Chapter 14
Green's function; fundamental
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George Green (14 July 1793 – 31 May 1841) was a British mathematician and physicist, who wrote *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism* (Green, 1828). The essay introduced several important concepts, among them a theorem similar to the modern Green's theorem, the idea of potential functions as currently used in physics, and the concept of what are now called Green's functions. George Green was the first person to create a mathematical theory of electricity and magnetism and his theory formed the foundation for the work of other scientists such as James Clerk Maxwell, William Thomson, and others. His work ran parallel to that of the great mathematician Gauss (potential theory).

http://en.wikipedia.org/wiki/George_Green

14.1 What is a Green's function?

We now consider the equation

$$L_x y + f(x) = 0,$$

where L_x is the self-adjoint differential operator,

$$L_x y = \frac{d}{dx}[p(x)y'] + q(x)y. \qquad \text{(self-adjoint)}$$

The solution of this equation is given by

$$y(x) = \int_a^b G(x,\xi) f(\xi) d\xi + \varphi(x).$$

Here the Green's function is defined by

$$L_x G(x,\xi) = -\delta(x-\xi),$$

and

$$L_x \varphi(x) = 0. \qquad \text{[arbitrary function } \varphi(x)\text{]}.$$

((**Proof**))

$$L_x y(x) = \int_a^b L_x G(x,\xi) f(\xi) d\xi + L_x \varphi(x) = -\int_a^b \delta(x-\xi) f(\xi) d\xi = -f(x).$$

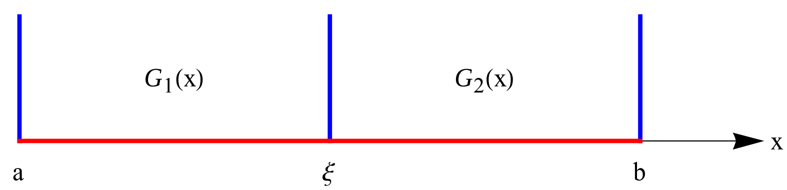
We need to find an explicit form of $G(x, \xi)$.

14.2 Construction of Green's function

$$L_x y = \frac{d}{dx}[p(x)y'(x)] + q(x)y(x)\,, \qquad \text{(self- adjoint)}$$

where L_x is the self-adjoint differential operator. The Green's function satisfies the differential equation given by

$$L_x G(x,\xi) = -\delta(x-\xi)\,,$$



(i) We define the Green's function

$$G(x) = G_1(x)\,, \qquad \text{for } a \le x < \xi$$

$$G(x) = G_2(x)\,. \qquad \text{for } a \le \xi < x \le b$$

(ii) $G_1(x)$ and $G_2(x)$ satisfy the following equations,

$$L_x G_1(x) = 0\,, \qquad \text{for } a \le x < \xi$$

$$L_x G_2(x) = 0\,. \qquad \text{for } a \le \xi < x \le b$$

(iii)

At $x = a$, $G_1(x)$ satisfies the homogeneous boundary condition;

$G_1(a) = 0$ or $G_1'(a) = 0$ or $\alpha\, G_1(a) + \beta\, G_1'(a) = 0$.

Similarly, at $x = b$, $G_2(x)$ satisfies the homogeneous boundary condition;

$G_2(b) = 0$ or $G_2'(b) = 0$ or $\alpha\, G_2(b) + \beta\, G_2'(b) = 0$.

(iv) Continuity at $x = \xi$.

$$\lim_{x\to\xi-\varepsilon} G_1(x) = \lim_{x\to\xi+\varepsilon} G_2(x).$$

where $\varepsilon \to 0$ ($\varepsilon>0$).

(v) We require that $G'(x)$ be discontinuous at $x = \xi$,

$$G_2'(x)|_{\xi+\varepsilon} - G_1'(x)|_{\xi-\varepsilon} = -\frac{1}{p(\xi)}.$$

((**Proof**))

$$\int_{\xi-\varepsilon}^{\xi+\varepsilon}[\frac{d}{dx}[p(x)G'(x)] + q(x)G(x)]dx = -\int_{\xi-\varepsilon}^{\xi+\varepsilon}\delta(x-\xi)dx = -1,$$

or

$$[p(x)G'(x)]_{\xi-\varepsilon}^{\xi+\varepsilon} + \int_{\xi-\varepsilon}^{\xi+\varepsilon} q(x)G(x)dx = -1$$

The second term is equal to zero in the limit of $\varepsilon\to0$ since the integrant is continuous at $x = \xi$.

Then we have

$$G_2'(x)|_{\xi+\varepsilon} - G_1'(x)|_{\xi-\varepsilon} = -\frac{1}{p(\xi)}$$

14.3 The symmetric nature in $G(x, \xi)$

We assume that

$L_x u(x) = 0$ for $a \le x < \xi$ with the homogeneous boundary condition at $x = a$.

$L_x v(x) = 0$ for $\xi < x < b$ with the homogeneous boundary condition at $x = b$.

Then the Green's function is expressed by

$G(x, \xi) = c_1 u(x)$ for $a \le x < \xi$,

$G(x, \xi) = c_2 v(x)$ for $\xi < x \le b$,

where c_1 and c_2 are constants.

(i) The continuity of $G(x, \xi)$ at $x = \xi$;

$$c_1 u(\xi) - c_2 v(\xi) = 0 .$$

(ii) The discontinuity in $G(x, \xi)$ at $x = \xi$,

$$c_1 u'(\xi) - c_2 v'(\xi) = \frac{1}{p(\xi)} .$$

This equation is closely related to the Wronskian determinant (see Chapter 5). The Wronskian determinant is defined as

$$W(\xi) = \begin{vmatrix} u(\xi) & v(\xi) \\ u'(\xi) & v'(\xi) \end{vmatrix} ,$$

$$L_x G = \frac{d}{dx}[p(x)G'] + q(x)G = pG'' + p'G' + qG = 0 .$$

We take the derivative,

$$W'(\xi) = \begin{vmatrix} u(\xi) & v(\xi) \\ u''(\xi) & v''(\xi) \end{vmatrix} = \begin{vmatrix} u(\xi) & v(\xi) \\ -\frac{1}{p}(p'u' + qu) & -\frac{1}{p}(p'v' + qv) \end{vmatrix} ,$$

or

$$W'(\xi) = -\frac{p'}{p} \begin{vmatrix} u(\xi) & v(\xi) \\ u'(\xi) & v'(\xi) \end{vmatrix} = -\frac{p'}{p} W(\xi) .$$

Then we have

$$W = u(\xi)v'(\xi) - u'(\xi)v(\xi) = \frac{A}{p(\xi)}$$

for the independence $u(x)$ and $v(x)$, where A is constant. Then we have the following equations,

$$c_1 u'(\xi) - c_2 v'(\xi) = \frac{1}{p(\xi)} ,$$

$$c_1 u(\xi) - c_2 v(\xi) = 0 ,$$

Here we define A as

$$A = p(\xi)[u(\xi)v'(\xi) - u'(\xi)v(\xi)] = p(\xi)W(\xi).$$

From these equations, we have

$$c_1 = -\frac{v(\xi)}{A}, \quad \text{and} \quad c_2 = -\frac{u(\xi)}{A}.$$

Thus we have

$$G(x,\xi) = -\frac{1}{A}u(x)v(\xi) \text{ for } a \le x < \xi,$$

$$G(x,\xi) = -\frac{1}{A}u(\xi)v(x) \text{ for } \xi < x \le b.$$

We find that

$$G(x,\xi) = G(\xi,x). \qquad \text{(Symmetry property)}$$

14.4 The check of the solution

Using the Green's function $G(x, \xi)$, we get the solution of

$$L_x y + f(x) = 0,$$

where L_x is the self-adjoint differential operator, and is given by

$$L_x y = \frac{d}{dx}[p(x)y'] + q(x)y. \qquad \text{(self-adjoint)}$$

The boundary condition (homogeneous) is given by

$$\alpha y(a) + \beta y'(a) = 0, \qquad \alpha y(b) + \beta y'(b) = 0.$$

The solution of this equation is obtained as

$$y(x) = \int_a^b G(x,\xi) f(\xi) d\xi.$$

Here we firm that this is a solution of the differential equation.

$$
\begin{aligned}
y(x) &= -\frac{1}{A}\int_a^x u(\xi)v(x)f(\xi)d\xi - \frac{1}{A}\int_x^b u(x)v(\xi)f(\xi)d\xi \\
&= -\frac{v(x)}{A}\int_a^x u(\xi)v(x)f(\xi)d\xi - \frac{u(x)}{A}\int_x^b v(\xi)f(\xi)d\xi
\end{aligned}
\qquad (1)
$$

For $y'(x)$, we have

$$
\begin{aligned}
y'(x) &= -\frac{1}{A}\int_a^x u(\xi)v'(x)f(\xi)dt - \frac{1}{A}u(x)v(x)f(x) \\
&-\frac{1}{A}\int_x^b u'(x)v(\xi)f(\xi)d\xi + \frac{1}{A}u(x)v(x)f(x)
\end{aligned}
$$

or

$$
y'(x) = -\frac{1}{A}v'(x)\int_a^x u(\xi)f(\xi)d\xi - \frac{1}{A}u'(x)\int_x^b v(\xi)f(\xi)d\xi\,. \qquad (2)
$$

For $y''(x)$, we also have

$$
\begin{aligned}
y''(x) &= -\frac{1}{A}v''(x)\int_a^x u(\xi)f(\xi)d\xi - \frac{1}{A}u''(x)\int_x^b v(\xi)f(\xi)d\xi \\
&-\frac{1}{A}[u(x)v'(x) - u'(x)v(x)]f(x)
\end{aligned}
$$

or

$$
y''(x) = -\frac{1}{A}v''(x)\int_a^x u(\xi)f(\xi)d\xi - \frac{1}{A}u''(x)\int_x^b v(\xi)f(\xi)d\xi - \frac{f(x)}{p(x)}\,. \qquad (3)
$$

From Eqs.(1), (2), and (3), we obtain

$$
\begin{aligned}
L_x y &= \frac{d}{dx}[p(x)y'] + q(x)y = p(x)y'' + p'(x)y' + q(x)y \\
&= -\frac{1}{A}L_x v\int_a^x u(\xi)f(\xi)dt - \frac{1}{A}L_x u\int_x^b v(\xi)f(\xi)d\xi - f(x)\,, \\
&= -f(x)
\end{aligned}
$$

or

$$
Ly = -f(x)\,.
$$

14.5 Boundary condition for the Green's function

We consider the Green's function $G(x, \xi)$ with the homogeneous boundary condition at $x = a$ and b.

$$y(x) = -\frac{v(x)}{A}\int_a^x u(\xi)f(\xi)d\xi - \frac{u(x)}{A}\int_x^b v(\xi)f(\xi)d\xi \,.$$

The values of $y(a)$, $y'(a)$, $y(b)$ and $y'(b)$ are obtained as

$$y(a) = -\frac{u(a)}{A}\int_a^b v(\xi)f(\xi)d\xi = -c_1 u(a)\,,$$

$$y(b) = -\frac{v(b)}{A}\int_a^b u(\xi)f(\xi)d\xi = -c_2 v(b)\,,$$

$$y'(a) = -\frac{u'(a)}{A}\int_a^b v(\xi)f(\xi)d\xi = -c_1 u(a)\,,$$

$$y'(b) = -\frac{v'(b)}{A}\int_a^b u(\xi)f(\xi)d\xi = -c_2 v'(b)\,,$$

where

$$c_1 = \frac{1}{A}\int_a^b v(\xi)f(\xi)d\xi\,, \qquad c_2 = \frac{1}{A}\int_a^b u(\xi)f(\xi)d\xi$$

Then we have

$$\alpha y(a) + \beta y'(a) = -c_1[\alpha u(a) + \beta u'(a)]$$

and

$$\alpha y(b) + \beta y'(b) = -c_2[\alpha u(b) + \beta u'(b)]$$

where α and β are constants. Thus the boundary conditions for the Green's function are given by the same boundary condition as $y(x)$,

$$\alpha u(a) + \beta u'(a) = 0\,, \qquad \alpha v(b) + \beta v'(b) = 0$$

In conclusion, given the linear differential operator L_x (acting on the variable x), the solution $y(x)$ of the differential equation $L_x y(x) = -f(x)$ can be obtained from the Green's function $G(x, \xi)$ by

$$y(x) = \int_{-\infty}^{\infty} G(x,\xi) f(\xi) d\xi .$$

The Green's function obeys the differential equation

$$L_x G(x,\xi) = -\delta(x-\xi) .$$

with the same boundary condition as the solution $y(x)$.

14.6 Example: homogeneous boundary condition

Show that

$$G(x,\xi) = x(1-\xi) \text{ for } 0 < x < \xi ,$$

$$G(x,\xi) = \xi(1-x) \text{ for } 0 < \xi < x < 1 ,$$

is the Green's function for

$$L_x y = y'' ,$$

where

$$y(0) = 0, \qquad y(1) = 0.$$

((**Solution**))

$$\begin{cases} L_x u = 0 & 0 \le x < \xi < 1 \\ L_x v = 0 & 0 \le \xi < x \le 1 \end{cases}$$

$$u'' = 0 \text{, or } u(x) = c_1 x + c_2 \qquad (0 \le x < \xi)$$

$$v'' = 0 \text{, or } v(x) = c_1' x + c_2' \qquad (\xi \le x \le 1)$$

Boundary condition:

$$u(0) = 0 \to c_2 = 0 \qquad u(x) = c_1 x$$

$$v(1) = 0 \to c_1' + c_2' = 0 \quad v(x) = c_1'(x-1)$$

$$G(x,\xi)=\begin{cases} c_1 x & 0\le x<\xi \\ c_1'(x-1) & \xi<x\le 1\end{cases}$$

The continuity of $G(x,\xi)$ at $x=\xi$:

$$c_1\xi = c_1'(\xi-1).$$

The discontinuity of $dG(x,\xi)/dx$ at $x=\xi$:

$$c_1'-c_1=-1.$$

Then we have

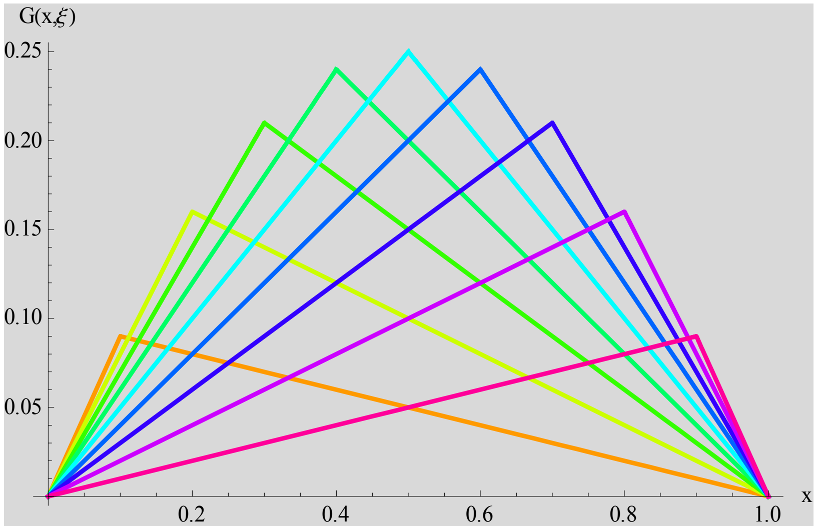
$$c_1 = 1-\xi, \qquad \text{and} \qquad c_1' = -\xi.$$

or

$$G(x,\xi)=\begin{cases} x(1-\xi) & 0\le x<\xi \\ \xi(1-x) & \xi<x\le 1\end{cases}.$$

((**Mathematica**))

```
Clear["Global`*"];

eq1 = DSolve[{G''[x] == -DiracDelta[x - ξ], G[0] == 0, G[1] == 0}, G[x], x] // Simplify

{{G[x] → (x - x ξ) HeavisideTheta[1 - ξ] +
    (-x + ξ) HeavisideTheta[x - ξ] + (-1 + x) ξ HeavisideTheta[-ξ]}}

G[x_] = G[x] /. eq1[[1]];

Simplify[G[x], 1 > x > ξ > 0] // Factor

-(-1 + x) ξ

Simplify[G[x], 0 < x < ξ < 1] // Factor

-x (-1 + ξ)
```



14.7 Example: boundary condition

$$L_x y = y'',$$

with

$$y(0) + y(1) = 0, \text{ and } y'(0) + y'(1) = 0$$

The Green's function is obtained as

$$G(x,\xi) = -\frac{1}{2}\left|x-\xi\right| + \frac{1}{4}.$$

((**Solution**))

$$\begin{cases} L_x u = 0 & 0 \le x < \xi < 1 \\ L_x v = 0 & 0 \le \xi < x \le 1 \end{cases}.$$

$$u'' = 0, \text{ or } u(x) = c_1 x + c_2 \qquad (0 \le x < \xi)$$

$$v'' = 0, \text{ or } v(x) = c_1' x + c_2' \qquad (\xi \le x \le 1)$$

Boundary condition:

$$u(0) + v(1) = 0 \to c_2 + c_1' + c_2' = 0,$$

$$u'(0) + v'(1) = 0 \to c_1 + c_1' = 0$$

Continuity of $G(x,\xi)$ at $x=\xi$:

$$c_1\xi + c_2 = c_1'\xi + c_2'.$$

Discontinuity of $dG(x,\xi)/dx$ at $x=\xi$:

$$c_1' - c_1 = -1.$$

Then we have

$$c_1 = \frac{1}{2}, \qquad c_2 = \frac{1}{4}(1-2\xi), \qquad c_1' = -\frac{1}{2}, \qquad c_2 = \frac{1}{4}(1+2\xi)$$

$$G(x,t) = \begin{cases} \frac{x}{2} + \frac{1}{4}(1\text{-}2\xi) & 0 \le x < \xi \\ -\frac{x}{2} + \frac{1}{4}(1+2\xi) & \xi < x \le 1 \end{cases}.$$

((**Mathematica**))

```
Clear["Global`*"]

eq1 = G''[x] == -DiracDelta[x - ξ];

eq2 = DSolve[{eq1, G[0] + G[1] == 0, G'[0] + G'[1] == 0}, G[x], x] //
  Simplify
```

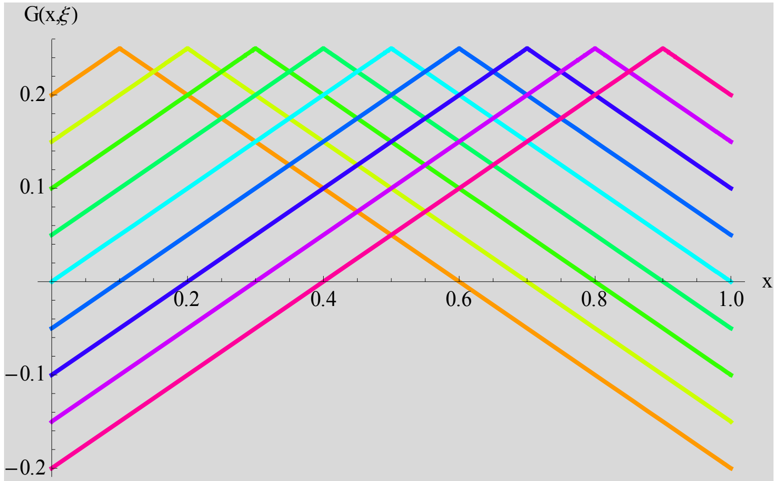
$$\left\{\left\{G[x] \to \frac{1}{4}\left((1+2x-2\xi)\,\text{HeavisideTheta}[1-\xi] - 4(x-\xi)\,\text{HeavisideTheta}[x-\xi] + (-1+2x-2\xi)\,\text{HeavisideTheta}[-\xi]\right)\right\}\right\}$$

```
G[x_] = G[x] /. eq2[[1]];

Simplify[G[x], 0 < x < ξ < 1]
```

$$\frac{1}{4}(1+2x-2\xi)$$

```
Simplify[G[x], 0 < ξ < x < 1]
```

$$\frac{1}{4}(1-2x+2\xi)$$



14.8 Example: inhomogeneous boundary condition

Arfken 10.5.1

Find the Green's function for

$$L_x y = y'' ,$$

with

$$y(0) = 0 , \qquad y'(1) = 0 .$$

((**Solution**))

$$\begin{cases} Lu = 0 & 0 \le x < \xi < 1 \\ Lv = 0 & 0 \le \xi < x \le 1 \end{cases} .$$

$$u'' = 0 \text{, or } u(x) = c_1 x + c_2 , \qquad (0 \le x < \xi)$$

$$v'' = 0 \text{, or } v(x) = c_1' x + c_2' , \qquad (\xi \le x \le 1)$$

Boundary condition:

$$u(0) = 0 \to c_2 = 0 \qquad u(x) = c_1 x .$$

$$v'(1) = 0 \to c_1' = 0 \qquad v(x) = c_2' .$$

$$G(x, \xi) = \begin{cases} c_1 x & 0 \le x < \xi \\ c_2' & \xi < x \le 1 \end{cases} .$$

Continuity of $G(x,\xi)$ at $x=\xi$:

$$c_1\xi = c_2'.$$

Discontinuity of $dG(x,\xi)/dx$ at $x=\xi$:

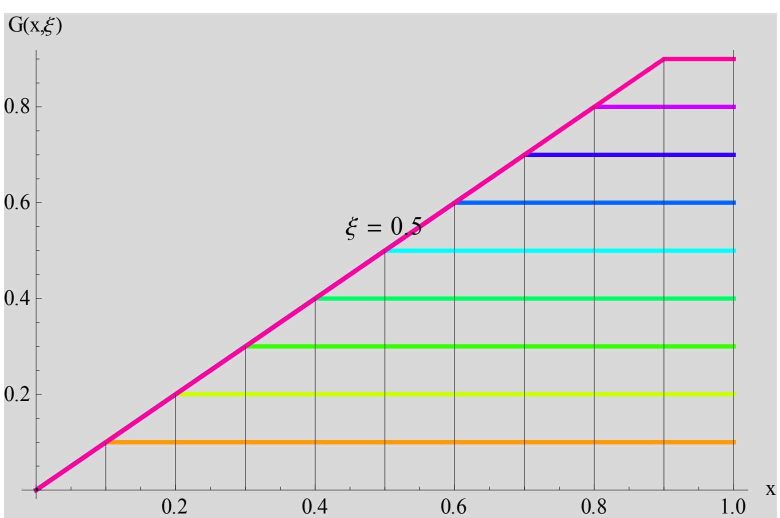
$$0 - c_1 = -1.$$

$$c_1 = 1,\ c_2' = \xi.$$

$$G(x,\xi) = \begin{cases} x & 0 \le x < \xi \\ \xi & \xi < x \le 1 \end{cases}.$$

((**Mathematica**))

```
Clear["Global`*"];

eq1 = G''[x] == -DiracDelta[x - ξ] // Simplify;

eq2 = DSolve[{eq1, G[0] == 0, G'[1] == 0}, G[x], x] // Simplify;

G[x_] = G[x] /. eq2[[1]]

x HeavisideTheta[1 - ξ] + (-x + ξ) HeavisideTheta[x - ξ] - ξ HeavisideTheta[-ξ]

eq3 = FullSimplify[G[x], 0 < x < ξ < 1]

x

eq4 = Simplify[G[x], 0 < ξ < x < 1]

ξ
```



14.9 Example: homogeneous boundary condition

Find the Green's function for

$$L_x y(x) = y'' + y,$$

with the boundary condition,

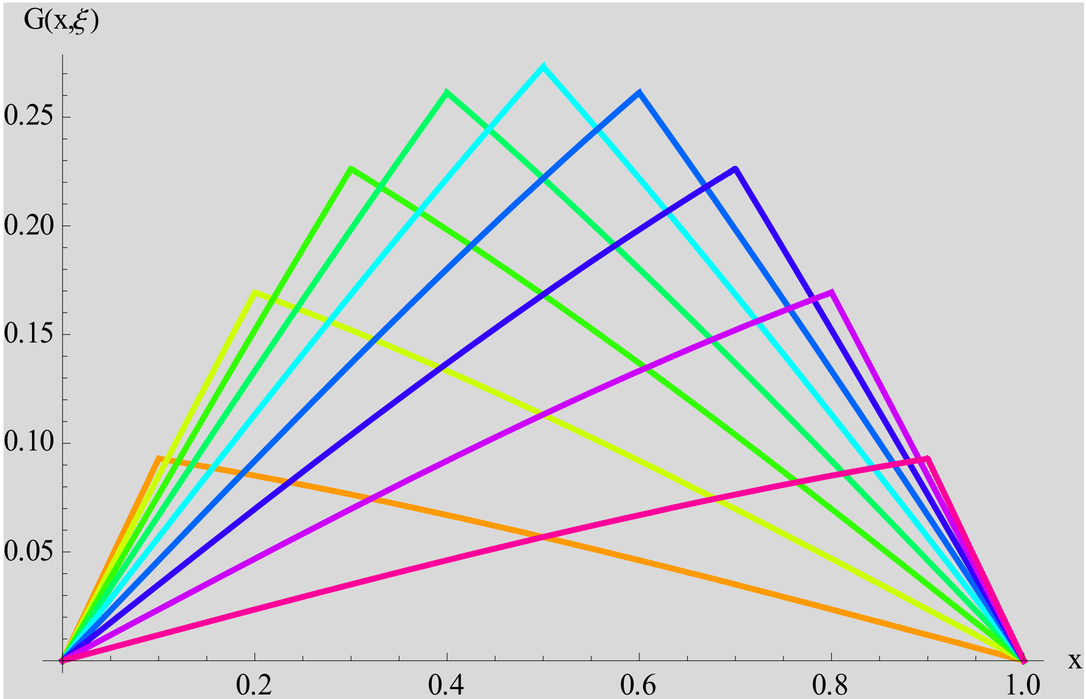
$$y(0) = 0, \text{ and } y(1) = 0.$$

The Green's function is obtained as

$$G(x,\xi) = \begin{cases} \csc(1)\sin(1-x)\sin(\xi) & 0 \le x < \xi \\ \csc(1)\sin(x)\sin(1-\xi) & \xi < x \le 1 \end{cases}$$

((**Mathematica**))

```
Clear["Global`*"];

eq1 = G''[x] + G[x] == - DiracDelta[x - ξ];

eq2 = DSolve[{eq1, G[0] == 0, G[1] == 0}, G[x], x];

G[x_] = G[x] /. eq2[[1]];

G1 = Simplify[G[x], 1 > x > ξ > 0] // TrigFactor

Csc[1] Sin[1 - x] Sin[ξ]

G2 = Simplify[G[x], 0 < x < ξ < 1] // TrigFactor

Csc[1] Sin[x] Sin[1 - ξ]
```



14.10 Example: inhomogeneous boundar condition

Arfken 10.5.2 (a)

Find the Green's function for

$$L_x y = y'' + y\,,$$

with

$$y(0) = 0\,, \qquad y'(1) = 0\,.$$

((**Solution**))

(i)

$$L_x u = 0 \qquad 0 \le x < \xi$$

$$u'' + u = 0$$

$$u = c_1 \sin x + c_2 \cos x$$

with the boundary condition

$$u(0) = 0\,,$$

$$c_2 = 0\,,$$

$$u = c_1 \sin x .$$

(ii)

$$L_x v = 0 \qquad \xi < x \le 1$$

$$v'' + v = 0$$

$$v = c_1' \sin x + c_2' \cos x$$

with the boundary condition,

$$v'(1) = 0 ,$$

$$c_2 = 0 ,$$

$$v' = c_1' \cos x - c_2' \sin x ,$$

$$c_1' \cos 1 = c_2' \sin 1 . \qquad (1)$$

(iii) Continuity of $G(x,\xi)$ at $x = \xi$:

$$c_1 \sin \xi = c_1' \sin \xi + c_2' \cos \xi . \qquad (2)$$

(iv) Discontinuity of $dG(x,\xi)/dx$ at $x = \xi$

$$-c_1 \cos \xi + (c_1' \cos \xi - c_2' \sin \xi) = -1 . \qquad (3)$$

From Eq.(1) , we have

$$c_2' = \cot 1 \cdot c_1'$$

Then

$$c_1 \sin \xi = c_1' \sin \xi + c_1' \cot 1 \cos \xi$$

$$c_1' = \frac{\sin \xi}{\sin \xi + \cot 1 \cos \xi} c_1$$

or

$$c_1 = \cos(1-\xi) \sec(1) , \quad c_2 = 0$$

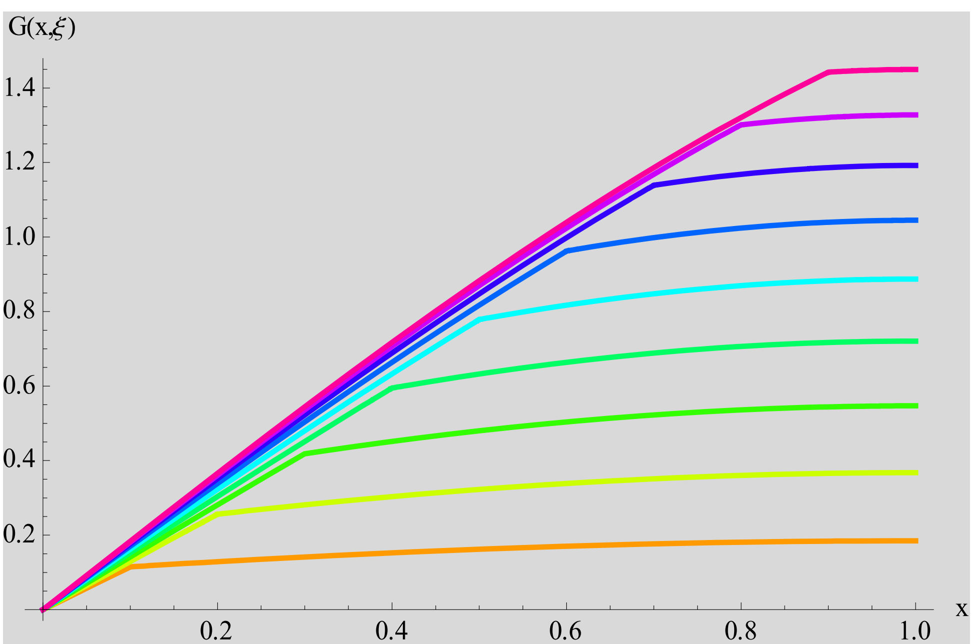
$$c_1' = \sin(\xi)\tan(1), \qquad c_2' = \sin(\xi),$$

and

$$G(x,\xi) = \begin{cases} \cos(1-\xi)\sec(1)\sin x & 0 \le x < \xi \\ \cos(1-x)\sec(1)\sin\xi & \xi < x \le 1 \end{cases}$$

((**Mathematica**))

```
Clear["Global`*"];

eq1 = G''[x] + G[x] == - DiracDelta[x - ξ];

eq2 = DSolve[{eq1, G[0] == 0, G'[1] == 0}, G[x], x];

G[x_] = G[x] /. eq2[[1]];

G1 = Simplify[G[x], 1 > x > ξ > 0] // TrigFactor

Cos[1 - x] Sec[1] Sin[ξ]

G2 = Simplify[G[x], 0 < x < ξ < 1] // TrigFactor

Cos[1 - ξ] Sec[1] Sin[x]
```



14.11 Eigenfunction and Green's function

We assume that

$$L_x u_n + \lambda_n w u_n = 0$$

where L_x is the Sturm-Liouville differential operator, $\{u_n\}$ is the eigenfunction, λ_n is the eigenvalue, and w is the weight function.

We now consider the problem

$$L_x y + f = 0.$$

Since the eigenfunctions of L forms a complete set, y may be written as a superposition of eigenfunction;

$$y = \sum_n c_n u_n .$$

Thus we have

$$f = -Ly = -L(\sum_n c_n u_n) = -\sum_n c_n L u_n = \sum_n c_n (\lambda_n w u_n)$$

$$\int_a^b u_m^* f dx = \int_a^b u_m^* \sum_n c_n (\lambda_n w u_n) dx$$

$$= \sum_n \lambda_n c_n \int_a^b u_m^* (w u_n) dx = c_m \lambda_m \int_a^b u_m^* (w u_m) dx$$

or

$$c_m = \frac{1}{\lambda_m} \frac{\int_a^b u_m^*(\xi) f(\xi) d\xi}{\int_a^b u_m^*(\xi)(w(\xi) u_m(\xi) d\xi}$$

If we work with normalized $u_n(x)$, so that

$$\int_a^b u_m^*(\xi)(w(\xi) u_m(\xi) d\xi = 1$$

or

$$c_n = \frac{1}{\lambda_n} \int_a^b u_n^*(\xi) f(\xi) d\xi$$

Then

$$y(x) = \sum_n \frac{1}{\lambda_n} \int_a^b u_n^*(\xi) u_n(x) f(z) dz = \int_a^b G(x, \xi) f(\xi) d\xi$$

where

$$G(x, \xi) = \sum_n \frac{1}{\lambda_n} u_n^*(\xi) u_n(x)$$

This is a Green's function. As a sanity check, we have

$$\begin{aligned} LG(x,\xi) &= \sum_n \frac{1}{\lambda_n} u_n^*(\xi) L u_n(x) \\ &= \sum_n \frac{1}{\lambda_n} u_n^*(\xi)[-\lambda_n \omega(x) u_n(x)] \\ &= -\sum_n u_n^*(\xi) \omega(x) u_n(x) \\ &= -\delta(x - \xi) \end{aligned}$$

((**Note**))

$$\delta(x - \xi) = \sum_n u_n *(x) w(x) u_n(\xi) = \sum_n u_n *(\xi) w(\xi) u_n(x) .$$

For any arbitrary function $\psi(x)$, we have

$$\psi(x) = \sum_n a_n u_n(x)$$

Using the relation

$$\begin{aligned} \int_a^b u_n *(x) w(x) \psi(x) dx &= \sum_m a_m \int_a^b u_n *(x) w(x) u_m(x) dx \\ &= \sum_m a_m \delta_{nm} = a_n \end{aligned}$$

Then

$$\psi(\xi)=\sum_n a_n u_n(\xi)$$

$$=\sum_n u_n(\xi)\int_a^b u_n{}^*(x)w(x)\psi(x)dx$$

$$=\int_a^b \sum_n u_n{}^*(x)u_n(\xi)w(x)\psi(x)dx$$

From the definition of the delta function,

$$\delta(x-\xi)=\sum_n u_n{}^*(x)w(x)u_n(\xi)=\sum_n u_n{}^*(\xi)w(\xi)u_n(x)\,.$$

14.12 Example

Find an appropriate Green's function for the equation

$$L_x y = y''+\frac{1}{4}y\,.$$

with the boundary condition $y(0)=y(\pi)=0$

$$L_x y_n+\lambda_n y_n=0\,,$$

where

$$L_x y=\frac{d}{dx}(y')+\frac{1}{4}y$$

$$u_n''+(\lambda_n+\frac{1}{4})u_n=0\,.$$

We put

$$u_n''+\omega_n{}^2 u_n=0$$

where

$$\omega_n^2=\lambda_n+\frac{1}{4}$$

and the boundary condition $u_n(0)=u_n(\pi)=0$

We have a solution for $u_n(x)$

$$u_n = \sqrt{\frac{2}{\pi}} \sin(nx)$$

with

$$\omega_n^2 = n^2 = \lambda_n + \frac{1}{4}$$

The Green's function is

$$G(x,\xi) = \sum_n \frac{1}{\lambda_n} u_n^*(\xi) u_n(x) = \frac{2}{\pi} \sum_{n=0} \frac{\sin(n\xi)\sin(nx)}{n^2 - \frac{1}{4}}$$

This should be equal to

$$G(x,\xi) = 2\cos(\frac{\xi}{2})\sin(\frac{x}{2}) \quad \text{for } 0<x<\xi<\pi$$

and

$$G(x,\xi) = 2\cos(\frac{x}{2})\sin(\frac{\xi}{2}) \quad \text{for } 0<\xi<x<\pi.$$

((**Mathematica**))

The Green's function can be derived using the Mathematica as follows.

$$G''(x) + \frac{1}{4} G(x) = -\delta(x-\xi) \,,$$

with $G(0) = 0$, and $G(\pi) = 0$.

```
Clear["Global`*"];

eq1 = G''[x] + 1/4 G[x] == -DiracDelta[x - ξ];

eq2 = DSolve[{eq1, G[0] == 0, G[π] == 0}, G[x], x];

G[x_] = G[x] /. eq2[[1]];

G[x] // Simplify[#, {0 < x < ξ < π}] &
```

$2\,\mathrm{Cos}\left[\frac{\xi}{2}\right]\,\mathrm{Sin}\left[\frac{x}{2}\right]$

```
G[x] // Simplify[#, {0 < ξ < x < π}] &
```

$2\,\mathrm{Cos}\left[\frac{x}{2}\right]\,\mathrm{Sin}\left[\frac{\xi}{2}\right]$

14.13 A useful generalization

We now consider a more generalized equation

$$L_x y + \mu w y + f(x) = 0$$

$$L_x u_n + \lambda_n w u_n = 0$$

The function $f(x)$ can be described by

$$f(x) = \sum_n a_n u_n(x)$$

with

$$a_n = \int_a^b u_n^*(\xi) w(\xi) f(\xi) d\xi$$

or

$$f(x) = \int_a^b \sum_n u_n{}^*(\xi) w(\xi) u_n(x) f(\xi) d\xi = \int_a^b \delta(x-\xi) f(\xi) d\xi$$

where

$$\delta(x-\xi) = \sum_n u_n^*(\xi) w(\xi) u_n(x)$$

Here we assume that

$$y(x) = \sum_n c_n u_n(x)$$

$$L_x \sum_n c_n u_n(x) + \mu w(x) \sum_n c_n u_n(x) + f(x) = 0$$

or

$$\sum_n c_n L_x u_n(x) + \mu w(x) \sum_n c_n u_n(x) + w(x) \sum_n u_n(x) \int_a^b u_n{}^*(\xi) f(\xi) d\xi = 0$$

Since $L_x u_n(x) + \lambda_n w(x) u_n(x) = 0$, we have

$$w(x) \sum_n (-\lambda_n + \mu) c_n u_n(x) + w(x) \sum_n u_n(x) \int_a^b u_n{}^*(\xi) f(\xi) d\xi = 0$$

Multiplying $\int_a^b dx u_m{}^*(x)$

$$\sum_n \int_a^b u_m{}^*(x) w(x) u_n(x) dx (-\lambda_n + \mu) c_n + \sum_n \int_a^b u_m{}^*(x) w(x) u_n(x) dx \int_a^b u_n{}^*(\xi) f(\xi) d\xi = 0$$

or

$$\sum_n \delta_{n.m} (-\lambda_n + \mu) c_n + \sum_n \delta_{n,m} \int_a^b u_n{}^*(\xi) f(\xi) d\xi = 0$$

or

$$(-\lambda_m + \mu) c_m + \int_a^b u_m{}^*(\xi) f(\xi) d\xi = 0$$

or

$$c_n = \frac{\int_a^b u_n^*(\xi) f(\xi) d\xi}{(\lambda_n - \mu)}.$$

Hence the solution is given by

$$y(x) = \int_a^b d\xi \sum_n \frac{u_n^*(\xi) u_n(x)}{(\lambda_n - \mu)} f(\xi) = \int_a^b d\xi G(x,\xi) f(\xi)$$

where the Green's function is defined by

$$G(x,\xi) = \sum_n \frac{u_n^*(\xi) u_n(x)}{(\lambda_n - \mu)}$$

14.14 Physical meaning of the Green function

We consider the impulse (Green's function) method for getting solutions for the harmonic oscillator with an arbitrary time dependent driving force. We write the solution as a superposition of solutions with zero initial displacement but velocities given by the impulses acting on the oscillator due to the external force. An arbitrary driving force is written as a sum of impulses, The single impulse $x(t)$ responses are added together in the form of a continuous integral.

We are going to work out a general expression for the response of a damped mass-spring system to an arbitrary force as a function of time making some very clever uses of Superposition. We will view the force as a sum of rectangular infinitesimal impulses and add the $x(t)$ solutions for each impulsive force. For an initially quiescent oscillator each impulse produces a solution equivalent to a free oscillator with initial velocity. The solution becomes a sum (integral) over such impulse responses.

14.15 Damped oscillator in Green function

We suppose that a damped harmonic oscillator is subjected to an external force $f(t)$ with finite duration and is at rest before the onset of the force. The displacement satisfies a differential equation of the form,

$$Lx(t) = x''(t) + 2\gamma x'(t) + \omega_0^2 x(t) = f(t),$$

where m is a mass, γ is the damping factor, and ω_0 is the natural angular frequency.

Using the Green function $G(t, \tau)$, we have

$$x(t) = -\int_{-\infty}^{\infty} G(t,\tau) f(\tau) d\tau ,$$

$$Lx(t) = -\int_{-\infty}^{\infty} LG(t,\tau) f(\tau) d\tau$$
$$= \int_{-\infty}^{\infty} \delta(t-\tau) f(\tau) d\tau$$
$$= f(t)$$

where G(t, τ) satisfies

$$LG(t,\tau) = -\delta(t-\tau) ,$$

with

$$G(t = 0, \tau) = G'(t = 0, \tau) = 0.$$

The form of $G(t, \tau)$ is given by

$$G(t,\tau) = -\frac{e^{-\gamma(t-\tau)}}{\omega_d} \sin[(t-\tau)\omega_d]\Theta(t-\tau)$$

and

$$x(t) = \int_{-\infty}^{\infty} f(\tau)[\frac{e^{-\gamma(t-\tau)}}{\omega_d} \sin[(t-\tau)\omega_d]\Theta(t-\tau)] d\tau ,$$

with

$$\omega_d = \sqrt{-\gamma^2 + \omega_0{}^2} .$$

((Note)) $\gamma = 0$ for un-damped oscillator

The Green function is given by

$$G(t,\tau) = -\frac{1}{\omega_0} \sin[(t-\tau)\omega_0]\Theta(t-\tau)$$

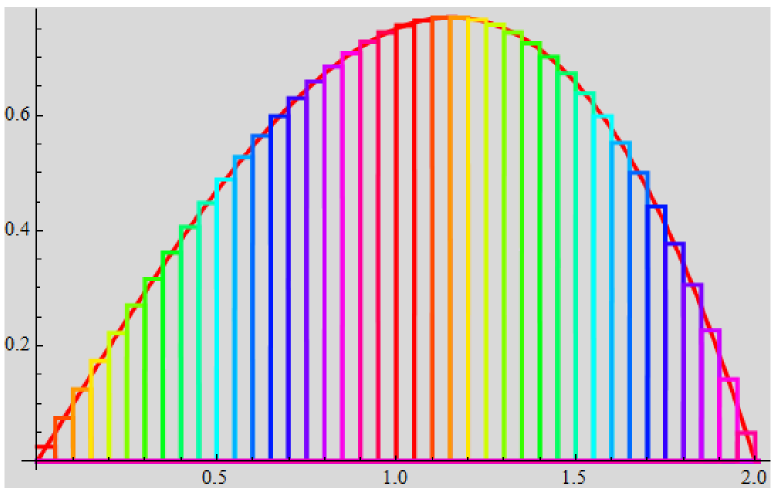
where G(t, τ) satisfies

$$G''(t,\tau) + \omega_0{}^2 G(t,\tau) = -\delta(t-\tau)$$

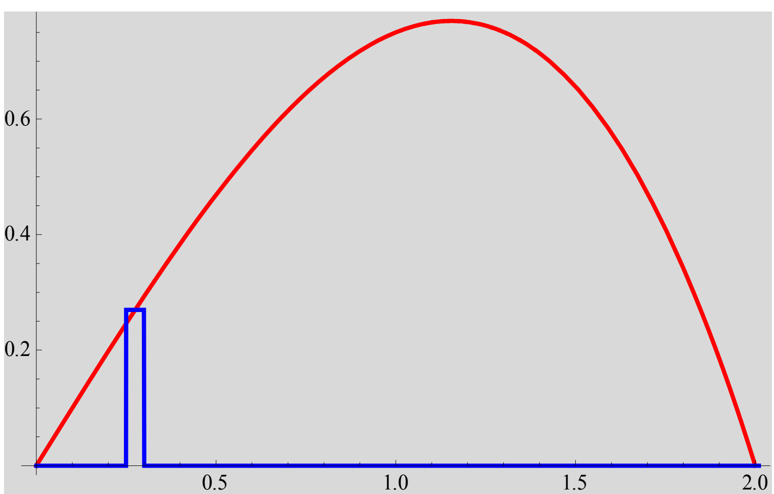
with

$$G(t=0,\tau)=G'(t=0,\tau)=0$$

We assume that $f(t)$ is a continuous function of t. This function consists of the combination of the square impulses with the time width $\Delta\tau$.



14.16 Example-1: one pulse



Suppose an impulse with a height $f(t_n)$ at $t = t_n$. This impulse can be defined by

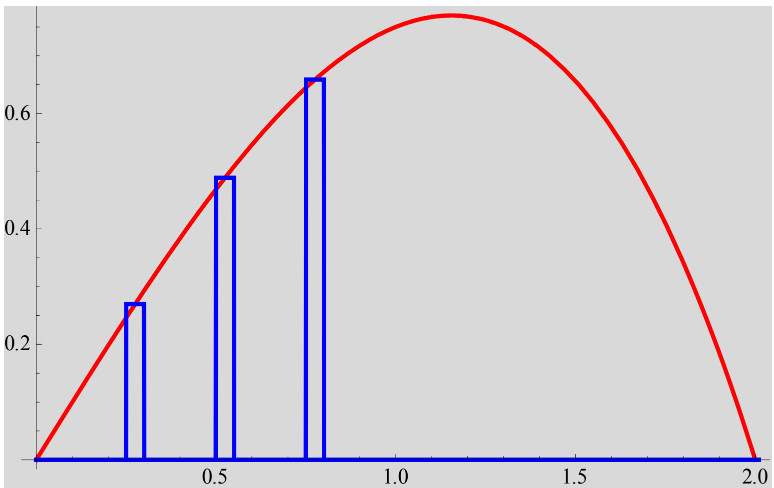
$$f(t_n)\delta(t-t_n)$$

Then we have

$$x(t) = -\int_{-\infty}^{\infty} G(t,\tau) f(t_n)\delta(\tau - t_n) d\tau$$
$$= -G(t,t_n) f(t_n)$$

where G(t, t_n) is equal to zero for $t<t_n$.

14.17 Example-2: many pulses



When there are many impulses at different times, $x(t)$ can be described by

$$x(t) = -\sum_n G(t,t_n) f(t_n)$$
$$= \sum_n \frac{\exp[-\gamma(t - t_n)]\sin[\omega_d (t - t_n)]}{\omega_d} \Theta[t - t_n] f[t_n]$$

14.18 Response to a step function

We consider the response $x(t)$ to the external force $f(t)$ which is given by the form

$$f(t) = a\Theta(t - t_0)$$

The response $x(t)$ is obtained as

$$x(t) = \int_{-\infty}^{\infty} f(\tau)[\frac{e^{-\gamma(t-\tau)}}{\omega_d} \sin[(t-\tau)\omega_d]\Theta(t-\tau)] d\tau \, ,$$

with

$$\omega_d = \sqrt{-\gamma^2 + \omega_0{}^2} \, .$$

We calculate the response using the Mathematica.

((**Mathematica**))

```
Clear["Global`*"];

G[t_, τ_] = - Exp[-γ (t - τ)]/ωd Sin[(t - τ) ωd] UnitStep[t - τ];

f[t_] = a UnitStep[t - t0];

x[t_] = Integrate[-G[t, τ] f[τ], {τ, -∞, ∞}] /. {(γ^2 + ωd^2) → ω0^2} // Simplify
```

$$\frac{a\left(\omega d-e^{(-t+t0)\gamma}\left(\omega d\,\mathrm{Cos}[(t-t0)\,\omega d]+\gamma\,\mathrm{Sin}[(t-t0)\,\omega d]\right)\right)\mathrm{UnitStep}[t-t0]}{\omega 0^2\,\omega d}$$

```
rule1 = {t0 → 0, a → 1, ω0 → 1, ωd → Sqrt[ω0^2 - γ^2]};

f1 = Plot[Evaluate[Table[x[t] //. rule1, {γ, 0, 0.8, 0.1}]], {t, 0, 30},
  PlotStyle → Table[{Hue[0.1 i], Thick}, {i, 0, 10}], PlotRange → All,
  AxesLabel → {"t", "x[t]"}];
f2 = Graphics[{Text[Style["γ=0", Black, 12], {3, 2.1}],
   Text[Style["γ=0.4", Black, 12], {3.5, 1.25}],
   Text[Style["γ=0.8", Black, 12], {4, 0.95}]}];
Show[f1, f2]
```

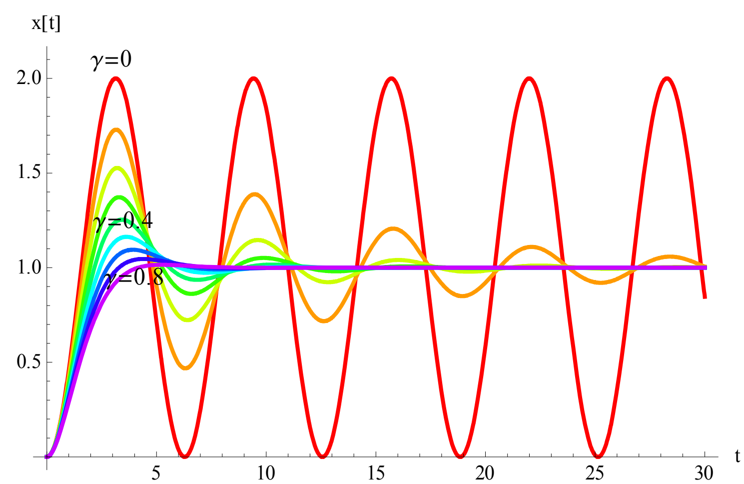
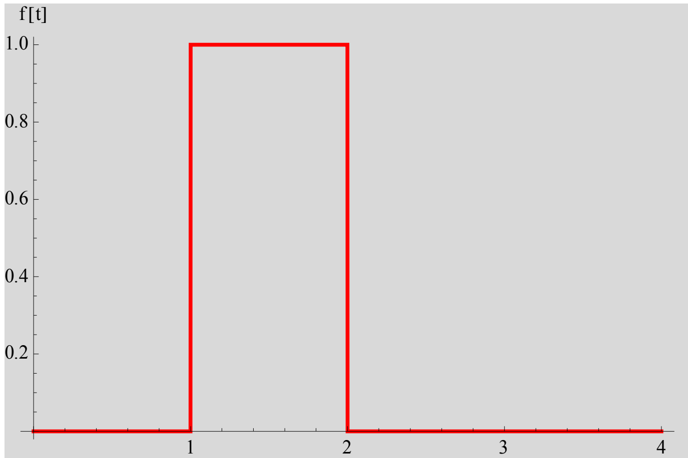


Fig. Step response for ω_0 =1, a = 1, t_0 = 0, $\omega_d = \sqrt{-\gamma^2 + \omega_0^{\,2}}$, γ is changed as a parameter. γ= 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, and 0.8

14.19 Square pulse (I)



What is the response to the square pulse given by

external force;

$$f(t) = a[\Theta(t - t_0) - \Theta(t - t_1)],$$

where a is the amplitude and $\Delta t = t_1 - t_0$ is the pulse width.

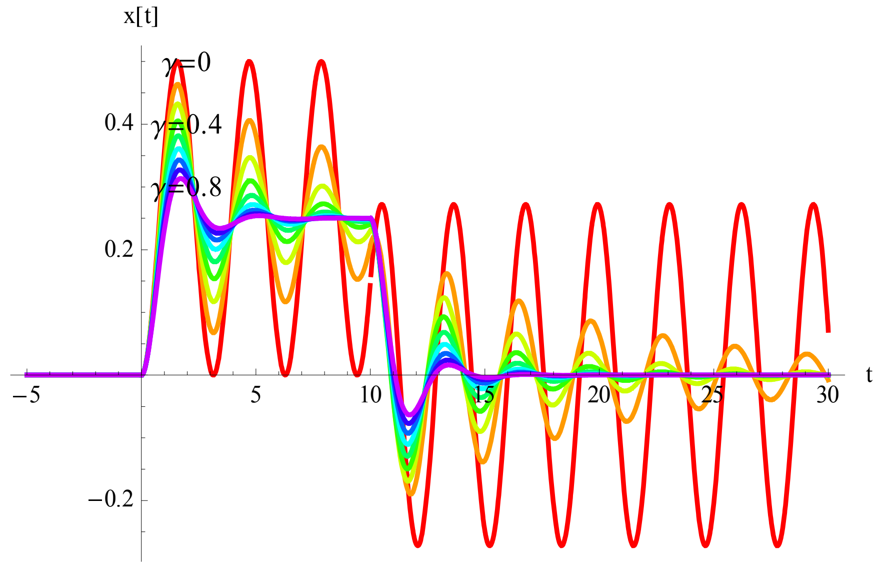
((**Mathematica**))

```
Clear["Global`*"];

G[t_, τ_] = - Exp[-γ (t - τ)]/ωd Sin[(t - τ) ωd] UnitStep[t - τ];

f[t_] = a (UnitStep[t - t0] - UnitStep[t - t1]);

x[t_] = Integrate[-G[t, τ] f[τ], {τ, -∞, ∞}] /. {(γ^2 + ωd^2) → ω0^2} // Simplify
```

$$\frac{1}{\omega0^2\,\omega d}\, a\left(\left(\omega d - e^{(-t+t0)\gamma}\left(\omega d \operatorname{Cos}[(t-t0)\,\omega d] + \gamma \operatorname{Sin}[(t-t0)\,\omega d]\right)\right)\operatorname{UnitStep}[t-t0] - \left(\omega d - e^{(-t+t1)\gamma}\left(\omega d \operatorname{Cos}[(t-t1)\,\omega d] + \gamma \operatorname{Sin}[(t-t1)\,\omega d]\right)\right)\operatorname{UnitStep}[t-t1]\right)$$

```
rule1 = {t0 → 0, t1 → 10, a → 1, ω0 → 2, ωd → Sqrt[ω0^2 - γ^2]};

f1 = Plot[Evaluate[Table[x[t] //. rule1, {γ, 0, 0.8, 0.1}]], {t, -5, 30},
  PlotStyle → Table[{Hue[0.1 i], Thick}, {i, 0, 10}], PlotRange → All,
  AxesLabel → {"t", "x[t]"}];
f2 = Graphics[{Text[Style["γ=0", Black, 12], {2, 0.5}],
   Text[Style["γ=0.4", Black, 12], {2, 0.4}],
   Text[Style["γ=0.8", Black, 12], {2, 0.3}]}];
Show[f1, f2]
```

Response:



14.20 Square pulse: Dirac delta function

$$f(t) = a[\Theta(t - t_0) - \Theta(t - t_1)],$$

with

$$b = a(t_1 - t_0) = a\tau = \text{const},$$

in the limit of $a \to \infty$ and $\tau \to 0$. In the limit, the f(t) can be described by a Dirac delta function,

External force:

$$f(t) = b\delta(t - t_0).$$

The response function is obtained as

$$x(t) = \frac{b}{\omega_d} e^{-\gamma(t-t_0)} \sin(\omega_d t)\Theta(t - t_0).$$

Response:

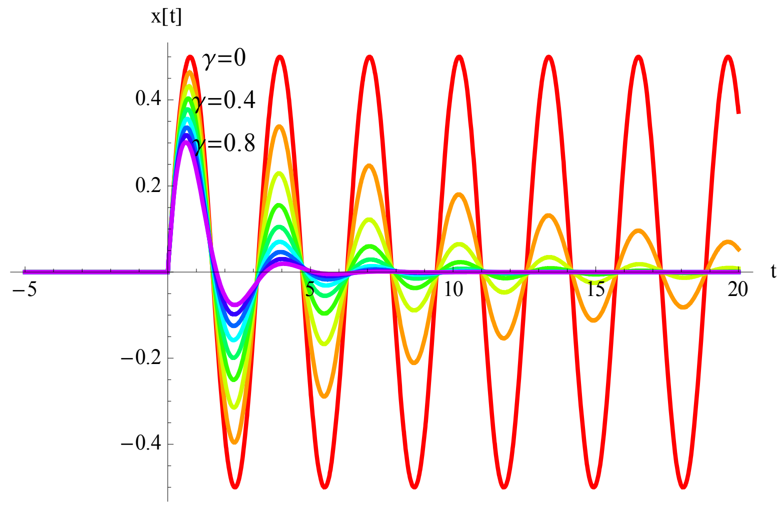


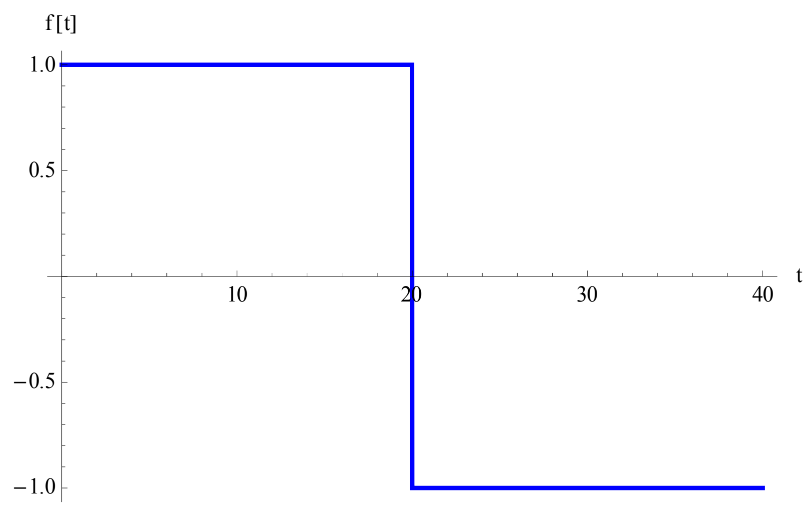
Fig. Response to the Dirac delta function for $\omega_0 = 2$, $b = 1$, $t_0 = 0$, $\omega_d = \sqrt{-\gamma^2 + \omega_0^2}$, γ is changed as a parameter. $\gamma = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7$, and 0.8

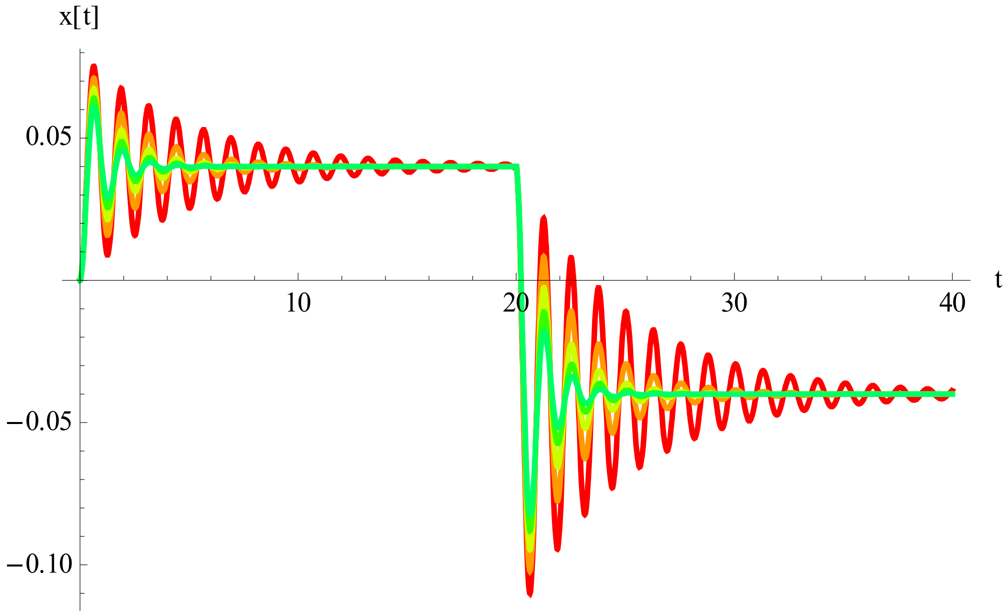
14.21 Step response (II)

External force:

$$f(t) = a[\Theta(t) - 2\Theta(t - t_0) + \Theta(t - 2t_0)]$$

Response:



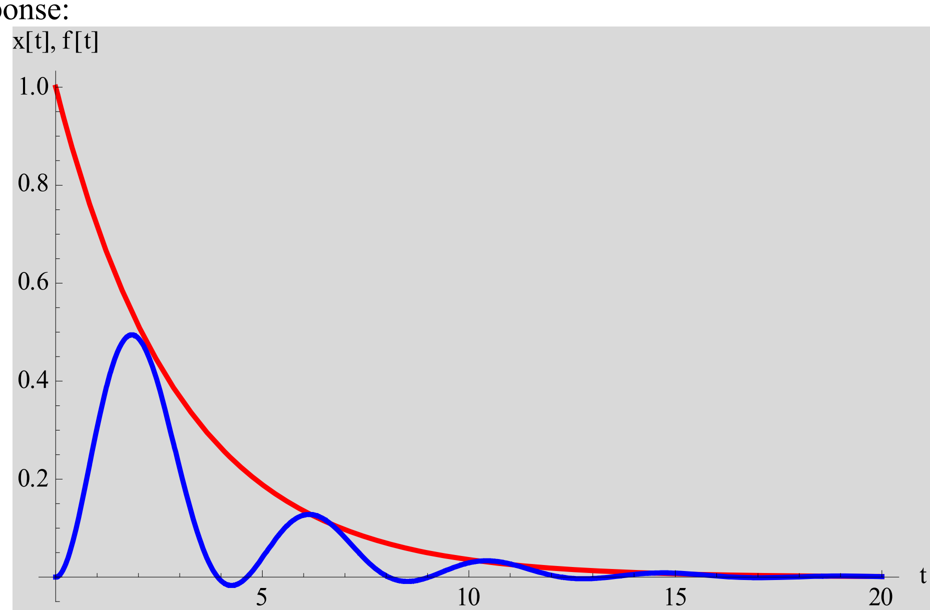


14.22 Exponential decay with time

External force;

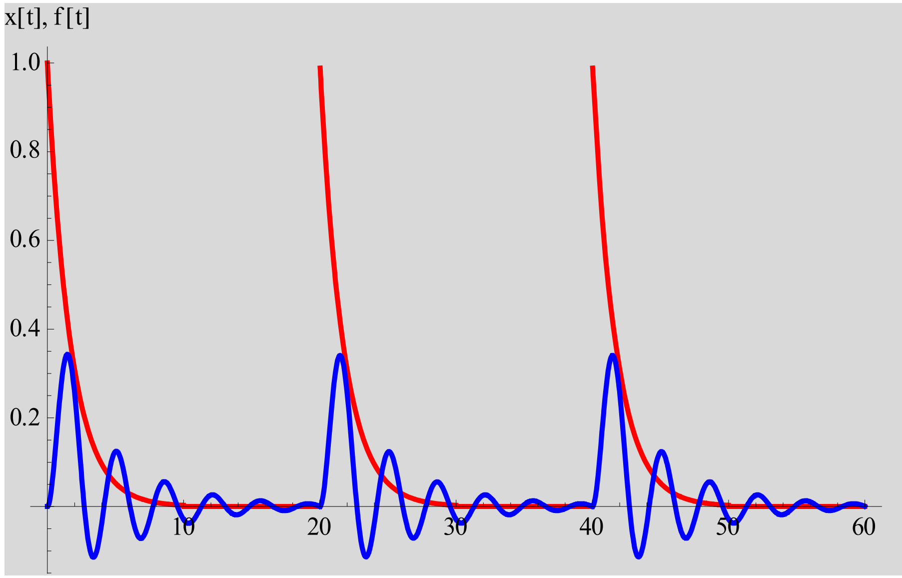
$$f(t) = f_0 e^{-t/t_0}\Theta(t)$$

and response:



External force and response

$$f(t) = f_0 e^{-t/t_0}[\Theta(t) - \Theta(t - T_0)] + f_0 e^{-(t-2T_0)/t_0}[\Theta(t - 2T_0) - \Theta(t - 3T_0)] +$$
$$f_0 e^{-(t-4T_0)/t_0}[\Theta(t - 4T_0) - \Theta(t - 5T_0)] +$$



14.23 Exponential decay with time

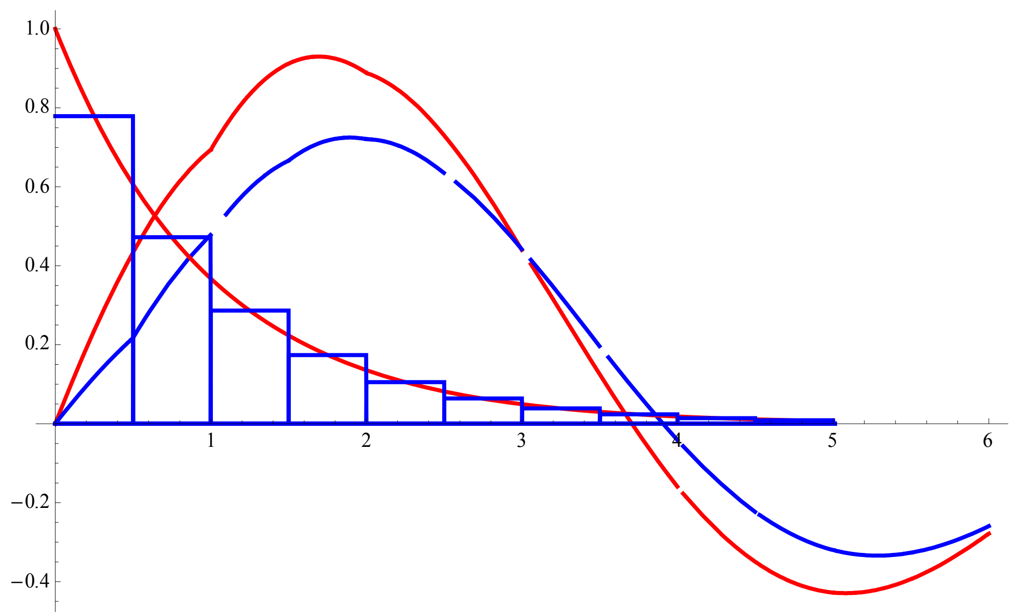
$$f(t) = f_0 e^{-t/t_0}\Theta(t)$$

$f(t)$ is approximated by the superposition of square pulses with a width $\Delta\tau$,

$$f(t) = \sum_{n=0}^{N-1} f(t_n)[\Theta(t-t_n) - \Theta(t-t_{n+1})\Delta\tau$$

where

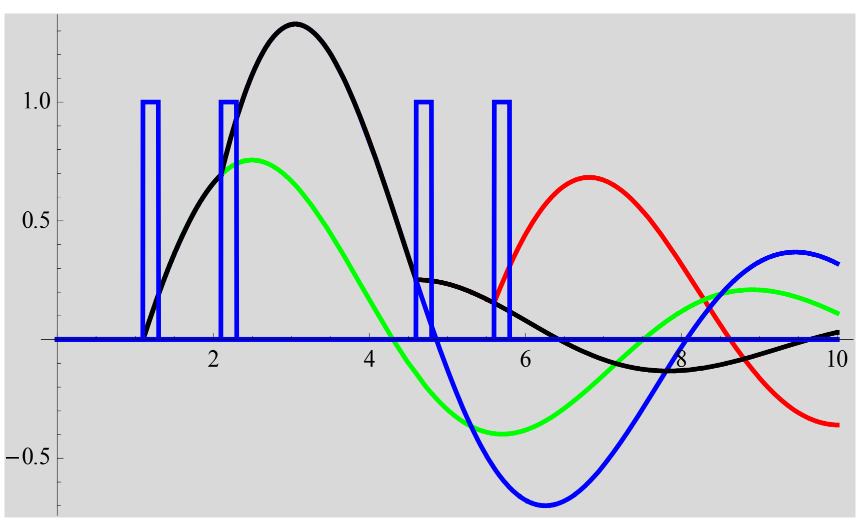
$$\Delta\tau = t_{n+1} - t_n$$



14.24 Example

Four square impulses with constant height f_0 at different times

```
Clear["Global`*"];

G[t_, τ_] = - Exp[- γ (t - τ)]/ω Sin[(t - τ) ω] UnitStep[t - τ];

rule1 = {f0 → 1, γ → 0.2, ω0 → 1, ω → Sqrt[1 - 0.2^2]};

χ[t_, t0_] = -G[t, t0] f0;

χ1[t_, t0_] = χ[t, t0] /. rule1;

g1 = χ1[t, 1.1] + χ1[t, 2.1] + χ1[t, 4.6] + χ1[t, 5.6];

g2 = χ1[t, 1.1];

g3 = χ1[t, 1.1] + χ1[t, 2.1];

g4 = χ1[t, 1.1] + χ1[t, 2.1] + χ1[t, 4.6];

pl1 = Plot[{g1, g2, g3, g4}, {t, 0, 10},
   PlotStyle → {{Red, Thick}, {Green, Thick}, {Blue, Thick}, {Black, Thick}},
   Background → LightGray];

Impulse[t_, t0_, t1_ ] = Which[t < t0, 0, t0 < t < t1, 1, t1 < t, 0];

pl2 =
  Plot[Evaluate[{Impulse[t, 1.1, 1.3], Impulse[t, 2.1, 2.3],
     Impulse[t, 4.6, 4.8], Impulse[t, 5.6, 5.8]}], {t, 0, 10},
   PlotStyle → {{Blue, Thick}, {Blue, Thick}, {Blue, Thick}, {Blue, Thick}} ,
   Background → LightGray];

Show[pl1, pl2]
```



14.25 Mathematica:Derivation of Green's function

$$G(t,\tau) = -\frac{e^{-\gamma(t-\tau)}}{\omega}\sin[(t-\tau)\omega]\Theta(t-\tau)$$

for the damped oscillator.

((**Mathematica**))

```
Clear["Global`*"];

eq11 = G''[t] + 2 γ G'[t] + ω0^2 G[t] == -DiracDelta[t - τ];

eq12 = DSolve[{eq11, G[0] == 0, G'[0] == 0}, G[t], t] // Simplify
```

$$\left\{\left\{G[t] \to -\frac{1}{2\sqrt{\gamma^2 - \omega 0^2}}\, e^{-(t-\tau)\left(\gamma+\sqrt{\gamma^2-\omega 0^2}\right)} \left(-1 + e^{2(t-\tau)\sqrt{\gamma^2-\omega 0^2}}\right)(\text{HeavisideTheta}[t-\tau] - \text{HeavisideTheta}[-\tau])\right\}\right\}$$

```
eq13 = eq12 /. {Sqrt[γ^2 - ω0^2] → ⅈ ω} // Simplify;

eq14 = eq13 /. {1/Sqrt[γ^2 - ω0^2] → -ⅈ/ω};

G11[t_, τ_] = G[t] /. eq14[[1]] // Simplify[#, 0 < t < τ] &
```

0

```
G22[t_, τ_] = G[t] /. eq14[[1]] // FullSimplify[#, t > τ > 0] &
```

$$\frac{i\, e^{-(t-\tau)(\gamma + i\omega)}\left(-1 + e^{2 i (t-\tau)\omega}\right)}{2\omega}$$

```
G22[t, τ] Exp[γ (t - τ)] // ExpToTrig // Simplify
```

$$-\frac{\text{Sin}[(t-\tau)\omega]}{\omega}$$

```
G[t_, τ_] = -Exp[-γ (t - τ)] Sin[(t - τ) ω]/ω UnitStep[t - τ]
```

$$-\frac{e^{-\gamma(t-\tau)}\,\text{Sin}[(t-\tau)\omega]\,\text{UnitStep}[t-\tau]}{\omega}$$

14.26 Derivation of Green's function using Fourier transform

$$LG(t) = G''(t) + 2\gamma G'(t) + \omega_0^{\ 2} G(t) = -\delta(t) .$$

We use the Fourier transform,

$$G(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(\omega) e^{-i\omega t} d\omega ,$$

$$G'(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [-i\omega G(\omega)] e^{-i\omega t} d\omega ,$$

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} d\omega ,$$

Then we have

$$-\omega^2 G(\omega) - i2\gamma\omega G(\omega) + \omega_0^{\ 2} G(\omega) = -\frac{1}{\sqrt{2\pi}} ,$$

$$G(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{\omega^2 + i2\gamma\omega - \omega_0^{\ 2}} .$$

The Green's function is obtained as

$$G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega^2 + i2\gamma\omega - \omega_0^{\ 2}} d\omega = \frac{1}{2\pi} \oint_{C_1} \frac{e^{-izt}}{z^2 + i2\gamma z - \omega_0^{\ 2}} dz ,$$

for $t>0$, using the Jordan's lemma for the contour Γ_1. There are two simple poles in the lower half plane. The contour C_1 has the clock-wise rotation. Tow simple poles are obtained as

$$z^2 + i2\gamma z - \omega_0^{\ 2} = 0 ,$$

or

$$z = -i\gamma \pm \omega_d ,$$

where

$$\omega_d = \sqrt{\omega_0^{\ 2} - \gamma^2} .$$

Using the residue theorem, we have

$$G(t) = -i[\mathrm{Res}(z = z_1) + \mathrm{Res}(z = z_2)]$$

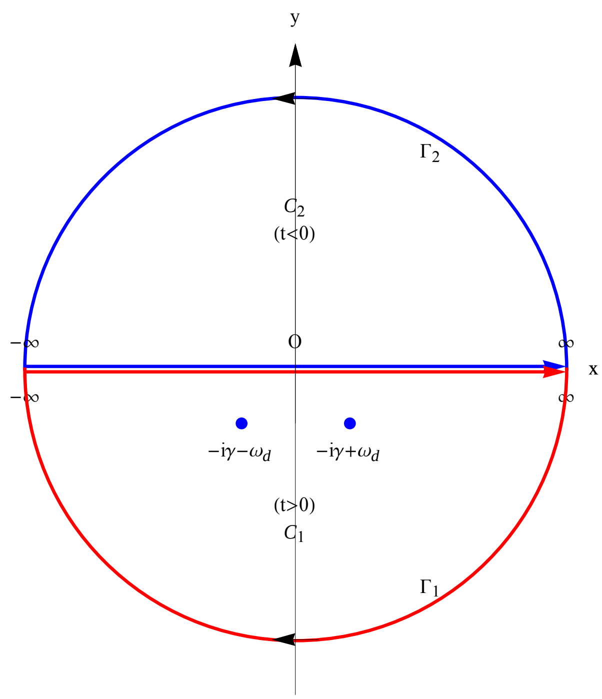
or

$$G(t) = -\frac{1}{\omega_d}\sin(\omega_d t)e^{-\gamma t}$$

The Green's function for $t<0$ is obtained as

$$G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\omega t}}{\omega^2 + i2\gamma\omega - \omega_0^{\,2}}d\omega = \frac{1}{2\pi}\oint_{C_2}\frac{e^{-izt}}{z^2 + i2\gamma z - \omega_0^{\,2}}dz$$

using the Jordan's lemma for the contour Γ_2. There is no pole in the upper half plane. Then we have $G(t) = 0$.



In summary we have

$$G(t) = -\frac{1}{\omega_d}\sin(\omega_d t)e^{-\gamma t}\Theta(t),$$

where $\Theta(t)$ is the step function; $\Theta(t) = 0$ for $t<0$ and 1 for $t>0$.

14.27 Green's function for the undamped oscillator

$$LG(t) = G''(t) + \omega_0{}^2 G(t) = -\delta(t).$$

We use the Fourier transform,

$$G(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(\omega) e^{-i\omega t} d\omega,$$

$$G''(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [-\omega^2 G(\omega)] e^{-i\omega t} d\omega,$$

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} d\omega,$$

Then we have

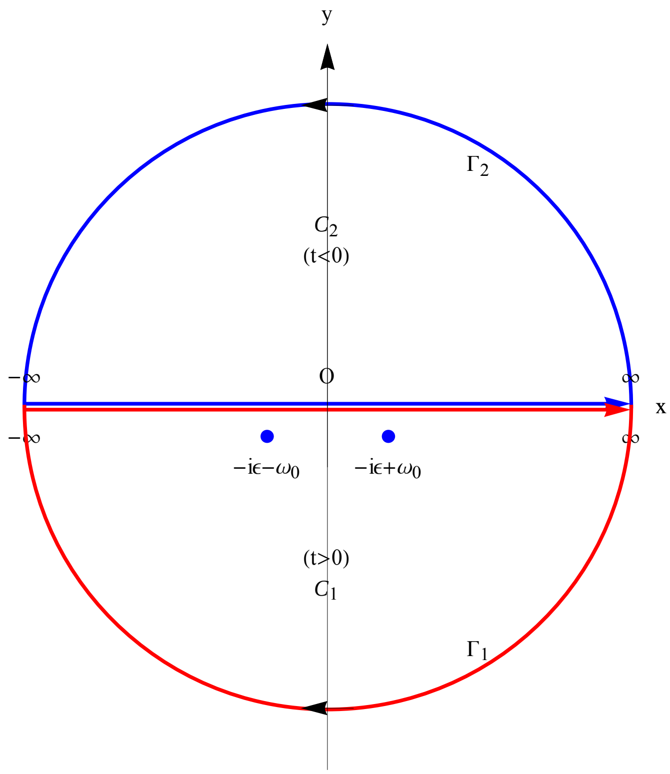
$$-\omega^2 G(\omega) - \omega_0{}^2 G(\omega) = -\frac{1}{\sqrt{2\pi}}.$$

$$G(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{\omega^2 - \omega_0{}^2}.$$

The Green's function is obtained as

$$\begin{aligned} G(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega^2 - \omega_0{}^2} d\omega \\ &= \frac{1}{2\pi} \frac{1}{2\omega_0} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \left(\frac{1}{\omega - \omega_0} - \frac{1}{\omega + \omega_0}\right) d\omega \end{aligned}.$$

(i) Retarded Green's function



For t>0, we need to choose the contour C_1 (in the lower-half plane). The integral along the Γ_1 is zero according to the Jordan's lemma. There is two simple poles inside the contour C_1. The contour C_1 has a clock-wise direction.

$$\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega^2 - \omega_0{}^2} d\omega + \int_{\Gamma_1} \frac{e^{-izt}}{z^2 - \omega_0{}^2} dz = \oint_{C_1} \frac{e^{-izt}}{z^2 - \omega_0{}^2} dz = -2\pi i[\mathrm{Re}\, s(z = \omega_0) + (z = -\omega_0)]$$

or

$$G_{ret}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega^2 - \omega_0{}^2} d\omega = -\frac{2\pi i}{2\pi}[\mathrm{Re}\, s(z = \omega_0) + (z = -\omega_0)]$$
$$= -\frac{1}{\omega_0} \sin(\omega_0 t)$$

For t<0, we need to choose the contour C_2 (in the upper-half plane). The integral along the Γ_2 is zero according to the Jordan's lemma. There is no pole inside the contour C_2.

$$\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega^2 - \omega_0{}^2} d\omega + \int_{\Gamma_2} \frac{e^{-izt}}{z^2 - \omega_0{}^2} dz = \oint_{C_2} \frac{e^{-izt}}{z^2 - \omega_0{}^2} dz = 0,$$

or

$$G_{ret}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega^2 - \omega_0{}^2} d\omega = 0 .$$

In summary, we have

$$G_{ret}(t) = -\frac{1}{\omega_0}\sin(\omega_0 t)\Theta(t) .$$

((Note))

$$G_{ret}(\omega) = \frac{1}{\sqrt{2\pi}}\frac{1}{2\omega_0}\left(\frac{1}{\omega - \omega_0 + i\varepsilon} - \frac{1}{\omega + \omega_0 + i\varepsilon}\right)$$

Using the formula,

$$\mathbf{F}^{-1}\left[\frac{1}{\sqrt{2\pi}}\frac{i}{\omega + i\varepsilon}\right] = \Theta(t) .$$

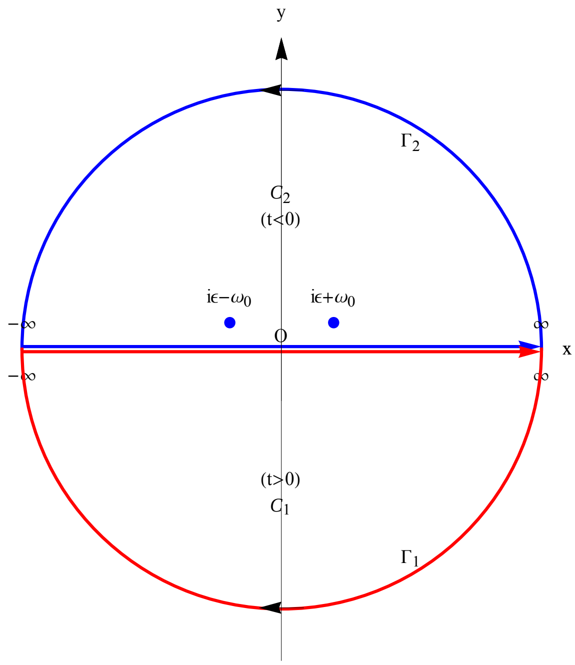
$$\mathbf{F}^{-1}\left[\frac{1}{\sqrt{2\pi}}\frac{i}{\omega - \omega_0 + i\varepsilon}]\right] = e^{-i\omega_0 t}\Theta(t) .$$

$$\mathbf{F}^{-1}\left[\frac{1}{\sqrt{2\pi}}\frac{i}{\omega + \omega_0 + i\varepsilon}]\right] = e^{i\omega_0 t}\Theta(t) .$$

we find that

$$\begin{aligned} G_{ret}(t) &= \frac{1}{\sqrt{2\pi}}\frac{1}{2\omega_0}\frac{\sqrt{2\pi}}{i}[e^{-i\omega_0 t}\Theta(t) - e^{i\omega_0 t}\Theta(t)] \\ &= -\frac{1}{\omega_0}\sin(\omega_0 t)\Theta(t) \end{aligned}$$

(ii) Advanced Green's function



For $t>0$, we need to choose the contour C_1 (in the lower-half plane). The integral along the Γ_1 is zero according to the Jordan's lemma. There is no pole inside the contour C_1.

$$\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega^2 - \omega_0^2} d\omega + \int_{\Gamma_1} \frac{e^{-izt}}{z^2 - \omega_0^2} dz = \oint_{C_1} \frac{e^{-izt}}{z^2 - \omega_0^2} dz = 0$$

or

$$G_{adv}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega^2 - \omega_0^2} d\omega = 0$$

For $t<0$, we need to choose the contour C_2 (in the upper-half plane). The integral along the Γ_2 is zero according to the Jordan's lemma. There is two simple poles inside the contour C_2. The contour C_2 has a counter clock-wise direction.

$$\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega^2 - \omega_0^2} d\omega + \int_{\Gamma_2} \frac{e^{-izt}}{z^2 - \omega_0^2} dz = \oint_{C_2} \frac{e^{-izt}}{z^2 - \omega_0^2} dz = 2\pi i[\mathrm{Re}\,s(z = \omega_0) + \mathrm{Re}\,s(z = -\omega_0)]$$

or

$$G_{adv}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega^2 - \omega_0^2} d\omega = \frac{1}{\omega_0} \sin(\omega_0 t)$$

In summary, we have

$$G_{adv}(t) = \frac{1}{\omega_0}\sin(\omega_0 t)\Theta(-t) .$$

14.28 Application of the un-damped Green's function

$$m\frac{dv(t)}{dt} + \alpha v(t) = F(t)$$

We define the Green's function

$$m\frac{dG(t)}{dt} + \alpha G(t) = -\delta(t) .$$

Then the form of $v(t)$ can be described using the Green's function as

$$v(t) = -\int_{-\infty}^{\infty} G(t,\tau)F(\tau)d\tau$$

We derive the Green's function using the Fourier transform.

$$G(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(\omega)e^{-i\omega t}d\omega$$

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}d\omega$$

Then we have

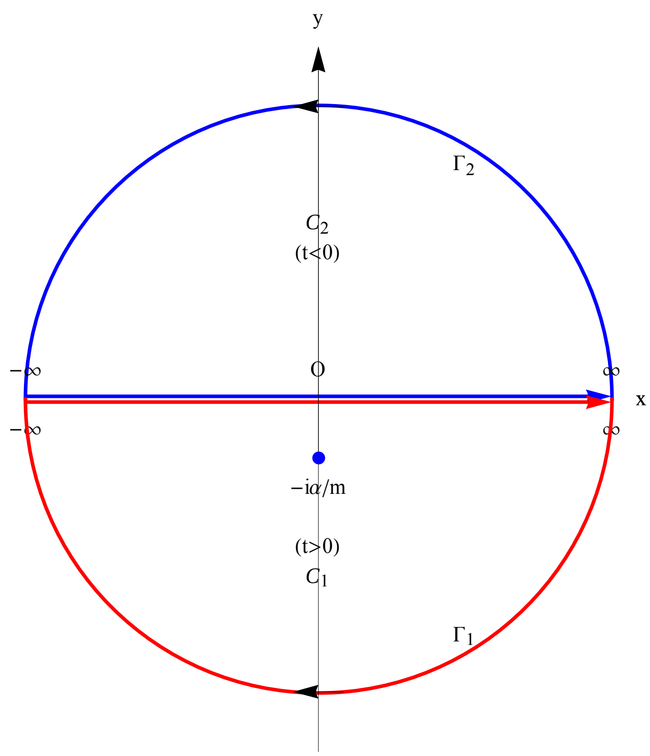
$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(-mi\omega + \alpha)G(\omega)e^{-i\omega t}dt = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}dt .$$

or

$$G(\omega) = \frac{1}{\sqrt{2\pi}}\frac{1}{mi}\frac{1}{(\omega + \frac{i\alpha}{m})} .$$

The inverse Fourier transform:

$$G(t) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \frac{1}{mi} \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega + \frac{i\alpha}{m}} d\omega$$



For $t>0$, we need to choose the contour C_1 (in the lower-half plane). The integral along the Γ_1 is zero according to the Jordan's lemma. There is a simple pole inside the contour C_1. The contour C_1 has a clock-wise direction.

$$\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega + \frac{i\alpha}{m}} d\omega + \int_{\Gamma_1} \frac{e^{-izt}}{z + \frac{i\alpha}{m}} dz = \oint_{C_1} \frac{e^{-izt}}{z + \frac{i\alpha}{m}} dz = -2\pi i \operatorname{Re}s(z = -\frac{i\alpha}{m})$$

or

$$\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega + \frac{i\alpha}{m}} d\omega = -2\pi i \operatorname{Re}s(z = -\frac{i\alpha}{m}) = -2\pi i \exp(-\frac{\alpha t}{m})$$

$$G(t) = \frac{1}{2\pi mi}(-2\pi i)\exp(-\frac{\alpha t}{m}) = -\frac{1}{m}\exp(-\frac{\alpha t}{m}) \qquad \text{for } t>0.$$

For $t<0$, we need to choose the contour C_2 (in the upper-half plane). The integral along the Γ_2 is zero according to the Jordan's lemma. There is no pole inside the contour C_2.

$$\int_{-\infty}^{\infty}\frac{e^{-i\omega t}}{\omega+\frac{i\alpha}{m}}d\omega+\int_{\Gamma_2}\frac{e^{-izt}}{z+\frac{i\alpha}{m}}dz=\oint_{C_2}\frac{e^{-izt}}{z+\frac{i\alpha}{m}}dz=0$$

or

$$\int_{-\infty}^{\infty}\frac{e^{-i\omega t}}{\omega+\frac{i\alpha}{m}}d\omega=0$$

or

$$G(t)=0 \qquad\qquad \text{for } t<0.$$

More generally we have

$$G(t,\tau)=-\frac{1}{m}\exp[-\frac{\alpha(t-\tau)}{m}]\Theta(t-\tau).$$

14.29 Example

Suppose that the input $F(t)$ is a force that is constant F_0 between $t = 0$ and $t = T$ and zero otherwise.

$$F(t)=F_0[\Theta(t)-\Theta(t-T)].$$

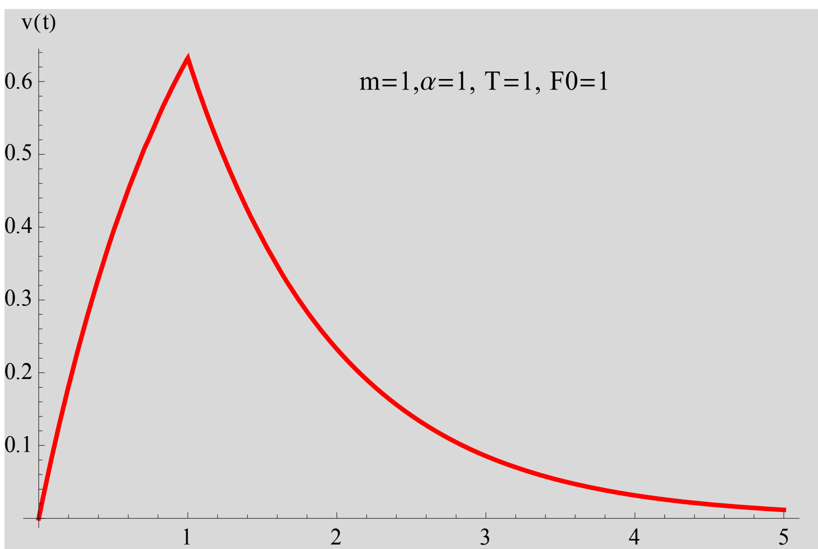
$$\begin{aligned}v(t)&=-\int_{-\infty}^{\infty}G(t,\tau)F(\tau)d\tau\\&=\frac{F_0}{m}\int_{-\infty}^{\infty}\exp[-\frac{\alpha(t-\tau)}{m}]\Theta(t-\tau)[\Theta(\tau)-\Theta(\tau-T)]d\tau\\&=\frac{F_0}{\alpha}[\{1-\exp(-\frac{\alpha t}{m})\}\Theta(t)-\{1-\exp(-\frac{\alpha(t-T)}{m})\}\Theta(t-T)]\end{aligned}$$

((**Mathematica**))

```
Clear["Global`*"];

v[t_] =
 F0/m ∫_{-∞}^{∞} Exp[(-α (t - τ))/m] UnitStep[t - τ]
   (UnitStep[τ] - UnitStep[τ - T]) dτ // FullSimplify
```

$$\frac{F0\left(\text{UnitStep}[t]-e^{-\frac{t\,\alpha}{m}}\text{UnitStep}[t]+\left(-1+e^{\frac{(-t+T)\,\alpha}{m}}\right)\text{UnitStep}[t-T]\right)}{\alpha}$$

```
rule1 = {m → 1, α → 1, T → 1, F0 → 1};

f1 = Plot[v[t] /. rule1, {t, 0, 5}, PlotStyle → {Red, Thick},
   AxesLabel → {"t", "v(t)"}, Background → LightGray];
f2 =
  Graphics[
   {Text[Style["m=1,α=1, T=1, F0=1", Black, 12], {3, 0.6}]}];
Show[f1, f2]
```



14.30 Stretched string problem: homogeneous boundary condition

The differential equation describing the displacement of the string is

$$T_0 y''(x) = f(x),$$

with the boundary condition $y(0) = y(L) = 0$ (homogeneous boundary condition). The solution can be described by using the Green function $G(x, \xi)$ as

$$y(x) = -\int_0^L G(x,\xi) f(\xi) d\xi$$

where $G(x, \xi)$ satisfies the differential equation

$$T_0 G''(x,\xi) = -\delta(x-\xi)$$

with the homogeneous boundary conditions, $G(0, \xi) = G(L, \xi) = 0$. Note that $G(x, \xi)$ is symmetric with respect to x and ξ.

The Green's function is obtained as

$$G(x,\xi) = \frac{1}{LT_0} x(L-\xi), \qquad \text{for } 0<x<\xi<L$$

and

$$G(x,\xi) = \frac{1}{LT_0} \xi(L-x) \qquad \text{for } 0<\xi<x<L$$

((**Mathematica**))

```
Clear["Global`*"]

eq1 = T0 G''[x] == -DiracDelta[x - ξ];

eq2 = DSolve[{eq1, G[0] == 0, G[L] == 0}, G[x], x] // Simplify
```

$$\left\{\left\{G[x] \to \frac{1}{L\,T0}\,(x\,(L-\xi)\,\text{HeavisideTheta}[L-\xi] + L\,(-x+\xi)\,\text{HeavisideTheta}[x-\xi] + (-L+x)\,\xi\,\text{HeavisideTheta}[-\xi])\right\}\right\}$$

```
G11[x_, ξ_] = G[x] /. eq2[[1]] // Simplify[#, 0 < x < ξ < L] &
```

$$\frac{x\,(L-\xi)}{L\,T0}$$

```
G22[x_, ξ_] = G[x] /. eq2[[1]] // Simplify[#, 0 < ξ < x < L] &
```

$$\frac{(L-x)\,\xi}{L\,T0}$$

```
y[x_] = ∫_0^x G22[x, ξ] (-f[ξ]) dξ + ∫_x^L G11[x, ξ] (-f[ξ]) dξ
```

$$\int_x^L -\frac{x\,(L-\xi)\,f[\xi]}{L\,T0}\,d\xi + \int_0^x -\frac{(L-x)\,\xi\,f[\xi]}{L\,T0}\,d\xi$$

14.31 Inhomogeneous boundary condition

The differential equation describing the displacement of the string is

$$T_0 y''(x) = f(x),$$

with the boundary condition $y(0) = 0$ and $y(L) = y_0$ (inhomogeneous boundary condition).

The Green's function $G(x, \xi)$ satisfies the differential equation given by

$$T_0 G''(x,\xi) = -\delta(x-\xi),$$

with the homogeneous boundary conditions,

$$G(0, \xi) = 0, G(L, \xi) = 0.$$

The solution can be described by using the Green's function as

$$y(x) = -\int_0^L G(x,\xi) f(\xi) d\xi + \phi(x),$$

where

$$T_0\phi''(x) = 0$$

with the

$\phi(0) = 0$ and $\phi(L) = y_0$.

The solution of $y(x)$ is obtained as

$$y(x) = \frac{xy_0}{L} + \int_x^L \frac{x(L-\xi)f(\xi)}{LT_0}d\xi + \int_0^x \frac{(L-x)\xi f(\xi)}{LT_0}d\xi$$

The validity of this method is given in the Appendix.

Ref: B.R Kusse and E.A. Westwig; Mathematical Physics
((**Mathematica**))

```
Clear["Global`*"];

eq1 = T0 G''[x] == -DiracDelta[x - ξ];

eq2 = DSolve[{eq1, G[0] == 0, G[L] == 0}, G[x], x] // Simplify
```

$$\left\{\left\{G[x] \to \frac{1}{L\,T0}\,(x\,(L-\xi)\,\text{HeavisideTheta}[L-\xi] + L\,(-x+\xi)\,\text{HeavisideTheta}[x-\xi] + (-L+x)\,\xi\,\text{HeavisideTheta}[-\xi])\right\}\right\}$$

```
G11[x_, ξ_] = G[x] /. eq2[[1]] // Simplify[#, 0 < x < ξ < L] &
```

$$\frac{x\,(L-\xi)}{L\,T0}$$

```
G22[x_, ξ_] = G[x] /. eq2[[1]] // Simplify[#, 0 < ξ < x < L] &
```

$$\frac{(L-x)\,\xi}{L\,T0}$$

```
eq3 = T0 ϕ''[x] == 0;

eq4 = DSolve[{eq3, ϕ[0] == 0, ϕ[L] == y0}, ϕ[x], x]
```

$$\left\{\left\{\phi[x] \to \frac{x\,y0}{L}\right\}\right\}$$

```
ϕ[x_] = ϕ[x] /. eq4[[1]];
```

$$y1[x\_] = \int_0^x G22[x, \xi]\,(f[\xi])\,d\xi + \int_x^L G11[x, \xi]\,(f[\xi])\,d\xi$$

$$\int_x^L \frac{x\,(L-\xi)\,f[\xi]}{L\,T0}\,d\xi + \int_0^x \frac{(L-x)\,\xi\,f[\xi]}{L\,T0}\,d\xi$$

$$y[x\_] = \phi[x] + \int_0^x G22[x, \xi]\,(f[\xi])\,d\xi + \int_x^L G11[x, \xi]\,(f[\xi])\,d\xi$$

$$\frac{x\,y0}{L} + \int_x^L \frac{x\,(L-\xi)\,f[\xi]}{L\,T0}\,d\xi + \int_0^x \frac{(L-x)\,\xi\,f[\xi]}{L\,T0}\,d\xi$$

```
{y[0], y[L], y1[0], y1[L]}
```

{0, y0, 0, 0}

14.32 Solving the differential equations using the Green's function

Solve the differential equation

$$y''(x) + y(x) = -e^{-x}$$

with the boundary condition

$y(0) = 0$, and $y'(0) = 1$.

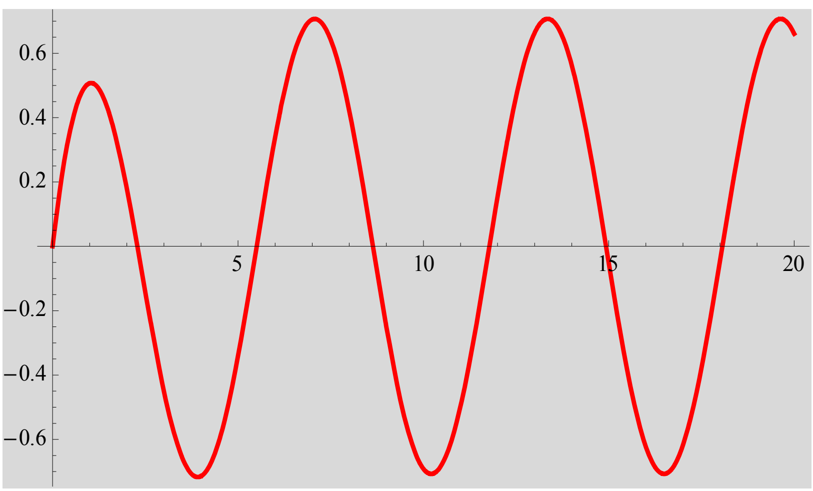
(1) Solving the differential equation without using the Green's function

((**Mathematica**))

```
Clear["Global`*"];

sol = DSolve[{y''[x] + y[x] == - Exp[-x],
      y[0] == 0, y'[0] == 1}, y[x], x];

y[x_] = y[x] /. sol[[1]] // Simplify
```

$$\frac{1}{2}\left(-e^{-x} + \mathrm{Cos}[x] + \mathrm{Sin}[x]\right)$$

```
Plot[y[x], {x, 0, 20}, PlotStyle → {Red, Thick}, Background → LightGray]
```



---

(2) Solution with the use of Green's function

We find the Green’s function;

$$G''(x) + G(x) = -\delta(x - \xi)$$

with the boundary condition

$$G(0) = 0, \qquad G'(0) = 1.$$

Using the Mathematica, we have

$$G_1(x,\xi) = \sin x \qquad \text{for } 0<x<\xi<1.$$

$$G_2(x,\xi) = \sin x - \sin(x-\xi) \qquad \text{for } 0<\xi<x<1.$$

The arbitrary function $\phi(x)$ satisfies the differential equation given by

$$\phi''(x) + \phi(x) = 0\,.$$

The solution of ϕ is

$$\phi = C_1 \sin x + C_2 \cos x\,.$$

Then the solution for $y(x)$ is

$$y(x) = \int_0^x G_2(x,\xi)e^{-\xi}d\xi + \int_x^\infty G_1(x,\xi)e^{-\xi}d\xi + C_1 \sin x + C_2 \cos x\,,$$

with the boundary condition,

$y(0) = 0$, and $y'(0) = 1$.

We find that $C_1 = 0$ and $C_2 = 0$.

$$y(x) = \frac{1}{2}(-e^{-x} + \sin x + \cos x)\,.$$

((**Mathematica**))

```
Clear["Global`*"];

Clear[G];

eq1 =
  DSolve[{G''[x] + G[x] == - DiracDelta[x - ξ],  G[0] == 0,
     G'[0] == 1}, G[x], x] // Simplify;


G11[x_] = G[x] /. eq1[[1]] // Simplify[#, 0 < x < ξ] &

Sin[x]

G22[x_] = G[x] /. eq1[[1]] // Simplify[#, x > ξ > 0] &

Sin[x] - Sin[x - ξ]
```

The arbitrary term satisfies the differential equation given by

$\phi''[x] + \phi[x] == 0$

or

$\phi(x)$=C1 Sin[x] + C2 Cos[x]

```
F[x_] :=
```

$$\int_0^x \text{G22}[x]\,(\text{Exp}[-\xi])\,d\xi + \int_x^\infty \text{G11}[x]\,(\text{Exp}[-\xi])\,d\xi + \text{C1 Sin}[x] +$$

```
   C2 Cos[x] // Simplify[#, x ∈ Reals] &

F[x]
```

$$\frac{1}{2}\left(-e^{-x} + \text{Cos}[x] + 2\,\text{C2 Cos}[x] + \text{Sin}[x] + 2\,\text{C1 Sin}[x]\right)$$

```
F'[x]
```

$$\frac{1}{2}\left(e^{-x} + \text{Cos}[x] + 2\,\text{C1 Cos}[x] - \text{Sin}[x] - 2\,\text{C2 Sin}[x]\right)$$

Boundary condition

```
eq1 = Solve[F[0] == 0, C2]

{{C2 → 0}}

eq2 = Solve[F'[0] == 1, C1]

{{C1 → 0}}

Simplify[F[x] /. eq1[[1]] /. eq2[[1]], x > 0]
```

$$\frac{1}{2}\left(-e^{-x} + \text{Cos}[x] + \text{Sin}[x]\right)$$

14.34 Example

Solve the differential equation

$$x''(t) + x(t) = \sin t$$

with the boundary condition

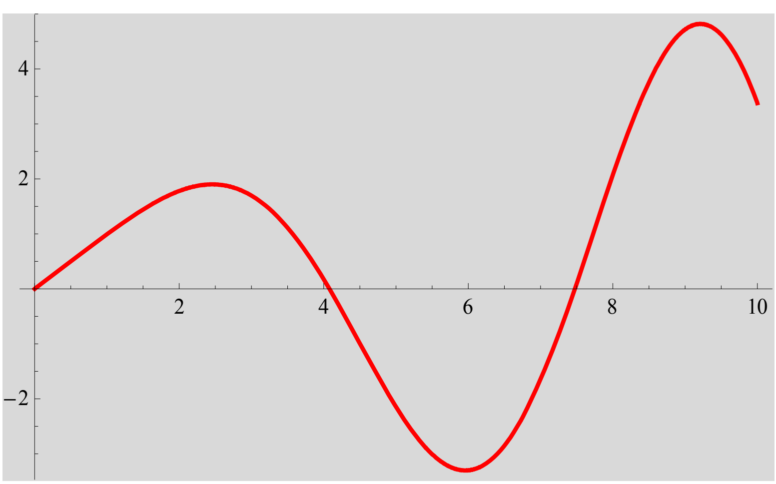
$$x(0) = 0, \text{ and } x'(0) = 1.$$

(1) Solving the differential equation without the use of the Green's function

$$x(t) = \frac{1}{2}(-t\cos t + 3\sin t)$$

((**Mathematica**))

```
Clear["Global`*"]

eq1 = x''[t] + x[t] == Sin[t]
x[t] + x''[t] == Sin[t]

eq2 = DSolve[{eq1, x[0] == 0, x'[0] == 1}, x[t], t] // Simplify
{{x[t] → 1/2 (-t Cos[t] + 3 Sin[t])}}

x[t_] = x[t] /. eq2[[1]]
1/2 (-t Cos[t] + 3 Sin[t])

p1 = Plot[x[t], {t, 0, 10}, PlotStyle → {Red, Thick}, Background → LightGray]
```



(2) Solution with the use of Green's function

We find the Green's function;

$$G''(t) + G(t) = -\delta(t-\tau)$$

with the boundary condition

$$G(0) = 0, \qquad G'(0) = 0.$$

Using the Mathematica, we have

$$G_1(t,\tau) = 0 \qquad \text{for } 0<t<\tau<1.$$

$$G_2(t,\tau) = -\sin(t-\tau) \qquad \text{for } 0<\tau<t<1.$$

The arbitrary function $\phi(t)$ satisfies the differential equation given by

$$\phi''(t) + \phi(t) = 0.$$

The solution of ϕ is

$$\phi = C_1 \sin t + C_2 \cos t.$$

Then the solution for $y(t)$ is

$$x(t) = \int_0^t G_2(t,\tau)(-\sin t)d\tau + C_1 \sin t + C_2 \cos t,$$

with the boundary condition,

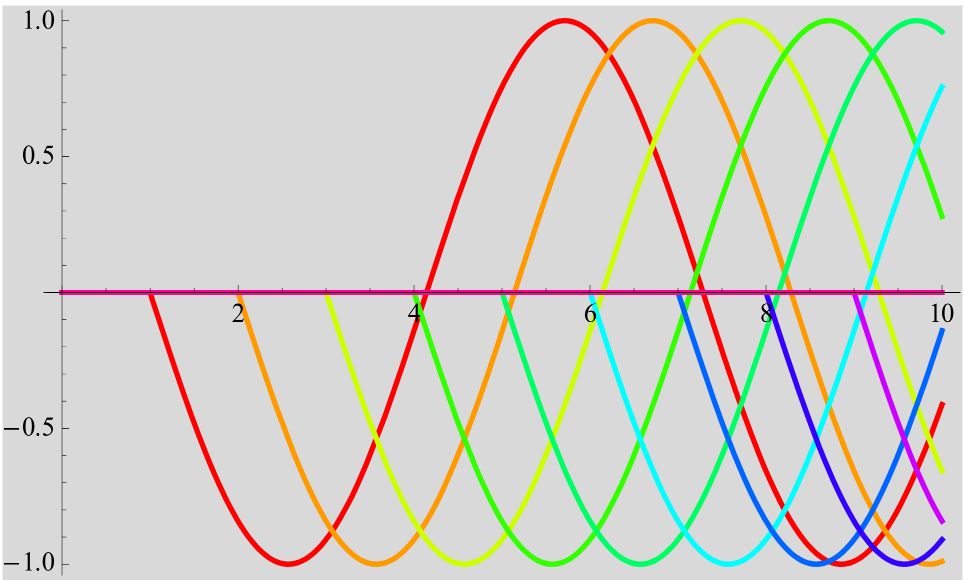
$x(0) = 0$, and $x'(0) = 1$.

We find that $C_1 = 0$ and $C_2 = 1$.

$$x(t) = \frac{1}{2}(-t\cos t + 3\sin t).$$

((**Mathematica**))

```
Clear["Global`*"];

seq1 = G''[t] + G[t] == -DiracDelta[t - τ];

seq2 = DSolve[{seq1, G[0] == 0, G'[0] == 0}, G[t], t];

seq3 = G[t] /. seq2[[1]];

Plot[Evaluate[Table[seq3, {τ, 1, 10, 1}]], {t, 0, 10},
 PlotStyle → Table[{Hue[0.1 i], Thick}, {i, 0, 10}],
 Background → LightGray]
```




```
G11[t_, τ_] = G[t] /. seq2[[1]] // Simplify[#, 0 < t < τ] &
```

```
0
```

```
G22[t_, τ_] = G[t] /. seq2[[1]] // Simplify[#, t > τ > 0] &
```

```
-Sin[t - τ]
```

The arbitrary term satisfies the differential equation given by

ϕ''[t] + ϕ[t] ==0
or
ϕ(t) = C1 Sin[t] + C2 Cos[t]

```
F[t_] := ∫_0^t G22[t, τ] (-Sin[τ]) dτ + C1 Sin[t] + C2 Cos[t] //
  Simplify[#, t ∈ Reals] &

F[t]

C2 Cos[t] + C1 Sin[t] + 1/2 (-t Cos[t] + Sin[t])

eq11 = F[0]

C2

eq12 = Solve[eq11 == 0]

{{C2 → 0}}

eq21 = D[F[t], t] /. t → 0 // Simplify

C1

eq22 = Solve[eq21 == 1, C1]

{{C1 → 1}}

F11[t_] = F[t] /. eq12[[1]] /. eq22[[1]] // Simplify

1/2 (-t Cos[t] + 3 Sin[t])
```

14.35 Poisson equation

Find the Green's function for the one dimensional Poisson equation

$$\frac{d^2\Phi(x)}{dx^2} = -\rho(x),$$

with boundary conditions;

$$\Phi(0) = 0, \qquad \Phi(1) = 0$$

Next, find the solution for Φ when

$$\rho(x) = \sin(\rho x).$$

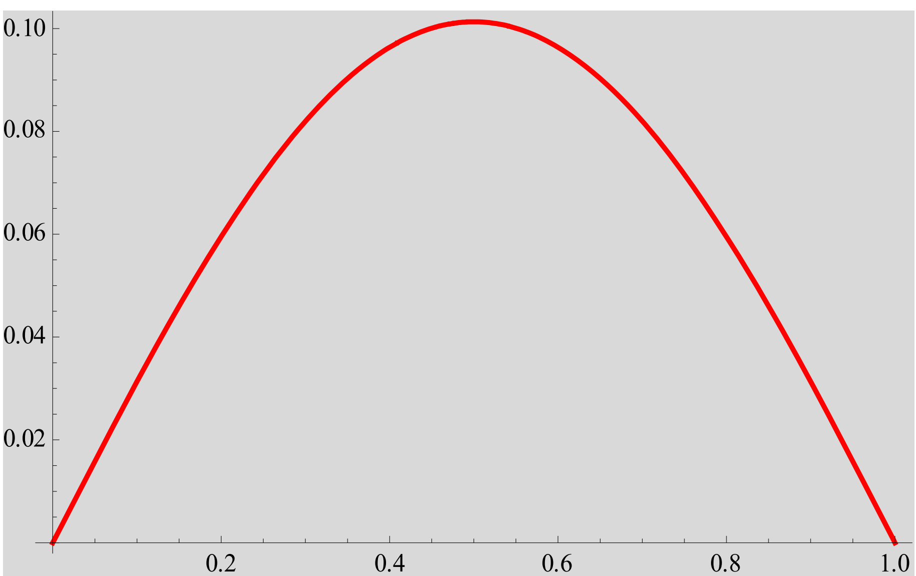
((Mathematica))

Method - 1 Ordinary differential equation

```
Clear["Global`*"];

eq1 = Φ''[x] == -Sin[π x];

eq2 = DSolve[{eq1, Φ[0] == 0, Φ[1] == 0}, Φ[x], x] // Simplify
```

$$\left\{\left\{\Phi[x] \to \frac{\text{Sin}[\pi x]}{\pi^2}\right\}\right\}$$

```
Φ[x_] = Φ[x] /. eq2[[1]];

p1 = Plot[Φ[x], {x, 0, 1}, PlotStyle → {Red, Thick},
  Background → LightGray]
```



Method - 2 Green's function method

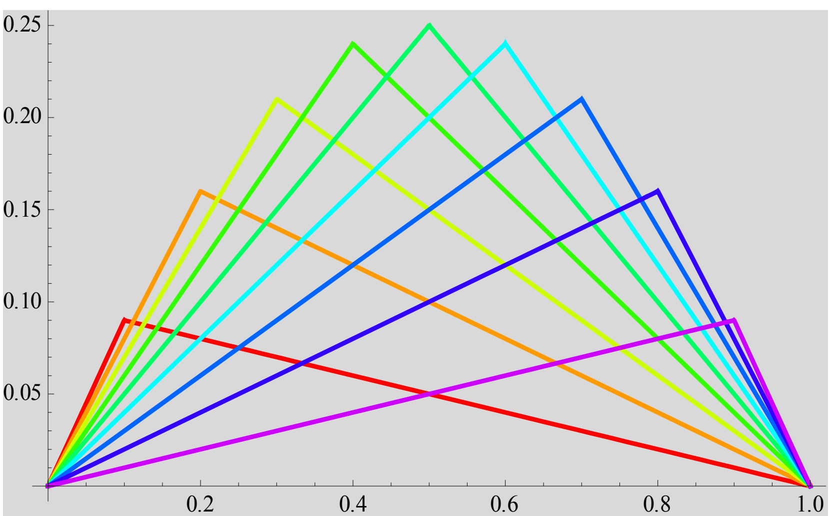
```
Clear["Global`*"];

seq1 = G''[x] == -DiracDelta[x - ξ];

seq2 = DSolve[{seq1, G[0] == 0, G[1] == 0}, G[x], x];

G[x_] = G[x] /. seq2[[1]];

G11[x_, ξ_] = G[x] // Simplify[#, 0 < x < ξ < 1] &
```

x - x ξ

```
G22[x_, ξ_] = G[x] // Simplify[#, 1 > x > ξ > 0] &
```

ξ - x ξ

```
Plot[Evaluate[Table[G[x], {ξ, 0.1, 1, 0.1}]], {x, 0, 1},
 PlotStyle → Table[{Hue[0.1 i], Thick}, {i, 0, 10}],
 Background → LightGray, PlotRange → All]
```



Φ[x_] := $\int_0^x$ G22[x, ξ] (Sin[π ξ]) dξ + $\int_x^1$ G11[x, ξ] (Sin[π ξ]) dξ //
  Simplify[#, x > 0] &

Φ[x]

$$\frac{\text{Sin}[\pi\, x]}{\pi^2}$$

APPENDIX-I

Stretched string problem: homogeneous boundary condition

We consider the solution of

$$L_x y + f(x) = 0,$$

where L_x is the self-adjoint differential operator, and is given by

$$L_x y = \frac{d}{dx}[p(x)y'] + q(x)y. \qquad \text{(self-adjoint)}$$

The boundary condition (inhomogeneous) is given by

$$y(a) = 0, \qquad y(b) = y_0.$$

Suppose that the solution of this equation is obtained using the Green's function $G(x, \xi)$ as

$$y(x) = \int_a^b G(x,\xi) f(\xi) d\xi + \varphi(x).$$

where G(x, ξ) satisfies the differential equation

$$L_x G(x,\xi) = -\delta(x-\xi),$$

with the homogeneous boundary condition

$$\alpha G(a,\xi) + \beta G'(a,\xi) = 0, \qquad \text{and} \qquad \alpha G(b,\xi) + \beta G'(b,\xi) = 0$$

or

$$\alpha u(a) + \beta u'(a) = 0, \quad \alpha u(b) + \beta u'(b) = 0$$

Note that

$$L_x \varphi(x) = 0.$$

What is the boundary condition imposed for $\varphi(x)$?

The solution of $y(x)$ can be rewritten as

$$y(x) = -\frac{v(x)}{A}\int_a^x u(\xi) f(\xi) d\xi - \frac{u(x)}{A}\int_x^b v(\xi) f(\xi) d\xi + \varphi(x).$$

$$y'(x) = -\frac{1}{A} v'(x) \int_a^x u(\xi) f(\xi) d\xi - \frac{1}{A} u'(x) \int_x^b v(\xi) f(\xi) d\xi + \varphi'(x) +$$

Thus the values of $y(a)$, $y(b)$, $y'(a)$, and $y'(b)$ are obtained as

$$y(a) = -\frac{u(a)}{A} \int_a^b v(\xi) f(\xi) d\xi + \varphi(a) = -c_1 u(a) + \varphi(a),$$

$$y(b) = -\frac{v(b)}{A} \int_a^b u(\xi) f(\xi) d\xi + \varphi(b) = -c_2 v(b) + \varphi(b),$$

$$y'(a) = -\frac{1}{A} u'(a) \int_a^b v(\xi) f(\xi) d\xi + \varphi'(a) = -c_1 u'(a) + \varphi'(a)$$

$$y'(b) = -\frac{1}{A} v'(b) \int_a^b u(\xi) f(\xi) d\xi + \varphi'(b) = -c_2 v'(b) + \varphi'(b)$$

Then we have

$$\alpha y(a) + \beta y'(a) = -c_1[\alpha u(a) + \beta u'(a)] + \alpha\varphi(a) + \beta\varphi'(a)$$

$$\alpha y(b) + \beta y'(b) = -c_1[\alpha u(b) + \beta u'(b)] + \alpha\varphi(b) + \beta\varphi'(b).$$

Since $\alpha u(a) + \beta u'(a) = 0$, and $\alpha u(b) + \beta u'(b) = 0$, we get

$$\alpha y(a) + \beta y'(a) = \alpha\varphi(a) + \beta\varphi'(a),$$

$$\alpha y(b) + \beta y'(b) = \alpha\varphi(b) + \beta\varphi'(b).$$

((**Simple case**))

When $\alpha = 1$, and $\beta = 0$,

$$y(a) = \varphi(a) = 0, \qquad y(b) = \varphi(b) = y_0$$

This is the boundary condition imposed on $\varphi(x)$.

APPENDIX-II

Table: Green's function with various boundary condition

(1)

(1) $L_x y = y''$ with $y(0) = 0$, $y(1) = 0$.

$G(x,\xi) = \xi(1-x)$ for $0<x<\xi<1$

$G(x,\xi) = x(1-\xi)$ for $0<\xi<x<1$

(2) $L_x y = y''$ with $y(0) = 0$, $y'(1) = 0$.

$G(x,\xi) = x$ for $0<x<\xi<1$

$G(x,\xi) = \xi$ for $0<\xi<x<1$

(3) $L_x y = y''$ with $y(0) + y(1) = 0$, $y'(0) + y'(1) = 0$.

$G(x,\xi) = \frac{1}{4}(1+2x-2\xi)$ for $0<x<\xi<1$

$G(x,\xi) = \frac{1}{4}(1-2x+2\xi)$ for $0<\xi<x<1$

(4) $L_x y = y''$ with $y(0) - y'(0) = 0$, $y(1) - y'(1) = 0$.

$G(x,\xi) = -\xi(1+x)$ for $0<x<\xi<1$

$G(x,\xi) = -x(1+\xi)$ for $0<\xi<x<1$

(5) $L_x y = y''$ with $y(0) + y'(0) = 0$, $y(1) - y'(1) = 0$.

$G(x,\xi) = -(2-\xi)(1-x)$ for $0<x<\xi<1$

$G(x,\xi) = -(2-x)(1-\xi)$ for $0<\xi<x<1$

APPENDIX III Green's function for un-damped oscillator

We consider the calculation

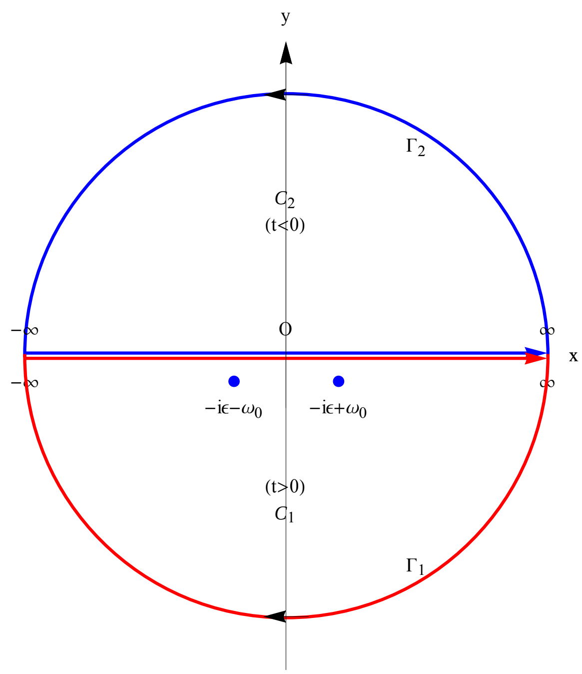
$$G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\omega t}}{\omega^2-\omega_0{}^2}d\omega$$

$$= \frac{1}{2\pi}\frac{1}{2\omega_0}\int_{-\infty}^{\infty}d\omega e^{-i\omega t}\left(\frac{1}{\omega-\omega_0}-\frac{1}{\omega+\omega_0}\right)d\omega$$

Here we add to ω_0 a positive or negative imaginary part $\pm i\varepsilon$, which amounts to the simple pole above or under the real axis. Note that $\varepsilon>0$ and $\omega_0>0$.

(i) Retarded Green's function

For t>0, we need to take the contour C_1 (clock-wise) in the lower half plane. The integral along the Γ_1 is equal to zero because of the Jordan's lemma. For t<0, we need to take the contour C_2 (counter-clock wise) in the upper half plane. The integral along the Γ_2 is equal to zero because of the Jordan's lemma.



For t>0

$$G(t)=\frac{1}{2\pi}\frac{1}{2\omega_0}\oint_{C_1}dze^{-izt}[\frac{1}{z-\omega_0+i\varepsilon}-\frac{1}{z+\omega_0+i\varepsilon}]$$

$$=\frac{1}{2\pi}\frac{1}{2\omega_0}(-2\pi i)[\mathrm{Re}\,s(z=\omega_0-i\varepsilon)+\mathrm{Re}\,s(z=-\omega_0-i\varepsilon)]$$

$$=-\frac{1}{\omega_0}\sin(\omega_0 t)$$

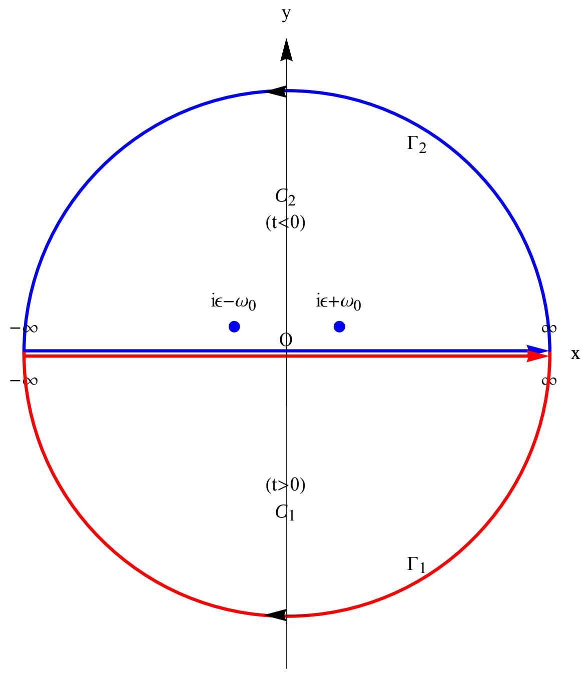
For t<0

$$G(t)=\frac{1}{2\pi}\frac{1}{2\omega_0}\oint_{C_2}dze^{-izt}[\frac{1}{z-\omega_0+i\varepsilon}-\frac{1}{z+\omega_0+i\varepsilon}]$$

$$=0$$

This Green's function is the same as that derived from the damped oscillator in the limit of $\gamma\rightarrow 0$.

(ii) Advanced Green's function



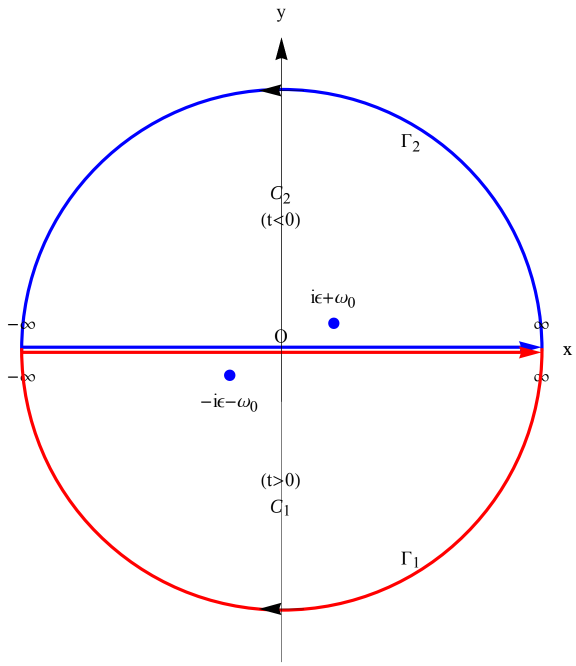
For t>0,

$$G(t) = \frac{1}{2\pi}\frac{1}{2\omega_0}\oint_{C_1} dz e^{-izt}[\frac{1}{z-\omega_0-i\varepsilon} - \frac{1}{z+\omega_0-i\varepsilon}]$$
$$= 0$$

For t<0,

$$G(t) = \frac{1}{2\pi}\frac{1}{2\omega_0}\oint_{C_2} dz e^{-izt}[\frac{1}{z-\omega_0-i\varepsilon} - \frac{1}{z+\omega_0-i\varepsilon}]$$
$$= \frac{1}{2\pi}\frac{1}{2\omega_0}(2\pi i)[\mathrm{Res}(z=\omega_0+i\varepsilon) + \mathrm{Res}(z=-\omega_0+i\varepsilon)]$$
$$= \frac{1}{\omega_0}\sin(\omega_0 t)$$

(iii) Other Green's function which is of limited interest as far as classical mechanics is concerned.



For t>0

$$G(t) = \frac{1}{2\pi}\frac{1}{2\omega_0}\oint_{C_1} dz e^{-izt}\left[\frac{1}{z-\omega_0-i\varepsilon} - \frac{1}{z+\omega_0+i\varepsilon}\right]$$

$$= \frac{1}{2\pi}\frac{1}{2\omega_0}(-2\pi i)[\mathrm{Res}(z=-\omega_0-i\varepsilon)]$$

$$= \frac{i}{2\omega_0}e^{i\omega_0 t}$$

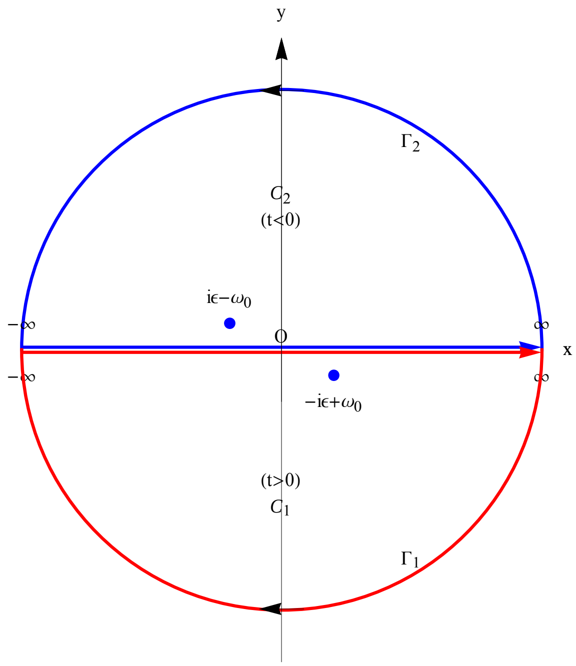
For t<0

$$G(t) = \frac{1}{2\pi}\frac{1}{2\omega_0}\oint_{C_2} dz e^{-izt}\left[\frac{1}{z-\omega_0-i\varepsilon} - \frac{1}{z+\omega_0+i\varepsilon}\right]$$

$$= \frac{1}{2\pi}\frac{1}{2\omega_0}(2\pi i)[\mathrm{Res}(z=\omega_0+i\varepsilon)]$$

$$= \frac{i}{2\omega_0}e^{-i\omega_0 t}$$

(iv) Other Green's function which is of limited interest as far as classical mechanics is concerned.



For t>0

$$
\begin{aligned}
G(t) &= \frac{1}{2\pi}\frac{1}{2\omega_0}\oint_{C_1} dz e^{-izt}[\frac{1}{z-\omega_0+i\varepsilon}-\frac{1}{z+\omega_0-i\varepsilon}] \\
&= \frac{1}{2\pi}\frac{1}{2\omega_0}(-2\pi i)[\mathrm{Re}\,s(z=\omega_0-i\varepsilon)] \\
&= -\frac{i}{2\omega_0}e^{-i\omega_0 t}
\end{aligned}
$$

For t<0

$$
\begin{aligned}
G(t) &= \frac{1}{2\pi}\frac{1}{2\omega_0}\oint_{C_2} dz e^{-izt}[\frac{1}{z-\omega_0+i\varepsilon}-\frac{1}{z+\omega_0-i\varepsilon}] \\
&= \frac{1}{2\pi}\frac{1}{2\omega_0}(2\pi i)[\mathrm{Re}\,s(z=-\omega_0+i\varepsilon)] \\
&= -\frac{i}{2\omega_0}e^{i\omega_0 t}
\end{aligned}
$$