direction perpendicular to that of the flow of current in the wire (whether the needle is held above or below the wire, or to either side), it would seem that the magnetic lines of force about a current-carrying wire appear in the form of concentric cylinders about the wire. If a cross section is taken perpendicularly through the wire, the lines of force will appear as concentric circles. This can be demonstrated by running a current-carrying wire upward through a small hole in a horizontal piece of cardboard. If iron filings are sprinkled on the cardboard and the cardboard is tapped, the filings will align themselves in a circular arrangement about the wire.

In the case of an ordinary magnet, the lines of force are considered to have a direction – one that travels from a north pole to a south pole. Since the north pole of a compass needle always points to the south pole of a magnet, it always points in the conventionally accepted direction of the line of forces. The direction of the north pole of a compass needle also indicates the direction of the lines of force in the neighborhood of a current-carrying wire, and this turns out to depend on the direction of the current-flow.

Ampère accepted Franklin's convention of current-flow from the positive electrode to the negative electrode. If, using this convention, a wire were held so that the current flowed directly toward you, the lines of force, as explored by a compass needle, would be moving around the wire in counterclockwise circles. If the current is flowing directly away from you, the lines of force would be moving around the wire in clockwise circles.

As an aid to memory, **Ampère** advanced what has ever since been called the "right-hand screw rule." Imagine yourself holding the current-carrying wire with your right hand; the fingers close about it and the thumb points along the wire in the direction in which the current is flowing. If you do that, then the sweep of the curving fingers, from palm to fingernails, indicates the direction of the magnetic lines of force.

4. Equivalence between magnetic moment and loop current (Purcell and Morin)

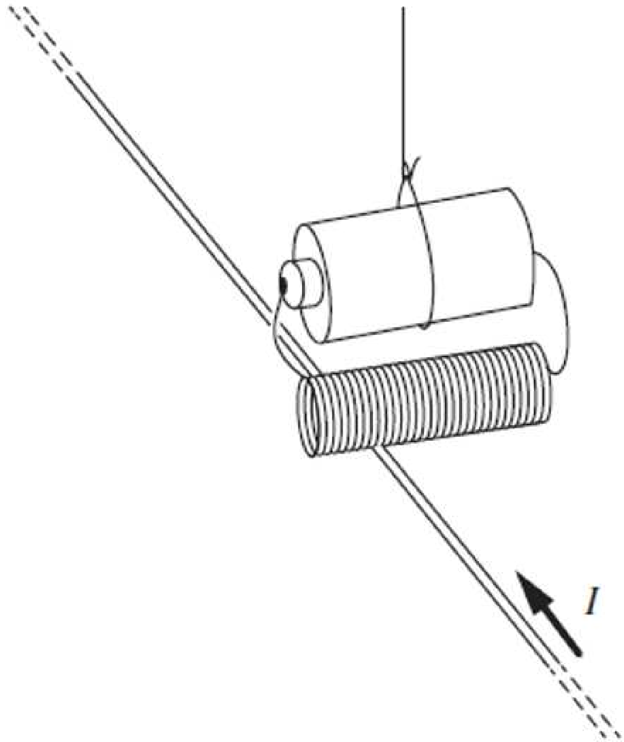


Fig.6 A compass needle (a) and a coil of wire carrying current (b) are similarly influenced by current in a nearby conductor. The direction of the current I is understood to be that in which positive ions would be moving if they were the carriers of the current. In the earth's magnetic field, the black end of the compass would point north (**Purcell and Morin**)

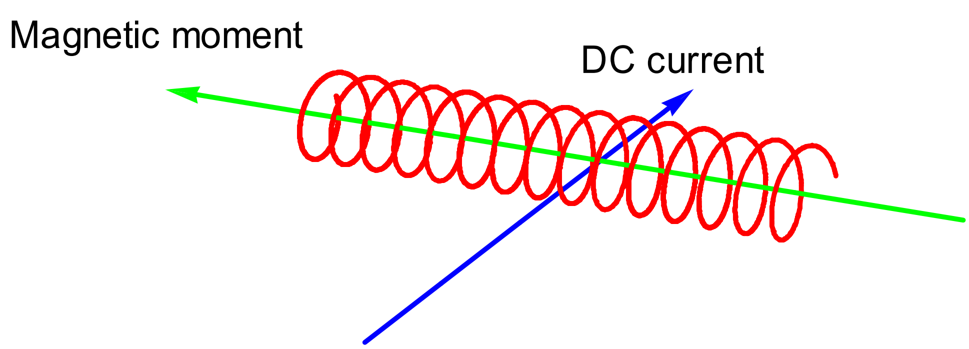


Fig.7 The compass needle (magnetic moment) is equivalent to the solenoid coil with a flowing current I_s. When the compass is perpendicular to the DC

current I (denoted by the blue straight line), the current direction of I becomes parallel to the current direction of I_s (denoted by red), leading to the attractive force between them.

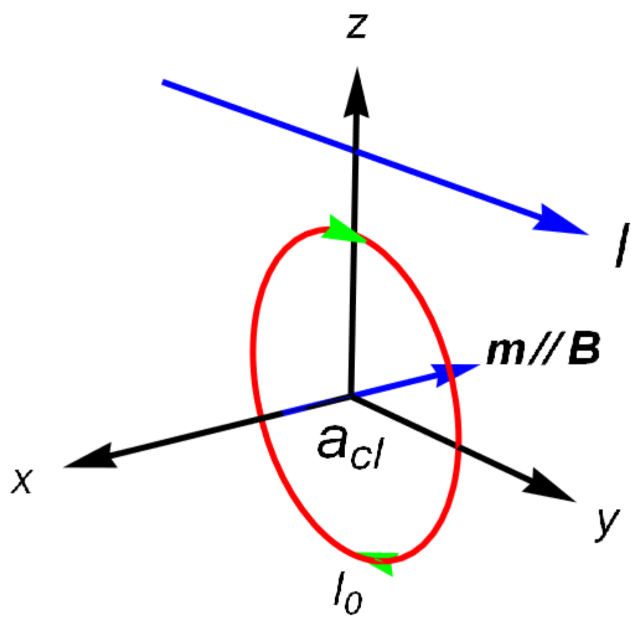


Fig.8 Schematic diagram of Oersted's experiment. This was discovered on 21 April 1820 by Danish physicist **Hans Christian Oersted** (1777–1851), when he noticed that the needle of a compass next to a wire carrying current turned so that the needle was perpendicular to the wire. **Oersted** investigated and found the physical law describing the magnetic field, now known as Oersted's law. The direction of the current I_0 in the loop current is parallel to the direction of the current I of the straight electric wire. The magnetic field is produced by the straight electric current wire at the center of compass, due to the Ampère's law. The compass needle is equivalent to the magnetic moment.

5. Bohr model for hydrogen atom

We consider a hydrogen atom (one electron around one proton).

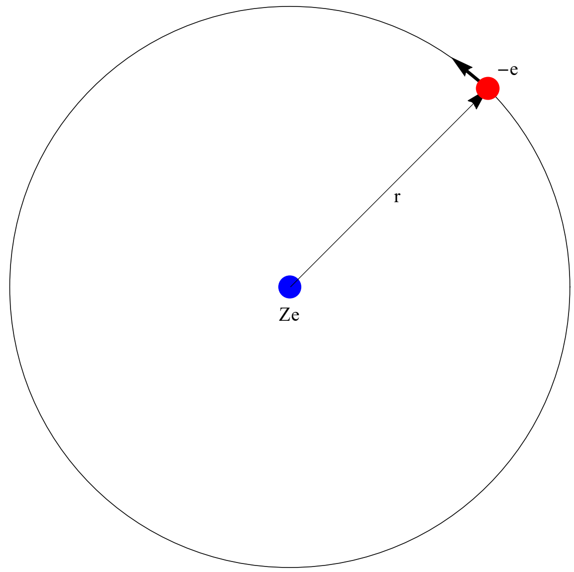


Fig.9 Hydrogen atom. Atomic number, $Z = 1$. Proton (+e charge, blue) and electron (-e charge, red). e>0.

Newton's second law:

$$m_e \frac{v^2}{r} = \frac{e^2}{4\pi\varepsilon_0 r^2} . \qquad \text{(SI units)}$$

The orbital angular momentum along the z axis is quantized.

$$L_z = m_e v r = n\hbar$$

which is equivalent to the de Broglie relation.

$$p = m_e\ v = \frac{h}{\lambda} = \frac{2\pi\hbar}{\lambda}$$

$$\frac{2\pi r}{\lambda} = n , \qquad (n;\ \text{integer})$$

or

$$\frac{2\pi r}{(\frac{2\pi\hbar}{m_e v})} = n , \qquad \text{or} \qquad L_z = m_e v r = n\hbar$$

From two equations, we have

$$m_e \frac{v^2}{r} = \frac{e^2}{4\pi\varepsilon_0 r^2} \qquad \text{or} \qquad m_e v^2 r = \frac{e^2}{4\pi\varepsilon_0} \qquad \text{(Newton's second equation)}$$

$$m_e v r = n\hbar \qquad \text{(Quantization of orbital angular momentum)}$$

we get the velocity

$$v_n = \frac{e^2}{4\pi\varepsilon_0 n\hbar} = \frac{1}{n}\frac{e^2}{4\pi\varepsilon_0 \hbar c}c = \frac{1}{n}\alpha c \qquad \text{(SI units)}$$

where n is a positive integer, and α is the fine structure constant,

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} = 0.007297352 = \frac{1}{137}$$

When $n = 1$, the radius is called the Bohr radius,

$$a_B = \frac{\hbar^2}{m_e e^2} = 5.29177210903(80)\times 10^{-11}\ \text{m}$$

where m_e is the electron mass, and the electron rest mass energy is

$$m_e c^2 = 0.51099895000(15)\ \text{MeV}$$

6. Quantization of magnetic moment from orbital motion of electron

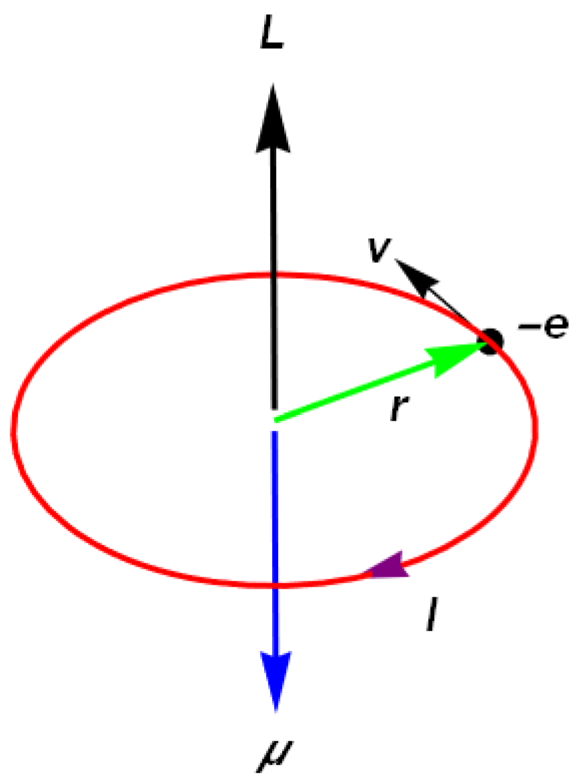


Fig.10 Orbital magnetic moment (μ) and orbital angular momentum of electron. The direction of the magnetic moment is antiparallel top that of the orbital angular momentum, since the charge of electron is negative as -*e* (*e*>0 in this note).

According to the Bohr model, the magnetic moment of orbital electron which undergoes a circular motion is expressed by

$$\mu = Ia_{cl} , \qquad \text{(SI units)}$$

where the charge of electron is -*e* (*e*>0), *I* is the loop current flowing around the perimeter of the circular orbit, and a_{cl} is the area of the Bohr orbit. The current *I* is defined by

$$I = \frac{dQ}{dt} = -\frac{e}{T} = -\frac{e}{(\frac{2\pi r}{v})} = -\frac{ev}{2\pi r} ,$$

where Q is the charge and T is the period, $T = \dfrac{2\pi r}{v}$. So that, the magnetic moment is obtained as

$$\mu = Ia_{cl} = -\frac{ev}{2\pi r}\pi r^2 = -\frac{evr}{2}.$$

The orbital angular momentum L_z along the z axis is

$$L_z = m_e vr = n\hbar ,$$

where $\hbar$ is the Planck's constant, from the duality of wave and particle. From the de Broglie relation, the circumference of the circle orbit is related to the wavelength ($\lambda = h/p = 2\pi\hbar/(m_e v)$) by

$$\frac{2\pi r}{\lambda} = n \qquad \text{(integer)} \qquad\qquad \text{(duality of wave and particle)}$$

where $p = m_e v$ is the linear momentum and v is the velocity.

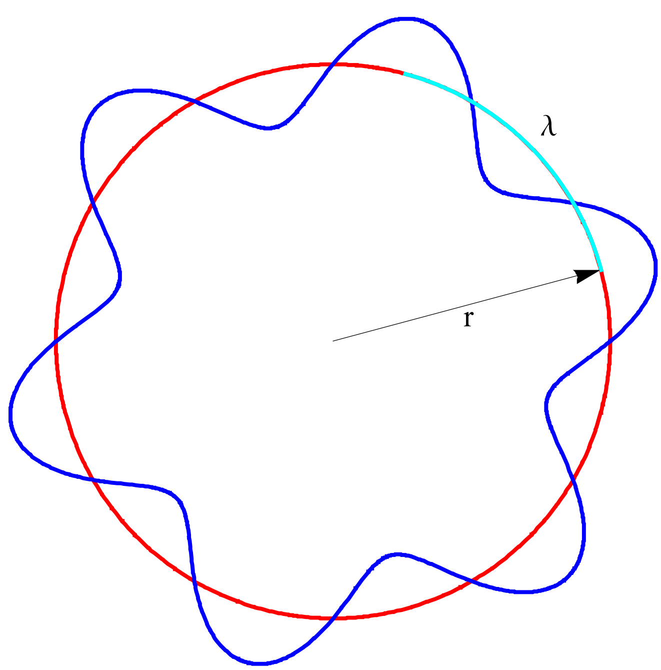


Fig.11 de Broglie relation. $l = 2\pi r = n\lambda$. $p = m_e v = \dfrac{2\pi\hbar}{\lambda}$. $n = 6$ in this case.

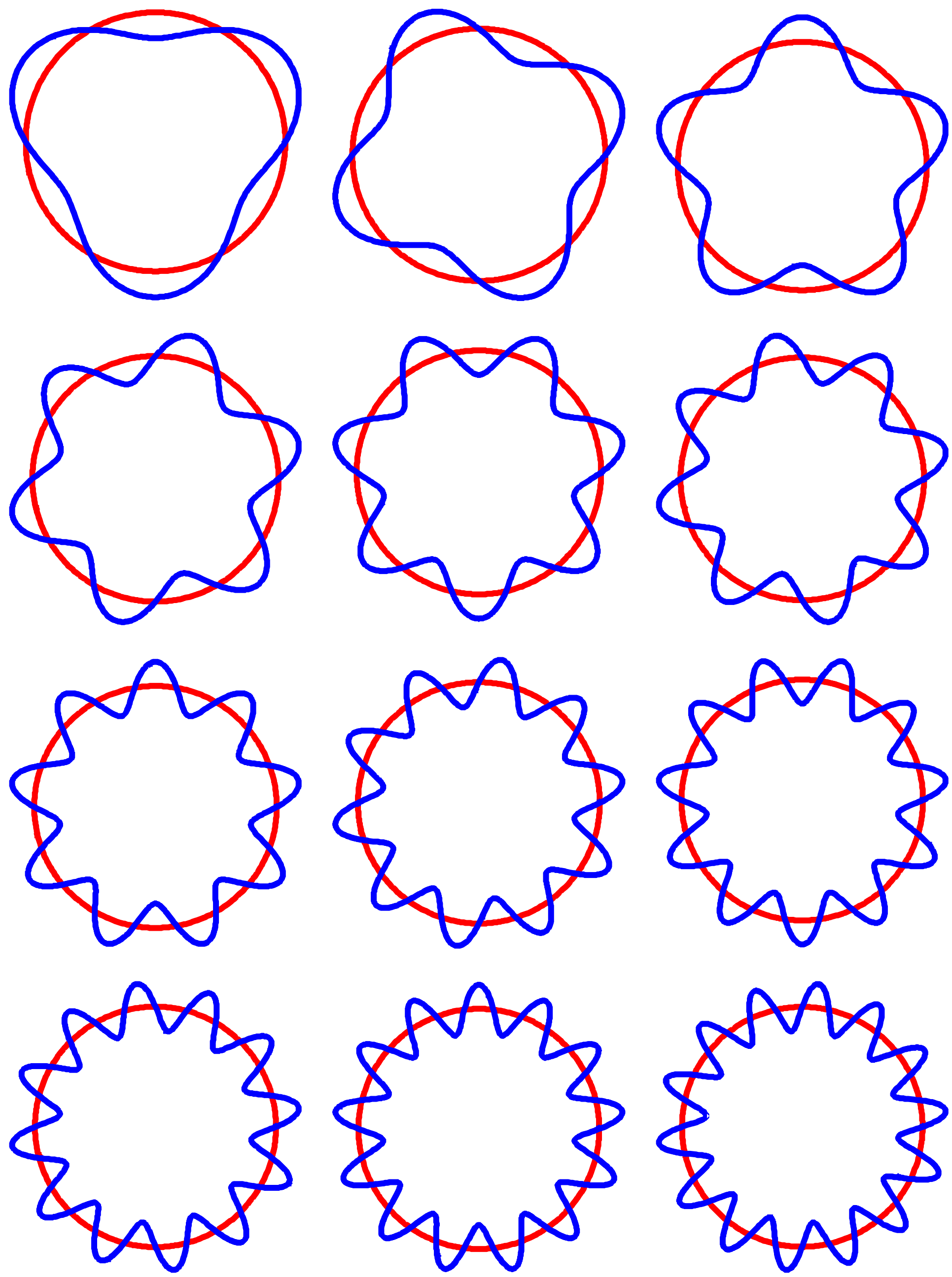


Fig.12 Stationary waves of electrons in the confinement of an atom. The electron wave fits an integral number of wavelength in each of the successive Bohr orbits. $l = 2\pi r = n\lambda$. n= 2 – 13. (n: integer)

This is equivalent to the quantization of orbital angular momentum. Using these results, we have the magnetic moment

$$\mu_z = -\frac{em_e vr}{2m_e} = -\frac{eL_z}{2m_e} = -\frac{e\hbar}{2m_e}\frac{L_z}{\hbar} = -\mu_B \frac{L_z}{\hbar}$$

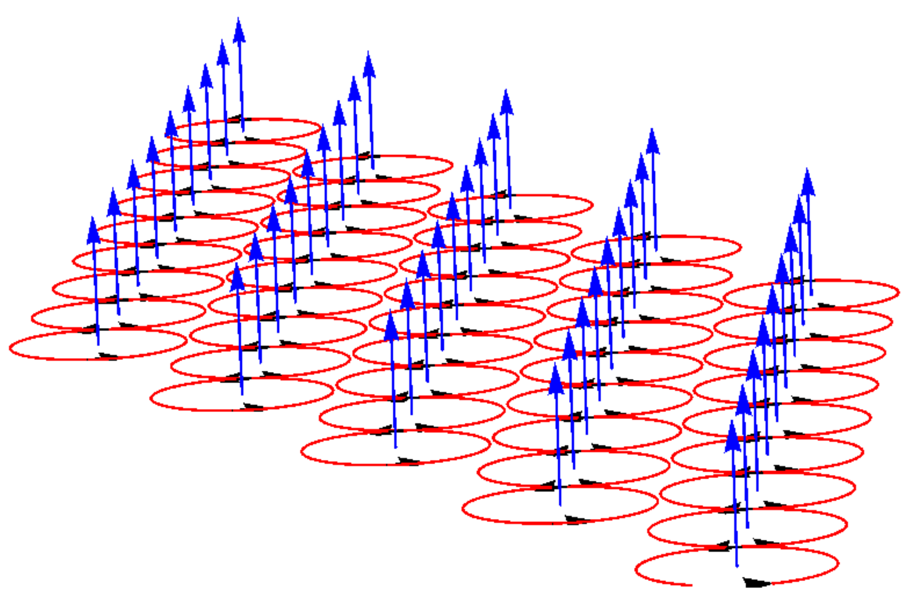
where m_e is the mass of electron, and $\mu_B = \frac{e\hbar}{2m_e}$ (SI units) is the Bohr magneton. The gyromagnetic ratio is the ratio of μ_z to L_z. The Bohr magneton is the fundamental unit of magnetic moment;

$$\mu_B = \frac{e\hbar}{2m_e} = 9.274009994(57) \times 10^{-24} \text{ J/T} \qquad \text{(SI units)}$$

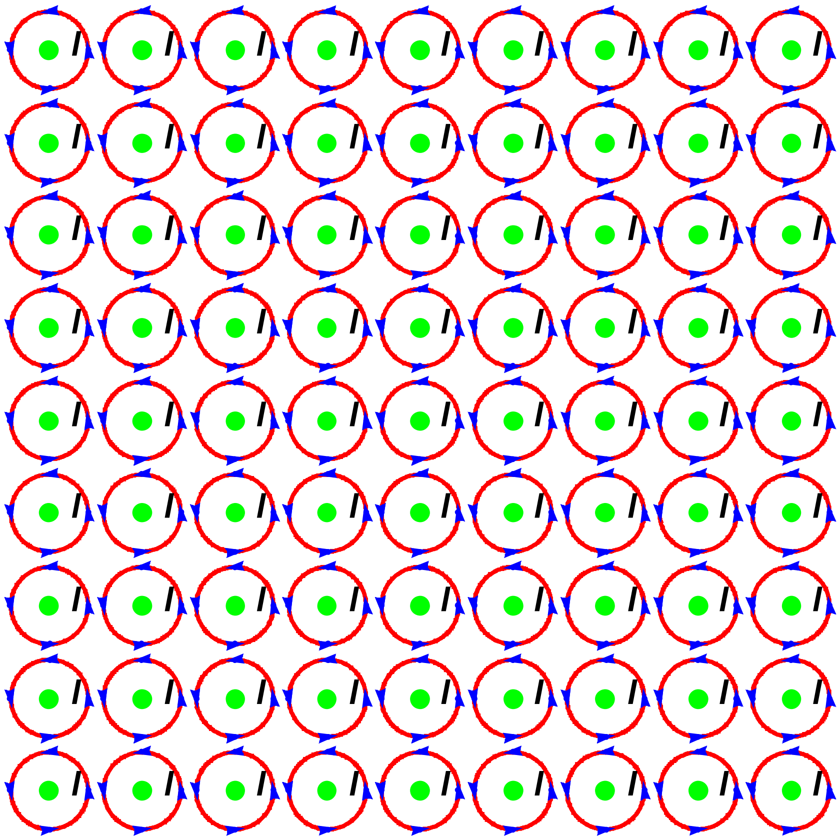
$$\mu_B = \frac{e\hbar}{2m_e c} = 9.274009994(57) \times 10^{-21} \text{ erg/G} \qquad \text{(cgs units)}$$

We note that erg/G = emu (cgs unit).

As shown in **Fig.13(a) and (b)**, a thin slab of uniformly magnetized material, with the atomic-scale magnetic moments indicated by tiny current loops. All the internal currents cancel. At the boundary of the system, there is no adjacent current loop to do the canceling. As a result, the system is equivalent to s single ribbon of current *I*, flowing around the boundary.



(a)



(b)

Fig.13 (a) and (b) Circulating atomic-scale loop currents as seen in across-section of magnetic system in the *z* direction. The direction of the magnetic moment follows the right-hand rule.

The orbital magnetic moment is related to the orbital angular momentum as

$$\boldsymbol{\mu}_O = -\frac{e\hbar}{2m_e}\frac{\mathbf{L}}{\hbar} = -\mu_B\frac{\mathbf{L}}{\hbar} = \gamma_o\mathbf{L} \qquad \text{(orbital magnetic moment)}$$

Note that γ_o is the gyromagnetic ratio. The negative sign of γ_o arises from the fact that the charge of electron is negative.

$$\gamma_o = -\frac{e}{2m_e} = -8.7941\times 10^{10} \ (\mathrm{s^{-1}T^{-1}}).$$

7. Spin magnetic moment and spin angular momentum (quantum mechanics)

The spin magnetic momentum $\boldsymbol{\mu}_s$ is related to the corresponding spin orbital momentum $\mathbf{S}$, from the analogy of the relation between the orbital magnetic moment and the orbital angular momentum, as

$$\boldsymbol{\mu}_s = -g_e \mu_B \frac{1}{\hbar}\mathbf{S},$$

where g_e is the Landé *g*-factor,

$$g_e = 2.0023193043617(15).$$

Conventionally, we use $g_e = 2.0$. The derivation of this expression can be naturally obtained only from the Dirac relativistic electron theory. It is not possible from the classical theory. Using these results, we have the spin magnetic moment. The gyromagnetic ratio is the ratio of $(\mu_s)_z$ to S_z

$$\mu_s = \gamma_s S_s,$$

with

$$\gamma_s = -\frac{e}{2m} g_s = -1.760859644(11)\times 10^{11}\ \mathrm{s}^{-1}\mathrm{T}^{-1} \qquad \text{(SI units)}$$

The negative sign of γ_s arises from the fact that the charge of electron is negative.

We have the spin magnetic moment as

$$\boldsymbol{\mu}_\mathrm{S} = -g_s \mu_B \frac{\mathbf{S}}{\hbar} \simeq -2\mu_B \frac{\mathbf{S}}{\hbar} = -\mu_B \boldsymbol{\sigma}. \qquad \text{(spin magnetic moment)}$$

where $S/\hbar = 1/2$, and $\boldsymbol{\sigma}$ is the Pauli operator. Thus, the minimum magnetic moment from spin is still μ_B. The Zeeman energy of the spin magnetic moment in the presence of magnetic field $\boldsymbol{B}$ is given by

$$\hat{H} = -\hat{\boldsymbol{\mu}}_S \cdot \mathbf{B} = \frac{2\mu_B}{\hbar}\hat{\mathbf{S}}\cdot\mathbf{B},$$

with

$$\boldsymbol{\mu}_S = -\frac{2\mu_B}{\hbar}\mathbf{S}.$$

We apply the inhomogeneous magnetic field along the z axis. The force is produced as

$$F_z = \mu_z \frac{\partial B_z}{\partial z},$$

(we will discuss below). Such a force is used for the experiment of Stern-Gerlach experiment with spin $S = 1/2$.

11. Total magnetic moment and total angular momentum

The total magnetic moment is a sum of the orbital magnetic moment and the spin magnetic moment.

$$\boldsymbol{\mu}_J = -\frac{\mu_B}{\hbar}(\mathbf{L} + 2\mathbf{S}).$$

where we use $g_e = 2$. The total angular momentum is

$$\mathbf{J} = \mathbf{L} + \mathbf{S}.$$

We note that

$$\begin{aligned}\boldsymbol{\mu}_J \cdot \mathbf{J} &= -\frac{\mu_B}{\hbar}(\mathbf{L} + 2\mathbf{S}) \cdot (\mathbf{L} + \mathbf{S}) \\ &= -\frac{\mu_B}{\hbar}(\mathbf{L}^2 + 2\mathbf{S}^2 + 3\mathbf{L} \cdot \mathbf{S}) \\ &= -g_J \frac{\mu_B}{\hbar}\mathbf{J}^2.\end{aligned}$$

and

$$\mathbf{J}^2 = (\mathbf{L} + \mathbf{S})^2 = \mathbf{L}^2 + \mathbf{S}^2 + 2\mathbf{L} \cdot \mathbf{S},$$

or

$$\mathbf{L} \cdot \mathbf{S} = \frac{1}{2}[\mathbf{J}^2 - (\mathbf{L}^2 + \mathbf{S}^2)],$$

where

$$\boldsymbol{\mu}_J = -g_J \frac{\mu_B}{\hbar} \mathbf{J} \,.$$

We define the Landé g-factor, g_J as

$$\begin{aligned}
-g_J \frac{\mu_B}{\hbar} \mathbf{J}^2 &= -\frac{\mu_B}{\hbar} (\mathbf{L} + 2\mathbf{S}) \cdot (\mathbf{L} + \mathbf{S}) \\
&= -\frac{\mu_B}{\hbar} (\mathbf{L}^2 + 2\mathbf{S}^2 + 3\mathbf{L} \cdot \mathbf{S}) \\
&= -\frac{\mu_B}{\hbar} \{\mathbf{L}^2 + 2\mathbf{S}^2 + \frac{3}{2}[\mathbf{J}^2 - (\mathbf{L}^2 + \mathbf{S}^2)]\} \\
&= -\frac{\mu_B}{\hbar} [\frac{1}{2}(-\mathbf{L}^2 + \mathbf{S}^2) + \frac{3}{2}\mathbf{J}^2]
\end{aligned}$$

$$g_J = \frac{3}{2} + \frac{\mathbf{S}^2 - \mathbf{L}^2}{2\mathbf{J}^2} = \frac{3}{2} + \frac{s(s+1) - l(l+1)}{2j(j+1)} \,.$$

We use the Dirac notation for the state $\left| j, m \right\rangle$, where $j = l + s, l + s - 1,$ ...., $\left| l - s \right|$ and $m = j, j-1, j-2, ..., -j$

with

$$\hat{\mathbf{J}}^2 \left| j, m \right\rangle = \hbar^2 j(j+1) \left| j, m \right\rangle, \qquad \hat{J}_z \left| j, m \right\rangle = m\hbar \left| j, m \right\rangle,$$

where $l = 0, 1, 2, 3...$ and s=1/2, 3/2,….

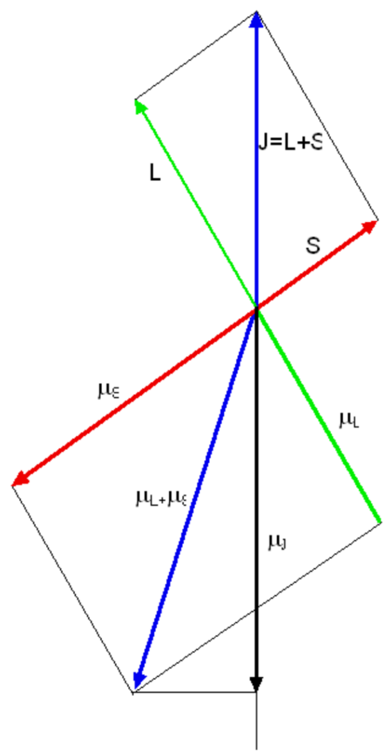


Fig.14 Total angular momentum (***J***) and total magnetic moment ($\boldsymbol{\mu}_J$).

8. Torque on a current-carrying loop (Young and Friedman)

We now discuss a torque on rectangular current loop in the presence of a uniform magnetic field ***B*** from a viewpoint of classical physics. A current loop—or magnetic moment—not only produces magnetic fields, but will also experience forces when placed in the magnetic field of other currents. We will look first at the forces on a rectangular current loop in a uniform magnetic field. Let the *z*-axis be along the direction of the field, and the plane of the loop be placed through the *y*-axis, making the angle θ with the *x-y* plane as in **Fig.15.** Then, the magnetic moment of the loop—which is normal to its plane—will make the angle θ with the magnetic field.

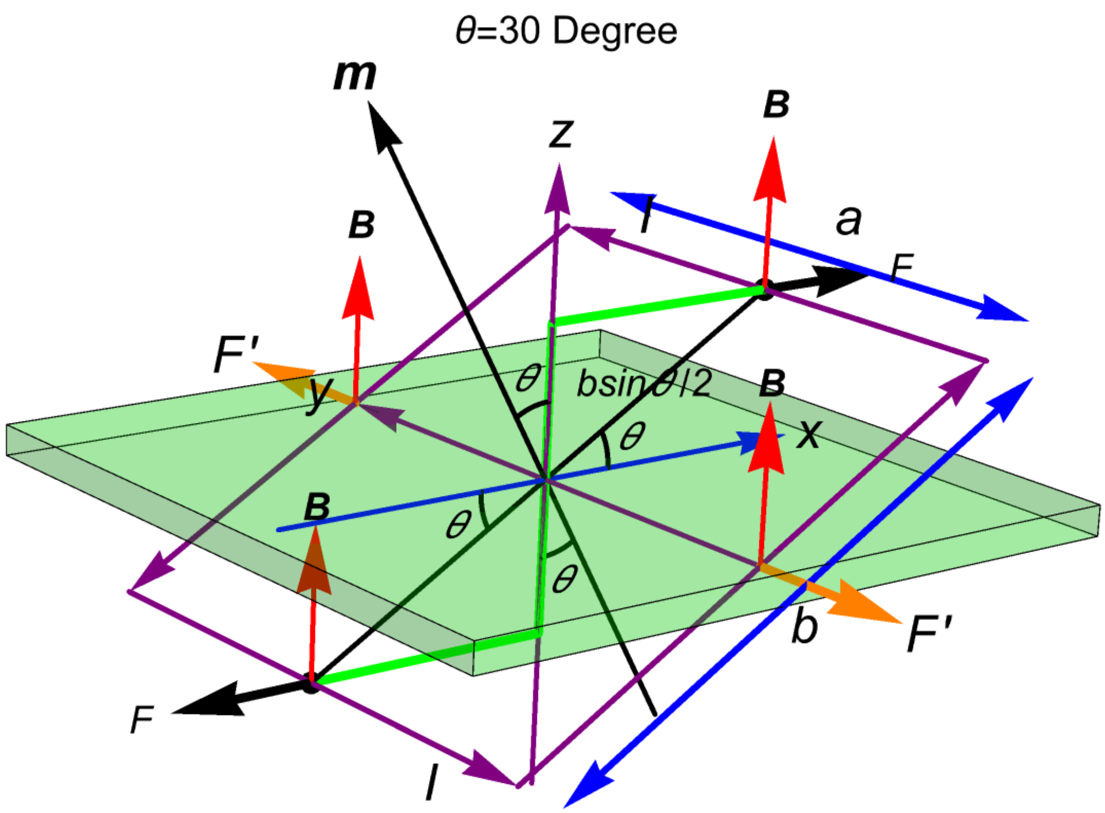


Fig.15 Finding the torque on the loop current carrying loop in a uniform magnetic field. The two pairs of forces acting on the loop (the *b* sides) cancel, so no net force acts on the loop. However, the forces on the *a* sides of the loop produces a torque. θ is the angle between a vector normal to the loop and the magnetic field. We choose $\theta = 30°$ in this figure. The area of rectangular current loop is $a_{cl} = ab$. The magnitude of the magnetic moment is denoted by $m = Ia_{cl}$.

Here we have a rectangular loop of wire with side lengths *a* and *b*. The total force on the loop is zero but that there can be a net torque acting on the loop, with some interesting properties. A line perpendicular to the plane of the loop (i.e., a normal to the plane) makes an angle θ with the direction of the magnetic field ***B***, and the loop carriers a current *I*. The wires leading the current into and out of the loop and the source of emf are omitted to keep the diagram simple. The force **F** on the right side of the loop (length *a*) is to the right, in the +*x*-direction. On this side, ***B*** is perpendicular to the current direction and the force on this side has magnitude

$$F = IaB .$$

A force $-\mathbf{F}$ with the same magnitude but opposite direction acts on the opposite side of the loop. The sides with length b make an angle $(\pi/2-\theta)$ with the direction of $\mathbf{B}$. The forces on these sides are the vectors $\mathbf{F}'$ and $-\mathbf{F}'$; their magnitude F' is given by

$$F' = IbB\sin(\frac{\pi}{2}-\theta) = IbB\cos\theta .$$

The lines of action of both forces lie along the *y*-axis. The total force on the loop is zero because the forces on opposite sides cancel out in pairs. The net force on a current loop in a uniform magnetic field is zero. However, the net torque is not in general equal to zero.

The two forces $\mathbf{F}'$ and $-\mathbf{F}'$ lie along the same line and so give rise to zero net torque with respect to any point. The two forces $\mathbf{F}$ and $-\mathbf{F}$ lie along different lines, and each gives rise to a torque about the *y*-axis. According to the right-hand rule for determining the direction of torques, the vector torques due to $\mathbf{F}$ and $-\mathbf{F}$ are both in the $\pm y$-direction; hence the net vector torque $\boldsymbol{\tau}$ is in the $+y$-direction as well. The moment arm for each of these forces (equal to the perpendicular distance from the rotation axis to the line of the action of force) is $(b/2)\sin\theta$. So, the torque due to each force has magnitude $F(b/2)\sin\theta$. Thus, the torque is in a clockwise direction. Note that the magnitude of net torque is obtained as

$$\begin{aligned}\tau &= \frac{Fb\ \sin\theta}{2}+\frac{Fb\ \sin\theta}{2}\\ &= Fb\ \sin\theta\\ &= IabB\sin\theta\\ &= mB\sin\theta\\ &= (\mathbf{m}\times\mathbf{B})_y\end{aligned}$$

The torque is the greatest when $\theta=\pi/2$. $\boldsymbol{B}$ is in the plane of the loop, and the normal to this plane is perpendicular to $\boldsymbol{B}$. The torque is zero when θ is 0 or π and the normal to the loop is parallel or antiparallel to the field. The value $\theta = 0$ is a stable equilibrium position because the torque is zero there, and when the loop is rotated slightly from this position, the resulting torque tends to rotate it back toward $\theta = 0$. The position $\theta = \pi$ is an unstable equilibrium position; if displaced slightly from this position, the loop tends to move farther away from $\theta = \pi$.

The area of the loop is equal to $a_{cl} = ab$. So, we get the torque

$$\tau = (Iab)B\sin\theta . \qquad \text{(magnitude of torque on a current loop).}$$

((**Right-hand rule**))

The product Iab is called the magnetic moment of the current loop, for which we use the symbol m (or the Greek letter μ in the quantum mechanics).

$$m = Iab$$

The right-hand rule determines the direction of the magnetic moment of a current-carrying loop.

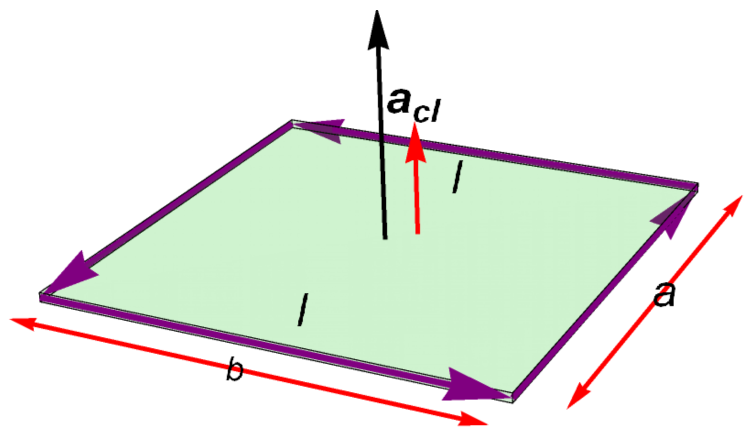


Fig.16 The right-hand rule determines the direction of the magnetic moment of a current-carrying loop. This is also the direction of the loop's area vector $\mathbf{a}_{cl}$; $\mathbf{m} = I\mathbf{a}_{cl}$ is a vector equation. $a_{cl} = ab$

((**Note**)) **Mathematica**

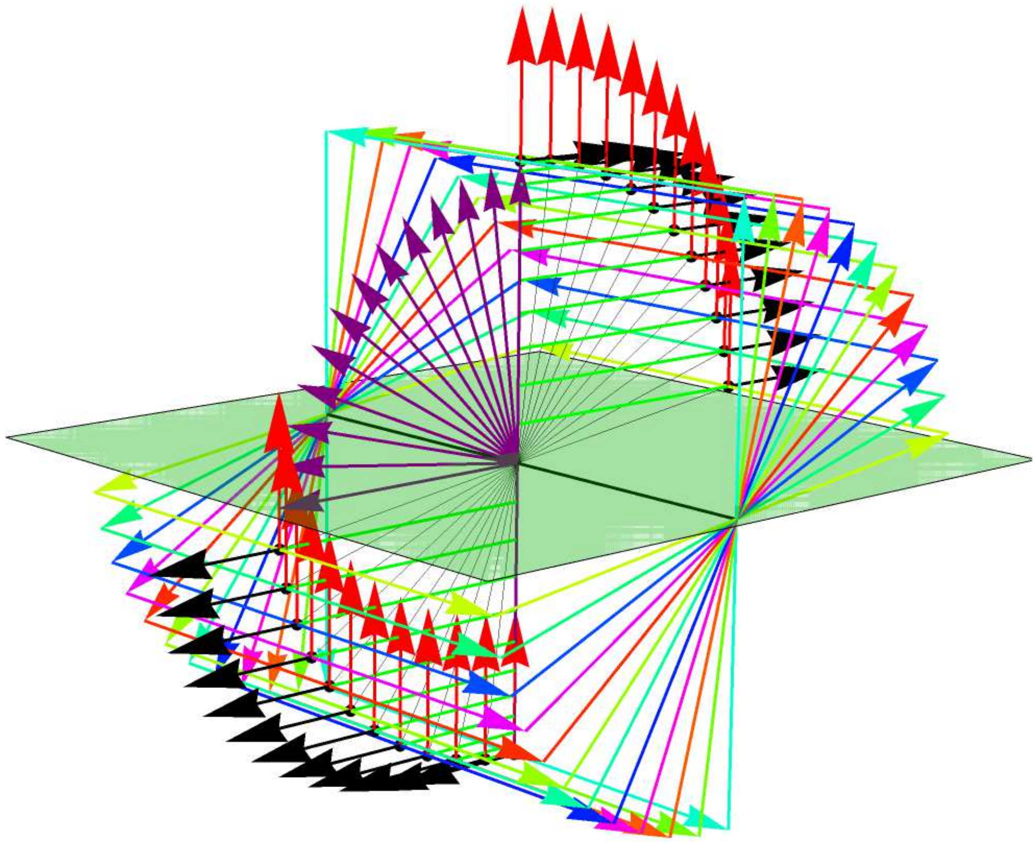
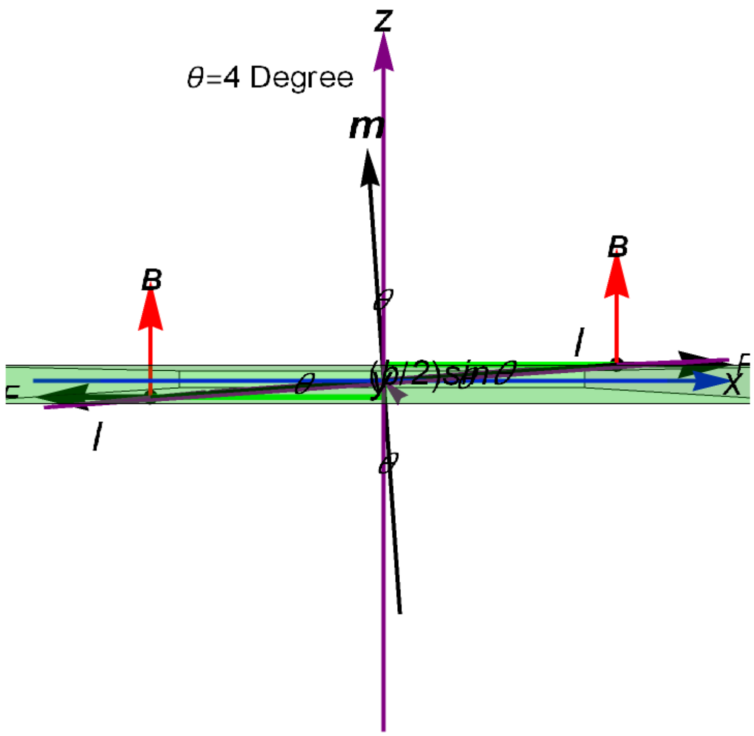
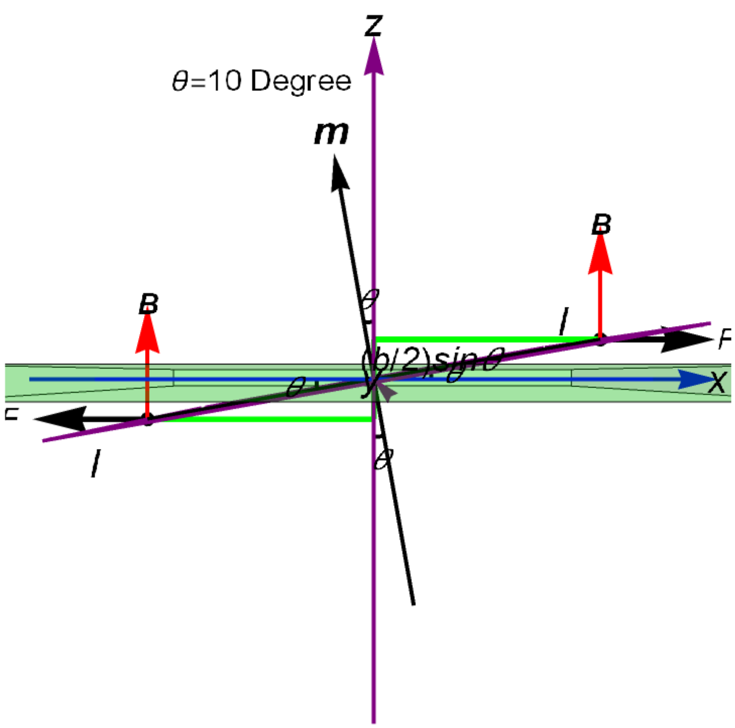


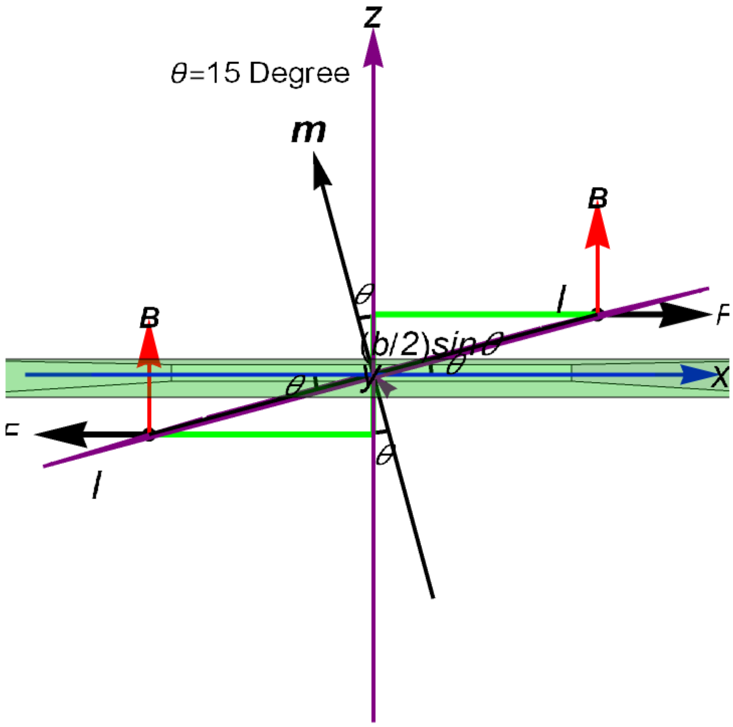
Fig.17 Torque on a current-carrying loop in a uniform field with various angle θ being changed as a parameter.



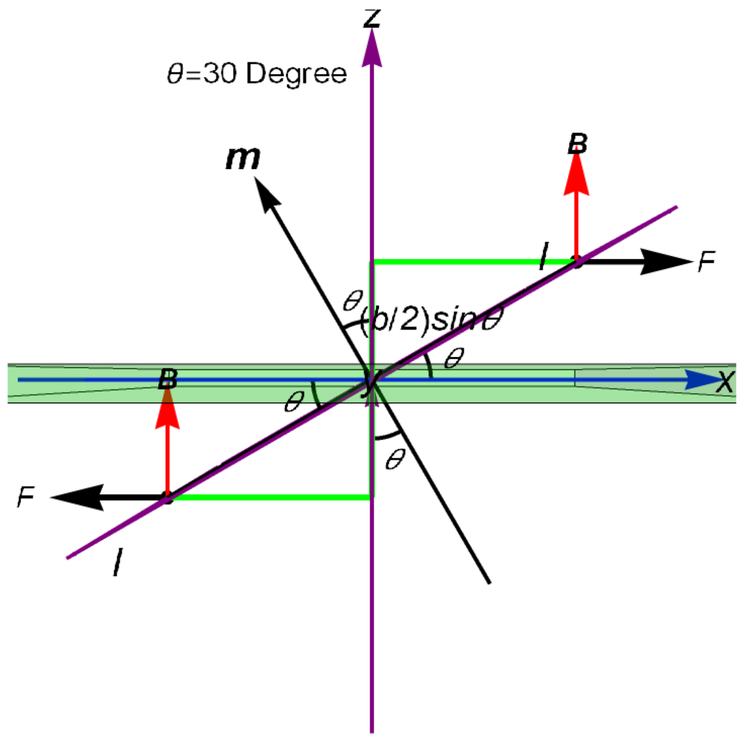
(a) $\theta = 4^{\circ}$



(b) $\theta = 10^{\circ}$.

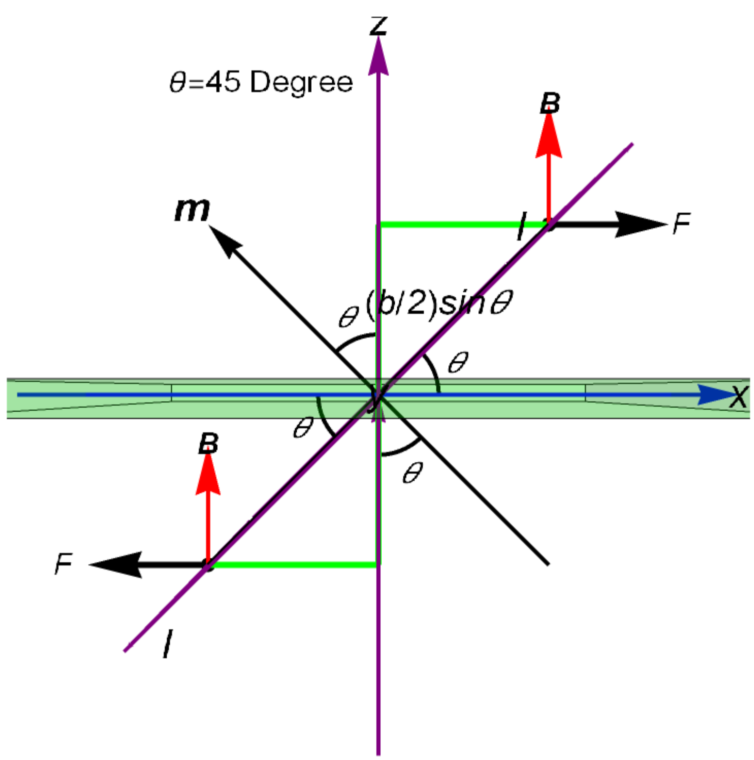


(c) $\theta = 15°$.



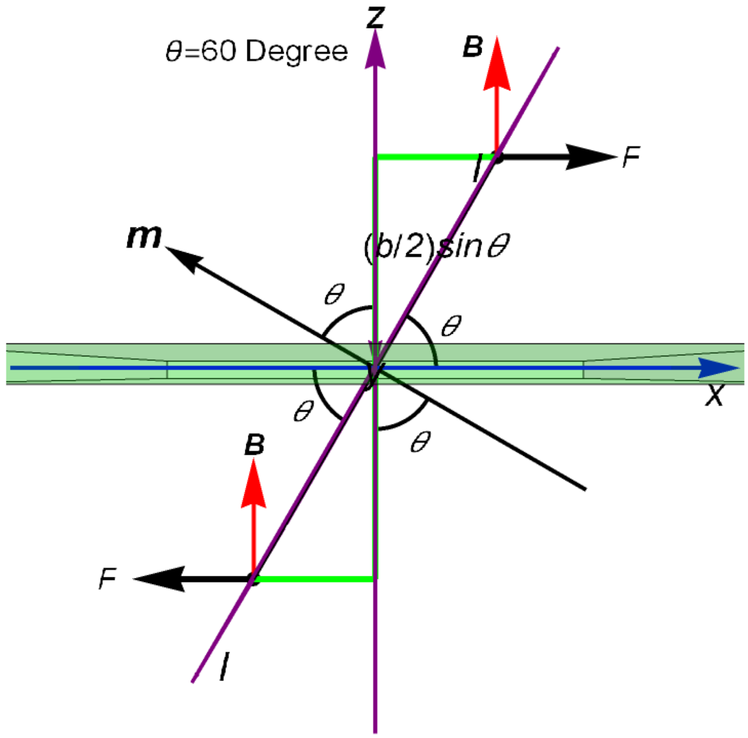
z
θ=30 Degree
m
B
I
F
θ
(b/2)sinθ
θ
B
X
θ
θ
F
I

(d) $\theta = 30°$.

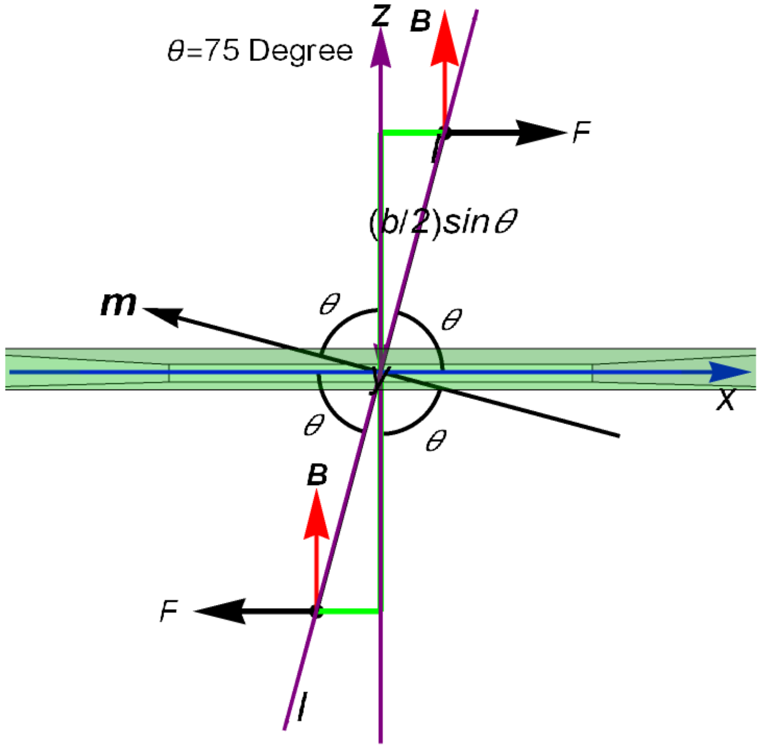


z
θ=45 Degree
B
m
I
F
θ
(b/2)sinθ
θ
X
θ
B
θ
F
I

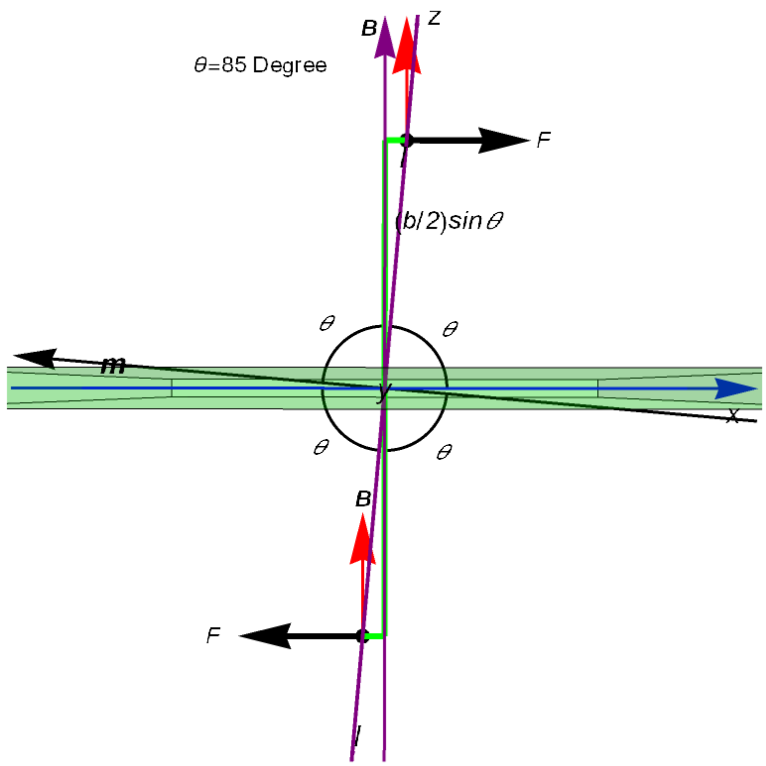
(e) $\theta = 45°$.



(f) $\theta = 60°$.


θ=75 Degree
Z
B
F
l
(b/2)sinθ
m
θ
θ
y
x
θ
θ
B
F
l

(g) $\theta = 75°$.


B
Z
θ=85 Degree
F
l
(b/2)sinθ
θ
θ
m
y
x
θ
θ
B
F
l

(h) $\theta = 85°$.

Fig.18 Torque on a current-carrying loop in a uniform field with various angle θ. (a) $\theta = 4°$, (b) $\theta = 10°$, (c) $\theta = 15°$, (d) $\theta = 30°$, (e) $\theta = 45°$, (f) $\theta = 60°$, (g) $\theta = 75°$, and (h) $\theta = 85°$

9. Work energy theorem for potential energy

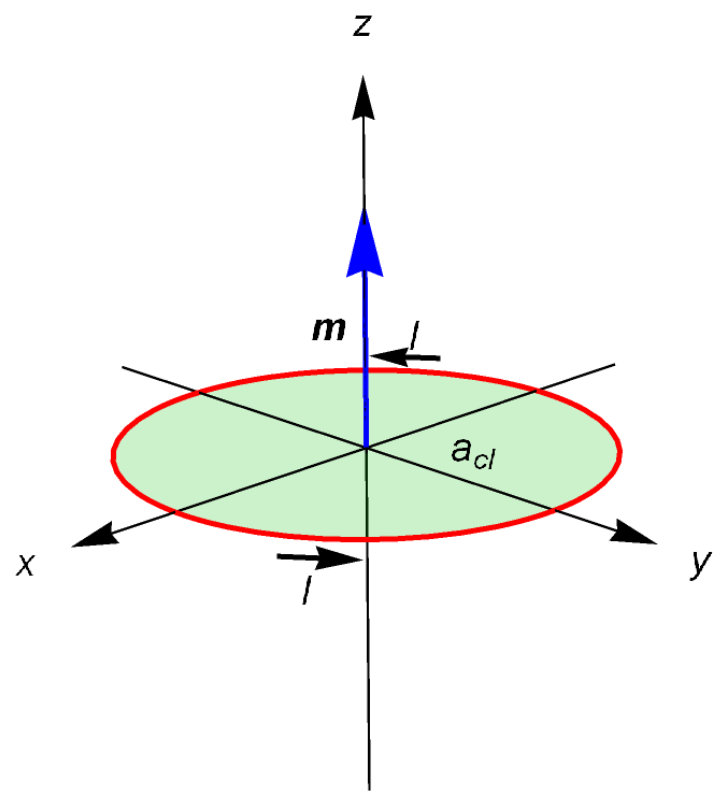


Fig.19 Magnetic moment vector ***m*** is related to the current by a right-hand rule. $m = Ia_{cl}$ (SI units). where a_{cl} is the area of circle (current loop).

The right-hand rule determines the direction of the magnetic moment $\mathbf{m}$ of a current-carrying loop. This is also the direction of the loop's area vector $\mathbf{a}_{cl}$;

$$\mathbf{m} = I\mathbf{a}_{cl},$$

is a vector equation. We can also define a vector magnetic moment $\mathbf{m}$ with magnitude Ia_{cl}. The direction of $\mathbf{m}$ is defined to be perpendicular to the plane of the current loop. with a sense determined by a right-hand rule. Wrap the fingers of your right hand around the perimeter of the current loop in the direction of the current. Then extend your thumb so that it is perpendicular to the plane of the loop; its direction is the direction of $\mathbf{m}$ (and of the vector area $\mathbf{a}_{cl}$ of the loop). The torque is the greatest when $\mathbf{m}$ and $\mathbf{B}$ are perpendicular and is zero when they are parallel or antiparallel. In the stable equilibrium position, $\mathbf{m}$ and $\mathbf{B}$ are parallel. The torque $\boldsymbol{\tau}$ can be expressed by

$$\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}. \qquad \text{(vector torque on a current loop)}$$

The direction of the magnetic moment is normal to the surface of the rectangular current loop. We use the work-energy theorem.

$$\Delta K = W = -\Delta U .$$

In this system, the work is expressed by

$$W = \int_0^\theta (-\tau) d\theta = -\int_0^\theta mB \sin\theta d\theta = mB\cos\theta ,$$

where the negative sign $(-\tau)$ indicates the direction of the torque is in counterclockwise. Note that $d\theta$ is in the counterclockwise. Then, the potential energy is derived as

$$U = -mB\cos\theta = -\mathbf{m}\cdot\mathbf{B} .$$

This result is directly analogous to the result we found for the torque exerted by an electric field $\mathbf{E}$ on an electric dipole with dipole moment $\mathbf{p}$.

When a magnetic dipole changes orientation in a magnetic field, the field does work on it. In an infinitesimal angular dependence $d\theta$, the work dW is given by $\tau d\theta$, and there is a corresponding change in potential energy. As the above discussion suggests, the potential energy is least when $\mathbf{m}$ and $\mathbf{B}$ are parallel and greatest when they are antiparallel. To find an expression for the potential energy U as a function of orientation, we can make use of the beautiful symmetry between the electric and magnetic dipole interactions. The torque on a magnetic dipole in a magnetic field is

$$\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B} .$$

So we can conclude that the corresponding energy is

$$U = -\mathbf{m}\cdot\mathbf{B} = -mB\cos\theta .$$

With this definition, U is zero when the magnetic dipole moment is perpendicular to the magnetic field.

10. Torque on electric dipole moment in the presence of electric field

The electric dipole is defined as

$$p = ql.$$

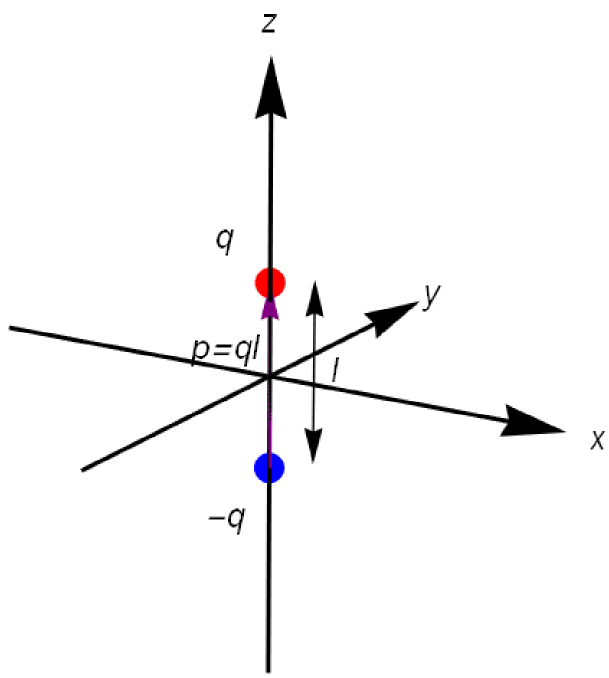


Fig.20 Electric dipole moment. $p = ql$. The direction of the electric dipole moment is along the z axis. $q > 0$.

The torque on electric dipole moment in the presence of uniform electric field is

$$\text{Torque} = 2qE\frac{d}{2}\sin\theta = qEd\sin\theta \qquad \text{(counterclockwise)}$$

The work energy theorem:

$$\Delta K = W = -\Delta U$$

$$W = -\int Eqd\sin\theta d\theta = Eqd\cos\theta$$

Then, the potential energy is

$$U = -W = -Eqd\cos\theta = -\mathbf{p}\cdot\mathbf{E}.$$

The torque is expressed by

$$\boldsymbol{\tau} = \mathbf{p}\times\mathbf{E} = pE\sin\theta(-\mathbf{e}_y).$$

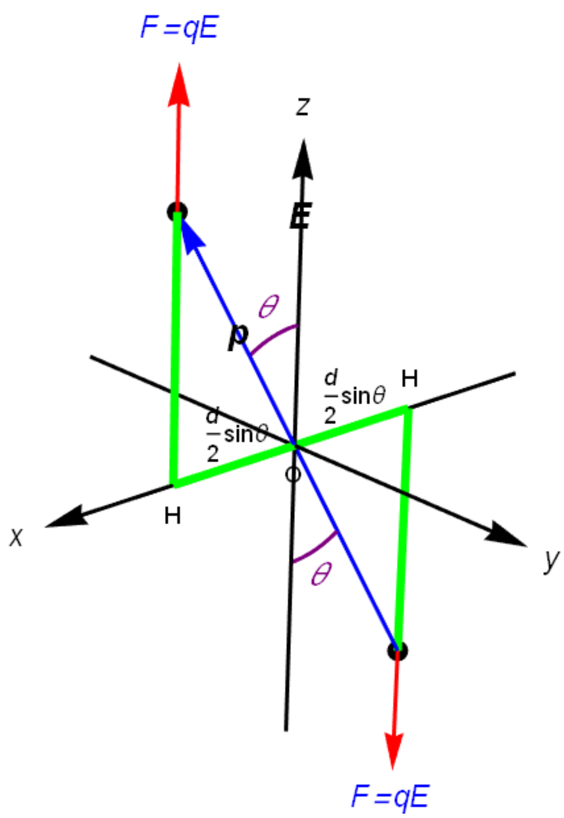


Fig.21 Torque of electric dipole moment in the presence of constant electric field ***E*** along the *z* axis.

11. The magnetic scalar potential $\phi_m{}^* = \dfrac{m\cos\theta}{4\pi r^2}$

$$\mathbf{E} = \frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\mathbf{e}_r + \sin\theta\mathbf{e}_\theta),$$

or

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}[-\frac{\mathbf{p}}{r^3} + \frac{3(\mathbf{p}\cdot\mathbf{r})\mathbf{r}}{r^5}].$$

This field can be expressed in terms of the electric potential V_P as

$$\mathbf{E} = -\nabla\phi_e,$$

where

$$\phi_e = \frac{p}{4\pi\varepsilon_0 r^2}\cos\theta = \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\varepsilon_0 r^3}.$$

Using this analogy, we have the magnetic scalar potential as

$$\phi_m^{\ *} = \frac{m\cos\theta}{4\pi r^2}.$$

The magnetic field ***B*** can be evaluated as

$$\begin{aligned}\mathbf{B} &= -\mu_0\nabla\phi_m^{\ *}\\ &= -\mu_0\nabla\frac{m\cos\theta}{4\pi r^2}\\ &= -\frac{\mu_0 m}{4\pi}\nabla\frac{\cos\theta}{r^2}\\ &= -\frac{\mu_0 m}{4\pi}[\mathbf{e}_r\cos\theta\frac{\partial}{\partial r}\frac{1}{r^2} + \mathbf{e}_\theta\frac{\partial}{r^3\partial\theta}\cos\theta]\\ &= \frac{\mu_0 m}{4\pi r^3}(\mathbf{e}_r 2\cos\theta + \mathbf{e}_\theta\sin\theta)\end{aligned}$$

or

$$\mathbf{B} = \frac{\mu_0}{4\pi}[-\frac{\mathbf{m}}{r^3} + \frac{3(\mathbf{m}\cdot\mathbf{r})\mathbf{r}}{r^5}].$$

This equation shows that the magnetic field of a distance loop current does not depend on its detailed geometry, but only on its magnetic moment ***m***. Because $\nabla\times\mathbf{B} = \mu_0\mathbf{J}$ is not a conserved field, and so it makes no sense to introduce a magnetic scalar potential in the same sense as we introduced an electrostatic potential. But $\nabla\times\mathbf{B}=0$ is zero whenever the current density **J** is zero. In this case, the magnetic field B in such a region, B can be written as the gradient of a magnetic scalar potential

$$\mathbf{B}=-\mu_0\nabla\phi_m^*.$$

Since $\nabla\times\mathbf{B}=0$, we get the Laplace equation for ϕ_m^* .The curl of the magnetic induction is zero wherever the current density is zero. When this is the case, the magnetic induction in such regions can be written as the gradient of a scalar potential:

$$\mathbf{B}=-\mu_0\nabla\phi_m^*.$$

However, the divergence of ***B*** is also zero, which means that

$$\nabla\cdot\mathbf{B}=-\mu_0\nabla^2\phi_m^*=0.$$

Thus ϕ_m^* , which is called the magnetic scalar potential, satisfies Laplace's equation. Much of the work of electrostatic can be taken over directly and used to evaluate ϕ_m^* for various situations; however, care must be taken in applying the boundary conditions. The expression for the scalar potential of a magnetic moment is particular useful. In the present case, ϕ_m^* can be obtained as

$$\phi_m^*(\mathbf{r})=\frac{\mathbf{m}\cdot\mathbf{r}}{4\pi r^3}=\frac{mr\cos\theta}{4\pi r^3}=\frac{m\cos\theta}{4\pi r^2}.$$

((**Note**)) **The expression of $\phi_m^*(\mathbf{r})$ in terms of the solid angle Ω**

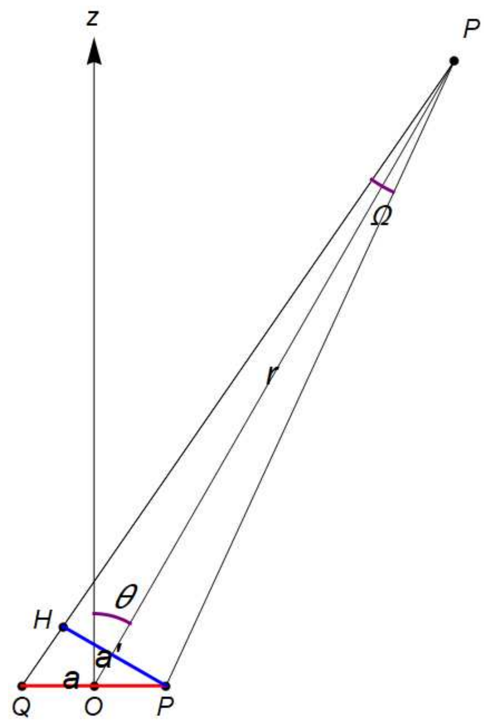


Fig.22 Solid angle $d\Omega$**;** $r^2 d\Omega = a_{cl}' = a_{cl}\cos\theta$

We note that

$$a_{cl}' = a_{cl}\cos\theta ,$$

$$r^2 d\Omega = a_{cl}' = a_{cl}\cos\theta .$$

Thus, we get the expression
as

$$\phi_m = \frac{I d\Omega}{4\pi} = \frac{I a_{cl}\cos\theta}{4\pi r^2} = \frac{m\cos\theta}{4\pi r^2} ,$$

with $m = I a_{cl}$.

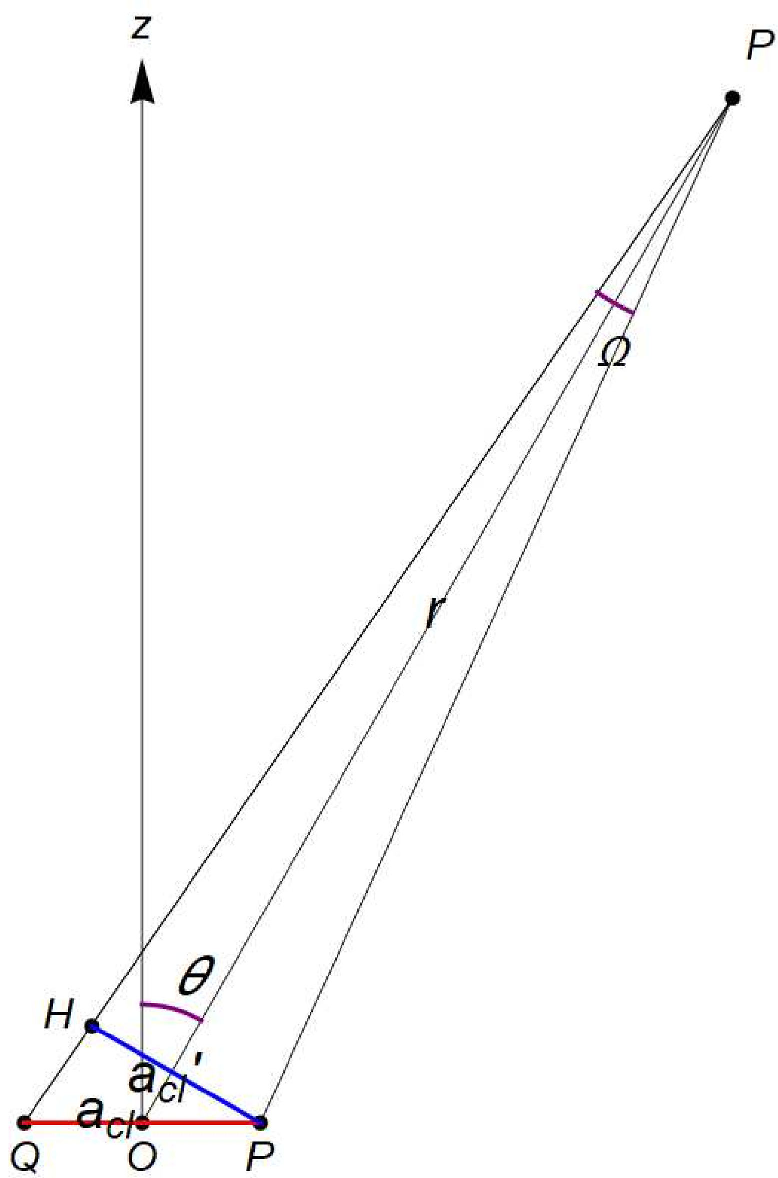


Fig.23 Magnetic scalar potential from a loop current (current I) with area a_{cl}. The magnetic moment is $m = Ia_{cl}$. Ω is the solid angle.

12. The discussion of the total mechanical energy (Feynman, Leighton, Sands)

We can show for our rectangular loop that U also corresponds to the mechanical work done in bringing the loop into the field. The total force on the loop is zero only in a uniform field; in a non-uniform field there *are* net forces on a current loop. In putting the loop into a region with a field, we must have gone through places where the field was not uniform, and so work was done. To make the calculation simple, we shall imagine that the loop is brought into the field with its moment pointing along the field. (It can be rotated to its final position after it is in place.) Imagine that we want to move the loop in the x-direction—

toward a region of stronger field—and that the loop is oriented as shown in **Fig.24**. We start somewhere where the field is zero and integrate the force times the distance as we bring the loop into the field.

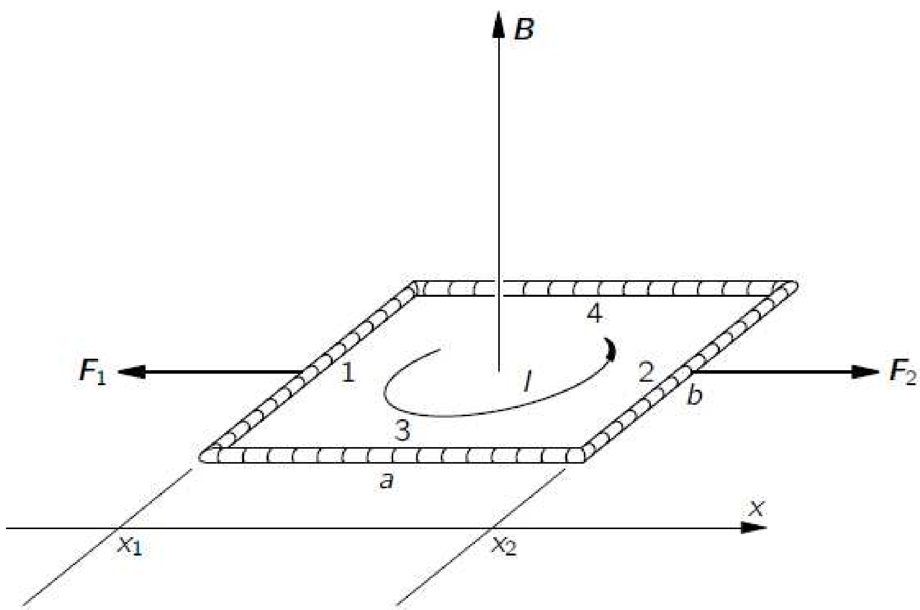


Fig.24 A loop is carried along the x- direction through the field $\boldsymbol{B}$, at right angles to x (**Feynman, Leighton, Sands**).

First, let's compute the work done on each side separately and then take the sum (rather than adding the forces before integrating). The forces on sides 3 and 4 are at right angles to the direction of motion, so no work is done on them. The force on side 2 is $IbB(x)$ in the x-direction, and to get the work done against the magnetic forces we must integrate this from some x where the field is zero, say at $x=-\infty$, to x_2, its present position:

$$W_2 = \int_{-\infty}^{x_2} F_2 dx = Ib \int_{-\infty}^{x_2} B(x) dx . \qquad (1)$$

Similarly, the work done against the forces on side 1 is

$$W_1 = -\int_{-\infty}^{x_2} F_1 dx = -Ib \int_{-\infty}^{x_2} B(x) dx . \qquad (2)$$

To find each integral, we need to know how $B(x)$ depends on x. But notice that side 1 follows along right behind side 2, so that its integral includes most of the work done on side 2. In fact, the sum of Eq.(1) and Eq.(2) is just

$$\Delta W = W_1 + W_2 = Ib\int_{x_1}^{x_2} B(x)dx \,. \tag{3}$$

But if we are in a region where B is nearly the same on both sides 1 and 2, we can write the integral as

$$\int_{x_1}^{x_2} B(x)dx = (x_2 - x_1)B = aB \,,$$

where B is the field at the center of the loop. The total mechanical energy we have put in is

$$U = -\Delta W = -IabB = -mB$$

using the work-energy theorem, where B is the field at the center of the loop.

We would, of course, have gotten the same result if we had added the forces on the loop before integrating to find the work. If we let B_1 be the field at side 1 and B_2 be the field at side 2, then the total force in the x-direction is

$$F_x = Ib(B_2 - B_1) \,.$$

If the loop is small, that is, if B_2 and B_1 are not too different, we can write

$$B_2 = B_1 + \frac{\partial B}{\partial x}a \,.$$

by using the Taylor expansion. So, the force is obtained as

$$F_x = Iab\frac{\partial B}{\partial x} = m\frac{\partial B}{\partial x} \,.$$

13. Discussion on force (Purcell and Morin)

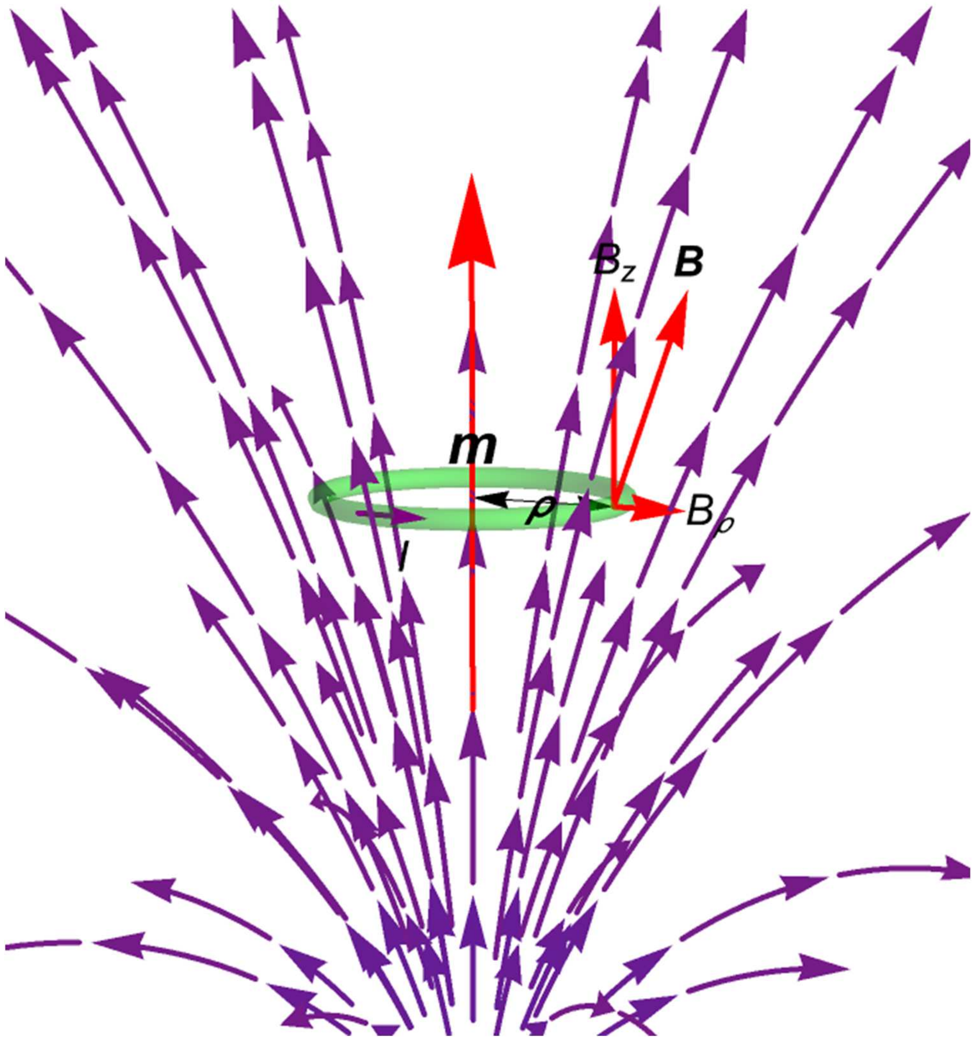


Fig.25 StreamPlot3D (Mathematica, Version 13). A current ring in an inhomogeneous magnetic field. (The field of the ring itself is not shown). Because of the radial component of the field, B_ρ, there is a force on the ring as a whole (based on the textbook of Purcell and Morin).

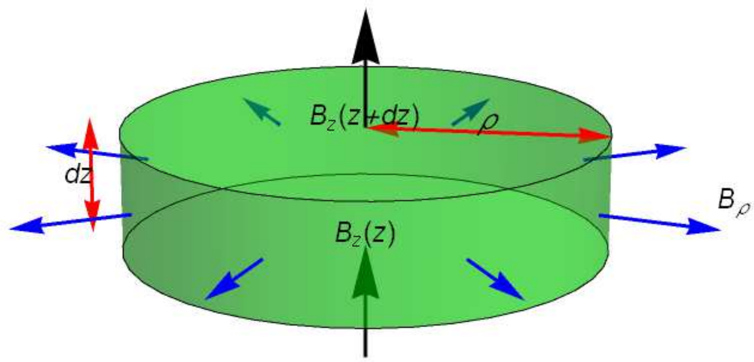


Fig.26 Gauss' theorem can be used to relate B_ρ and $\frac{\partial B_z}{\partial z}$, leading to the relation $B_\rho = -\frac{\rho}{2}\frac{\partial}{\partial z}B_z(z)$. (**Purcell and Morin**).

The force exerted on the ring current I is

$$\begin{aligned} d\mathbf{F} &= I\rho d\phi[\mathbf{e}_\phi \times (B_z\mathbf{e}_z + B_\rho\mathbf{e}_\rho) \\ &= I\rho d\phi(B_z\mathbf{e}_\rho - B_\rho\mathbf{e}_z) \end{aligned}$$

using the cylindrical coordinates. We note that $\nabla\cdot\mathbf{B} = 0$. This can be done by

$$\frac{\partial}{\partial\rho}(\rho B_\rho) + \rho\frac{\partial}{\partial z}B_z(z) = 0$$

or

$$B_\rho = -\frac{\rho}{2}\frac{\partial}{\partial z}B_z(z).$$

where we assume that $B_z(z)$ depends only on z. This can be also derived form the Gauss's law, applied on a pancake-like cylinder. We have

$$0 = \int(\nabla\cdot\mathbf{B})d^3\mathbf{r} = \int\mathbf{B}\cdot d\mathbf{a} = \pi\rho^2[B_z(z+dz) - B_z(z)] + 2\pi\rho dz B_\rho.$$

Since

$$B_z(z+dz)-B_z(z)=dz\frac{\partial B_z(z)}{\partial z}, \qquad \text{(Taylor expansion)}$$

we get

$$B_\rho=-\frac{\rho}{2}\frac{\partial B_z(z)}{\partial z}.$$

The resultant force is directed along the z axis

$$\begin{aligned}F_z&=-I\rho(2\pi)B_\rho\\&=-I\rho(2\pi)\left(-\frac{\rho}{2}\frac{\partial B_z(z)}{\partial z}\right)\\&=I\pi\rho^2\frac{\partial B_z(z)}{\partial z}\\&=m\frac{\partial B_z(z)}{\partial z}\end{aligned}$$

Note that because of the symmetry, the resultant force in the x-y plane is cancelled out. In general, this can be rewritten as

$$\mathbf{F}=-\nabla U=-\nabla(-\mathbf{m}\cdot\mathbf{B})=\nabla(\mathbf{m}\cdot\mathbf{B}),$$

and

$$U=-\mathbf{m}\cdot\mathbf{B}.$$

14. Calculation of the force $\mathbf{F}=\nabla(\mathbf{m}\cdot\mathbf{B})$ by using Mathematica

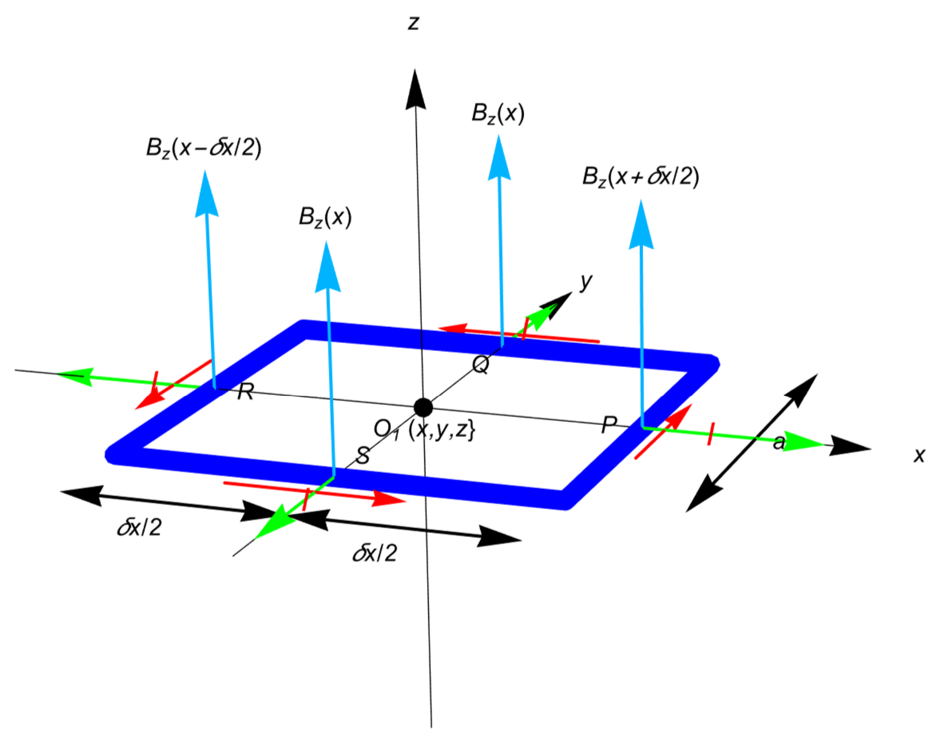


Fig.27 A rectangular loop of wire with the current *I*. The magnetic field is applied along the *z* axis. $\mathbf{B} = \mathbf{e}_z B_z(x)$.

We consider a rectangular loop of wire with current *I*. The area of the square wire is $a_{cl} = a\delta x$. The magnetic moment is given by $m = Ia_{cl} = Ia\delta x$. Suppose that the magnetic field (which depends on the position coordinate *x*) is applied along the *z* axis; $\mathbf{B} = \mathbf{e}_z B_z(x)$. We find that the net force on the top side (Q) and bottom side (S) of the rectangular loop is zero. The force on the side (Q) cancels the force on the side (S). The force on the right side (P) is given by

$$\mathbf{F}(P) = (\mathbf{e}_x) aIB_z(x + \frac{\delta x}{2}),$$

while the force on the left side (R) is given by

$$\mathbf{F}(R) = (-\mathbf{e}_x) aIB_z(x - \frac{\delta x}{2}).$$

The net force exerted on the rectangular loop of the wire with current I is directed along the x axis,

$$\begin{aligned} F_x &= aI[B_z(x+\frac{\delta x}{2}) - B_z(x-\frac{\delta x}{2})] \\ &= aI\delta x \frac{\partial}{\partial x} B_z(x) \\ &= m\frac{\partial}{\partial x} B_z(x) \end{aligned}$$

with the magnetic moment m

$$m = Ia\delta x ,$$

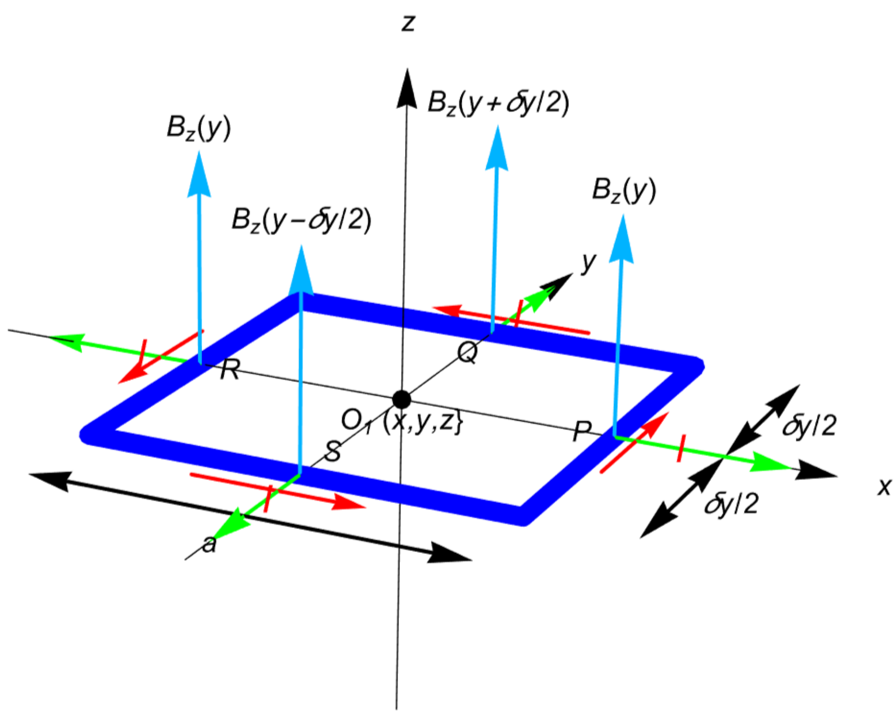


Fig.28 A rectangular loop of wire with the current I. The magnetic field is applied along the z axis. $\mathbf{B} = \mathbf{e}_z B_z(y)$.

We consider a rectangular loop of wire with current I. The area of the square wire is $a_{cl} = a\delta y$. The magnetic moment is given by $m = Ia_{cl} = Ia\delta y$. Suppose that the magnetic field (which depends on the position coordinate y) is applied along the z axis; $\mathbf{B} = \mathbf{e}_z B_z(y)$. We find that the net force on the right side (P) and left side (R) is zero. The force on the right side (P) cancels the force on the left side (R). The force on the top side (Q) is given by

$$\mathbf{F}(Q) = (\mathbf{e}_y) aIB_z(y + \frac{\delta y}{2}),$$

while the force on the bottom side (S) is given by

$$\mathbf{F}(R) = (-\mathbf{e}_y) aIB_z(y - \frac{\delta y}{2}).$$

The net force exerted on the rectangular loop of the wire with current I is directed along the y axis,

$$\begin{aligned} F_y &= aI[B_z(y + \frac{\delta y}{2}) - B_z(y - \frac{\delta y}{2})] \\ &= aI\delta y \frac{\partial}{\partial y} B_z(y) \\ &= m \frac{\partial}{\partial y} B_z(y) \end{aligned}$$

with the magnetic moment m

$$m = Ia\delta y.$$

15. Feynman subscript notations (Mathematica)

(a)

We start with the vector identity.

$$\begin{aligned} \nabla(\mathbf{m}\cdot\mathbf{B}) &= [(\mathbf{m}\cdot\nabla)\mathbf{B} + \mathbf{m}\times(\nabla\times\mathbf{B})] + [(\mathbf{B}\cdot\nabla)\mathbf{m} + \mathbf{B}\times(\nabla\times\mathbf{m})] \\ &= \nabla_{\mathbf{B}}(\mathbf{m}\cdot\mathbf{B}) + \nabla_{\mathbf{m}}(\mathbf{m}\cdot\mathbf{B}) \end{aligned}$$

Here, the *Feynman subscript notations* are defined as

$$\nabla_{\mathbf{B}}(\mathbf{m}\cdot\mathbf{B}) = (\mathbf{m}\cdot\nabla)\mathbf{B} + \mathbf{m}\times(\nabla\times\mathbf{B}),$$

and

$$\nabla_{\mathbf{m}}(\mathbf{m}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{m} + \mathbf{B}\times(\nabla\times\mathbf{m}) .$$

We realize that one may have some difficulty in calculating the terms such as $(\mathbf{m}\cdot\nabla)\mathbf{B}$ and $(\mathbf{B}\cdot\nabla)\mathbf{m}$. Unlike the differential operators (div, grad, ∇^2, curl), these types of operators are not available in the Mathematica. Here, we define the Feynman subscript notations in the Mathematica program.

First, we need to define the operator such that

$$(\mathbf{A}\cdot\nabla)\mathbf{B} ,$$

where ***A*** and ***B*** are vectors defined by

$$\mathbf{A} = (A_x(x,y,z), A_y(x,y,z), A_z(x,y,z)) ,$$

$$\mathbf{B} = (B_x(x,y,z), B_y(x,y,z), B_z(x,y,z)) ,$$

in the Cartesian coordinates. The operator $(\mathbf{A}\cdot\nabla)\mathbf{B}$ (which is vector) can be expressed

$$\begin{aligned}(\mathbf{A}\cdot\nabla)\mathbf{B} &= (A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z})\mathbf{B} \\ &= (A_x\frac{\partial}{\partial x})\mathbf{B} + (A_y\frac{\partial}{\partial y})\mathbf{B} + (A_z\frac{\partial}{\partial z})\mathbf{B}\end{aligned}$$

Correspondingly, we define by a differential operator in the Mathematica,

$Pd[\mathbf{A}\_,\mathbf{B}\_] := (A[[1]]\ D[\mathbf{B},x] + A[[2]]\ D[\mathbf{B},y] + A[[3]]\ \mathrm{D}[\mathbf{B},\mathrm{z}])$

where

$A[[1]] = \mathrm{A}_x[x,y,z]$, $A[[2]] = \mathrm{A}_y[x,y,z]$, $A[[3]] = \mathrm{A}_z[x,y,z]$

If you want to calculate $(\mathbf{A}\cdot\nabla)\mathbf{B}$ for the given forms of **A** and ***B***, we can use the replacement using Function such that

$$ruleA = \{A_x \to Function[\{x, y, z\},\ x^2 yz^2], A_y \to Function[\{x, y, z\},\ 3xyz],$$
$$A_z \to Function[\{x, y, z\},\ x + y + z^2]$$

$$ruleB = \{B_x \to Function[\{x, y, z\},\ x^2 + y - z^2], B_y \to Function[\{x, y, z\},\ 3x + y + z],$$
$$B_z \to Function[\{x, y, z\},\ xy + z^2]$$

Thus, we get

$$Pd[\mathbf{A}, \mathbf{B}] /.ruleA /.ruleB$$

(b)

$$\begin{aligned}\nabla(\mathbf{A}\cdot\mathbf{B}) &= (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}\times(\nabla\times\mathbf{B}) + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{B}\times(\nabla\times\mathbf{A}) \\ &= [(\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}\times(\nabla\times\mathbf{B})] + [(\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{B}\times(\nabla\times\mathbf{A})] \\ &= \mathrm{FIGS}[\mathbf{A}, \mathbf{B}] + \mathrm{FIGS}[\mathbf{B}, \mathbf{A}]\end{aligned}$$

where

$$\mathrm{FIGS}[\mathbf{A}, \mathbf{B}] = (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}\times(\nabla\times\mathbf{B}).$$

$$\mathrm{FIGS}[\mathbf{B}, \mathbf{A}] = (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{B}\times(\nabla\times\mathbf{A}).$$

((**Mathematica**))

Using the Mathematica program made above, we solve typical problems including the Feynman subscript notations.

Feynman identities-1

Grad[A.B]=FIGS[A,B]+FIGS[B,A]

Pd[A, B]=(A·∇)B

Pd[B, A]=(B·∇)A

FIGS[A,B]=(A·∇)B+Ax(∇xB)

FIGS[B,A]=(B·∇)A+Bx(∇xA)

```
Clear["Global`"];
ux = {1, 0, 0};
uy = {0, 1, 0};
uz = {0, 0, 1};
r = {x, y, z};
R = √(r.r) ;
Lap := Laplacian[#, { x, y, z}, "Cartesian"] &;
Gra := Grad[#, { x, y, z}, "Cartesian"] &;
Curla := Curl[#, {x, y,  z}, "Cartesian"] &;
Diva := Div[#, {x, y,  z}, "Cartesian"] &;
Pd[a_, b_] := (a[[1]] × D[b, x] + a[[2]] × D[b, y] + a[[3]] × D[b, z]);
FIGS[a_, b_] := Pd[a, b] + Cross[a, Curla[b]];
```

Definition of vectors m and B;
m1={mx[x,y,z], my[x,y,z], mz[x,y,z]}
B1={Bx[x,y,z], By[x,y,z], Bz[x,y,z]}

```
m1 = {mx[x, y, z], my[x, y, z], mz[x, y, z]};
B1 = {Bx[x, y, z], By[x, y, z], Bz[x, y, z]};
```

```
j11 = FIGS[m1, B1]
```

$$\{mx[x, y, z]\, Bx^{(1,0,0)}[x, y, z] + my[x, y, z]\, By^{(1,0,0)}[x, y, z] + mz[x, y, z]\, Bz^{(1,0,0)}[x, y, z],\ mx[x, y, z]\, Bx^{(0,1,0)}[x, y, z] + my[x, y, z]\, By^{(0,1,0)}[x, y, z] + mz[x, y, z]\, Bz^{(0,1,0)}[x, y, z],\ mx[x, y, z]\, Bx^{(0,0,1)}[x, y, z] + my[x, y, z]\, By^{(0,0,1)}[x, y, z] + mz[x, y, z]\, Bz^{(0,0,1)}[x, y, z]\}$$

```
j12 = FIGS[B1, m1]
```

$$\{Bx[x, y, z]\, mx^{(1,0,0)}[x, y, z] + By[x, y, z]\, my^{(1,0,0)}[x, y, z] + Bz[x, y, z]\, mz^{(1,0,0)}[x, y, z],\ Bx[x, y, z]\, mx^{(0,1,0)}[x, y, z] + By[x, y, z]\, my^{(0,1,0)}[x, y, z] + Bz[x, y, z]\, mz^{(0,1,0)}[x, y, z],\ Bx[x, y, z]\, mx^{(0,0,1)}[x, y, z] + By[x, y, z]\, my^{(0,0,1)}[x, y, z] + Bz[x, y, z]\, mz^{(0,0,1)}[x, y, z]\}$$

```
Gra[m1.B1] - (j11 + j12) // Simplify
```

{0, 0, 0}

(1)

B11=(0,0,Bz(x)); m11={mx1,my1,mz1};

mx1, my1, and mz1 are independent of x,y,z

```
rule1 = {Bx → Function[{x, y, z}, 0], By → Function[{x, y, z}, 0 ],
  Bz → Function[{x, y, z}, Bz[x ]], mx → Function[{x, y, z}, mx1],
  my → Function[{x, y, z}, my1], mz → Function[{x, y, z}, mz1]};
B11 = B1 /. rule1;
m11 = m1 /. rule1;
FIGS[m11, B11]
```

{mz1 Bz′[x], 0, 0}

```
FIGS[B11, m11]
```

{0, 0, 0}

```
Gra[m11.B11]
```

{mz1 Bz′[x], 0, 0}

(2)

B12=(0,0,Bz(y)); m12={mx1,my1,mz1}

```
rule2 = {Bx → Function[{x, y, z}, 0], By → Function[{x, y, z}, 0 ],
  Bz → Function[{x, y, z}, Bz[y ]], mx → Function[{x, y, z}, mx1],
  my → Function[{x, y, z}, my1], mz → Function[{x, y, z}, mz1]};
B12 = B1 /. rule2;
m12 = m1 /. rule2;
```

```
FIGS[m12, B12]
```

{0, mz1 Bz′[y], 0}

```
FIGS[B12, m12]
```

{0, 0, 0}

```
Gra[m12.B12]
```

{0, mz1 Bz′[y], 0}

(3)

B12=(0,0,Bz[z]); m12={mx1,my1,mz1}

```
rule3 = {Bx → Function[{x, y, z}, 0], By → Function[{x, y, z}, 0 ],
  Bz → Function[{x, y, z}, Bz [z]], mx → Function[{x, y, z}, mx1],
  my → Function[{x, y, z}, my1], mz → Function[{x, y, z}, mz1]};
B13 = B1 /. rule3;
m13 = m1 /. rule3;

FIGS[m13, B13]

{0, 0, mz1 Bz′[z]}

FIGS[B13, m13]

{0, 0, 0}

Gra[m13.B13]

{0, 0, mz1 Bz′[z]}
```

16. Current density due to magnetization; $\mathbf{J}_M = \nabla \times \mathbf{M}$

If the magnetization is nonuniform, the cancellation is not complete. We consider the abrupt change of magnetization shown in Fig. It is evident that between the two broken lines there is more current moving down than that moving up. There is a resultant current in the interior. To find the relationship between $\mathbf{J}_M$ and $\mathbf{M}$ we consider two small volume elements located next to each other in the direction of the y-axis, each element of volume abc. If the magnetization in the first volume element is $\mathbf{M}(x, y, z)$, then the magnetization in the second volume is

$$\mathbf{M}(x, y, z) + \frac{\partial \mathbf{M}(x, y, z)}{\partial y} \Delta y + ...$$

The x-component of magnetic moment of the first element, $M_x(x, y, z)abc$, may be written in term of a circulating current, I_c'

$$(M_x + \frac{\partial M_x}{\partial y})abc = I_c' bc .$$

Using the same argument, we have just made, you can show that this surface will contribute to J_y the amount $\partial M_x / \partial z$. These are the only surfaces which can contribute to the *y*-component of the current, so we have that the total current density in the *y*-direction is

$$J_y = \frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x}.$$

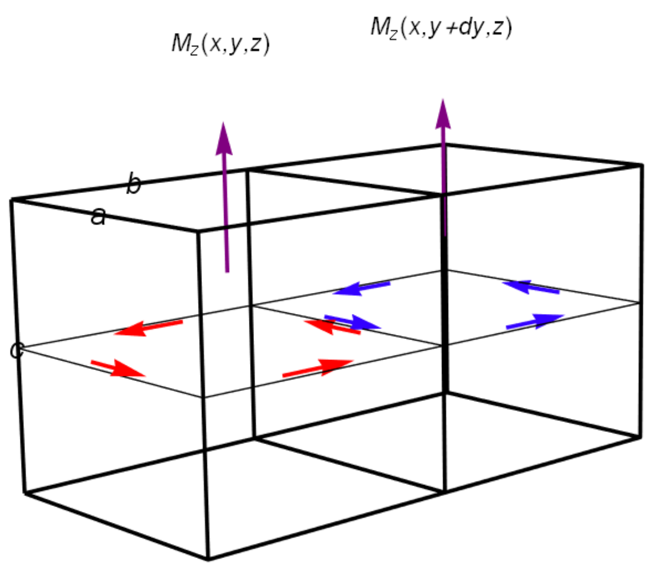
Working out the currents on the remaining faces of a cube—or using the fact that our *z*-direction is completely arbitrary—we can conclude that the vector current density is indeed given by the equation

$$\mathbf{J}_b = \nabla \times \mathbf{M}.$$

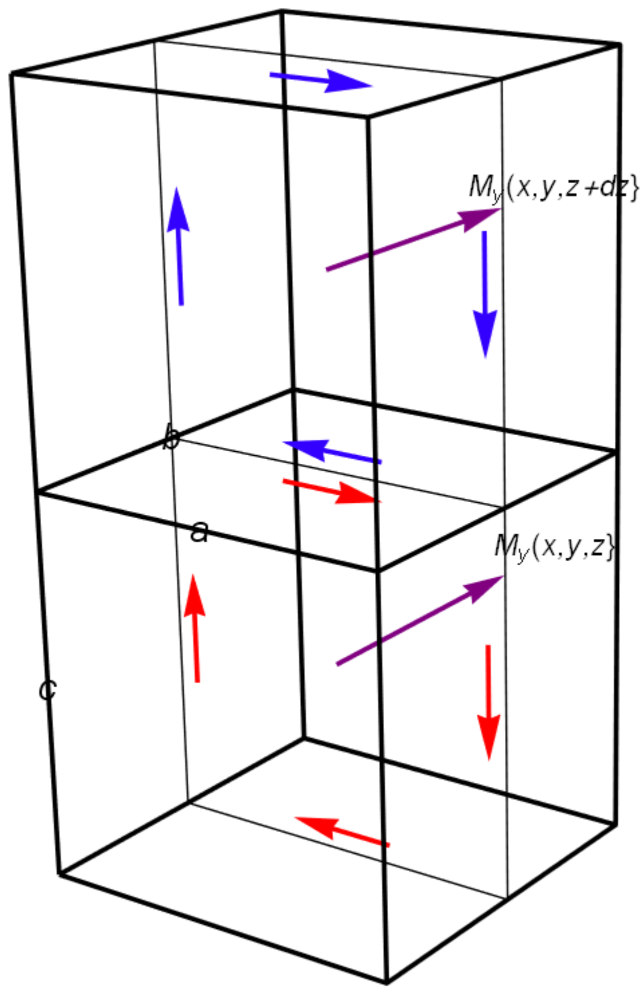
So if we choose to describe the magnetic situation in matter in terms of the average magnetic moment per unit volume ***M***, we find that the circulating atomic currents are equivalent to an average current density in matter given by $\mathbf{J}_b = \nabla \times \mathbf{M}$.

(a) Current density along the *x* axis for axis for the system with non-uniform magnetization

$$J_x = (\nabla \times \mathbf{M})_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}.$$



(a)

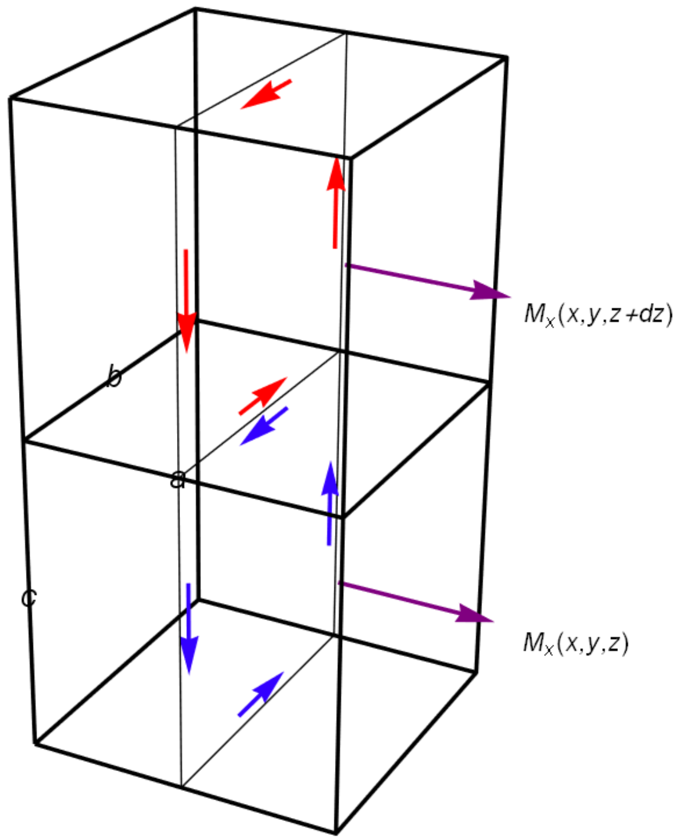


(b)

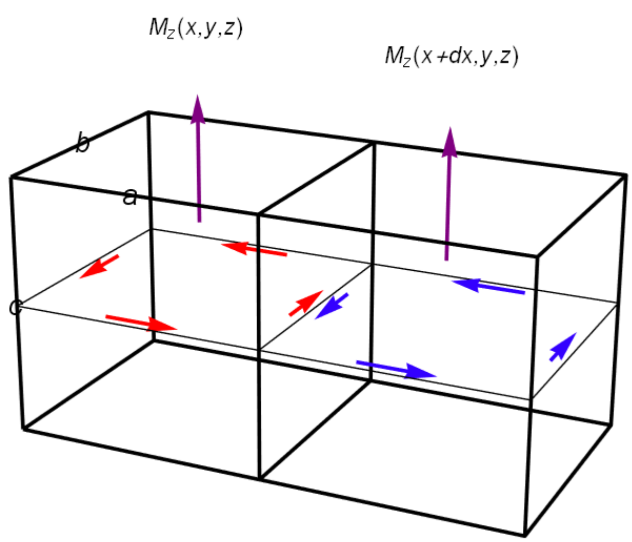
Fig.29 Equivalent Amperian current loops in a magnetized medium, showing cancellation effect on internal boundary. The current density J_x from (a) dM_z/dy and (b) dM_y/dz. $J_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}$.

(b) Current density along the *y* axis for the system with non-uniform magnetization

$$J_y = (\nabla \times \mathbf{M})_y = \frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x}$$



$M_x(x,y,z+dz)$
b
a
c
$M_x(x,y,z)$



$M_z(x,y,z)$
$M_z(x+dx,y,z)$
b
a
c

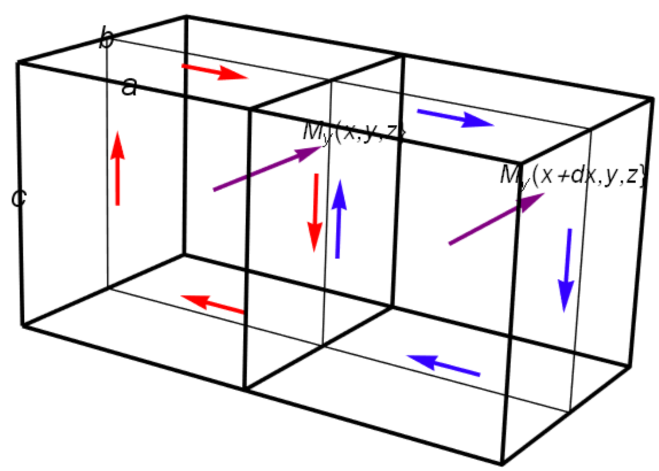
Fig.30 The current density J_y from (a) dM_x / dz and (b) dM_z / dx.

$$J_y = \frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x}.$$

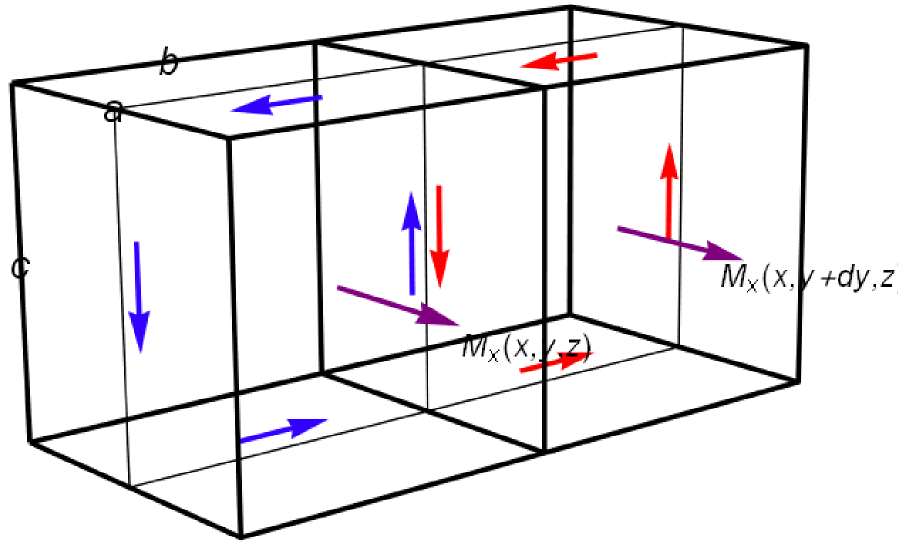
(c) Current density along the *z* axis for the system with non-uniform magnetization

$$J_z = (\nabla \times \mathbf{M})_z = \frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y}$$

If the magnetization of two neighboring blocks is not the same, there is a net surface current in between.



(a)



(b)

Fig.31 The current density J_z from (a) dM_y / dx and (b) dM_x / dy .

$$J_z = \frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y}.$$

17. Surface current and magnetization

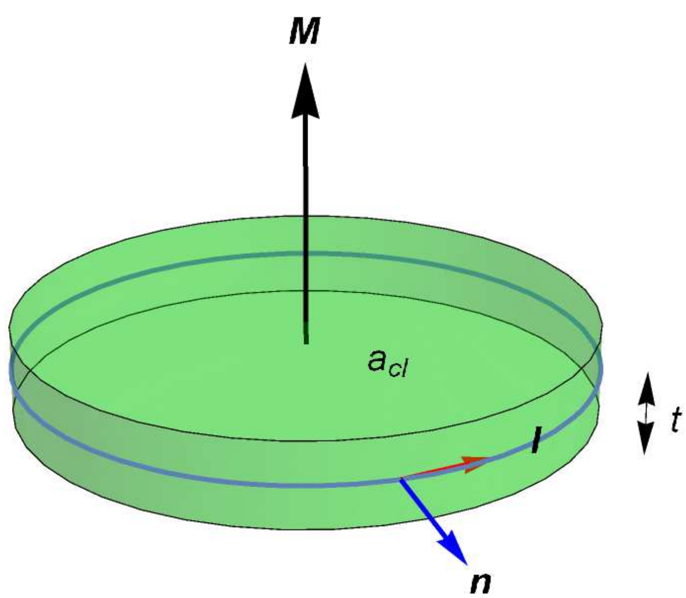


Fig.32 The magnetization $\mathbf{M}$ is related to the magnetic moment m as $\mu = Ma_{cl}t = Ia_{cl}$. The loop current I is $I = Mt$; $\mathbf{I} = (\mathbf{M}\times\mathbf{n})t = \mathbf{K}t$. a_{cl} is the area of cylinder. t is the thickness.

$$m = M(a_{cl}t) = Ia_{cl}.$$

$$Mt = I.$$

$$\mathbf{I} = (\mathbf{M}\times\mathbf{n})t.$$

$$\frac{\mathbf{I}}{t} = \mathbf{K} = \mathbf{M}\times\mathbf{n}, \qquad \text{(surface current)}$$

where ***M*** is the magnetization (magnetic moment per unit volume). If the magnetization is uniform, the current in the loop tends to cancel each other out, and there is not net effect current in the interior of the system. But at the surface there are no adjacent currents to produce a cancellation, and because the currents in the whirls are all circulating in the same sense, the result in effect is that of a surface current circulating on the surface.

Note that $\mathbf{K} = \mathbf{M}\times\mathbf{n}$ is tangent to the surface because $\mathbf{M}$ is perpendicular to $\mathbf{n}$. All the internal currents cancel. At the edge there is no adjacent loop to do the cancelling .

18. Vector potential of known current

Here we discuss the vector potential ***A***. To this end, we start with the Maxwell's equations;

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J},$$

with

$$\mathbf{B} = \nabla\times\mathbf{A}.$$

So that, we have

$$\begin{aligned}\nabla\times\mathbf{B} &= \nabla\times(\nabla\times\mathbf{A})\\ &= \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}\\ &= \mu_0 J\end{aligned}$$

We assume that

$$\nabla\cdot\mathbf{A} = 0. \qquad \text{(Coulomb gauge)}$$

So that we have

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} .$$

We use the three-dimensional Green function such that

$$\nabla^2 G(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}') ,$$ (we use the notation used by Arfken et al.)

with

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} .$$

So that, our solution for ***A*** is obtained as

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \mu_0 \int d^3\mathbf{r}' G(\mathbf{r},\mathbf{r}')\mathbf{J}(\mathbf{r}') \\ &= \frac{\mu_0}{4\pi}\int d^3\mathbf{r}' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\end{aligned}$$

Note that

$$\begin{aligned}\nabla^2 \mathbf{A}(\mathbf{r}) &= \mu_0 \int d^3\mathbf{r}' \nabla^2 G(\mathbf{r},\mathbf{r}')\mathbf{J}(\mathbf{r}') \\ &= -\mu_0 \int d^3\mathbf{r}' \delta(\mathbf{r}-\mathbf{r}')\mathbf{J}(\mathbf{r}') \\ &= -\mu_0 \mathbf{J}(\mathbf{r})\end{aligned}$$

---

19. The magnetic field of a small current loop: definition of magnetic moment

We start with the vector potential due to the current loop (two-dimension)

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d\mathbf{s} \frac{I}{|\mathbf{r}-\mathbf{r}'|} .$$

$$\begin{aligned}\frac{1}{|\mathbf{r}-\mathbf{r}'|} &= \frac{1}{r - r'\cos\theta} \\ &\simeq \frac{1}{r}(1+\frac{r'}{r}\cos\theta) \\ &= \frac{1}{r} + \frac{1}{r^3}(\mathbf{r}\cdot\mathbf{r}')\end{aligned}$$

where

$$\cos\theta = \frac{\mathbf{r}\cdot\mathbf{r}'}{rr'}.$$

Thus, we get

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int Id\mathbf{s}[\frac{1}{r} + \frac{1}{r^3}(\mathbf{r}\cdot\mathbf{r}')] = \frac{\mu_0 I}{4\pi r^3}\int d\mathbf{s}(\mathbf{r}\cdot\mathbf{r}').$$

We note the two things.

(a)

$$\begin{aligned} d[(\mathbf{r}\cdot\mathbf{r}')\mathbf{r}'] &= (\mathbf{r}\cdot d\mathbf{r}')\mathbf{r}' + (\mathbf{r}\cdot\mathbf{r}')d\mathbf{r}' \\ &= (\mathbf{r}\cdot d\mathbf{s})\mathbf{r}' + (\mathbf{r}\cdot\mathbf{r}')d\mathbf{s} \end{aligned}$$

and

$$\oint d[(\mathbf{r}\cdot\mathbf{r}')\mathbf{r}'] = \oint(\mathbf{r}\cdot d\mathbf{s})\mathbf{r}' + \oint d\mathbf{s}(\mathbf{r}\cdot\mathbf{r}') = 0$$

(b)

$$\mathbf{r}\times(\mathbf{r}'\times d\mathbf{s}) = (\mathbf{r}\cdot d\mathbf{s})\mathbf{r}' - d\mathbf{s}(\mathbf{r}\cdot\mathbf{r}')$$

$$d\mathbf{s}(\mathbf{r}\cdot\mathbf{r}') = (\mathbf{r}'\times d\mathbf{s})\times\mathbf{r} + (\mathbf{r}\cdot d\mathbf{s})\mathbf{r}'$$

$$\begin{aligned} \oint d\mathbf{s}(\mathbf{r}\cdot\mathbf{r}') &= \oint(\mathbf{r}'\times d\mathbf{s})\times\mathbf{r} + \oint(\mathbf{r}\cdot d\mathbf{s})\mathbf{r}' \\ &= \oint(\mathbf{r}'\times d\mathbf{s})\times\mathbf{r} - \oint d\mathbf{s}(\mathbf{r}\cdot\mathbf{r}') \end{aligned}$$

or

$$\oint d\mathbf{s}(\mathbf{r}\cdot\mathbf{r}') = \frac{1}{2}\oint(\mathbf{r}'\times d\mathbf{s})\times\mathbf{r}$$

The magnetic moment:

$$\frac{I}{2}\int(\mathbf{r}'\times d\mathbf{s}) = I\mathbf{a}_{cl} = \mathbf{m}$$

with

$$\frac{1}{2}\int(\mathbf{r}'\times d\mathbf{s}) = \mathbf{a}_{cl}$$

Thus, the vector potential **A** is

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi r^3}(\mathbf{m}\times\mathbf{r})},$$

where ***m*** is the magnetic moment located at the origin, along the **z** axis.

The magnetic field ***B*** is evaluated as

$$\begin{aligned}
\mathbf{B} &= \nabla\times\mathbf{A} \\
&= \frac{\mu_0}{4\pi}\nabla\times\left(\mathbf{m}\times\frac{\mathbf{r}}{r^3}\right) \\
&= \frac{\mu_0}{4\pi}[\mathbf{m}(\nabla\cdot\frac{\mathbf{r}}{r^3}) - (\mathbf{m}\cdot\nabla)\frac{\mathbf{r}}{r^3}]
\end{aligned}$$

Since $\nabla\cdot\frac{\mathbf{r}}{r^3} = 0$, we have

$$\mathbf{B} = -\frac{\mu_0}{4\pi}(\mathbf{m}\cdot\nabla)\frac{\mathbf{r}}{r^3} = -\frac{\mu_0}{4\pi}[\frac{\mathbf{m}}{r^3} - \frac{3(\mathbf{m}\cdot\mathbf{r})\mathbf{r}}{r^3}].$$

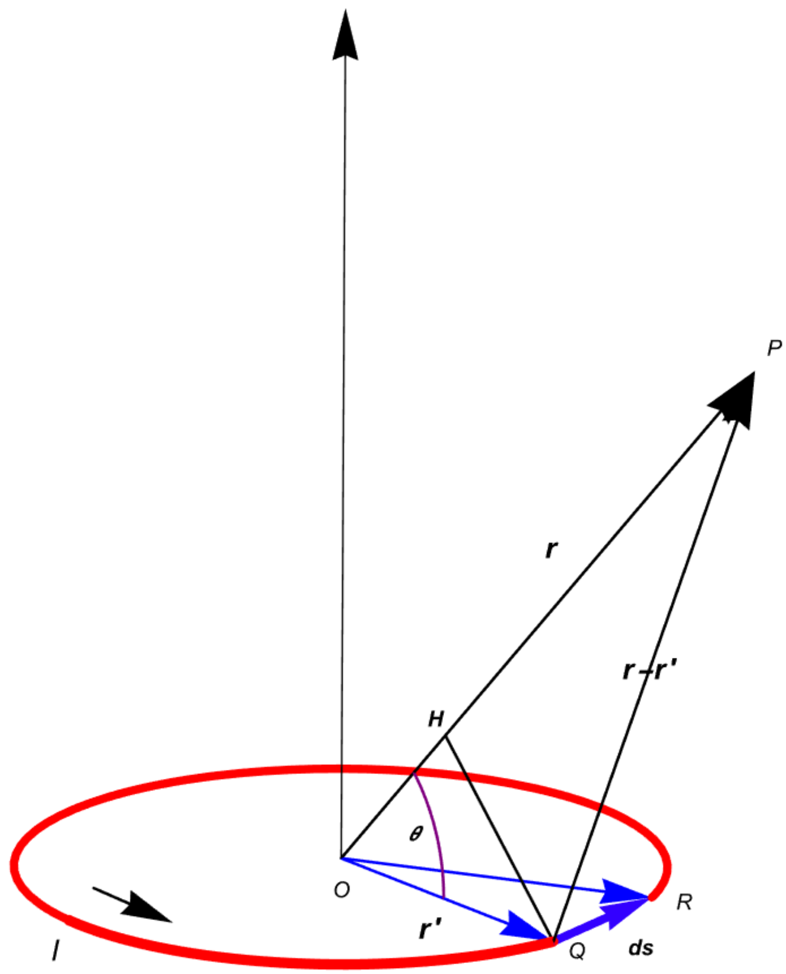


Fig.33 Vector potential ***A*** ay a point far from the current loop (which is not always a circle loop). The magnetic moment ***m*** is $m = Ia_{cl}$. a_{cl} is the area of the current loop. $|\mathbf{r}-\mathbf{r}'| \simeq r - \overline{OH} = r - r'\cos\theta = r - \frac{1}{r}(\mathbf{r}\cdot\mathbf{r}')$.

((The interaction between magnetic moments))

The interaction between magnetic moments $\mathbf{m}_1$ and $\mathbf{m}_2$ is expressed by

$$\begin{aligned} H_{in} &= -\mathbf{m}_2 \cdot \mathbf{B} \\ &= \frac{\mu_0}{4\pi r_{12}^{\ 3}} \cdot [\mathbf{m}_1 \cdot \mathbf{m}_2 - \frac{3(\mathbf{m}_1 \cdot \mathbf{r}_{12})(\mathbf{m}_2 \cdot \mathbf{r}_{12})}{r_{12}^{\ 2}}] \end{aligned}$$

where $\mathbf{r}_{12}$ is a position vector connecting between the moments $\mathbf{m}_1$ and $\mathbf{m}_2$.

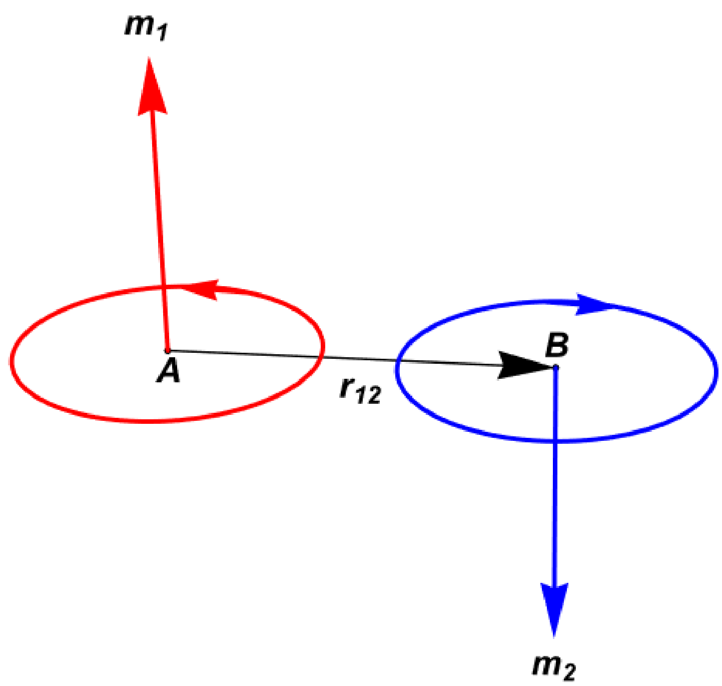


Fig.34 The magnetic interaction between the magnetic moments $\mathbf{m}_1$ and $\mathbf{m}_2$. The magnetic moments are directed along the z axis. When the position vector $\mathbf{r}_{12}$ is perpendicular to the magnetic moments, the direction of $\mathbf{m}_1$ is antiparallel to that of $\mathbf{m}_2$. $H_{dipole} = \frac{\mu_0}{4\pi r_{12}{}^3}(\mathbf{m}_1 \cdot \mathbf{m}_2)$, where $\mathbf{m}_1 \cdot \mathbf{r}_{12} = 0$ and $\mathbf{m}_2 \cdot \mathbf{r}_{12} = 0$. The Hamiltonian favors an antiferromagnetic alignment of adjacent magnetic moments.

20. Vector potential and magnetic field due to magnetic moment; Stokes, theorem

The vector potential at the position vector $\boldsymbol{r}$ due to the magnetic moment $\boldsymbol{m}$ along the z axis at the origin O,

$$\mathbf{A} = \frac{\mu_0}{4\pi r^2}(\mathbf{m} \times \mathbf{e}_r) = \frac{\mu_0}{4\pi r^3}(\mathbf{m} \times \mathbf{r}),$$

where

$$\mathbf{m} = I a_{cl}\ \mathbf{e}_z .$$

In the spherical coordinate, the vector potential is

$$\mathbf{A} = \frac{\mu_0 m}{4\pi r^2}(\mathbf{e}_z \times \mathbf{e}_r) = \frac{\mu_0 m \sin\theta}{4\pi r^2}\mathbf{e}_\phi ,$$

where

$$\mathbf{e}_\phi = \frac{1}{r\sin\theta}\frac{\partial \mathbf{r}}{\partial \phi} = -\sin\phi \mathbf{e}_x + \cos\phi \mathbf{e}_y .$$

In the cartesian coordinates,

$$\mathbf{A} = \frac{\mu_0 m}{4\pi r^3}(\mathbf{e}_z \times \mathbf{r}) = \frac{\mu_0 m}{4\pi r^3}\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 0 & 0 & 1 \\ x & y & z \end{vmatrix} = \frac{\mu_0 m}{4\pi r^3}(-y, x, 0),$$

where

$$r = (x^2 + y^2 + z^2)^{1/2} .$$

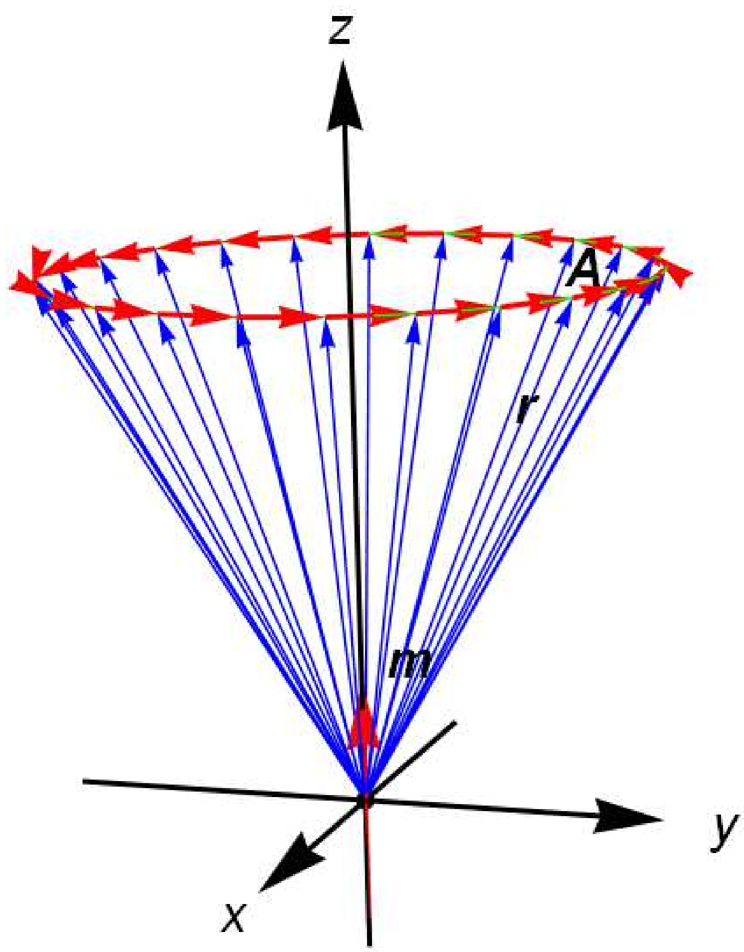


Fig.35 A magnetic moment ***m*** located at the origin. At every point far from the origin, the vector potential ***A*** is a vector parallel to the *x*-*y* plane, tangent to a circle around the *z* axis (the vector potential ***A*** denoted by red arrows.

The magnetic field ***B*** is obtained as

$$\mathbf{B} = \nabla\times\mathbf{A}$$

$$= \frac{\mu_0 m}{4\pi r^5}(3zx,\ 3yz,\ 3z^2 - r^2) \qquad \text{(Cartesian coordinates)}$$

$$= \frac{\mu_0 m}{4\pi r^5}[3zx\mathbf{e}_x + 3yz\mathbf{e}_y + (3z^2 - r^2)\mathbf{e}_z)$$

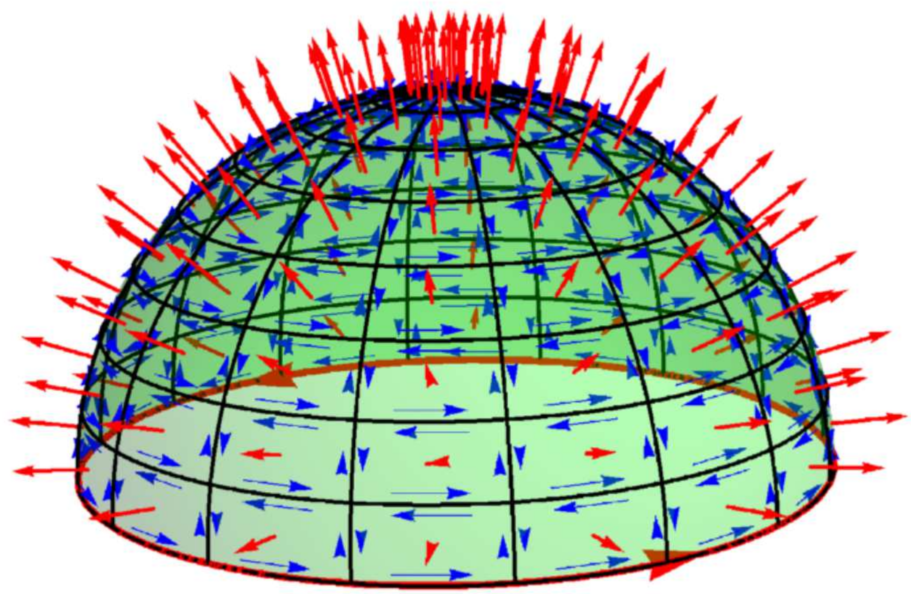
or

$$\mathbf{B} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta \mathbf{e}_r + \sin\theta \mathbf{e}_\theta).$$ (Spherical coordinates)

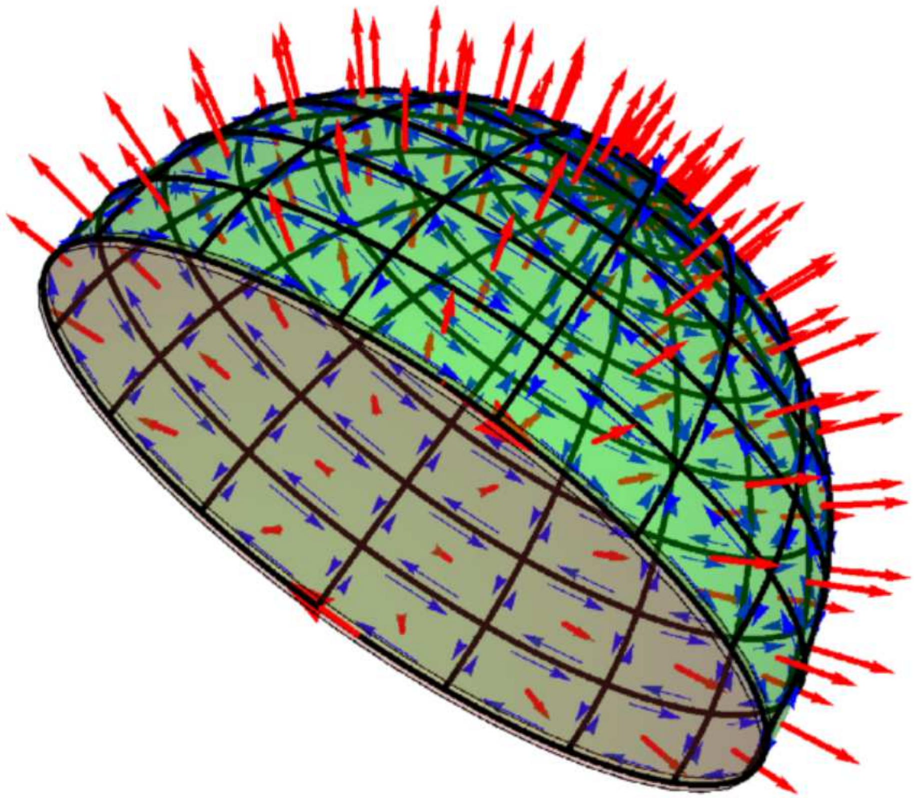
Note that

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} = 0.$$

We now apply the Stokes' theorem for the magnetic field ***B*** from the magnetic moment.



(a)



(b)

Fig.36 (a) and (b), $\oint(\nabla\times\mathbf{A})\cdot d\mathbf{a}=\oint\mathbf{A}\cdot d\mathbf{s}$ (Stokes' theorem). $d\mathbf{s}$ is the perimeter. The direction of $d\mathbf{a}$ for each rectangle is normal and out of the plane. The line integral of a vector field over a loop (a perimeter) is equal to the *flux of its curl* through the enclosed surface.

We have the Stokes' theorem for the vector potential ***A*** as

$$\oint(\nabla\times\mathbf{A})\cdot d\mathbf{a}=\oint\mathbf{A}\cdot d\mathbf{s},$$

The magnetic flux is obtained as

$$\Phi=\oint\mathbf{B}\cdot d\mathbf{a}=\oint\mathbf{A}\cdot d\mathbf{s}.$$

We note that

$$\begin{aligned}\oint \mathbf{A}\cdot d\mathbf{s} &= \int_0^{2\pi} \frac{\mu_0 m \sin\theta}{4\pi r^2}\mathbf{e}_\phi \cdot (r\sin\theta d\phi)\mathbf{e}_\phi \\ &= \int_0^{2\pi} \frac{\mu_0 m \sin\theta}{4\pi r^2} r\sin\theta d\phi \\ &= \frac{\mu_0 m}{2r}\sin^2\theta\end{aligned}$$

where $\quad d\mathbf{s} = \mathbf{e}_\phi r\sin\theta d\phi$.

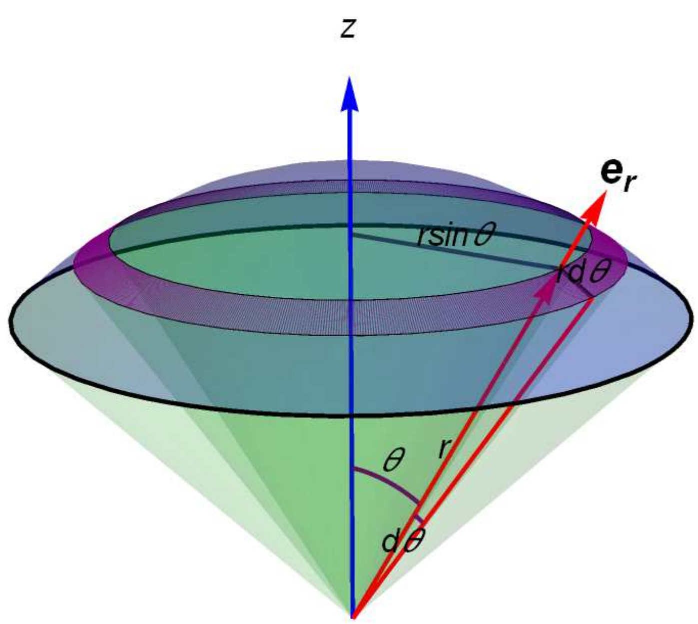


Fig.37 Evaluation of magnetic flux $\Phi = \int_S \mathbf{B}\cdot d\mathbf{a} = \oint (\nabla\times\mathbf{A})\cdot d\mathbf{a} = \int \mathbf{A}\cdot d\mathbf{s}$

Next, we calculate the surface integral

$$
\begin{aligned}
\oint (\nabla \times \mathbf{A}) \cdot d\mathbf{a} &= \int_S \mathbf{B} \cdot d\mathbf{a} \\
&= \int_S \frac{\mu_0 m \cos\theta}{2\pi r^3} \mathbf{e}_r \cdot (2\pi r^2 \sin\theta d\theta) \mathbf{e}_r \\
&= \int_0^\theta \frac{\mu_0 m}{2r} \sin 2\theta d\theta \\
&= \frac{\mu_0 m}{2r} \frac{1}{2} [1 - \cos(2\theta)] \\
&= \frac{\mu_0 m}{2r} \sin^2\theta
\end{aligned}
$$

So that the Stokes' theorem is verified from this example.

21. The *B* field from a magnetized system

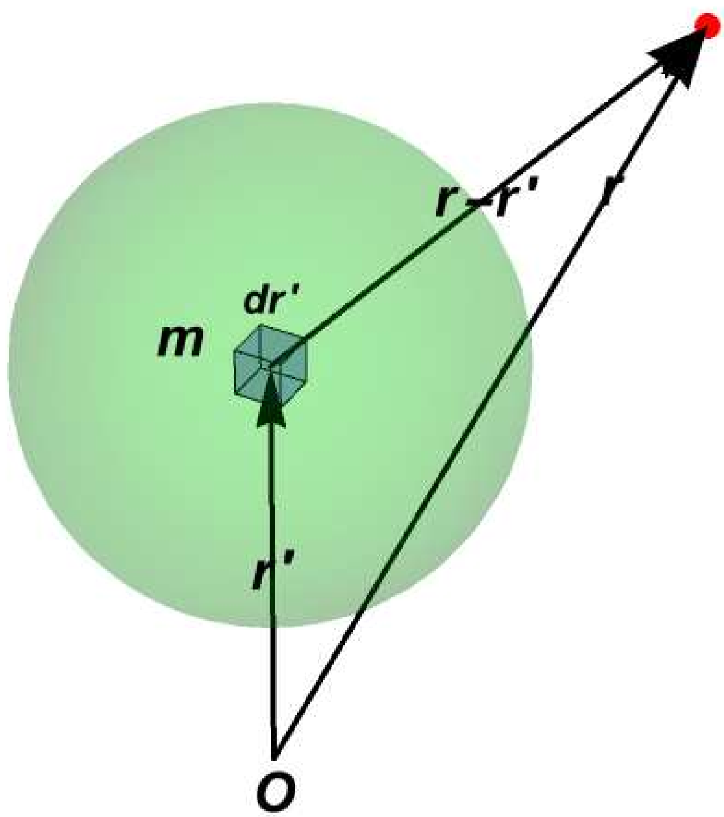


Fig.38 Vector potential $\mathbf{A}$ at the point $\boldsymbol{r}$ due to the magnetic moment at the source coordinates $\boldsymbol{r}$'.

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

$\mathbf{m} = \mathbf{M}(\mathbf{r}')d^3\mathbf{r}'$. (magnetic noment)

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d^3\mathbf{r}'.$$

$$\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}.$$

Using this relation, we get

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\mathbf{M}(\mathbf{r}')\times\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}'.$$

Noting that

$$\mathbf{M}(\mathbf{r}')\times\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\nabla'\times\mathbf{M}(\mathbf{r}') - \nabla'\times\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}.$$

we have

$$\begin{aligned}
\mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\nabla'\times\mathbf{M}(\mathbf{r}')d^3\mathbf{r}' - \frac{\mu_0}{4\pi}\int\nabla'\times\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' \\
&= \frac{\mu_0}{4\pi}\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\nabla'\times\mathbf{M}(\mathbf{r}')d^3\mathbf{r}' + \frac{\mu_0}{4\pi}\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}[\mathbf{M}(\mathbf{r}')\times d\mathbf{a}'] \\
&= \frac{\mu_0}{4\pi}\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{J}_b(\mathbf{r}')d^3\mathbf{r}' + \frac{\mu_0}{4\pi}\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{K}_b(\mathbf{r}')da'
\end{aligned}$$

with

$$\mathbf{J}_b = \nabla\times\mathbf{M}(\mathbf{r})$$

$$\mathbf{K}_b = \mathbf{M}(\mathbf{r})\times\mathbf{n}$$

where $\boldsymbol{n}$ is the normal unit vector. We use the formula

$$\int \nabla' \times \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}' = -\int \frac{1}{|\mathbf{r}-\mathbf{r}'|}[\mathbf{M}(\mathbf{r}')\times d\mathbf{a}'] \qquad \text{(the proof is given below)}$$

and the relation

$$\begin{aligned}\int \frac{1}{|\mathbf{r}-\mathbf{r}'|}[\mathbf{M}(\mathbf{r}')\times d\mathbf{a}'] &= \int \frac{1}{|\mathbf{r}-\mathbf{r}'|}[\mathbf{M}(\mathbf{r}')\times \mathbf{n}']da' \\ &= \int \frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{K}_b(\mathbf{r}')da'\end{aligned}$$

Proof (Griffiths)

$$\int_V (\nabla \times \mathbf{v}) d^3\mathbf{r} = -\int_A \mathbf{v}\times d\mathbf{a} .$$

We assume that $\boldsymbol{c}$ is a constant vector.

Gauss's theorem:

$$\int_V \nabla \cdot (\mathbf{v}\times\mathbf{c}) d^3\mathbf{r} = \int_A (\mathbf{v}\times\mathbf{c})\cdot d\mathbf{a} .$$

Note that

$$\begin{aligned}\nabla\cdot(\mathbf{v}\times\mathbf{c}) &= \mathbf{c}\cdot(\nabla\times\mathbf{v}) - \mathbf{v}\cdot(\nabla\times\mathbf{c}) \\ &= \mathbf{c}\cdot(\nabla\times\mathbf{v})\end{aligned}$$

and

$$d\mathbf{a}\cdot(\mathbf{v}\times\mathbf{c}) = \mathbf{c}\cdot(d\mathbf{a}\times\mathbf{v}) = -\mathbf{c}\cdot(\mathbf{v}\times d\mathbf{a}) .$$

The first term of Gauss' theorem;

$$\int_V \nabla\cdot(\mathbf{v}\times\mathbf{c}) d^3\mathbf{r} = \mathbf{c}\cdot\int_V (\nabla\times\mathbf{v}) d^3\mathbf{r} .$$

The second term of Gauss' theorem:

$$\int_A (\mathbf{v}\times\mathbf{c})\cdot d\mathbf{a} = -\mathbf{c}\cdot\int_A \mathbf{v}\times d\mathbf{a} .$$

Thus, we have

$$\mathbf{c}\cdot\int_V (\nabla\times\mathbf{v})d^3\mathbf{r} = -\mathbf{c}\cdot\int_A \mathbf{v}\times d\mathbf{a}\,,$$

or

$$\int_V (\nabla\times\mathbf{v})d^3\mathbf{r} = -\int_A \mathbf{v}\times d\mathbf{a}\,.$$

((**Note**)) **Units of physical quantities**

m: magnetic moment
M: magnetization (magnetic moment per unit volume)

$$[M] = \frac{Am^2}{m^3} = \frac{A}{m} \qquad [J_b] = \frac{A}{m^2}$$

$$[K_b] = \frac{A}{m} \qquad [m] = Am^2$$

22. Vector potential and magnetic field from uniformly magnetized sphere (Griffiths)

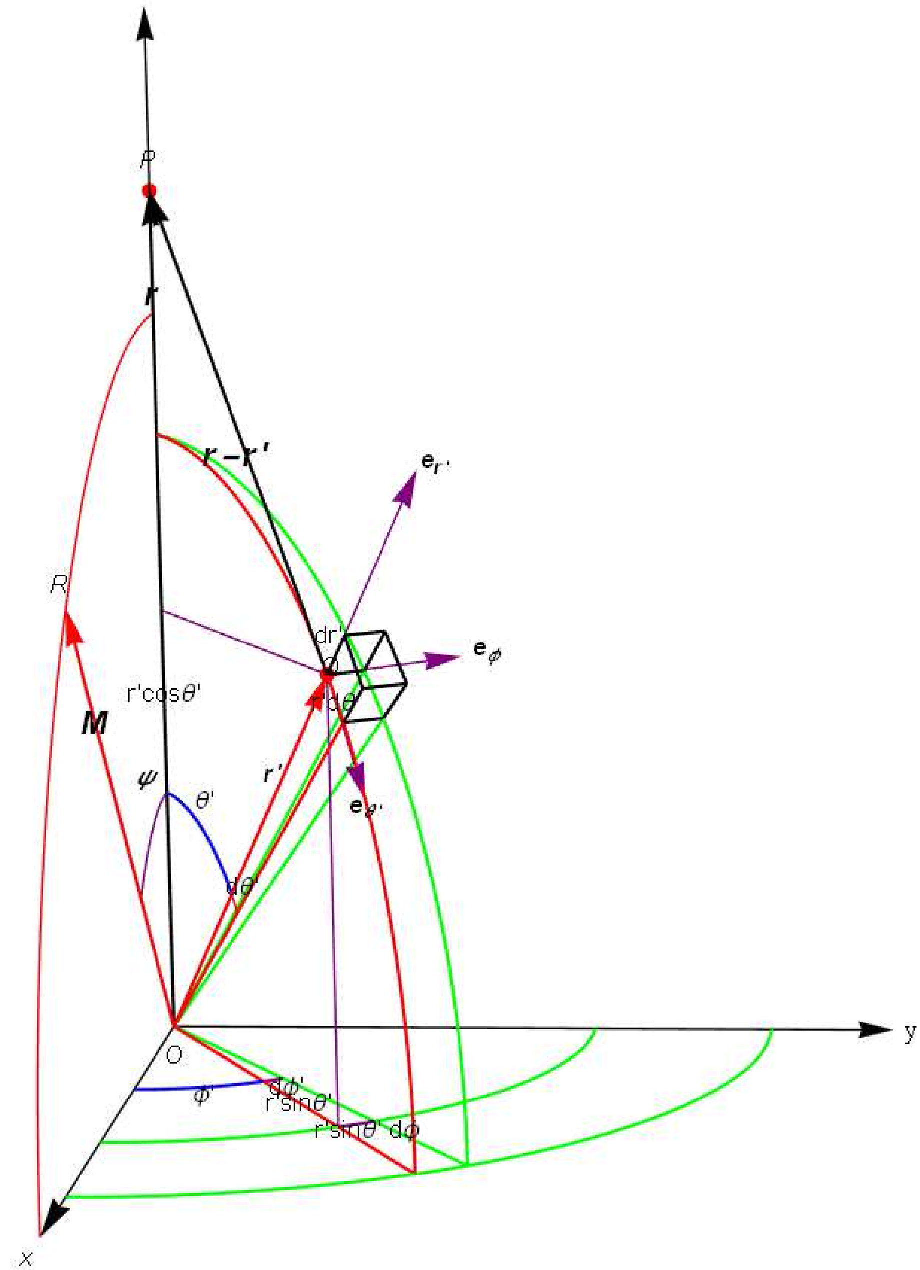


Fig.39 The vector potential and the magnetic field at a point P (at the position vector ***r***) from a sphere (with radius *R*) having the uniform magnetization **M** directed in the *x*-*z* plane.

Here we discuss the vector potential and magnetic field from uniform magnetization inside a sphere with radius *R*.

The magnetization vector of the sphere is in the *x*-*z* plane,

$$\mathbf{M} = (M\sin\psi, 0, M\cos\psi)$$

The unit vectors:

$$\mathbf{e}_{r'} = (\sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta')$$

Since $\mathbf{n}' = \mathbf{e}_{r'}$ the surface current due to the magnetization **M**,

$$\begin{aligned}
\mathbf{K}_b' &= \mathbf{M}\times\mathbf{n}' \\
&= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ M\sin\psi & 0 & M\cos\psi \\ \sin\theta'\cos\phi' & \sin\theta'\sin\phi' & \cos\theta' \end{vmatrix} \\
&= -M\cos\psi\sin\theta'\sin\phi'\mathbf{e}_x + (-M\sin\psi\cos\theta' + M\cos\psi\sin\theta'\cos\phi')\mathbf{e}_y \\
&\quad + M\sin\psi\sin\theta'\sin\phi'\mathbf{e}_z
\end{aligned}$$

We need to calculate the vector potential at the point P $\mathbf{r} = (0, 0, r)$.

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{r^2 + r'^2 - 2\mathbf{r}\cdot\mathbf{r}'} = \sqrt{r^2 + R^2 - 2rR'\cos\theta'}$$

The vector potential:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_S \frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathbf{K}_b(\mathbf{r}')da',$$

where

$$da' = R^2\sin\theta' d\theta' d\phi'.$$

$$\begin{aligned}
\mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' \frac{\mathbf{K}_b(\mathbf{r}')R^2\sin\theta'}{\sqrt{r^2 + R^2 - 2rR\cos\theta'}} \\
&= \frac{\mu_0 R^2}{4\pi}\int_0^{\pi} d\theta' \frac{\sin\theta'}{\sqrt{r^2 + R^2 - 2rR\cos\theta'}} \int_0^{2\pi} d\phi' \mathbf{K}_b(\mathbf{r}')
\end{aligned}$$

We note that

$$\int_0^{2\pi} \mathbf{K}_b{}' d\phi' = \int_0^{2\pi} d\phi'(-M\cos\psi\sin\theta'\sin\phi')\mathbf{e}_x + \int_0^{2\pi} d\phi'(-M\sin\psi\cos\theta' + M\cos\psi\sin\theta'\cos\phi')\mathbf{e}_y$$

$$+\int_0^{2\pi} d\phi'(-M\cos\psi\sin\theta'\sin\phi')\mathbf{e}_z$$

$$= \int_0^{2\pi} d\phi'(-M\sin\psi\cos\theta')\mathbf{e}_y$$

$$= -2\pi M\sin\psi\cos\theta'\mathbf{e}_y$$

since

$$\int_0^{2\pi} d\phi'\sin\phi' = \int_0^{2\pi} d\phi'\mathrm{cosn}\,\phi' = 0\,.$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 R^2}{4\pi}(-2\pi M)\sin\psi\mathbf{e}_y\int_0^{\pi} d\theta'\frac{\sin\theta'\cos\theta'}{\sqrt{r^2+R^2-2rR\cos\theta'}}$$

$$= \frac{\mu_0 R^2}{2r}(\mathbf{M}\times\mathbf{r})\int_0^{\pi} d\theta'\frac{\sin\theta'\cos\theta'}{\sqrt{r^2+R^2-2rR\cos\theta'}}$$

or

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{3}(\mathbf{M}\times\mathbf{r})\begin{cases} 1 & (r<R) \\ (\dfrac{R}{r})^3 & (r>R) \end{cases}$$

$$\int_0^{\pi} d\theta'\frac{\sin\theta'\cos\theta'}{\sqrt{r^2+R^2-2rR\cos\theta'}} = \begin{cases} \dfrac{2r}{3R^2} & (r<R) \\ \dfrac{2R}{3r^2} & (r>R) \end{cases}$$

where

$$\mathbf{M}\times\mathbf{r} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ M\sin\psi & 0 & M\cos\psi \\ 0 & 0 & r \end{vmatrix} = -Mr\sin\psi\mathbf{e}_y .$$

((**Mathematica**))

```
Clear["Global`*"];
f1 = Integrate[Sin[θ] Cos[θ] / Sqrt[R^2 + r^2 - 2 R r Cos[θ]], {θ, 0, π},
   Assumptions -> {R > 0, r > 0}];

H11 = Simplify[f1, r > R]
```

$$\frac{2\,R}{3\,r^2}$$

```
H12 = Simplify[f1, r < R]
```

$$\frac{2\,r}{3\,R^2}$$

For $r > R$ (outside of the sphere)

$$\boxed{\begin{aligned}\mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\left(\frac{\mathbf{m}\times\mathbf{r}}{r^3}\right) \\ &= \frac{m\mu_0}{4\pi r^2}\sin\theta\mathbf{e}_\phi\end{aligned}}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{m}{r^3}(2\cos\theta\mathbf{e}_r + \sin\theta\mathbf{e}_\theta) .$$

Note that at $\theta = 0$ and $r = R$

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{m}{R^3}2\mathbf{e}_r = \frac{2}{3}\mu_0 M\mathbf{e}_r .$$

For $r < R$ (inside of the sphere)

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi R^3}(\mathbf{m}\times\mathbf{r})$$
$$= \frac{m\mu_0 r}{4\pi R^3}\sin\theta\mathbf{e}_\phi$$

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{m\mu_0}{2\pi R^3}\mathbf{e}_z = \frac{2}{3}\mu_0 M\mathbf{e}_z = \frac{2}{3}\mu_0\mathbf{M} ,$$

which is constant

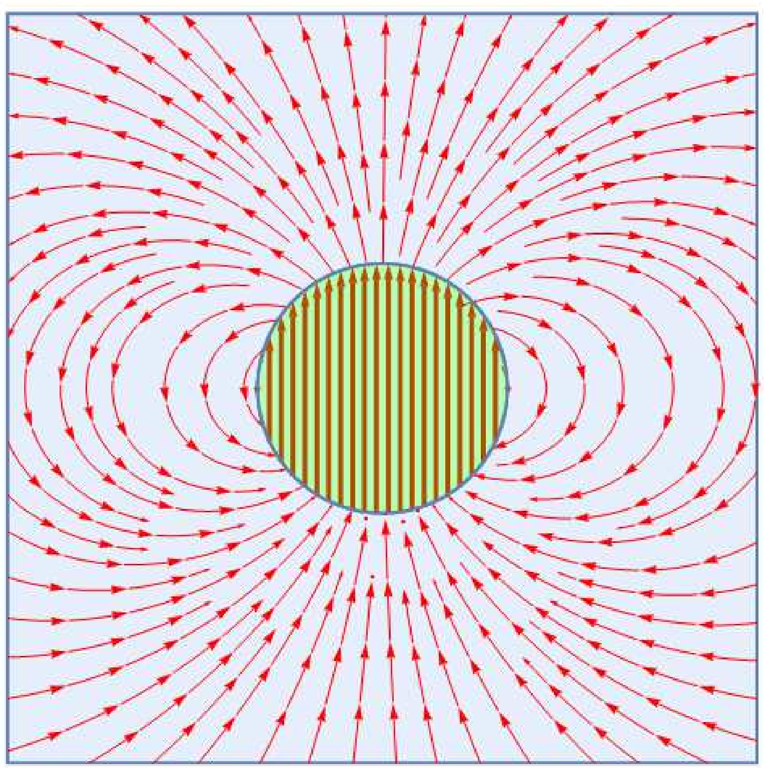


Fig.40 StreamPlot (Mathematica version 13). Magnetic distribution from sphere with uniform magnetization M along the z axis. The magnetic field is constant; $B = \frac{2}{3}\mu_0 M$ along the z axis, inside the sphere. The normal

components of $\boldsymbol{B}$ on the surface of sphere is continuous since $\nabla \cdot \mathbf{B} = 0$. The *y-z* plane.

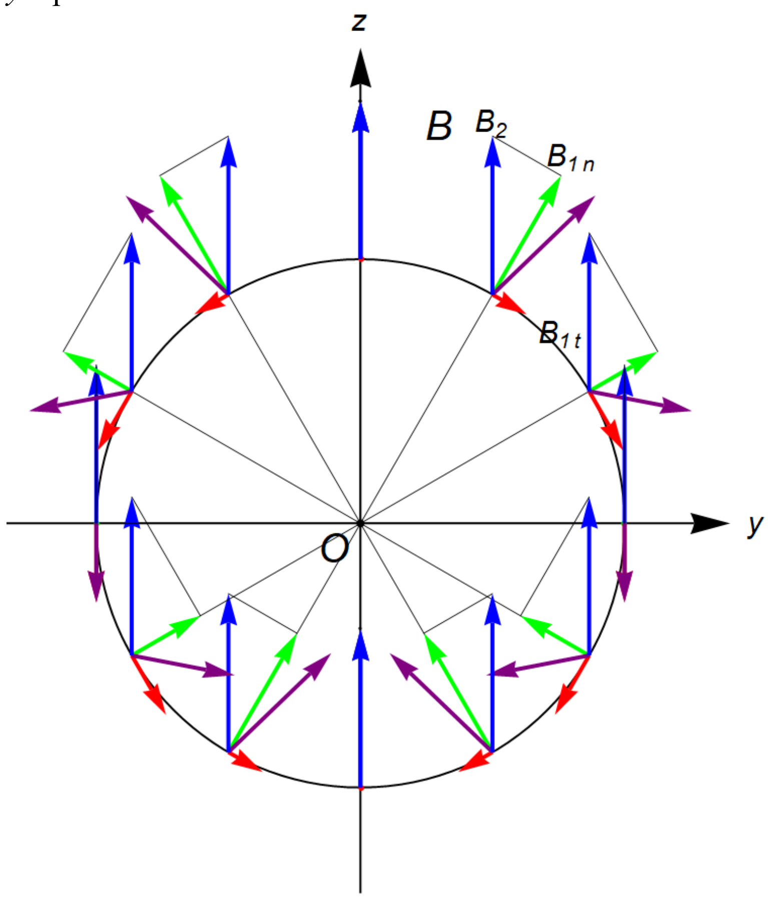


Fig.41 Boundary condition for the field $\mathbf{B}$ inside (blue) and outside (green, red, purple) sphere with uniform magnetization *M*. The normal component of the field $\mathbf{B}$ is continuous on the boundary of the sphere. $\mathbf{B}_2$ (blue) is the resultant field inside the sphere. B_{1n} (green) is the normal component of the field $\mathbf{B}_2$ outside the sphere. B_{1t} (red) is the tangential component of the field $\mathbf{B}_2$ outside the sphere. The boundary condition: $B_{1_n} = B_{2_n}$.

23. Boundary conditions of B and auxiliary field H

$$\nabla \cdot \mathbf{B} = 0 , \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\nabla \cdot \mathbf{A} = 0 \qquad \text{(Coulomb gauge)}$$

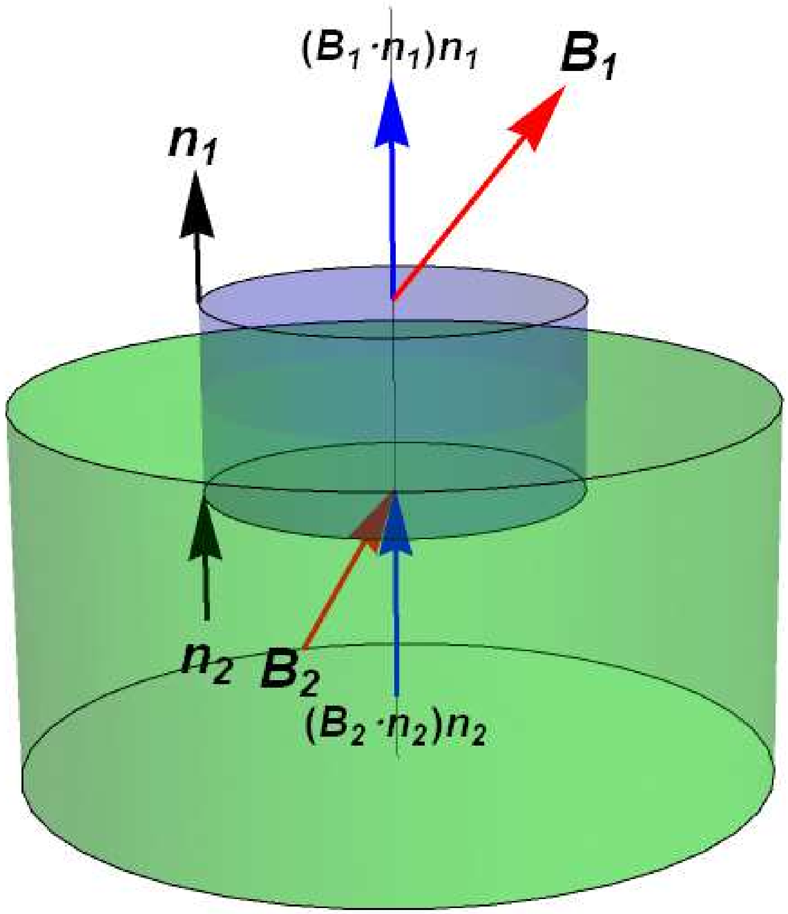


Fig.42 Boundary condition for ***B***. $\nabla \cdot \mathbf{B} = 0$ (indicating no magnetic monopole exists in nature). The Gauss' law leads to the boundary condition for the normal components $B_{1n} = B_{2n}$.

$$\nabla \cdot \mathbf{B} = 0 .$$

$$\int (\nabla \cdot \mathbf{B}) d^3\mathbf{r} = \int \mathbf{B} \cdot d\mathbf{a} = 0 . \qquad \text{(Gauss' law)}$$

$$B_{1n} = B_{2n} .$$

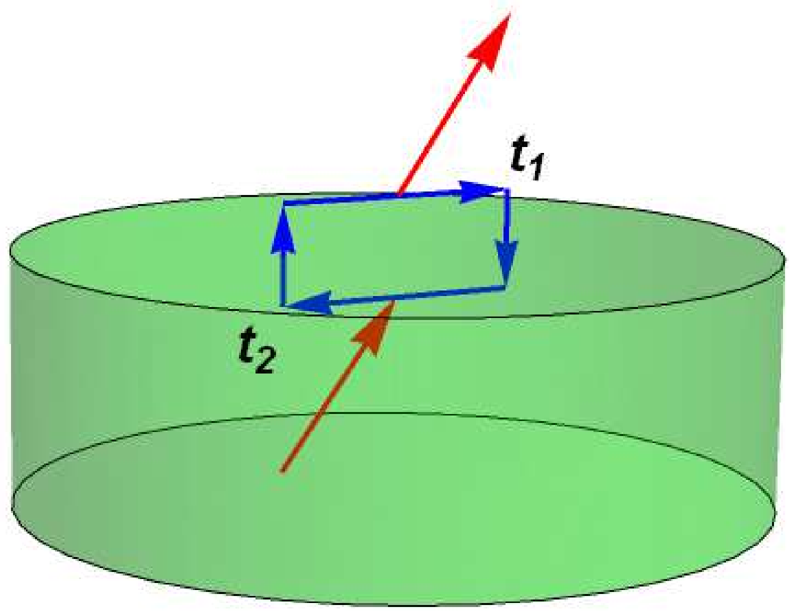


Fig.43 When the free current density $\mathbf{J}_f$=0, we have the relation $\nabla\times\mathbf{H}=0$ for the auxiliary field ***H***. The Stokes' law leads to the boundary condition for the tangential components $H_{1t}=H_{2t}$.

We define the auxiliary field ***H*** as

$$\mathbf{J}=\mathbf{J}_b+\mathbf{J}_f=\nabla\times\mathbf{M}+\mathbf{J}_f$$

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}=\mu_0(\mathbf{J}_b+\mathbf{J}_f)$$

$$\nabla\times(\frac{\mathbf{B}}{\mu_0}-\mathbf{M})=\mathbf{J}_f$$

We introduce the auxiliary field ***H*** as

$$\mathbf{B}=\mu_0(\mathbf{M}+\mathbf{H}).$$

$$\nabla\times\mathbf{H}=\mathbf{J}_f$$

When $\mathbf{J}_f=0$, we have

$$\nabla\times\mathbf{H}=0$$

$$\int (\nabla \times \mathbf{H}) \cdot d\mathbf{a} = \oint \mathbf{H} \cdot d\mathbf{s} = 0, \qquad \text{(Stokes' law)}$$

$$H_{1t} = H_{2t}.$$

We consider the auxiliary field ***H*** from the uniform magnetization from sphere. The auxiliary field ***H*** outside the sphere is expressed by

$$\begin{aligned}
\mathbf{H} &= \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \\
&= \frac{\mathbf{B}}{\mu_0} \qquad\qquad \text{for } r > R \\
&= \frac{\mu_0}{4\pi} \frac{m}{r^3} (2\cos\theta \mathbf{e}_r + \sin\theta \mathbf{e}_\theta) \\
&= \frac{1}{3} \mu_0 M \frac{R^3}{r^3} (2\cos\theta \mathbf{e}_r + \sin\theta \mathbf{e}_\theta)
\end{aligned}$$

since ***M*** = 0 outside the sphere. The auxiliary field ***H*** is a magnetic field inside the magnetic system. At $r = R$, we have

$$\mathbf{H} = \frac{1}{3} \mu_0 M (2\cos\theta \mathbf{e}_r + \sin\theta \mathbf{e}_\theta).$$

$$\begin{aligned}
\mathbf{H} &= \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \\
&= \frac{1}{\mu_0} \frac{2}{3} \mu_0 \mathbf{M} - \mathbf{M} \\
&= -\frac{1}{3} \mathbf{M}
\end{aligned}$$

or

$$\mathbf{H} = -\frac{1}{3} \mathbf{M} = -\frac{1}{3} M \mathbf{e}_z = -\frac{1}{3} M (\cos\theta \mathbf{e}_r - \sin\theta \mathbf{e}_\theta).$$

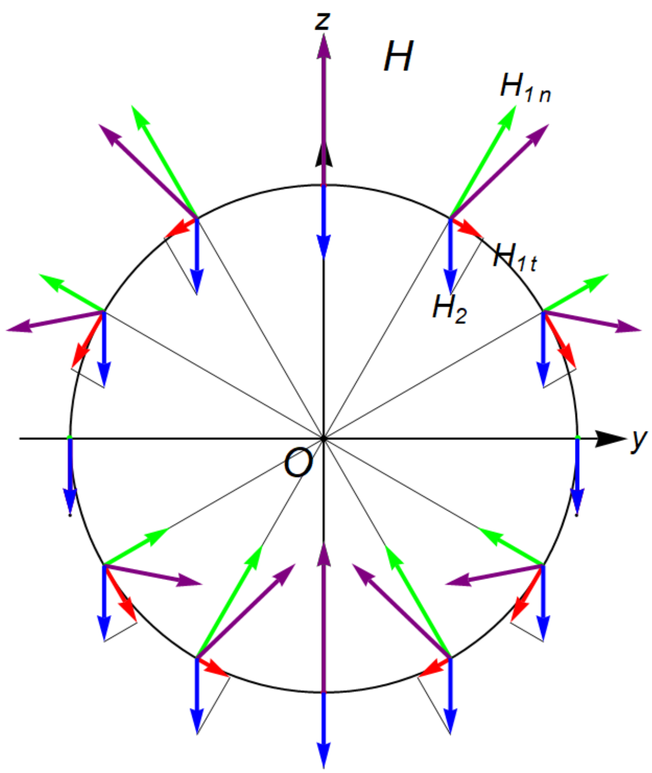


Fig.44 Boundary condition for the auxiliary field ***H*** inside (blue) and outside (green, red, purple) sphere with uniform magnetization *M*. The tangential component of the field H is continuous on the boundary of the sphere. $\boldsymbol{H}_2$ (blue) is the resultant field inside the sphere. $\boldsymbol{H}_1$ (purple) is the resultant field outside the sphere. $H_{1t} = H_{2t}$.

24. Vector potential and magnetic field from uniformly magnetized systems

(a) The cylindrical uniform magnetization.

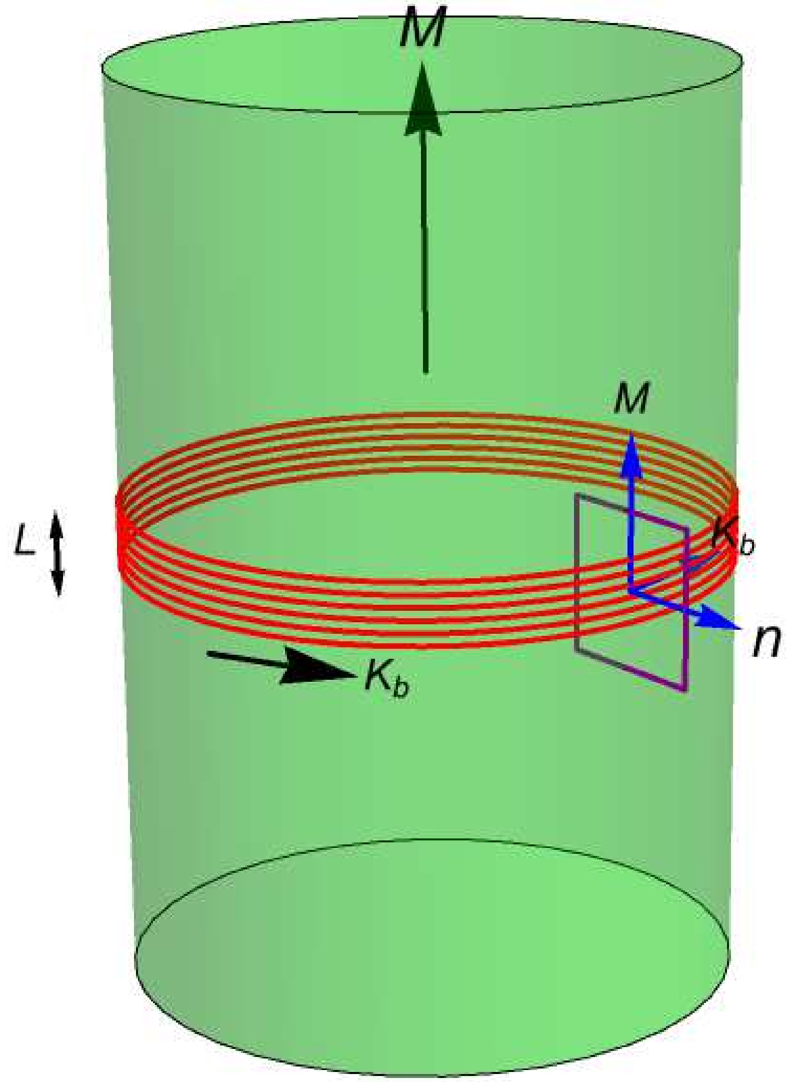


Fig.45 Ampère’s law for the surface current $\mathbf{K}_b = \mathbf{M} \times \mathbf{n}$ due to the cylindrical uniform magnetization.

$$\mathbf{J}_b = \nabla \times \mathbf{M} = 0 .$$

$$\mathbf{K}_b = \mathbf{M} \times \mathbf{n} = M \mathbf{e}_\phi$$

The magnetic field ***B*** inside of the cylinder. We apply the Ampere’s law

$$BL = \mu_0 K_b L = \mu_0 ML ,$$

or

$$B = \mu_0 M \ .$$

Note that the magnetic field outside the cylinder is zero.

(b) Circular ring with uniform magnetization inside

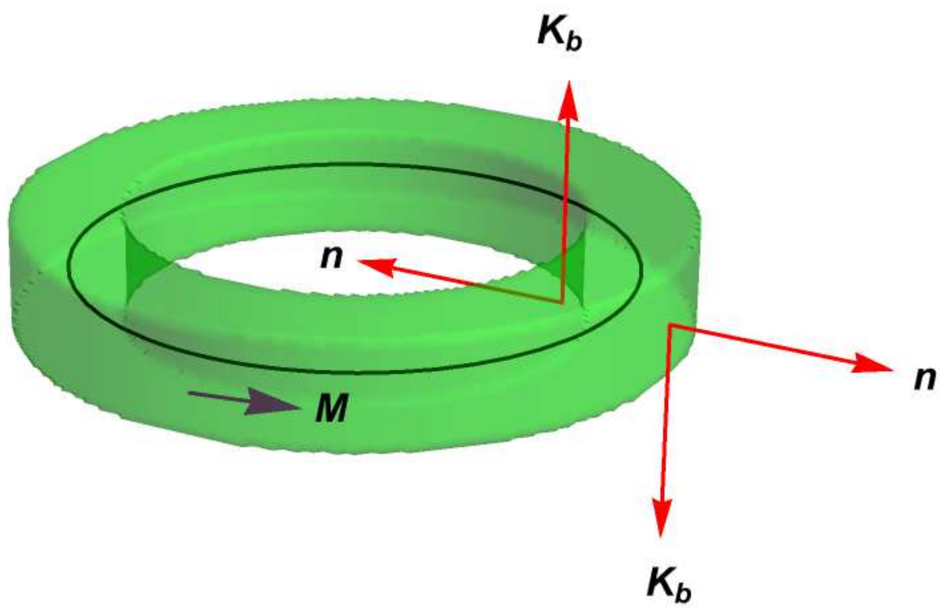


Fig.46 Ampère's law for the surface current $\mathbf{K}_b = \mathbf{M} \times \mathbf{n}$ due to the circular ring with uniform magnetization.

Ampère's law

$$B(2\pi R) = \mu_0 K_b (2\pi R)$$

or

$$B = \mu_0 K_b = \mu_0 M$$

Because of $\nabla \cdot \mathbf{B} = 0$, on the boundary, the component of $\boldsymbol{B}$ normal to the plane is continuous.

(c) Cylindrical shell with uniform magnetization

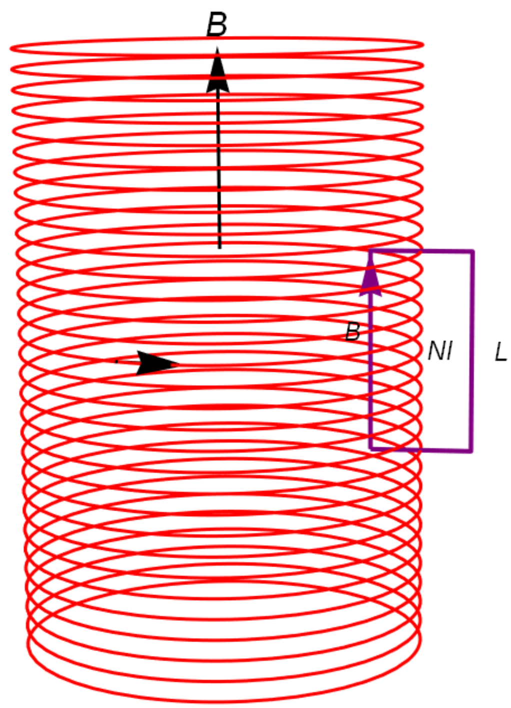


Fig.47 Cylindrical shell with uniform magnetization $\mathbf{M}$. Ampère's law for the surface current $\mathbf{K}_b = \mathbf{M} \times \mathbf{n}$ due to the solenoid. N is the total number of turns.

We apply the Ampère's law;

$$BL = \mu_0 NI \,, \qquad B = \mu_0 \frac{N}{L} I = \mu_0 M \,.$$

The magnetic moment:

$$m = NIA \qquad (\mathrm{Am}^2)$$

The magnetization:

$$M = \frac{m}{V} = \frac{NIA}{AL} = \frac{NI}{L} \qquad (\mathrm{A/m})$$

25. Summary

All the essential properties of the electricity and magnetism are included in the beautiful four Maxwell's equations. Mainly, we discussed the Maxwell's equation (I) and (III) in the stationary state above, where no time dependence of ***E***- and ***B***-field is taken into account. The ***B***- field lines cannot stop or start anywhere but must endlessly circulate in loops. The situation may change when the time dependence of ***E***- and ***B***-field is considered in Maxwell's equations (II) [the Faraday's law] and (IV) [Ampere-Maxwell law]. The circulation of the ***E***-field can be produced by the change of the ***B***-field with time (Faraday's law). The circulation of the ***B***-field can be produced by the change of ***E***-field with time, as well as an electrical current, flow of electric charge (Ampère's law);

(IV) $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ (Ampere's law)

However, Maxwell realized that an extra term (the displacement current) should be inserted in the Ampere's law, changing it as follows:

(IV') $\nabla \times \mathbf{B} = \mu_0 (\mathbf{J} + \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t})$ (Ampere-Maxwell's law)

This turned out to be the final piece of the jigsaw. It was already known that electricity and magnetism were connected. When we assume that $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\dfrac{\partial \mathbf{A}}{\partial t}$ with $\nabla \cdot \mathbf{A} =$ 0, $\mathbf{J} = 0$ and $\rho = 0$ (in vacuum), we get the wave equation for the vector potential ***A*** as

$$\begin{aligned}\nabla \times \mathbf{B} &= \nabla \times (\nabla \times \mathbf{A}) \\ &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \\ &= -\nabla^2 \mathbf{A}\end{aligned}$$

$$\varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} = -\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

leading to the wave equation for ***A*** as

$$(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \mathbf{A} = 0$$

In other words, the light (transverse electromagnetic wave) propagates with the velocity $c = 1/\sqrt{\varepsilon_0 \mu_0}$, as was predicted first by Maxwell. The direction of the electric field (the polarization vector) is perpendicular to that of propagation of the wave.

((**Nomenclature**))

A Vector potential

$\mathbf{M}$	Magnetization (magnetic moment per unit volume)	
$\mathbf{r}'$	Source coordinates	

$\mathbf{J}_b = \nabla \times \mathbf{M}$	Bound current density
$\mathbf{K}_b = \mathbf{M} \times \mathbf{n}$	Bound surface current density
I	Circulating current (loop current)
$\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f$	Total current density
$m = Ia_{cl}$	Magnetic moment
$\phi_m^{\ *}$	Magnetic scalar potential
ϕ_e	Electric scalar potential
$\mathbf{H} = \dfrac{\mathbf{B}}{\mu_0} - \mathbf{M}$	Auxiliary field
$\nabla \cdot \mathbf{B} = 0$	Maxwell's equation
$\nabla \cdot \mathbf{H} = \mathbf{J}_f$	($\mathbf{J}_f$; free current density)
$\varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}$	Displacement current

TABLE-1 Comparison of properties between magnetic moment and electric dipole moment

For comparison, here we show a list of the relation between the magnetic moment and electric dipole moment.

Electric dipole moment	**Magnetic moment**
$p = ql$ (electric dipole moment)	$m = Ia_{cl}$ (magnetic moment)

$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}$ (torque)	$\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}$ (torque)
$U_e = -\mathbf{p} \cdot \mathbf{E}$ (potential energy)	$U_m = -\mathbf{m} \cdot \mathbf{B}$ (potential energy)
	$\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}$ (vector potential)
$\mathbf{E} = -\nabla \phi_e$ (electric field)	$\mathbf{B} = -\mu_0 \nabla \phi_m{}^*$ (magnetic field)
	($\mathbf{H} = -\nabla \phi_m{}^*$)
$\phi_e = \frac{\mathbf{p} \cdot \mathbf{r}}{4\pi\varepsilon_0 r^3}$ (electric scalar potential)	$\phi_m{}^* = \frac{\mathbf{m} \cdot \mathbf{r}}{4\pi r^3}$ (magnetic scalar potential)
$\nabla \cdot \mathbf{E} = 0$	$\nabla \cdot \mathbf{B} = 0$
$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}[-\frac{\mathbf{p}}{r^3} + \frac{3(\mathbf{p} \cdot \mathbf{r})\mathbf{r}}{r^5}]$	$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0}{4\pi}[-\frac{\mathbf{m}}{r^3} + \frac{3(\mathbf{m} \cdot \mathbf{r})\mathbf{r}}{r^5}]$
$\mathbf{E} = \frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta \mathbf{e}_r + \sin\theta \mathbf{e}_\theta)$ (Spherical coordinates)	$\mathbf{B} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta \mathbf{e}_r + \sin\theta \mathbf{e}_\theta)$ (Spherical coordinates)
$\mathbf{E} = \frac{p}{4\pi\varepsilon_0 r^5}[3zx\mathbf{e}_x + 3yz\mathbf{e}_y + (x^2 + y^2 - 2z^2)\mathbf{e}_z]$ (Cartesian coordinates)	$\mathbf{B} = \frac{\mu_0 m}{4\pi r^5}[3zx\mathbf{e}_x + 3yz\mathbf{e}_y + (x^2 + y^2 - 2z^2)\mathbf{e}_z]$ (Cartesian coordinates)

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APPENDIX-I

Vector potential *A* and magnetic field due to magnetic moment (Cartesian coordinates) Mathematica Program

Magnetic field, Vector potential due to current loop (Cartesian coordinates)

```
Clear["Global`"];
ux = {1, 0, 0};
uy = {0, 1, 0};
uz = {0, 0, 1};
r = {x, y, z};
R = √(r.r) ;
Lap := Laplacian[#, { x, y, z}, "Cartesian"] &;
Gra := Grad[#, { x, y, z}, "Cartesian"] &;
Curla := Curl[#, {x, y,  z}, "Cartesian"] &;
Diva := Div[#, {x, y,  z}, "Cartesian"] &;
```

M1 = m1 uz; A1 = $\frac{\mu 0}{4\pi R^3}$ Cross[M1, r] // Simplify

$$\left\{-\frac{m1\, y\, \mu 0}{4\pi\left(x^2+y^2+z^2\right)^{3/2}},\ \frac{m1\, x\, \mu 0}{4\pi\left(x^2+y^2+z^2\right)^{3/2}},\ 0\right\}$$

```
B1 = Curla[A1] // Simplify
```

$$\left\{\frac{3\, m1\, x\, z\, \mu 0}{4\pi\left(x^2+y^2+z^2\right)^{5/2}},\ \frac{3\, m1\, y\, z\, \mu 0}{4\pi\left(x^2+y^2+z^2\right)^{5/2}},\ -\frac{m1\left(x^2+y^2-2z^2\right)\mu 0}{4\pi\left(x^2+y^2+z^2\right)^{5/2}}\right\}$$

```
Diva[B1] // Simplify
0

Curla[B1] // Simplify
{0, 0, 0}
```

APPENDIX-II

Vector potential A and magnetic field due to magnetic moment

Magnetic field, vector potential due to current loop (spherical coordinates)

```
Clear["Global`"];
SetCoordinates[Spherical[r, θ, ϕ]];
ux = {Sin[θ] Cos[ϕ], Cos[θ] Cos[ϕ], -Sin[ϕ]};
uy = {Sin[θ] Sin[ϕ], Cos[θ] Sin[ϕ], Cos[ϕ]};
uz = {Cos[θ], -Sin[θ], 0};
ur = {1, 0, 0};
Lap := Laplacian[#, { r, θ, ϕ}, "Spherical"] &;
Gra := Grad[#, { r, θ, ϕ}, "Spherical"] &;
Diva := Div[#, {r, θ, ϕ}, "Spherical"] &;
Curla := Curl[#, {r, θ, ϕ}, "Spherical"] &;
```

m1 = m 1 uz; r1 = r ur; A1 = $\frac{\mu 0}{4 \pi r^3}$ Cross[m1, r1]

$\left\{0, 0, \frac{m\, \mu 0 \,\mathrm{Sin}[\theta]}{4 \pi r^2}\right\}$

B1 = Curla[A1] // Simplify

$\left\{\frac{m\, \mu 0 \,\mathrm{Cos}[\theta]}{2 \pi r^3}, \frac{m\, \mu 0 \,\mathrm{Sin}[\theta]}{4 \pi r^3}, 0\right\}$

Diva[B1]

0

Curla[B1]

{0, 0, 0}