Chapter 15S
Fresnel-Kirchhoff diffraction

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Augustin-Jean Fresnel (10 May 1788 – 14 July 1827), was a French physicist who contributed significantly to the establishment of the theory of wave optics. Fresnel studied the behaviour of light both theoretically and experimentally. He is perhaps best known as the inventor of the Fresnel lens, first adopted in lighthouses while he was a French commissioner of lighthouses, and found in many applications today.

http://en.wikipedia.org/wiki/Augustin-Jean_Fresnel

Christiaan Huygens, FRS (14 April 1629 – 8 July 1695) was a prominent Dutch mathematician, astronomer, physicist, horologist, and writer of early science fiction. His work included early telescopic studies elucidating the nature of the rings of Saturn and the discovery of its moon Titan, the invention of the pendulum clock and other investigations in timekeeping, and studies of both optics and the centrifugal force. Huygens achieved note for his argument that light consists of waves, now known as the Huygens–Fresnel principle, which became instrumental in the understanding of wave-particle duality. He generally receives credit for his discovery of the centrifugal force, the laws for collision of bodies, for his role in the development of modern calculus and his original observations on sound perception (see repetition pitch). Huygens is seen as the first theoretical physicist as he was the first to use formulae in physics.

http://en.wikipedia.org/wiki/Christiaan_Huygens

Green's theorem
Fresnel-Kirchhhoff diffraction
Huygens-Fresnel principle
Fraunhofer diffraction
Fresel diffraction
Fresnel's zone
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Cornu spiral

15S.1 Green's theorem

First we will give the proof of the Green's theorem given by

$$\int_V (\psi\nabla^2\phi - \phi\nabla^2\psi)d\tau = \int_A (\psi\nabla\phi - \phi\nabla\psi)\cdot d\mathbf{a}\ .$$

This theorem is the prime foundation of scalar diffraction theory. However, only an prudent choice of the Green's function and a closed surface A will allow its direct application to the diffraction theory: $\mathrm{d}\tau$ is volume element and $\mathrm{d}\boldsymbol{a}$ is the surface element.
((**Proof**)) In the Gauss's theorem, we put

$$\xi = \psi\nabla\phi$$

Then we have

$$I_1 = \int_V \nabla \cdot \xi d\tau = \int_V \nabla \cdot (\psi \nabla \phi) d\tau = \int_A (\psi \nabla \phi) \cdot d\mathbf{a} .$$

Noting that

$$\nabla \cdot (\psi \nabla \phi) = \psi \nabla^2 \phi + \nabla \psi \cdot \nabla \phi ,$$

we have

$$I_1 = \int_V (\psi \nabla^2 \phi + \nabla \psi \cdot \nabla \phi) d\tau = \int_A (\psi \nabla \phi) \cdot d\mathbf{a} .$$

By replacing $\psi \leftrightarrow \phi$, we also have

$$I_1 = \int_V (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) d\tau = \int_A (\phi \nabla \psi) \cdot d\mathbf{a} ,$$

Thus we find the Green's theorem

$$I_1 - I_2 = \int_V (\psi \nabla^2 \phi - \phi \nabla^2 \psi) d\tau = \int_A (\psi \nabla \phi - \phi \nabla \psi) \cdot d\mathbf{a} ,$$

or

$$\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) d\tau = \int_A (\phi \nabla \psi - \psi \nabla \phi) \cdot \mathbf{n} da ,$$

15S.2 Fresnel-Kirchhoff diffraction theory

According to the Huygen's construction, every point of a wave-front may be considered as a center of a secondary disturbance which gives rise to spherical wavelets, and the wave-front at any later instant may be regarded as the envelop of these wavelets. Fresnel was able to account for diffraction by supplementing Huygen's construction with the postulate that the secondary wavelets mutually interfere. This combination of the Huygen's construction with the principle of interference is called the Huygens-Fresnel principle. (Born and Wolf, Principles of optics, 7-th edition).

The basic idea of the Huygens-Fresnel theory is that the light disturbance at an observation point P arises from the superposition of secondary waves that proceed from a surface (aperture) situated between the point P and the light source S.

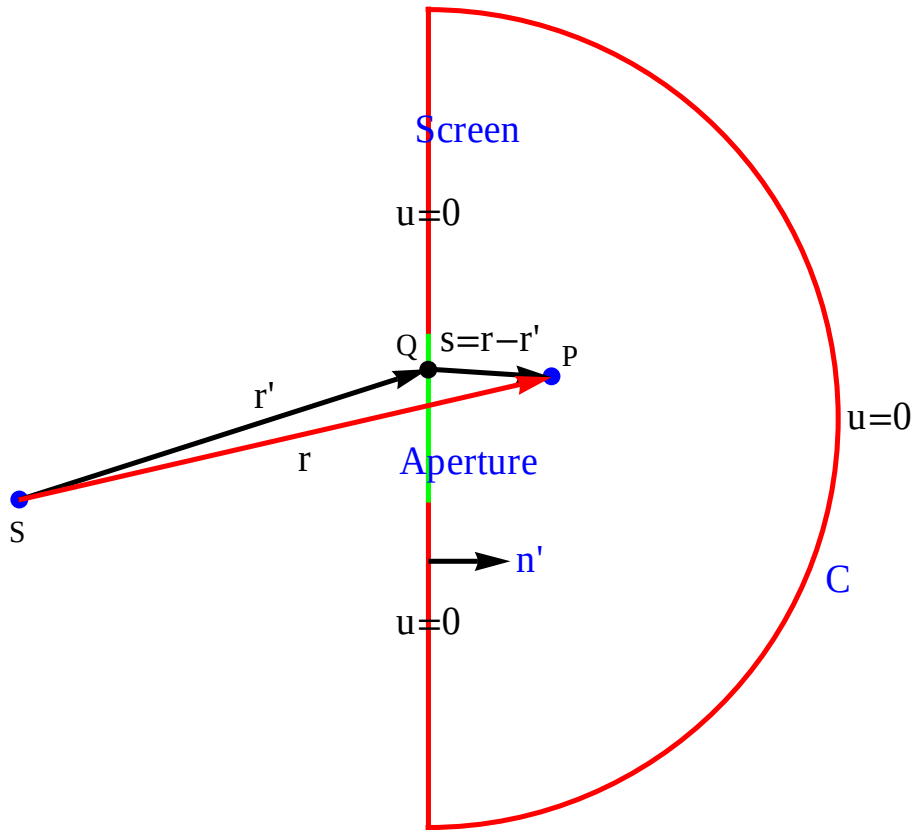


Fig. Illustrating the deviation of the Fresnel-Kirchhoff diffraction formula. ***r***' is the position vector from the source point S (at the origin). The aperture is denoted by the green line. The volume integral is taken over a region bounded by red lines (consisting a screen with an aperture and the surface *C*). $\overrightarrow{SP} = \mathbf{r}$. $\overrightarrow{PQ} = \mathbf{r} - \mathbf{r}'$. The green line denotes an aperture.

We now start with the Green's theorem given by

$$\int_V [\phi(\mathbf{r}')\nabla'^2\psi(\mathbf{r}') - \psi(\mathbf{r}')\nabla'^2\phi(\mathbf{r}')]d^3\mathbf{r}' = \int_A [\phi(\mathbf{r}')\nabla'\psi(\mathbf{r}') - \psi(\mathbf{r}')\nabla'\phi(\mathbf{r}')]\cdot\mathbf{n}da',$$

Here we assume that

$$\phi(\mathbf{r}') = u(\mathbf{r}')$$

and

$$\psi(\mathbf{r}') = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi\,|\,\mathbf{r}-\mathbf{r}'|} = G(\mathbf{r},\mathbf{r}')\,, \qquad (\nabla'^2 + k^2)G(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}')\,,$$

with

$$\nabla' G(\mathbf{r},\mathbf{r}') = -\frac{\hat{s}}{4\pi}\frac{e^{iks}}{s}(ik - \frac{1}{s}) \approx -ik\frac{\hat{s}}{4\pi}\frac{e^{iks}}{s}\,,$$

where k (= $2\pi/\lambda$) is the wavenumber, $G(\mathbf{r},\mathbf{r}')$ is the Green's function, $G(\mathbf{r},\mathbf{r}') = G(\mathbf{s})$ with $\mathbf{s} = \mathbf{r} - \mathbf{r}'$, and $\hat{s}$ is the unit vector of $\boldsymbol{s}$. Then we have

$$\int_V [u(\mathbf{r}')\nabla'^2 G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\nabla'^2 u(\mathbf{r}')]d^3\mathbf{r}' = \int_{Aperture+screen+C} [u(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\nabla' u(\mathbf{r}')]\cdot\mathbf{n}da'\,,$$

or

$$\int_V [-u(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\nabla'^2 u(\mathbf{r}')]d^3\mathbf{r}' = \int_{Aperture+screen+C} [u(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\nabla' u(\mathbf{r}')]\cdot\mathbf{n}da'\,.$$

Further we assume that $u(\boldsymbol{r}') = 0$ everywhere but in the aperture. In the aperture, $u(\boldsymbol{r}')$ is the field due to a point source at the point S. So we get

$$\begin{aligned} u(\mathbf{r}) &= \int_{Aperture} [u(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\nabla'\mathbf{u}(\mathbf{r}')]\cdot\mathbf{n}'da' \\ &= \frac{1}{4\pi}\int_{Aperture} [u(\mathbf{r}')\nabla'\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} - \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\nabla'\mathbf{u}(\mathbf{r}')]\cdot\mathbf{n}'da' \end{aligned}$$

where $\boldsymbol{n}'$ (=$-\boldsymbol{n}$) is an inwardly directed normal to the aperture surface. This is one form of *the integral theorem of Helmholtz and Kirchhoff.* It is reasonable to suppose that everywhere on the aperture A, $u(\mathbf{r}')$ and $\nabla' u(\mathbf{r}')$ will not appreciably differ from the

values obtained in the free space. So we assume that $u_{inc}(\mathbf{r}')$ satisfies the Maxwell's equation in the free space. $u_{inc}(\mathbf{r}')$ is the component of the electric field for the spherical wave (incident wave);

$$(\nabla'^2 + k^2)u_{inc}(\mathbf{r}') = 0 .$$

The solution of this differential equation is given by

$$u(\mathbf{r}') = u_{inc}(\mathbf{r}') = E_0 \frac{e^{ikr'}}{r},$$

and

$$\nabla' u_{inc}(\mathbf{r}') = -E_0 \frac{\hat{r}'}{4\pi}\frac{e^{ikr'}}{r'}(ik - \frac{1}{r'}) \approx -ikE_0 \frac{\hat{r}'}{4\pi}\frac{e^{ikr'}}{r'} = -\frac{ik}{4\pi}\hat{r}' u_{inc}(\mathbf{r}'),$$

on the aperture surface, where E_0 is a electric field constant amplitude and $\hat{r}'$ is the unit vector of $\boldsymbol{r}'$. Then we have

$$u(\mathbf{r}) = \frac{1}{4\pi}\int_{Aperture} E_0\{\frac{e^{ikr'}}{r'}[-\hat{\mathbf{s}}\cdot\mathbf{n}'\frac{e^{iks}}{s}(ik - \frac{1}{s})] - \frac{e^{iks}}{s}[-\hat{\mathbf{r}}'\cdot\hat{\mathbf{n}}'\frac{e^{ikr'}}{r'}(ik - \frac{1}{r'})]\}da'$$

$$\approx \frac{-ik}{4\pi}\int_{Aperture}\frac{e^{iks}}{s}u_{inc}(\mathbf{r}')(\hat{\mathbf{s}} + \hat{\mathbf{r}}')\cdot\hat{\mathbf{n}}' da'$$

$$= -\frac{ik}{2\pi}\int_{Aperture}\frac{e^{iks}}{s}u_{inc}(\mathbf{r}')I(\hat{\mathbf{r}}', \hat{\mathbf{s}})da'$$

(Fresnel-Kirchhoff diffraction formula)

where the inclination factor is defined as

$$I(\hat{\mathbf{r}}', \hat{\mathbf{s}}) = \frac{1}{2}(\hat{\mathbf{s}} + \hat{\mathbf{r}}')\cdot\hat{\mathbf{n}}'.$$

for the convenience. This expression is consistent with the Huyghens' principle. $u(\boldsymbol{r})$ is the superposition of the spherical waves e^{iks}/s emanating from the wavefront e^{ikr}/r produced by a point source.

15S.3 The Huygens-Fresnel principle

In order to explain the essence of the Huygens-Fresnel principle, we use the simple model as shown below.

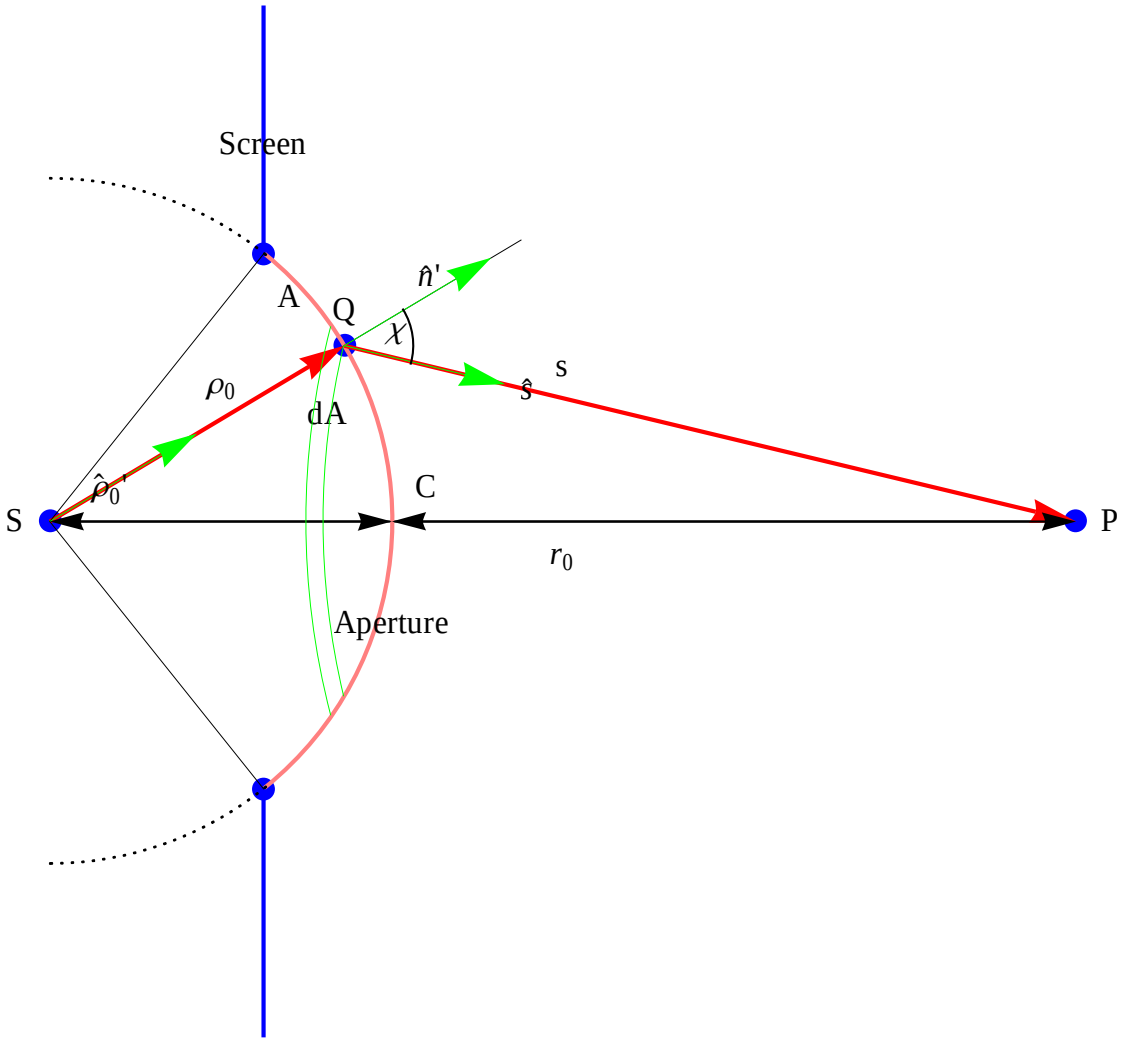


Fig. Illustrating the diffraction formula. We assume that the shape of the aperture in the screen is a sphere.

Let A be the instantaneous position of a spherical monochromatic wave-front of radius ρ_0 which proceeds from a point source S, and let P be a point at which the light disturbance at a point Q on the wave-front may be represented by

$$u(\mathbf{Q}) = E_0 \frac{e^{ik\rho_0}}{\rho_0},$$

where E_0 is the amplitude at unit distance from the source S. According to the Huygens-Fresnel principle, each element of the wave-front is the center of a secondary disturbance, which is propagated in the form of spherical wavelets. The contribution du(***P***) due to the element dA at the point Q is expressed by

$$du(P) = -\frac{ik}{2\pi} I(\chi) E_0 \frac{e^{ik\rho_0}}{\rho_0} \frac{e^{iks}}{s} dA$$

where s =QP. The inclination factor $I(\chi)$ is given by

$$I(\chi) = \frac{1}{2}(\hat{\mathbf{s}} + \hat{\boldsymbol{\rho}}_0) \cdot \hat{\mathbf{n}}' = \frac{1}{2}(1 + \cos\chi),$$

where χ is the angle (often called the angle of diffraction) between the normal at Q and the direction of propagation, and $\hat{\boldsymbol{\rho}}_0 = \hat{\mathbf{n}}'$ in the present case. The total disturbance at the point P is given by

$$u(P) = -E_0 \frac{ik}{2\pi} \frac{e^{ik\rho_0}}{\rho_0} \iint_{Aperture} I(\chi) \frac{e^{iks}}{s} dA.$$

15S.4 Fresnel zone

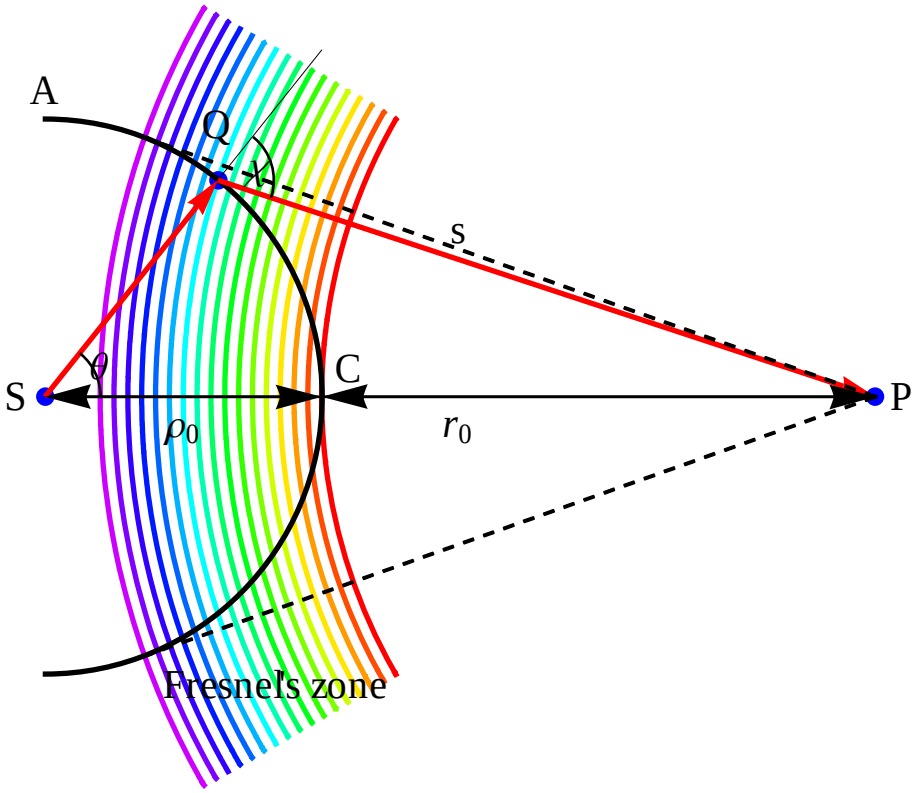


Fig. Fresnel's zone construction (Born and Wolf, Principles of optics, p.413)

In order to evaluate u(P), we use the so-called Fresnel's zone. With the center at P, we construct the spheres of radii,

$$r_0,\ r_0+\frac{\lambda}{2},\ r_0+\frac{2\lambda}{2},\ r_0+\frac{3\lambda}{2},\ r_0+\frac{4\lambda}{2},\ r_0+\frac{5\lambda}{2},.....$$

where r_0 = CP. C is the point of the intersection of SP with the wave-front S. The spheres divide A into a number of zones

$Z_1\,(r_0\ -\ r_0+\frac{\lambda}{2})$, the Fresnel's 1st zone

$Z_2\,(r_0+\frac{\lambda}{2}\ -\ r_0+\frac{2\lambda}{2})$, the 2-nd zone

$Z_3\,(r_0+\frac{2\lambda}{2}\ -\ r_0+\frac{3\lambda}{2})$, the 3-rd zone

.....

$Z_j\,(r_0+\frac{(j-1)\lambda}{2}\ -\ r_0+\frac{j\lambda}{2})$, the j-th zone

.....

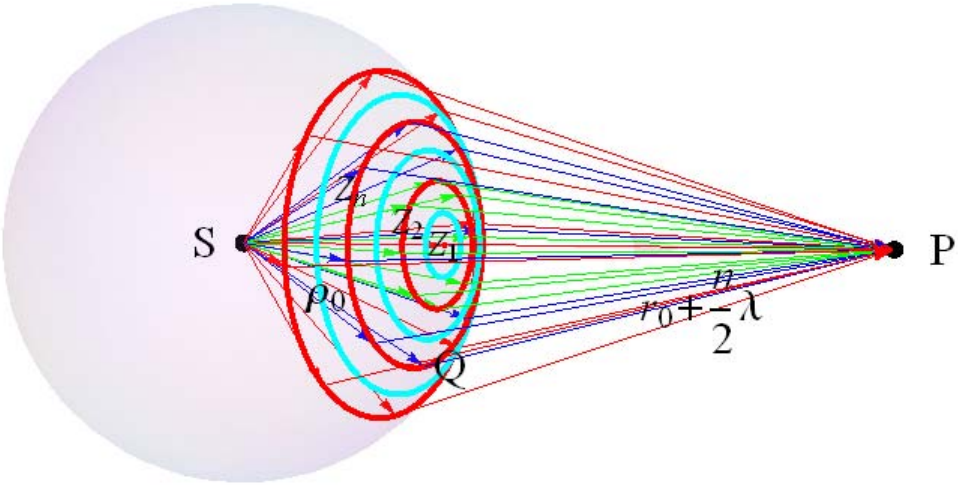


Fig. Fresnel's zone (n = 1, 2, 3, ---). The distance PQ is equal to $r_0 + n\lambda/2$ $r_0 + (n-1)\lambda/2$ and for the n-th zone. This figure is drawn by using Graphics3D of the Mathematica.

Since ρ_0 and r_0 are much larger than the wavelength λ, the inclination factor may be assume to have the same value (I_j), for points for the j-th zone. From the figure, we have

$$s^2 = (\rho_0 + r_0)^2 + \rho_0{}^2 - 2\rho_0(\rho_0 + r_0)\cos\theta .$$

So that

$$sds = \rho_0(\rho_0 + r_0)\sin\theta d\theta ,$$

and the surface element dA is given by

$$dA = 2\pi\rho_0{}^2 \sin\theta d\theta = 2\pi\rho_0{}^2 \frac{sds}{\rho_0(\rho_0 + r_0)} .$$

((**Note-1**))

The area of the end cap is given by

$$A = \int_0^{\theta} dA = 2\pi\rho_0{}^2 \int_0^{\theta} \sin\theta d\theta = 2\pi\rho_0{}^2 (1-\cos\theta)$$
$$= 2\pi\rho_0{}^2 - 2\pi\rho_0{}^2 \frac{(\rho_0 + r_0)^2 + \rho_0{}^2 - s^2}{2\rho_0(\rho_0 + r_0)}$$

The area of the *j*-th zone is obtained as

$$A_j = [2\pi\rho_0{}^2 - 2\pi\rho_0{}^2 \frac{(\rho_0 + r_0)^2 + \rho_0{}^2 - (r_0 + \frac{j\lambda}{2})^2}{2\rho_0(\rho_0 + r_0)}]$$
$$-[2\pi\rho_0{}^2 - 2\pi\rho_0{}^2 \frac{(\rho_0 + r_0)^2 + \rho_0{}^2 - (r_0 + \frac{(j-1)\lambda}{2})^2}{2\rho_0(\rho_0 + r_0)}]$$
$$= \frac{2\pi\rho_0{}^2}{2\rho_0(\rho_0 + r_0)}[(r_0 + \frac{j\lambda}{2})^2 - (r_0 + \frac{(j-1)\lambda}{2})^2]$$
$$= \frac{\lambda\pi\rho_0}{\rho_0 + r_0}[r_0 + \frac{(2j-1)\lambda}{4}]$$

The mean distance from the *j*-th zone to the point P is denoted as $\bar{s}_j$

$$\bar{s}_j = \frac{(r_0 + \frac{j\lambda}{2}) + (r_0 + \frac{(j-1)\lambda}{2})}{2} = r_0 + \frac{(2j-1)}{4}\lambda$$

Then we have

$$\frac{A_j}{\bar{s}_j} = \frac{\lambda\pi\rho_0}{\rho_0 + r_0},$$

which is independent of *j*.

((Note-2)) Inclination factor

We note that

$$\hat{\mathbf{s}} \cdot \hat{\mathbf{n}}' = \cos\chi = \frac{2\rho_0 r_0 + r_0{}^2 - s^2}{2\rho_0 s}.$$

Then the inclination factor is obtained as

$$I(\chi) = \frac{1}{2}(1 + \frac{2\rho_0 r_0 + {r_0}^2 - s^2}{2\rho_0 s}) = -\frac{(s + r_0)(s - r_0 - 2\rho_0)}{4\rho_0 s},$$

since

$$\overrightarrow{SP} = (\rho_0 + r_0, 0), \qquad \overrightarrow{SQ} = (\rho_0 \cos\theta, \rho_0 \sin\theta), \qquad \hat{\mathbf{n}}' = (\cos\theta, \sin\theta)$$

$$\mathbf{s} = \overrightarrow{QP} = \overrightarrow{SP} - \overrightarrow{SQ} = (\rho_0 + r_0 - \rho_0 \cos\theta, -\rho_0 \sin\theta),$$

$$\cos\chi = \frac{\mathbf{s} \cdot \hat{\mathbf{n}}'}{s} = \frac{(\rho_0 + r_0)\cos\theta - \rho_0}{s} = \frac{2\rho_0 r_0 + {r_0}^2 - s^2}{2\rho_0 s}.$$

Then the contribution of the *j*-th Fresnel's zone to *u*(P) is

$$\begin{aligned} u_j(P) &= 2\pi \frac{ik}{2\pi} E_0 \frac{e^{ik\rho_0}}{\rho_0} \iint_{Z_j} I(\chi) \frac{e^{iks}}{s} {\rho_0}^2 \frac{s\,ds}{\rho_0(\rho_0 + r_0)} \\ &\approx ikE_0 \frac{e^{ik\rho_0}}{(\rho_0 + r_0)} I_j \int_{r_0 + (j-1)\lambda/2}^{r_0 + j\lambda/2} e^{iks}\, ds \\ &= ikE_0 \frac{e^{ik(\rho_0 + r_0)}}{(\rho_0 + r_0)} I_j \int_{(j-1)\lambda/2}^{j\lambda/2} e^{iks'}\, ds' \end{aligned}$$

where $s' = s - r_0$, and $I(\chi)$ is approximated by I_j, the value of $I(\chi)$ at $s = r_0 + (j-1)\frac{\lambda}{2}$;

$$I_j = \frac{[2r_0 + (j-1)\frac{\lambda}{2}][-2\rho_0 + (j-1)\frac{\lambda}{2}]}{4\rho_0[r_0 + (j-1)\frac{\lambda}{2}]}$$

and $I_1 = -1$. Noting that

$$\int_{(j-1)\lambda/2}^{j\lambda/2} e^{iks'}\, ds' = -\frac{i(-1)^j \lambda}{\pi}$$

we get

$$u_j(P) = 2E_0 \frac{e^{ik(\rho_0+r_0)}}{(\rho_0+r_0)} I_j (-1)^j$$

The total effect at the point P is obtained by summing all the contributions:

$$u(P) = \sum_{j=1}^{n} u_j(P) = 2E_0 \frac{e^{ik(\rho_0+r_0)}}{(\rho_0+r_0)} \sum_{j=1}^{n} I_j (-1)^j$$

For $n = 2m+1$ (odd)

$$\begin{aligned} u(P) &= -2E_0 \frac{e^{ik(\rho_0+r_0)}}{(\rho_0+r_0)} [\frac{I_1}{2} + (\frac{I_1}{2} - I_2 + \frac{I_3}{2}) + ... + (\frac{I_{2m-3}}{2} - I_{2m-2} + \frac{I_{2m-1}}{2}) \\ &\quad + (\frac{I_{2m-1}}{2} - I_{2m} + \frac{I_{2m+1}}{2}) + \frac{I_{2m+1}}{2}] \\ &\approx -2E_0 \frac{e^{ik(\rho_0+r_0)}}{(\rho_0+r_0)} (\frac{I_1}{2} + \frac{I_{2m+1}}{2}) \\ &= -E_0 \frac{e^{ik(\rho_0+r_0)}}{(\rho_0+r_0)} (I_1 + I_{2m+1}) \end{aligned}$$

since I_j changes slowly with j, and

$$I_j \approx \frac{I_{j-1} + I_{j+1}}{2}.$$

For $n = 2m$ (even)

$$u(P) = -2E_0 \frac{e^{ik(\rho_0+r_0)}}{(\rho_0+r_0)}[I_1 - \frac{I_2}{2} - (\frac{I_2}{2} - I_3 + \frac{I_4}{2}) + ... +$$
$$-(\frac{I_{2m-2}}{2} - I_{2m-1} + \frac{I_{2m}}{2}) - \frac{I_{2m}}{2}]$$
$$\approx -2E_0 \frac{e^{ik(\rho_0+r_0)}}{(\rho_0+r_0)}(I_1 - \frac{I_2}{2} - \frac{I_{2m}}{2})$$
$$\approx -E_0 \frac{e^{ik(\rho_0+r_0)}}{(\rho_0+r_0)}(I_1 - I_{2m})$$

where $I_2 \approx I_1$. Suppose that we allow m to become large enough so the entire spherical wave is divided into zones: $I_{2m} = 0$ and $I_{2m+1} = 0$. Then we get

$$u(P) = -E_0 \frac{e^{ik(\rho_0+r_0)}}{\rho_0+r_0} I_1 = -E_0 \frac{e^{ik(\rho_0+r_0)}}{\rho_0+r_0}$$

since $I_1 = 1$. We note that the disturbance at P only due to the first zone is

$$u_1(P) = -2E_0 \frac{e^{ik(\rho_0+r_0)}}{(\rho_0+r_0)} I_1 = -2E_0 \frac{e^{ik(\rho_0+r_0)}}{(\rho_0+r_0)}$$

Therefore we have

$$u(P) = \frac{1}{2} u_1(P).$$

In other words, the total disturbance at P is equal to half of the disturbance due to the first zone.

15S.5 Reformulation of the Fresnel-Kirchhoff diffraction:

We consider the new coordinate system for the above Fresnel-Kirchhoff diffraction formula, where the origin of the system is moved from the source point S to a specific point in the screening with an aperture (in the above figure, we put the new origin O_1 at the center of the aperture). Note that the shape of the aperture is two-dimensional (such as a square aperture or a circle aperture).

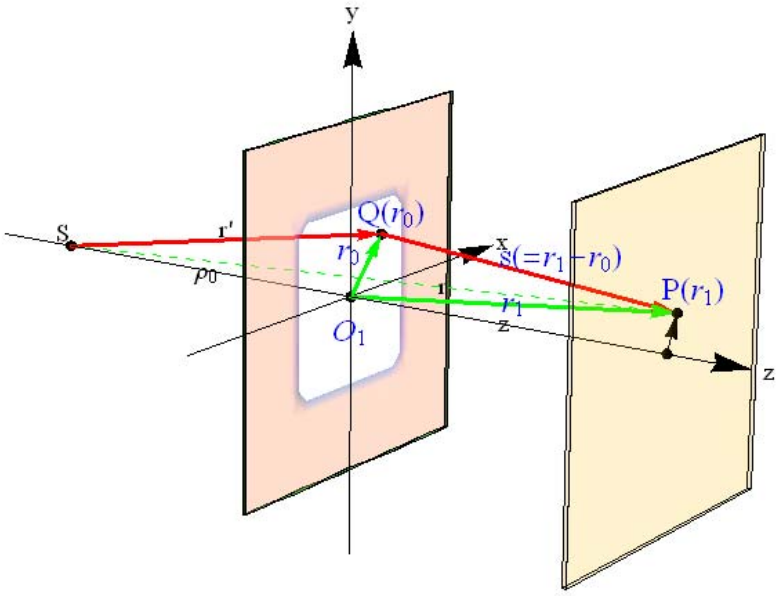


Fig. New coordinate system. The origin is moved to the center of aperture. $\overrightarrow{O_1Q} = \mathbf{r}_0$. $\overrightarrow{O_1P} = \mathbf{r}_1$. $\overrightarrow{PQ} = \mathbf{r}_1 - \mathbf{r}_0$. The point Q is on the aperture and the point P is on the observation plane.

We define the new origin O_1 in a specific point in the screen with an aperture. The vector vector ***r***' is expressed by

$$\mathbf{r}' = \boldsymbol{\rho}_0 + \mathbf{r}_0 ,$$

where $\boldsymbol{r}_0$ is a position vector located on the aperture. The vector ***r*** is expressed by

$$\mathbf{r} = \boldsymbol{\rho}_0 + \mathbf{r}_1 ,$$

where $\boldsymbol{\rho}_0$ is the vector connecting the source point S and the new origin O_1 in the aperture, and $\mathbf{r}_1$ is a position vector from the new origin O_1 to the position on the observation plane. The vector ***s*** is given by

$$\mathbf{s} = \mathbf{r} - \mathbf{r}' = \mathbf{r}_1 - \mathbf{r}_0 , \qquad s = |\mathbf{r}_1 - \mathbf{r}_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2 + z^2} .$$

For simplicity we assume that

$$s \approx z ,$$

and

$$I(\hat{\mathbf{r}}', \mathbf{s}) \approx 1,$$

for $\hat{\mathbf{s}} // \hat{\mathbf{r}}' // \hat{\mathbf{n}}'$. Then we get the formula of the Fresnel-Kirchhoff diffraction,

$$u(\mathbf{r}_1) \approx -\frac{ik}{2\pi z} \int_{Aperture} e^{ik|\mathbf{r}_1 - \mathbf{r}_0|} u_{inc}(\mathbf{r}_0 + \boldsymbol{\rho}_0) dA_0 ,$$

where dA_0 is the surface area element in the aperture; $dA_0 = dx_0 dy_0$. We assume that

$$u_{inc}(\mathbf{r}_0 + \boldsymbol{\rho}_0) \approx u_{inc}(\boldsymbol{\rho}_0) ,$$

Then we get

$$u(\mathbf{r}_1) \approx -\frac{ik}{2\pi z} u_{inc}(\boldsymbol{\rho}_0) \int_{Aperture} e^{ik|\mathbf{r}_1 - \mathbf{r}_0|} dA_0 = -\frac{i}{\lambda z} u_{inc}(\boldsymbol{\rho}_0) \int_{Aperture} e^{ik|\mathbf{r}_1 - \mathbf{r}_0|} dA_0 .$$

15S.6 Fresnel diffraction

Imagine that we have an opaque shield with a single small aperture which is illuminated by plane waves from a very distance point source. First we also have a screen parallel with, and very close to the aperture. In this case, an image of the aperture is projected on the screen. As the screen moves further away from the aperture, the image of the aperture becomes increasing more structured as the fringes become more prominent. This is called the Fresnel diffraction. The moving of the screen at a very great distance from the aperture results in the drastic chage of the projected pattern which is the two-dimensional (2D) Fourier transform image of the aperture pattern. This is called the Fraunhofer diffraction.

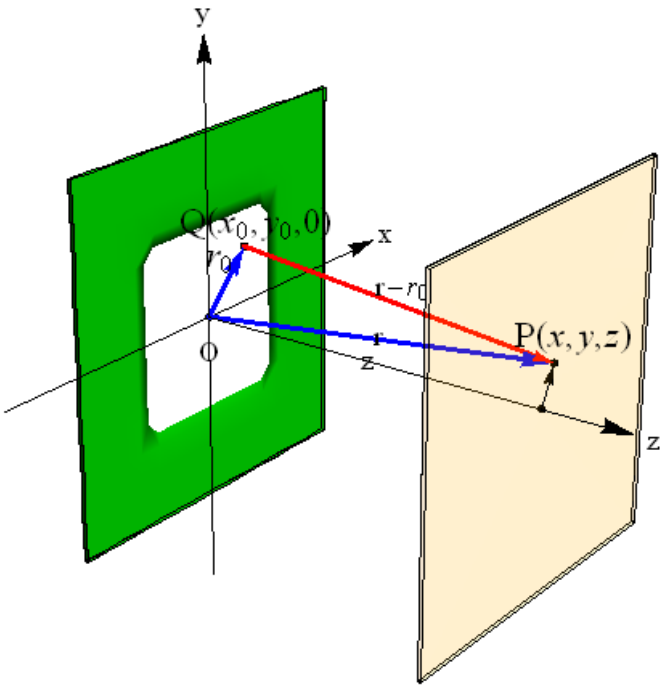


Fig. Rectangular aperture for the Fresnel diffraction and Fraunhofer diffraction. $\boldsymbol{r}_0 = (x_0, y_0, 0)$. $\overrightarrow{OP} = \mathbf{r} = (x, y, z)$

The electric field $E(x, y, z)$ at the point P is expressed by

$$E(x,y,z) \approx -\frac{ik}{2\pi z} E_{inc}(\boldsymbol{\rho}_0) \iint_{Aperture} e^{ik|\mathbf{r}-\mathbf{r}_0|} dx_0 dy_0 \,,$$

where $\boldsymbol{r} = (x, y, z)$ at the point P of the observation plane, $\boldsymbol{r}_0 = (x_0, y_0, 0)$ at the point Q in the aperture, as shown in Fig.

$$|\mathbf{r}-\mathbf{r}_0| = \sqrt{(x-x_0)^2 + (y-y_0)^2 + z^2}$$

This equation describes how the light travels from the aperture to the observation plane, a distance z apart. All points from the aperture contribute to the intensity at the point P. Suppose that

$$z^2 >> \rho^2 = (x - x_0)^2 + (y - y_0)^2$$

Then we can expand $|\mathbf{r} - \mathbf{r}_0|$ around $\rho = 0$, as

$$|\mathbf{r} - \mathbf{r}_0| = z + \frac{\rho^2}{2z} - \frac{\rho^4}{8z^3} + \frac{\rho^6}{16z^5} + ...$$

Then we have

$$k|\mathbf{r} - \mathbf{r}_0| = kz + \frac{k\rho^2}{2z} - \frac{k\rho^4}{8z^3} + \frac{k\rho^6}{16z^5} + ...$$

We assume that

$$\frac{k\rho^4}{8z^3} << 2\pi ,$$

or

$$\frac{\rho^4}{\lambda z^3} << 8.$$

When ρ is on the same order as the size of aperture *a*, this condition can be rewritten as

$$\frac{a^4}{\lambda z^3} << 8.$$

((**Note**))

Instead of this condition, we use the *Fresnel's number F*. When *F*>>1, the Fresnel diffraction can occur. On the other hand, when *F*<<1, the Fraun*hofer diffraction can occur. The Definition of F will be given below. Note that the condition (F*>>1) is not so restrictive compared to the above condition; $\frac{a^4}{\lambda z^3} << 8$. The Fresnel diffraction may o ccur when the factor $\exp(-i\frac{k\rho^4}{8z^3})$ slowly changes compared to the other factors $\exp[i(kz + \frac{k\rho^2}{2z})]$.

Under such a condition, the distance $|\mathbf{r} - \mathbf{r}_0|$ is approximated by

$$\left|\mathbf{r}-\mathbf{r}_0\right| = z + \frac{\rho^2}{2z}.$$

This is called the Fresnel approximation.

$$E(x,y,z) = -i\frac{e^{ikz}}{\lambda z}E_{inc}(\boldsymbol{\rho}_0)\iint_{Aperture} dx_0 dy_0 \exp[ik\frac{(x-x_0)^2+(y-y_0)^2}{2z}]$$
$$= -i\frac{e^{ikz}}{\lambda z}E_{inc}(\boldsymbol{\rho}_0)\iint \Theta(x_0,y_0,0)dx_0 dy_0 \exp[ik\frac{(x-x_0)^2+(y-y_0)^2}{2z}]$$

where $\Theta(x_0,y_0,0)=1$ for the inside of the aperture, and 0 for the outside of the aperture. Mathematically, this integral corresponds to the convolution of $\Theta(x_0,y_0,0)$ and $h(x_0,y_0)$, and can be expressed by

$$E(x,y,z) = \Theta(x,y,0)\otimes \exp h(x,y)$$

where

$$h(x,y) = -i\frac{e^{ikz}}{\lambda z}\exp(ik\frac{x^2+y^2}{2z}).$$

15S.7 Fraunhofer diffraction

The above equation can be rewritten as

$$E(x,y,z) = -i\frac{e^{ikz}}{\lambda z}\exp[ik\frac{(x^2+y^2)}{2z}]\iint dx_0 dy_0 \Theta(x_0,y_0,0)\exp[ik\frac{x_0{}^2+y_0{}^2}{2z}]\exp[-i\frac{k}{z}(x_0x+y_0y)]$$

Suppose that

$$k\left(\frac{x_0{}^2+y_0{}^2}{2z}\right) << 1.$$

Then the quadratic phase factor $\exp(ik\frac{x_0{}^2+y_0{}^2}{2z})$ is approximately unity over the entire aperture. So we get

$$E(x,y,z) = -i\frac{e^{ikz}}{\lambda z}\exp[ik\frac{(x^2+y^2)}{2z}]\iint dx_0 dy_0 \Theta(x_0,y_0,0)\exp[-i\frac{k}{z}(x_0 x + y_0 y)].$$

In other words, the field distribution can be found directly from a Fourier transform of the aperture distribution itself. Aside from the multiplicative factors, this expression is simply the Fourier transform of the aperture.

$$\frac{1}{\left(\sqrt{2\pi}\right)^2}\iint dx_0 dy_0 \Theta(x_0,y_0,0)\exp[-i(k_x x_0 + k_y y_0)] = \mathbf{F}[\Theta(x_0,y_0)$$

with

$$k_x = \frac{k}{z}x = \frac{2\pi}{\lambda z}x, \qquad k_y = \frac{k}{z}y = \frac{2\pi}{\lambda z}y$$

where ***F*** is the operation of Fourier transform. This is called the Fraunhofer diffraction.

15S.8 Direct derivation for the Fraunhofer diffraction

We start with the formula given by

$$E(x,y,z) \approx -\frac{ik}{2\pi z}E_{inc}(\boldsymbol{\rho}_0)\iint \Theta(\mathbf{r}_0)e^{ik|\mathbf{r}-\mathbf{r}_0|}d^2\mathbf{r}_0$$

As shown in Fig., the distance *s* is approximated by

$$s = |\mathbf{r} - \mathbf{r}_0| = r - (\hat{r}\cdot\mathbf{r}_0),$$

for $r>>r_0$.

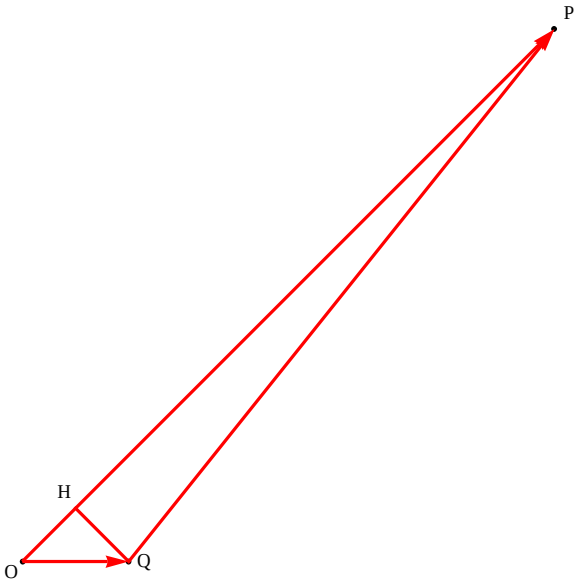


Fig. $\overrightarrow{OQ} = \mathbf{r}_0$ $\overrightarrow{OP} = \mathbf{r}$. $\overrightarrow{QP} = \mathbf{s} = \mathbf{r} - \mathbf{r}_0$. $\overrightarrow{OH} = (\hat{r} \cdot \mathbf{r}_0)\hat{r}$

$$\left|\overrightarrow{QP}\right| = s = |\mathbf{r} - \mathbf{r}_0| \approx \left|\overrightarrow{QH}\right| = r - (\hat{r} \cdot \mathbf{r}_0).$$

Then we get the expression,

$$E(x,y,z) \approx -\frac{ik}{2\pi z} E_{inc}(\boldsymbol{\rho}_0) \iint \Theta(\mathbf{r}_0) e^{ikr - ik(\hat{r}\cdot\mathbf{r}_0)} d^2\mathbf{r}_0$$
$$= -\frac{ik}{2\pi z} E_{inc}(\boldsymbol{\rho}_0) e^{ikr} \iint \Theta(\mathbf{r}_0) e^{-ik(\hat{r}\cdot\mathbf{r}_0)} d^2\mathbf{r}_0 .$$

Here we note that

$$k\hat{r} \cdot \mathbf{r}_0 = \frac{k}{r}(\mathbf{r} \cdot \mathbf{r}_0) \approx \frac{k}{z}(xx_0 + yy_0) = k(\frac{x}{z})x_0 + k(\frac{y}{z})x_0 .$$

Then $E(x, y, z)$ is proportional to the Fourier transform of the aperture,

$$\frac{1}{\left(\sqrt{2\pi}\right)^2} \iint \Theta(\mathbf{r}_0) e^{-ik(\hat{r}\cdot\mathbf{r}_0)} d^2\mathbf{r}_0 = \mathbf{F}[\Theta(\mathbf{r}_0), \{k_x = k\frac{x}{z}, k_y = k\frac{y}{z}\}]$$

with the wave vector given by

$$k_x = k\frac{x}{z}, k_y = k\frac{y}{z}.$$

The x and y coordinates of the observation point P are proportional to the wave numbers k_x, and k_y, respectively.

15S.9 Fresnel's number; *F*

Fresnel's number F is a dimensionless number occurring in optics, in particular in diffraction theory. For an electromagnetic wave passing through an aperture and hitting a screen, the Fresnel number F is defined as

$$F = \frac{a^2}{z\lambda},$$

where a is the characteristic size (the side of the square aperture) of the aperture, z is the distance of the observation plane from the aperture and λ is the incident wavelength. Depending on the value of F the diffraction theory can be simplified into two special cases: Fraunhofer diffraction for $F<<1$ and Fresnel diffraction for $F>>1$. We note that the condition for the appearance of the Fraunhofer diffraction is evaluated as

$$F = \frac{a^2}{\lambda z} << \frac{1}{\pi} = 0.318.$$

On the other hand, the condition for the appearance of the Fresnel diffraction can be rewritten as

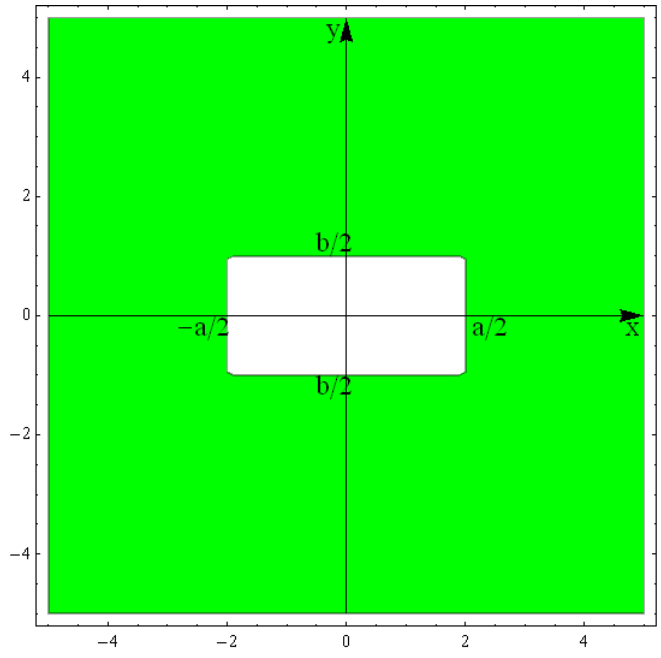
$$F = \frac{a^2}{\lambda z} << 8\frac{z^2}{a^2}.$$

Suppose that z = 100 mm and a = 5 mm. Then we get the inequality as

$F<<3200$.

15S.10 Fresnel diffraction with a rectangular aperture

The Fresnel diffraction occurs when the condition $z >> \rho >> \lambda$ is satisfied.



Suppose that a rectangular aperture of ($-a \le x \le a$, $-b \le x \le b$) is normally illuminated by a monochromatic plane wave of unit amplitude and the wavelength λ.

$$U(x,y) = -i\frac{e^{ikz}}{\lambda z}\iint \exp\{i\frac{k}{2z}[(x_1-x)^2+(y_1-y)^2]\}dx_1dy_1$$
$$= -i\frac{e^{ikz}}{\lambda z}\int_{-a}^{a}\exp[i\frac{k}{2z}(x_1-x)^2]dx_1\int_{-b}^{b}\exp[i\frac{k}{2z}(y_1-x)^2]dy_1$$

where k is the wavenumber and is given by $k = 2\pi/\lambda$. It follows that the expression can be separated into the product of two integrals over x_1 and y_1,

$$F(x) = \int_{-a}^{a}\exp[i\frac{k}{2z}(x_1-x)^2]dx_1$$
$$F(y) = \int_{-b}^{b}\exp[i\frac{k}{2z}(y_1-y)^2]dy_1$$

These integrals are simplified by the change of variables,

$$\xi = \sqrt{\frac{k}{\pi z}}(x_1-x), \qquad \mu = \sqrt{\frac{k}{\pi z}}(y_1-y)$$

yielding

$$F(x) = \sqrt{\frac{\pi z}{k}} \int_{\xi_1}^{\xi_2} \exp(i\frac{\pi}{2}\xi^2) d\xi$$

$$F(y) = \sqrt{\frac{\pi z}{k}} \int_{\mu_1}^{\eta_2} \exp(i\frac{\pi}{2}\eta^2) d\eta$$

where the limits of integration are

$$\xi_1 = \sqrt{\frac{k}{\pi z}}(-a - x), \qquad \xi_2 = \sqrt{\frac{k}{\pi z}}(a - x)$$

$$\eta_1 = \sqrt{\frac{k}{\pi z}}(-b - y), \qquad \eta_2 = \sqrt{\frac{k}{\pi z}}(b - y)$$

The integrals $F(x)$ and $F(y)$ can be evaluated in terms of the Fresnel integrals, which are defined by

$$C(\alpha) = \int_0^{\alpha} \cos(\frac{\pi}{2}t^2) dt$$

$$S(\alpha) = \int_0^{\alpha} \sin(\frac{\pi}{2}t^2) dt$$

We note that

$$\begin{aligned} \int_0^{\alpha} \exp[i(\frac{\pi}{2}\xi^2)] d\xi &= \int_0^{\alpha} [\cos(\frac{\pi}{2}\xi^2) + i\sin(\frac{\pi}{2}\xi^2)] d\xi \\ &= C(\alpha) + iS(\alpha) \end{aligned}$$

and

$$\begin{aligned} \int_{\xi_1}^{\xi_2} \exp[i(\frac{\pi}{2}\xi^2)] d\xi &= \int_0^{\xi_2} \exp[i(\frac{\pi}{2}\xi^2)] d\xi - \int_0^{\xi_1} \exp[i(\frac{\pi}{2}\xi^2)] d\xi \\ &= C(\xi_2) + iS(\xi_2) - C(\xi_1) - iS(\xi_1) \\ &= C(\xi_2) - C(\xi_1) + i[S(\xi_2) - S(\xi_1)] \end{aligned}$$

Finally we have the complex field distribution

$$U(x,y)=\frac{e^{ikz}}{2i}\{C(\xi_2)-C(\xi_1)+i[S(\xi_2)-S(\xi_1)]\}\{C(\eta_2)-C(\eta_1)+i[S(\eta_2)-S(\eta_1)]\}$$

and the corresponding intensity distribution

$$I(x,y)=\frac{1}{4}\{[C(\xi_2)-C(\xi_1)]^2+[S(\xi_2)-S(\xi_1)]^2\}\{[C(\eta_2)-C(\eta_1)]^2+[S(\eta_2)-S(\eta_1)]^2\}$$

Note that

$$C(\infty)=\frac{1}{2},\qquad C(-\infty)=-\frac{1}{2},\qquad C(-\xi)=-C(\xi)$$

$$S(\infty)=\frac{1}{2},\qquad S(-\infty)=-\frac{1}{2}\qquad S(-\xi)=-S(\xi)$$

15S.11 Cornu spiral

Marie Alfred Cornu (March 6, 1841 – April 12, 1902) was a French physicist. The French generally refer to him as Alfred Cornu. Cornu was born at Orléans and was educated at the École polytechnique and the École des mines. Upon the death of Émile Verdet in 1866, Cornu became, in 1867, Verdet's successor as professor of experimental physics at the École polytechnique, where he remained throughout his life. Although he made various excursions into other branches of physical science, undertaking, for example, with Jean-Baptistin Baille about 1870 a repetition of Cavendish's experiment for determining the gravitational constant *G*, his original work was mainly concerned with optics and spectroscopy. In particular he carried out a classical redetermination of the speed of light by A. H. L. Fizeau's method (see Fizeau-Foucault Apparatus), introducing various improvements in the apparatus, which added greatly to the accuracy of the results. This achievement won for him, in 1878, the *prix Lacaze* and membership of the French Academy of Sciences (*l'Académie des sciences*), and the Rumford Medal of the Royal Society in England. In 1892, he was elected a member of the Royal Swedish Academy of Sciences. In 1896, he became president of the French Academy of Sciences. In 1899, at the jubilee commemoration of Sir George Stokes, he was Rede lecturer at Cambridge, his subject being the wave theory of light and its influence on modern physics; and on that occasion the honorary degree of D.Sc. was conferred on him by the university. He died at Romorantin on April 12, 1902. The Cornu spiral, a graphical device for the computation of light intensities in Fresnel's model of near-field diffraction, is named after him. The spiral is also used in geometrical road design. The Cornu depolarizer is also named after him.

http://en.wikipedia.org/wiki/Marie_Alfred_Cornu

Here we make a plot of Cornu's spiral.

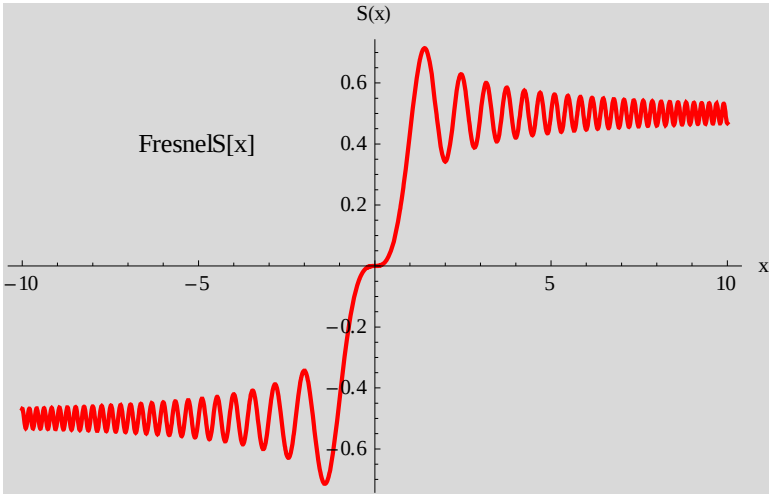


Fig. Plot of $S(x)$ as a function of x.

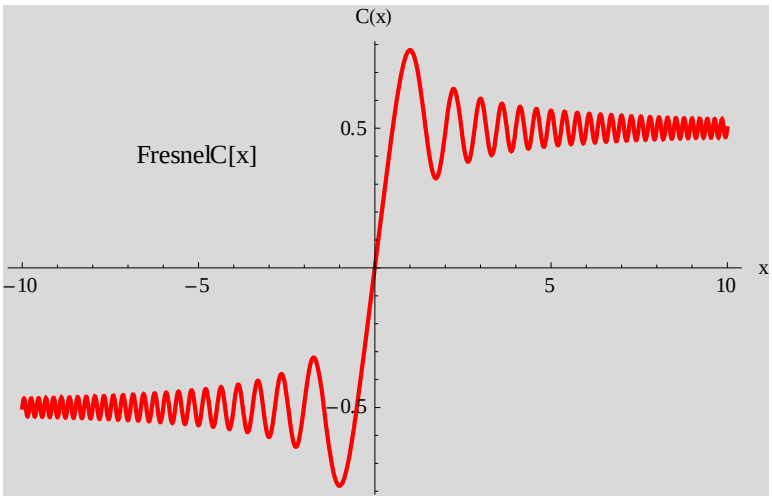


Fig. Plot of $C(x)$ as a function of x.

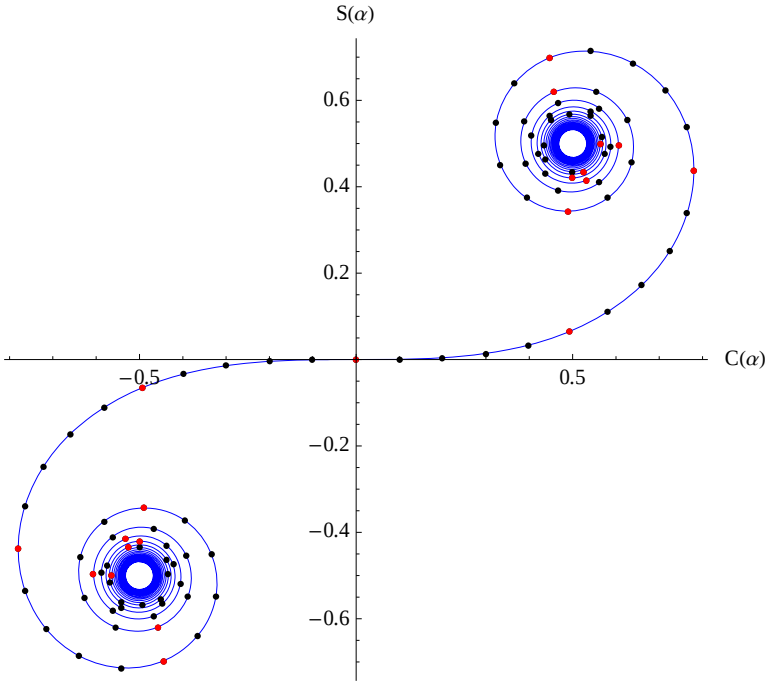


Fig. ParametricPlot of {C(α), S(α)} when a is changed as α parameter.

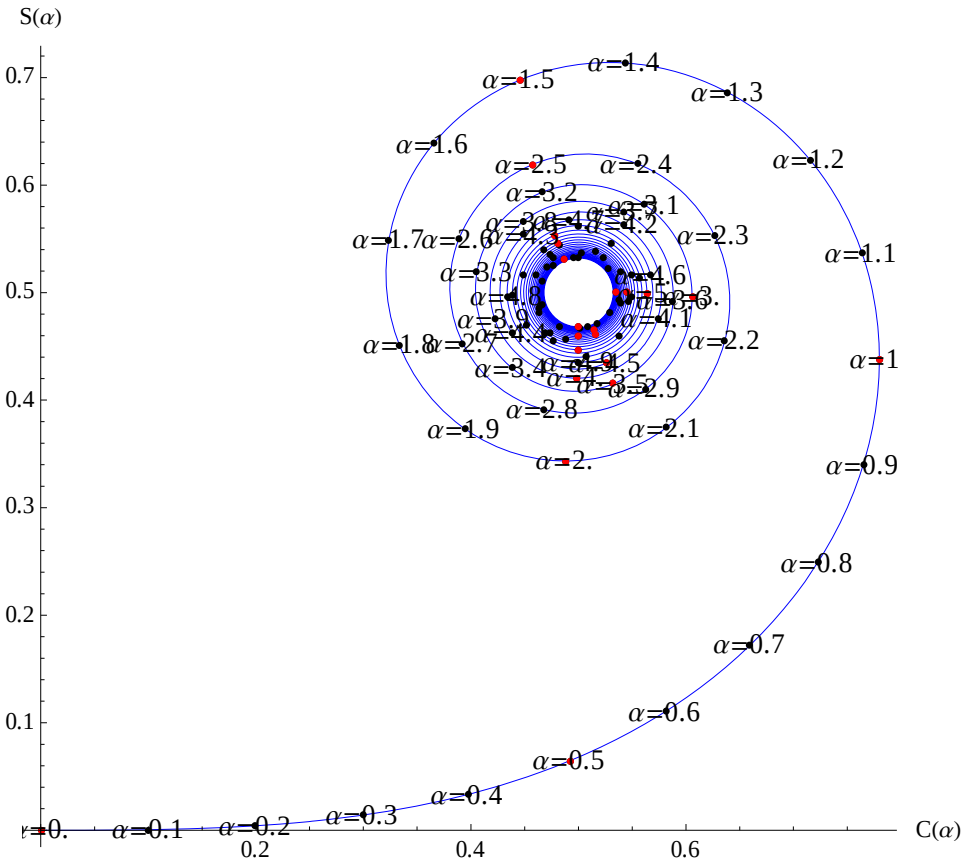


Fig. ParametricPlot of {C(α), S(α)} around the point Z (1/2, 1/2) when a is changed as α parameter.

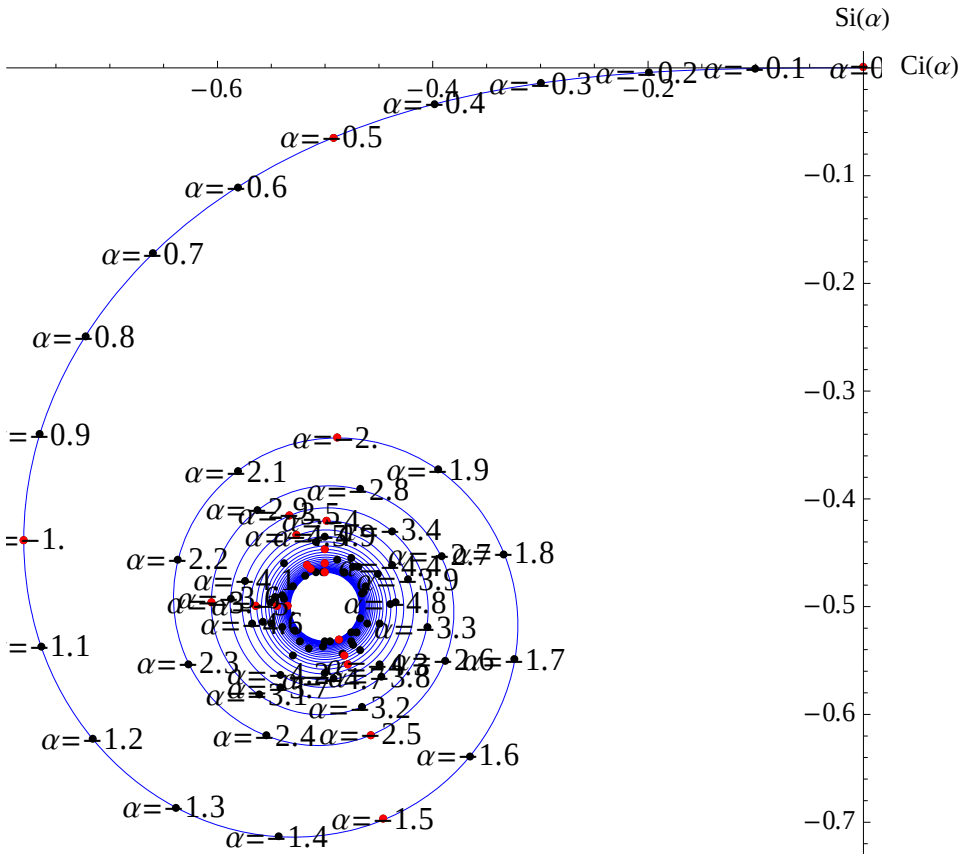
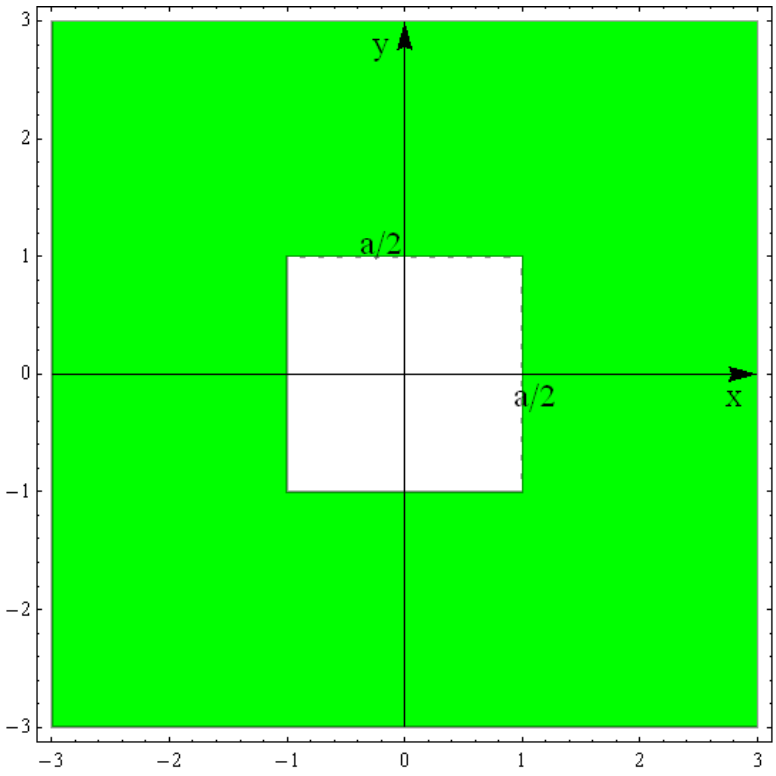


Fig. ParametricPlot of {C(α), S(α)} around the point Z' (-1/2, -1/2) when a is changed as α parameter.

15S.12 Fresnel diffraction with a square aperture

We calculate the intensity of the Fresnel diffraction with the square aperture by using the Mathematica (Plot3D, ContourPlot). λ and z are fixed. The size of the square ($L = a$)



(a) ***F* = 22.7848**

λ = 632 nm, z = 400 mm, $L = a$ = 2.4 mm

x(mm), y(mm)

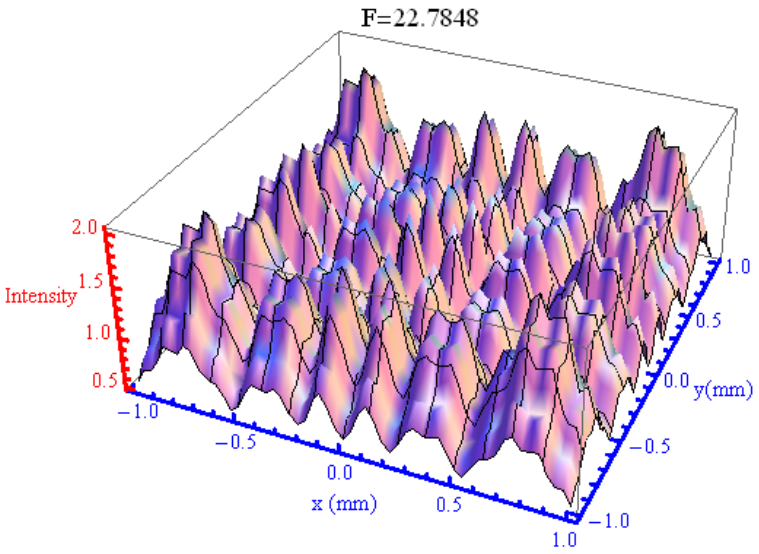


Fig. Plot3D of the intensity $I(x, y)$.

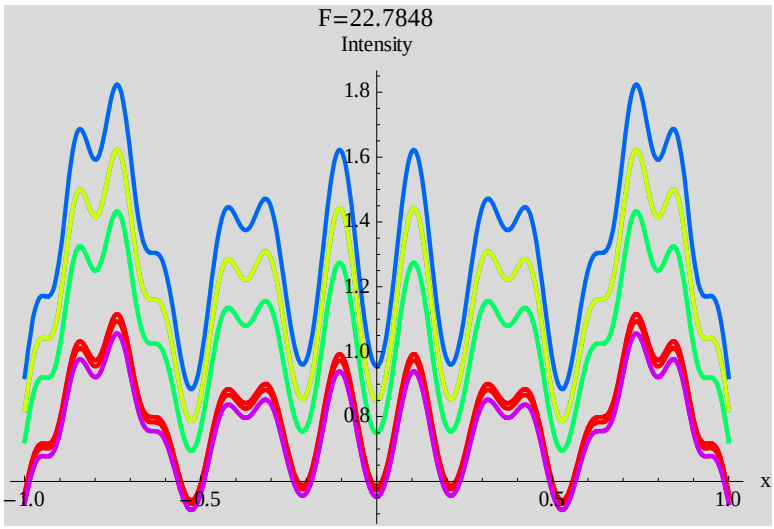


Fig. Plot of intensity vs x (mm). y (mm) is changed as a parameter. y = -1, -0.8, -0.6, -0.4, -0.2, 0, 0.2, 0.4, 0.6, 0.8, and 1.0 mm

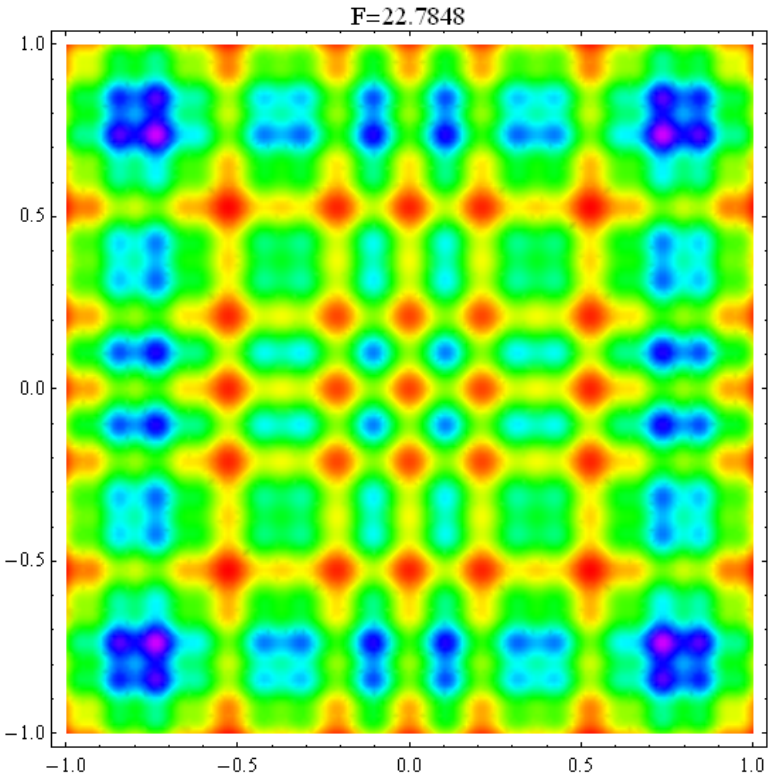
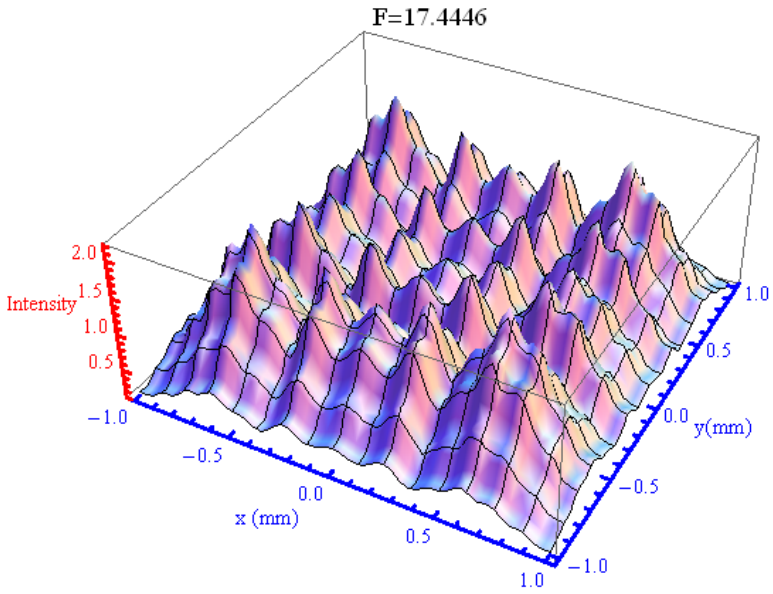


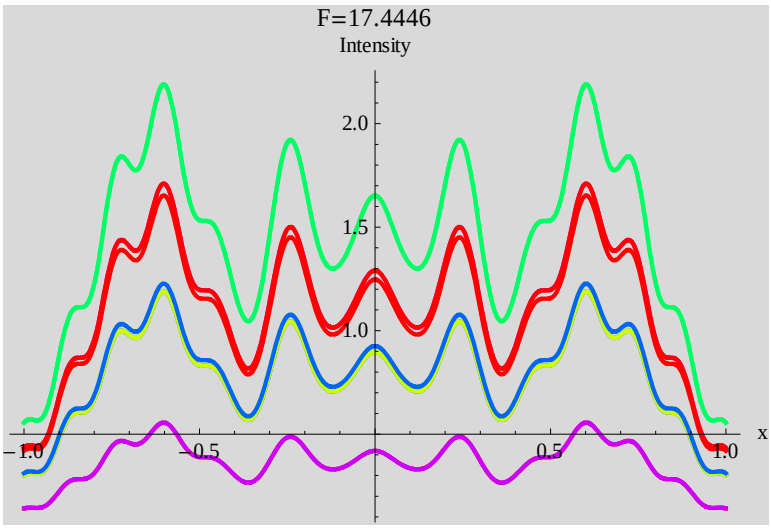
Fig. DensityPlot. The distribution of the intensity in the x (mm) and y (mm) plane.

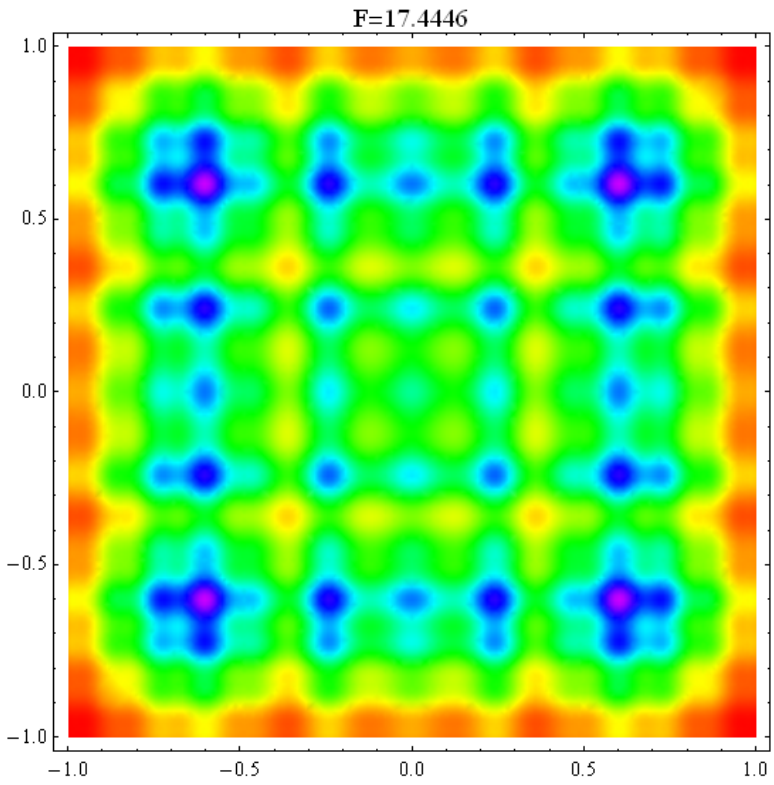
(b) **F = 17.4446**

λ = 632 nm, z = 400 mm, $L = a$ = 2.1 mm

x(mm), y(mm)


F=17.4446
Intensity
2.0
1.5
1.0
0.5
−1.0
−0.5
0.0
0.5
1.0
x (mm)
1.0
0.5
0.0
−0.5
−1.0
y(mm)

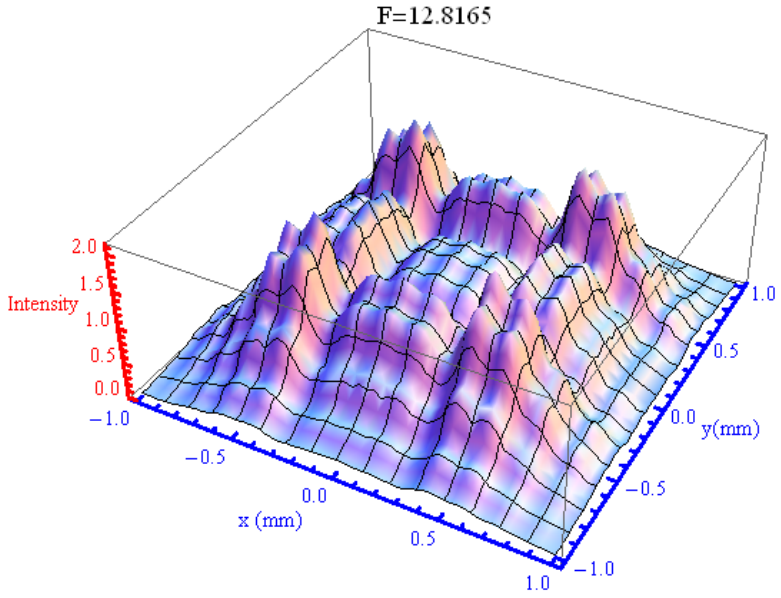

F=17.4446
Intensity
2.0
1.5
1.0
−1.0
−0.5
0.5
1.0
x

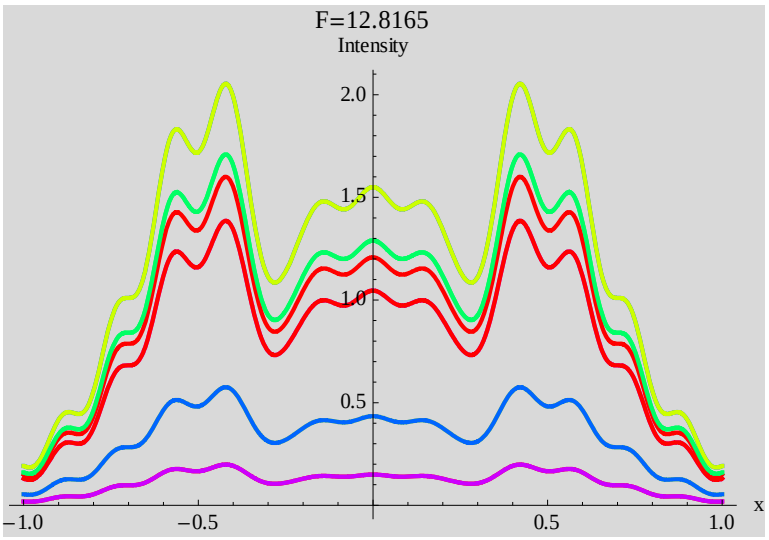


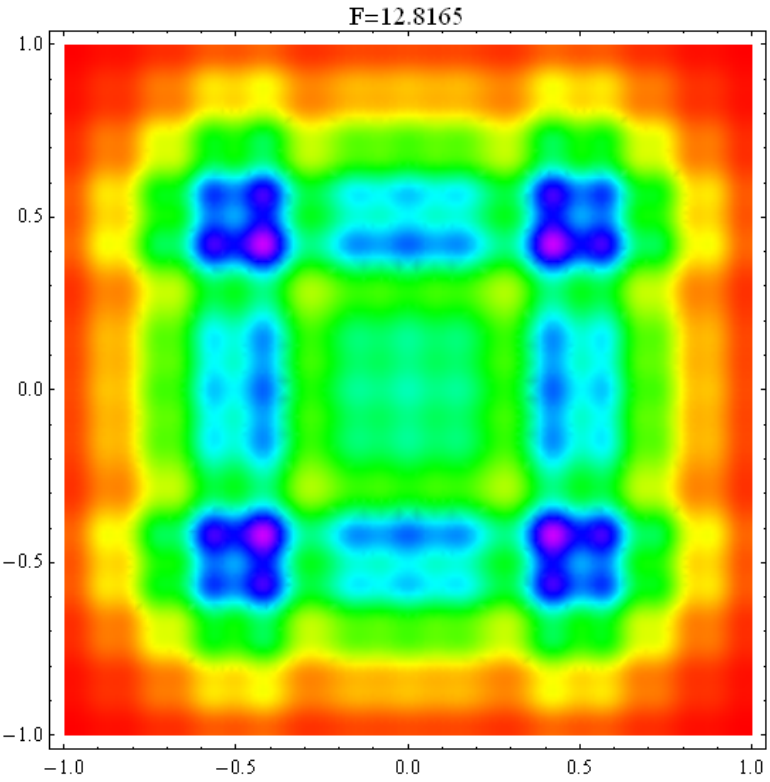
(c) ***F* = 12.8165**

λ = 632 nm, z = 400 mm, $L = a$ = 1.8 mm

x(mm), y(mm)



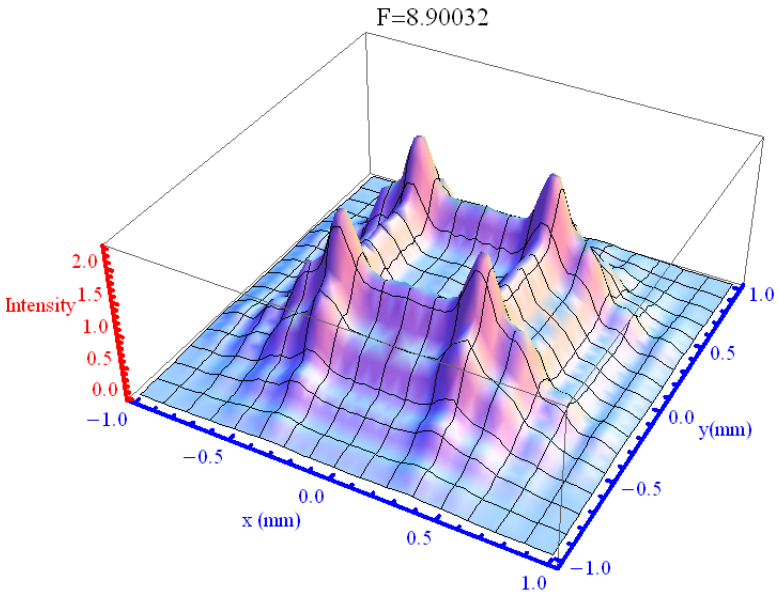


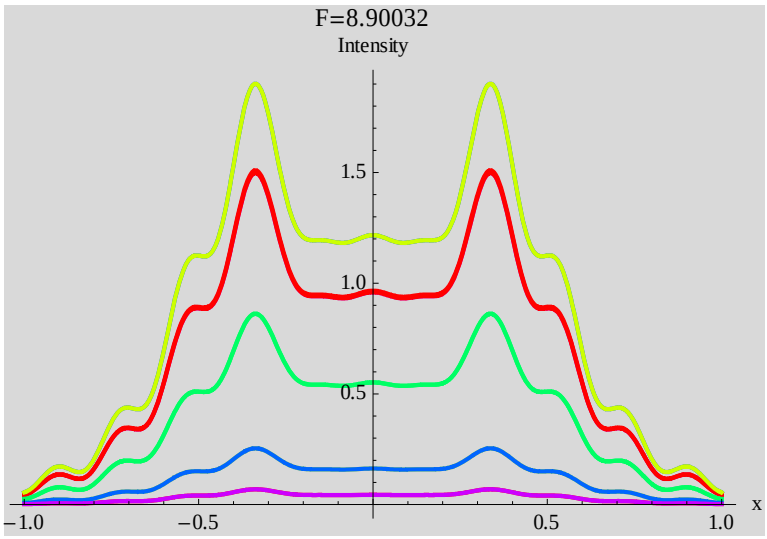


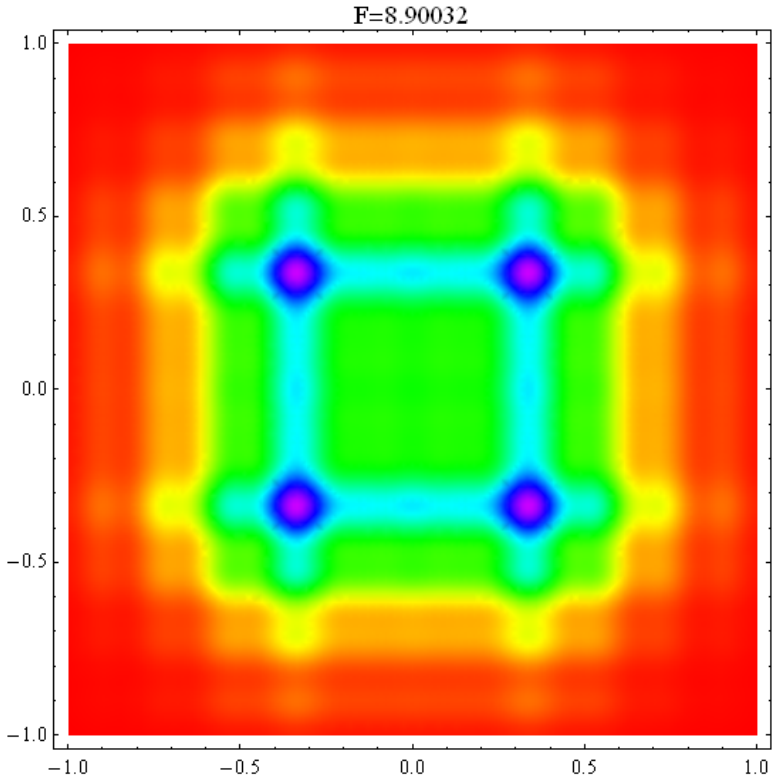
(d) ***F* = 8.9032**

λ = 632 nm, z = 400 mm, $L = a$ = 1.5 mm

x(mm), y(mm)


F=8.90032
Intensity
2.0
1.5
1.0
0.5
0.0
−1.0
−0.5
0.0
0.5
1.0
x (mm)
1.0
0.5
0.0
−0.5
−1.0
y(mm)

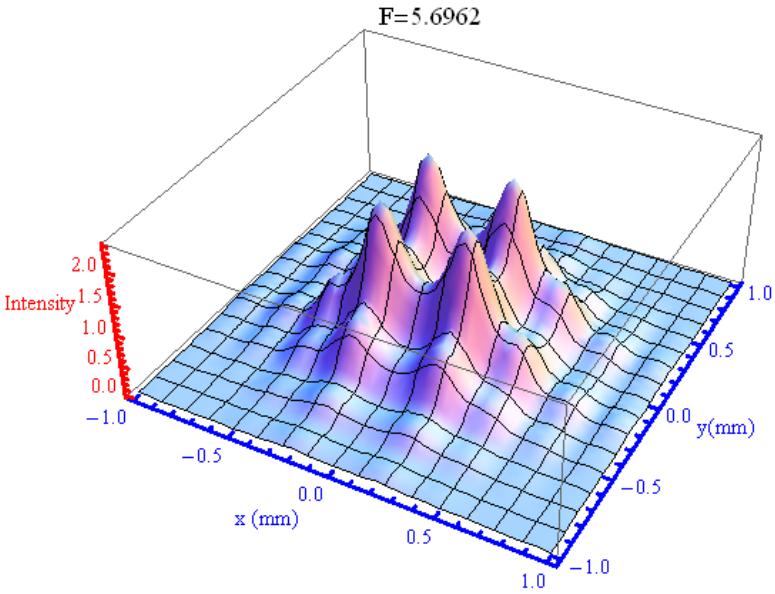

F=8.90032
Intensity
1.5
1.0
0.5
−1.0
−0.5
0.5
1.0
x

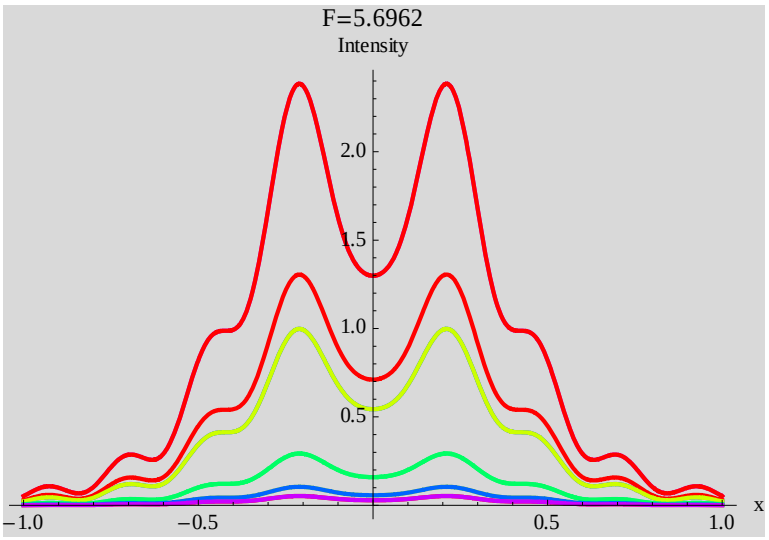


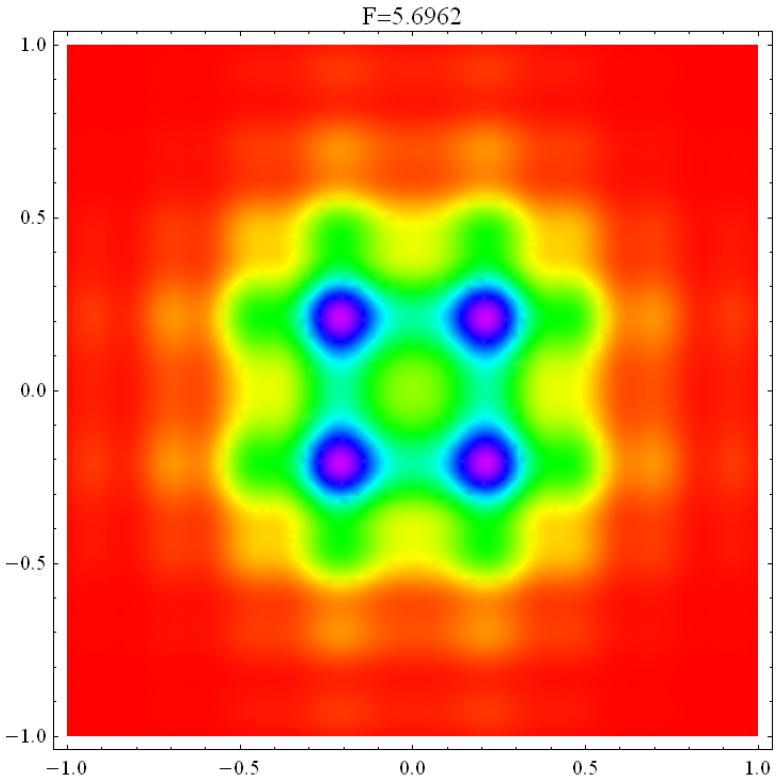
(e) ***F* = 5.6962**

λ = 632 nm, z = 400 mm, $L = a$ = 1.2 mm

x(mm), y(mm)



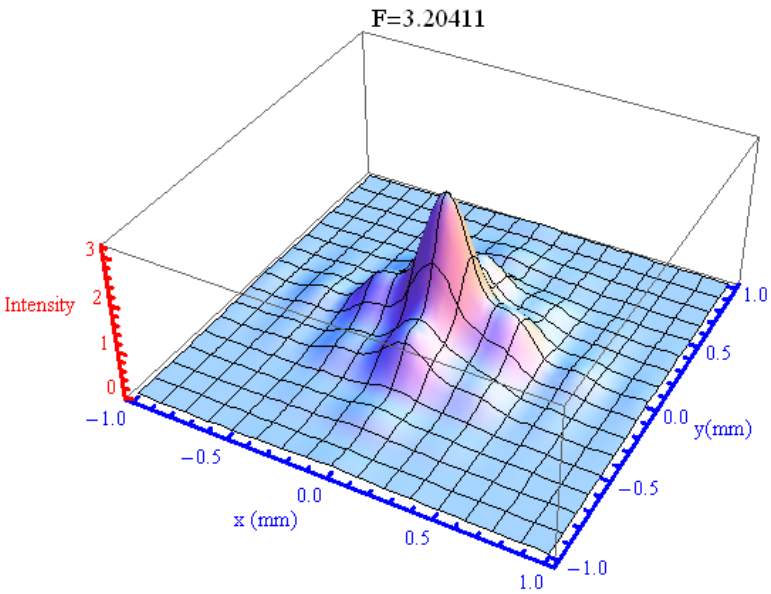


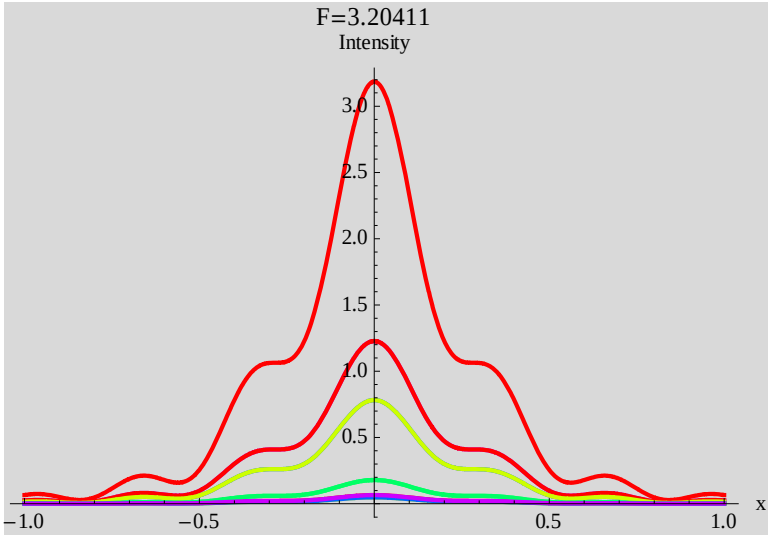


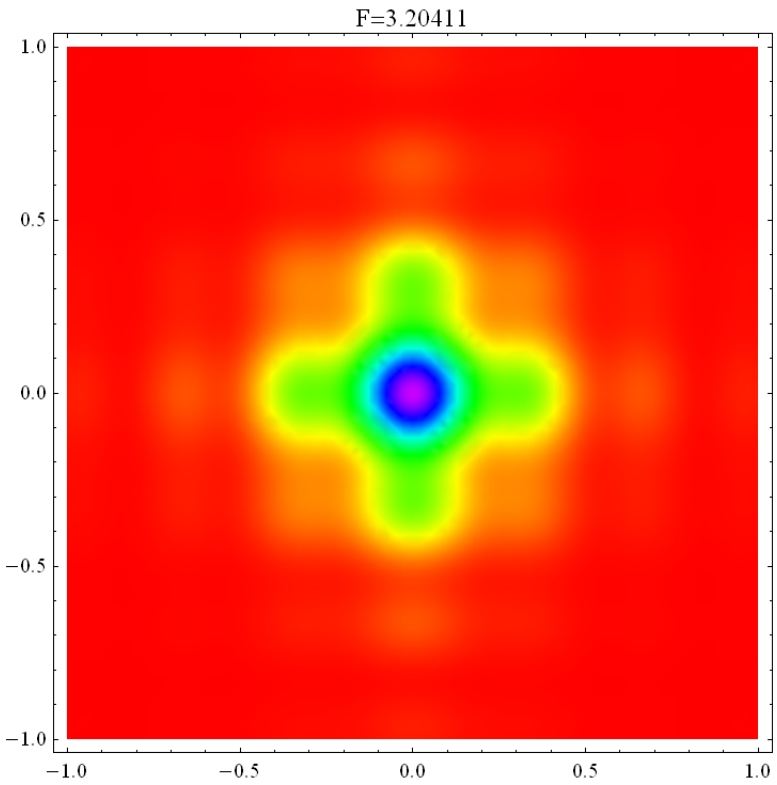
(f) ***F* = 3.20411**

λ = 632 nm, z = 400 mm, $L = a$ = 0.9 mm

x(mm), y(mm)


F=3.20411
Intensity
3
2
1
0
−1.0
−0.5
0.0
0.5
1.0
x (mm)
1.0
0.5
0.0
−0.5
−1.0
y(mm)

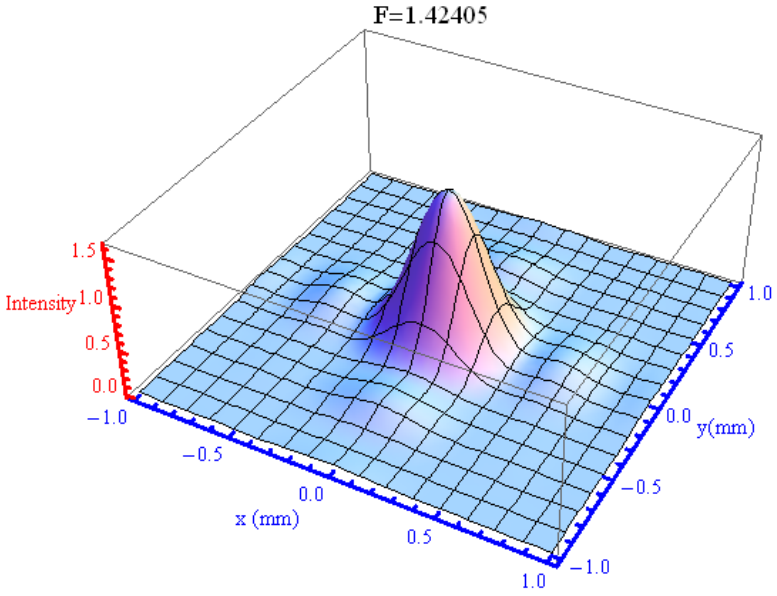

F=3.20411
Intensity
3.0
2.5
2.0
1.5
1.0
0.5
−1.0
−0.5
0.5
1.0
x

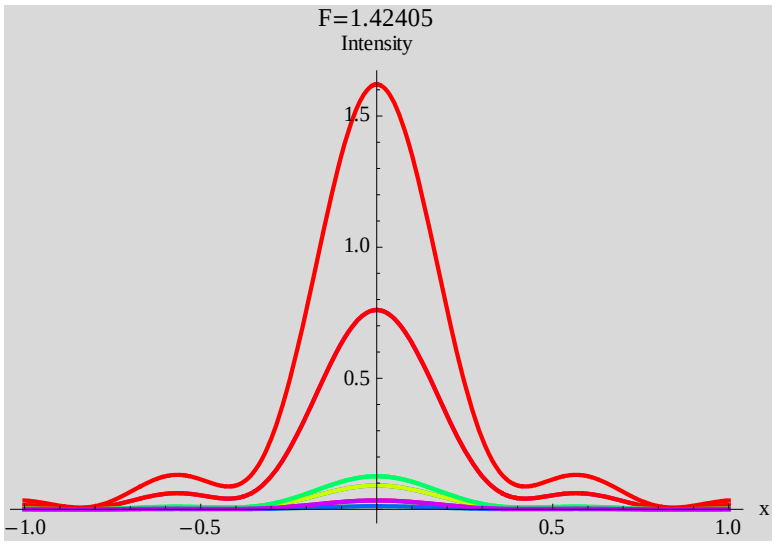


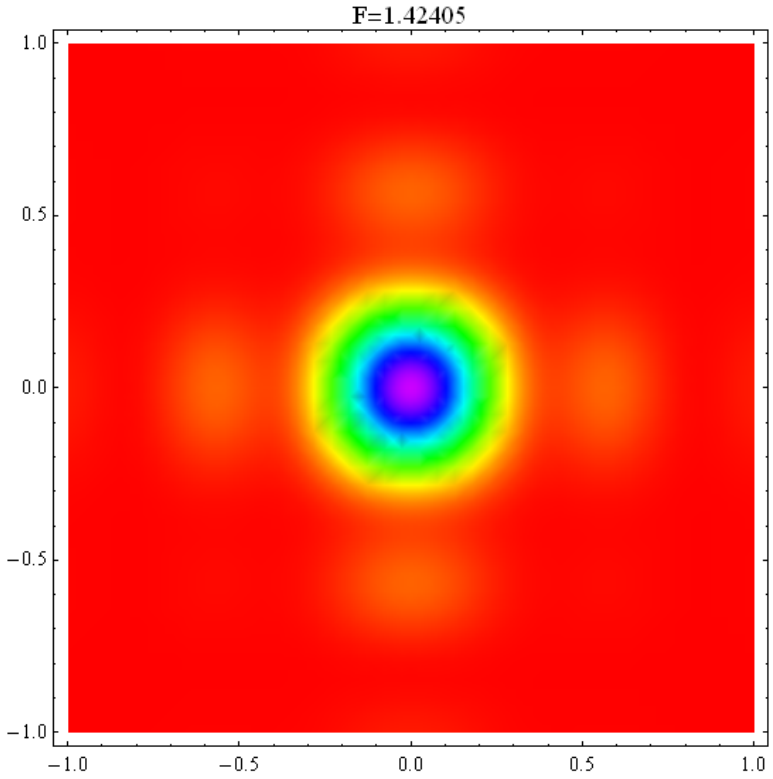
(g) ***F* = 1.42405**

λ = 632 nm, z = 400 mm, L = a = 0.6 mm

x(mm), y(mm)







15S.13 A semi-infinite planar opaque screen with infinte ρ_0

We calculate the intensity of the Fresnel diffraction with the semi-infinite planar opaque screen by using the Mathematica (Plot3D, ContourPlot). λ and z are fixed. The size of the square ($L = a$)

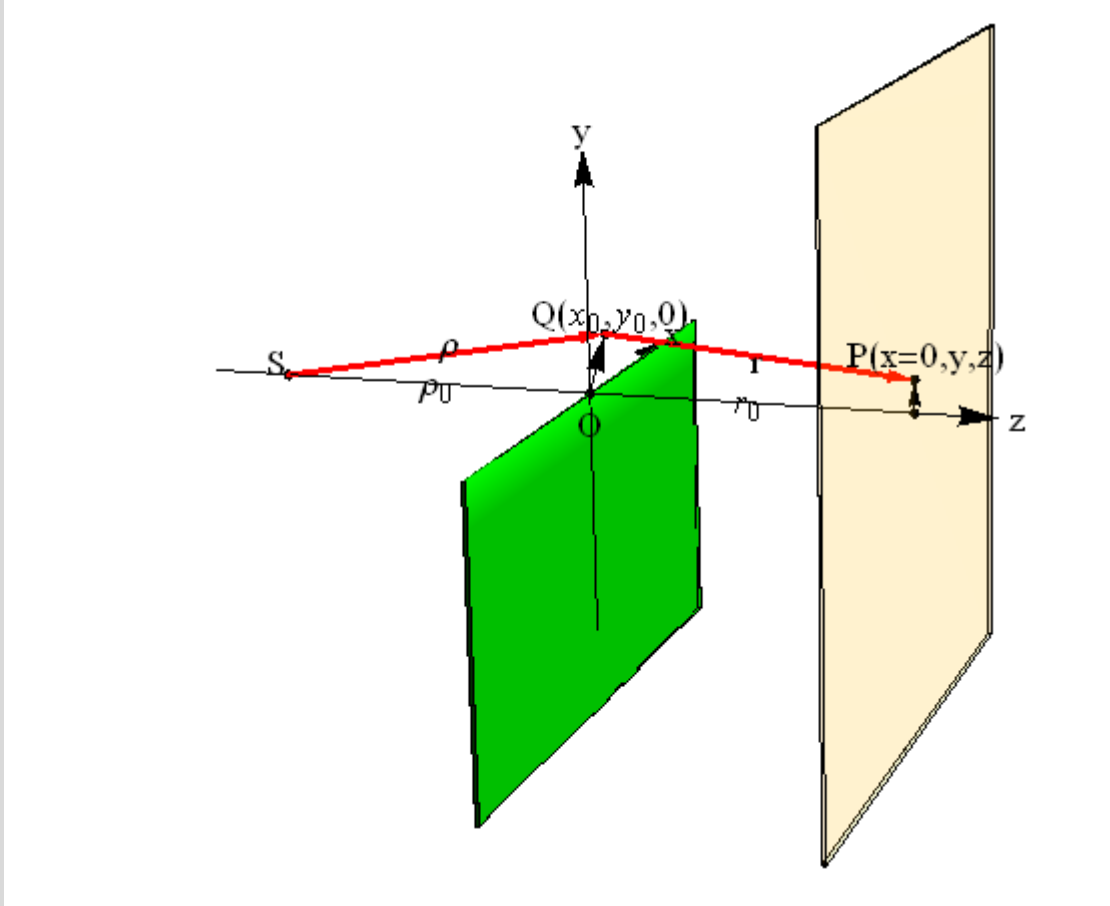


Fig. A semi-infinite planar opaque screen with finte ρ_0. $r_0 = z$.

We assume that the plane wave arrives at a semi infinite opaque screen. As shown in the above figure, the distance r (=QP) is given

$$r = \sqrt{z^2 + x_0^{\,2} + (y - y_0)^2}$$

where P at ($x = 0$, y, z) and Q at (x_0, y_0, $z = 0$). The distance r is approximated as

$$r = z + \frac{x_0^{\,2} + (y_0 - y)^2}{2z}.$$

Then the wave arriving at the point P ($x = 0$, y, z) on the screen is expressed in the form

$$\exp(ikz)\int_{-\infty}^{\infty}\exp(i\frac{k}{2z}x_0^{\,2})dx_0\int_{0}^{\infty}\exp[i\frac{k}{2z}(y_0 - y)^2]dy_0$$

We put

$$t = \sqrt{\frac{k}{\pi z}}(y_0 - y)$$

and

$$dt = \sqrt{\frac{k}{\pi z}}dy_0$$

Then we have

$$\int_0^\infty \exp[i\frac{k}{2z}(y_0 - y)^2]dy_0 = \sqrt{\frac{\pi z}{k}}\int_{-w}^\infty \exp(i\frac{\pi t^2}{2})dt$$

where

$$w = y\sqrt{\frac{k}{\pi z}}\,.$$

The resultant electric field at the point P is given by

$$\frac{\pi z}{k}\exp(ikz)\int_{-\infty}^\infty \exp[i\frac{\pi t^2}{2}]dt\int_{-w}^\infty \exp[i\frac{\pi t^2}{2}]dt$$

where

$$\int_0^\alpha \exp[i\frac{\pi t^2}{2}]dt = C(\alpha) + iS(\alpha)$$

$$\int_{-w}^{\infty} \exp[i\frac{\pi t^2}{2}]dt = \int_{0}^{\infty} \exp[i\frac{\pi t^2}{2}]dt - \int_{0}^{-w} \exp[i\frac{\pi t^2}{2}]dt$$

$$= \frac{1}{2} + i\frac{1}{2} - [C(-w) + iS(-w)]$$

$$= \frac{1}{2} + i\frac{1}{2} + [C(w) + iS(w)]$$

$$= [C(w) + \frac{1}{2}] + i[S(w) + \frac{1}{2}]$$

The intensity is given by

$$I = \frac{1}{2} I_0([C(w) + \frac{1}{2}]^2 + [S(w) + \frac{1}{2}]^2) .$$

15S.14 Example: a semi-infinite planar opaque screen

λ = 632 nm (He-Ne laser) and z = 400 mm (the distance between the screen and the aperture). The intemsity oscillates with the distance y. The intensity has a local maximum at the points P_1, P_2, P_3, P_4, P_5, P_6, and so on, and a local minimum at the points Q_1, Q_2, Q_3, Q_4, and so on.

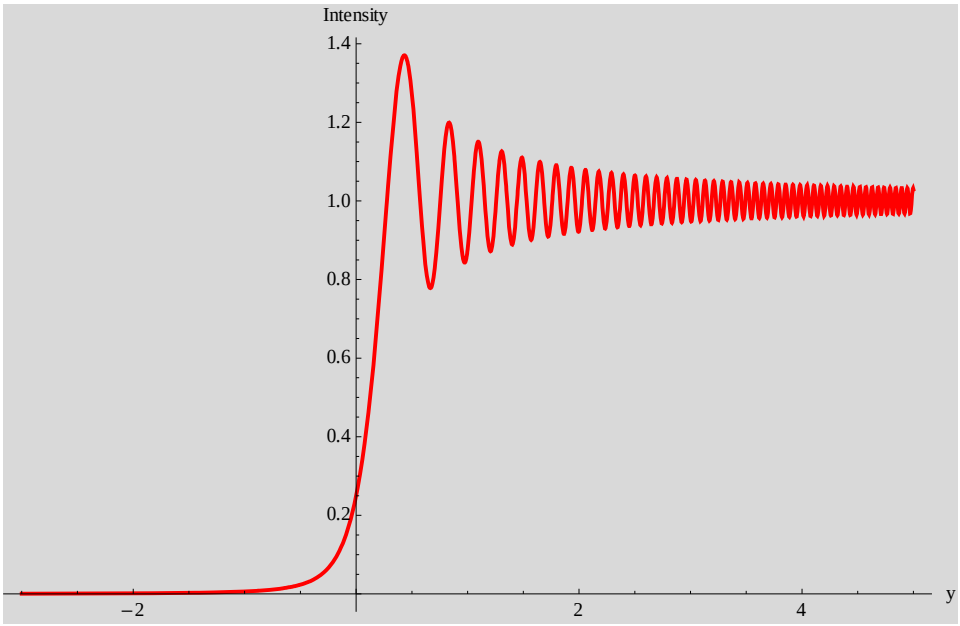


Fig. Intensity vs the distance y. y<0 (shadow region). I_0 = 1. λ = 632 nm (He-Ne laser) and z = 400 mm. y is in the units of mm. The geometrical shadow edge is at y = 0 (the intensity = 1/4); At w = 0, $C(w)$ = $S(w)$ = 0.

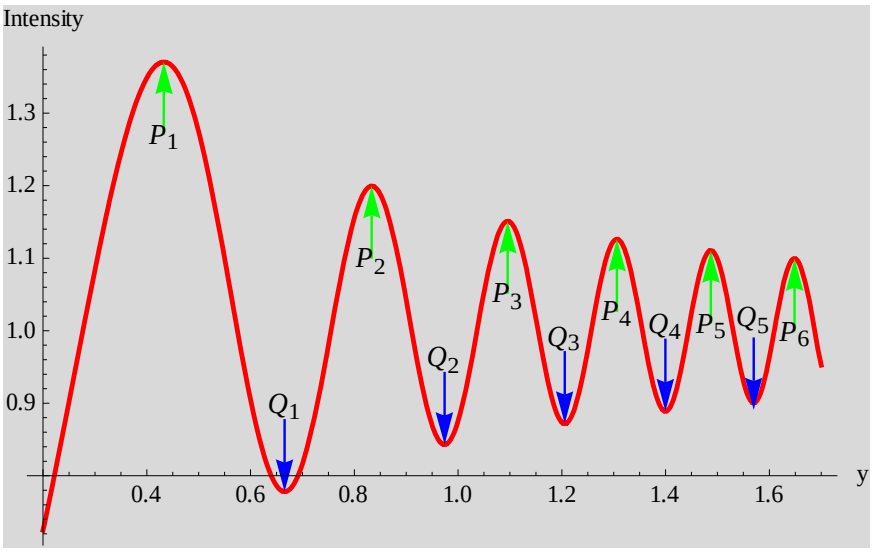


Fig. Intensity I vs y (mm). $I_0 = 1$.

P_1(y = 0.432748, I = 1.37044), P_2(y = 0.83353, I = 1.19927),
P_3(y = 1.09572, I = 1.15088), P_4(y = 1.30625, I = 1.12606),
P_5(y = 1.48725, I = 1.11039), P_6(y = 1.6485, I = 1.09937).

Q_1(y = 0.665733, I = 0.778251), Q_2(y = 0.973793, I = 0.843162),
Q_3(y = 1.20571, I = 0.871942), Q_4(y = 1.39975, I = 0.889064),
Q_5(y = 1.56999, I = 0.900735).

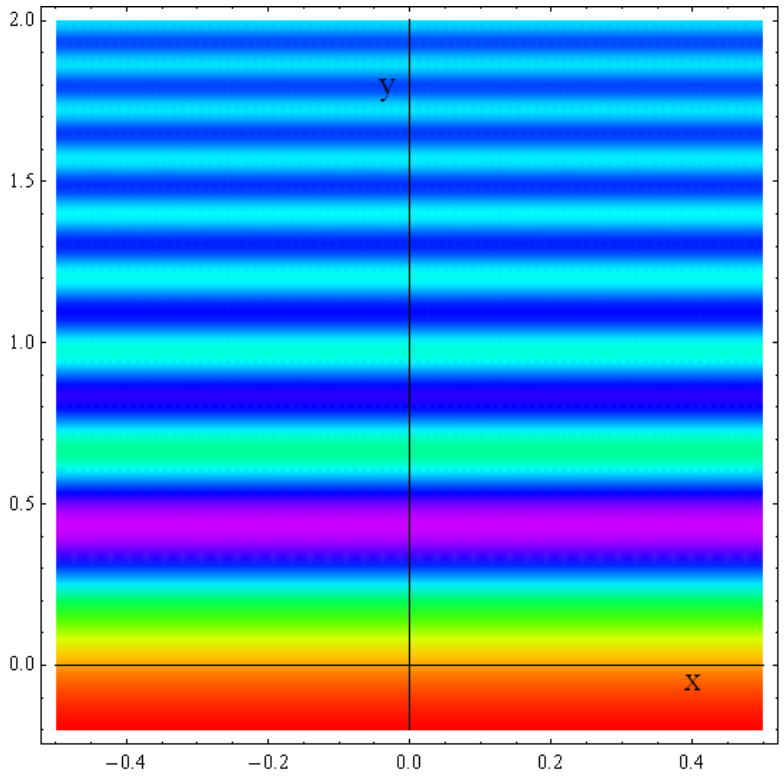


Fig. DensityPlot of the intensity vs the distance y from $y = 0$.

Using the value of y in the units of mm, we get

$$w = \alpha = \frac{25}{\sqrt{79}}\, y\,.$$

The intensity corresponds to the square of the distance between (-1/2, -1/2) point and (C(α),S(α)). The intensity has a maximum at y = 0.432748, 0.83353, 1.09572, 1.30625 (mm). In the Cornu spiral.

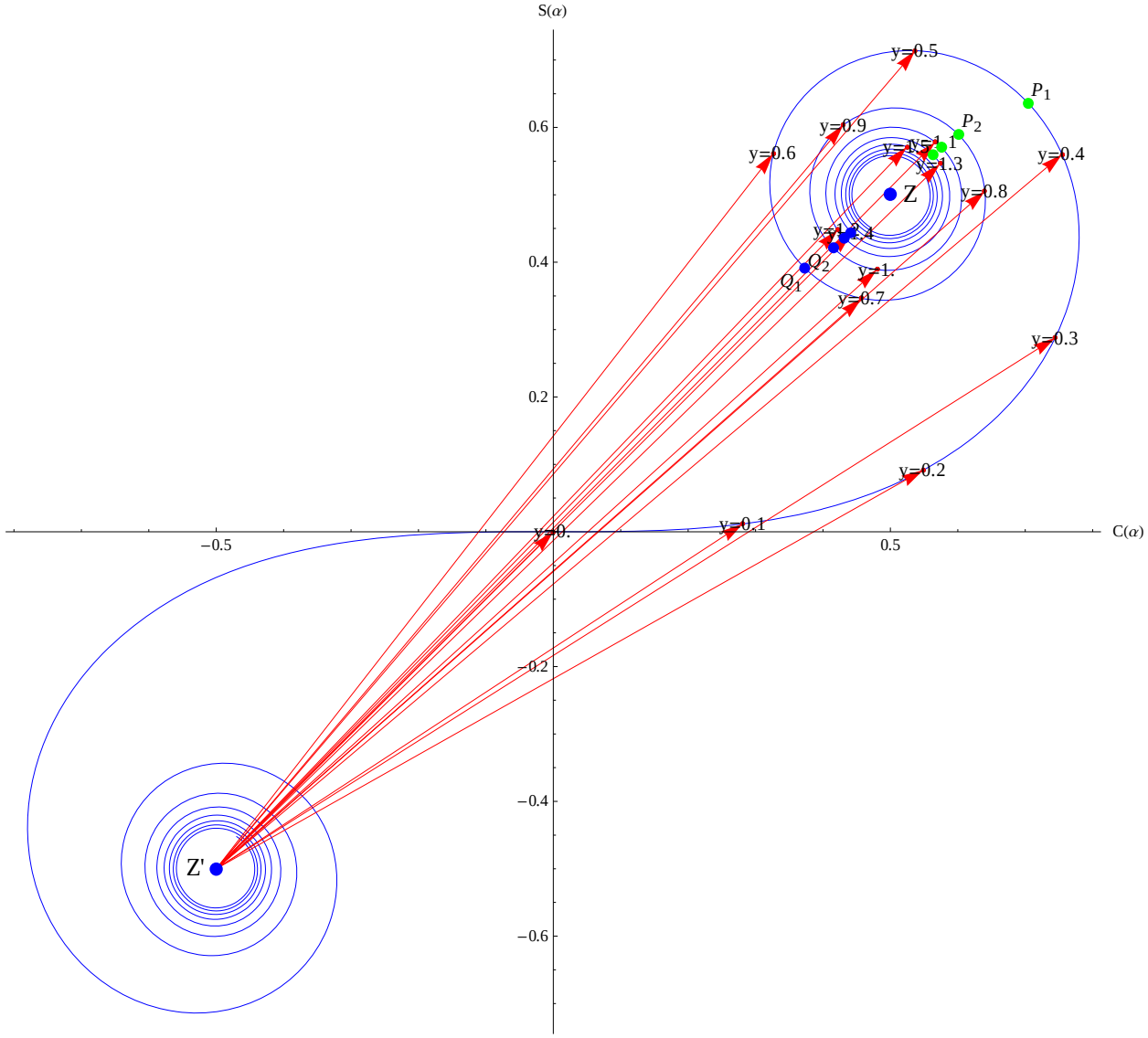


Fig. Cornu plot. Z' at (-1/2, -1/2). Z at (1/2, 1/2).

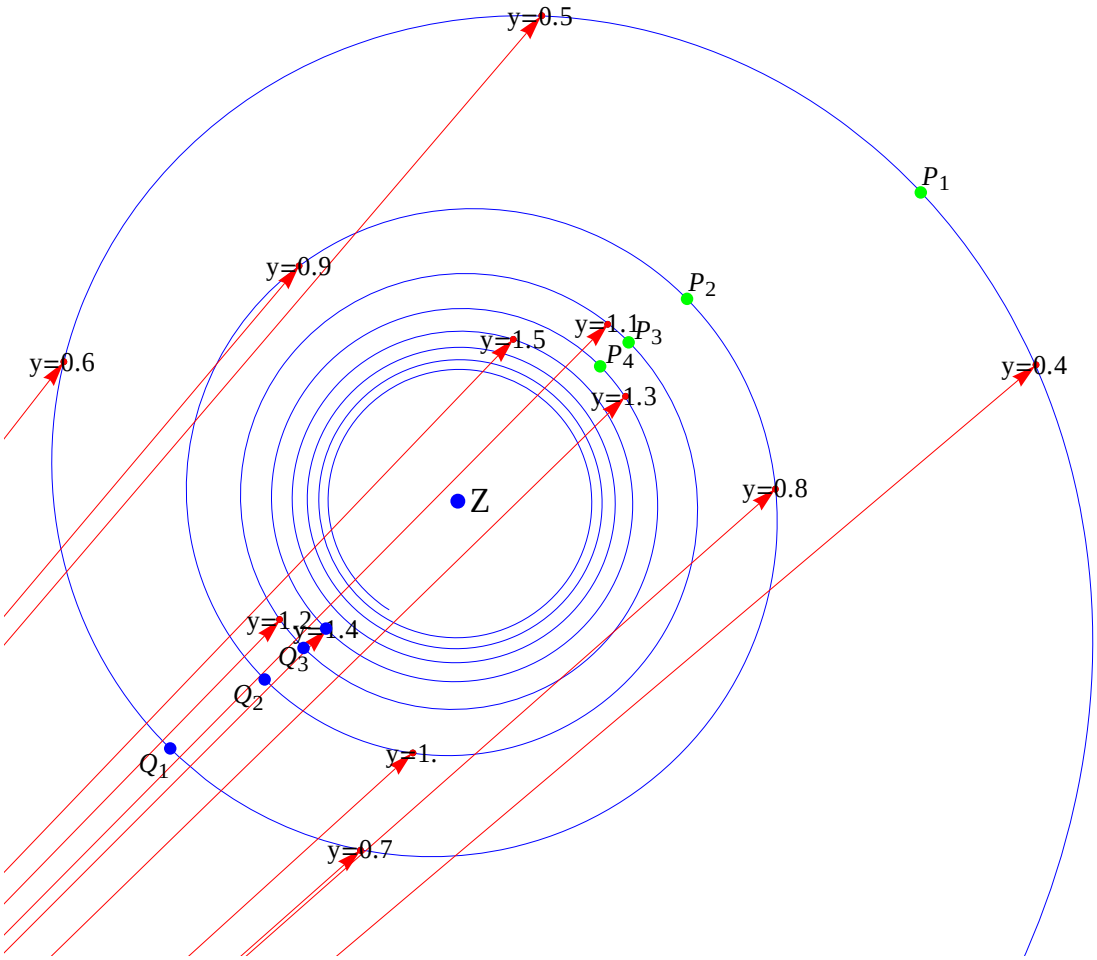
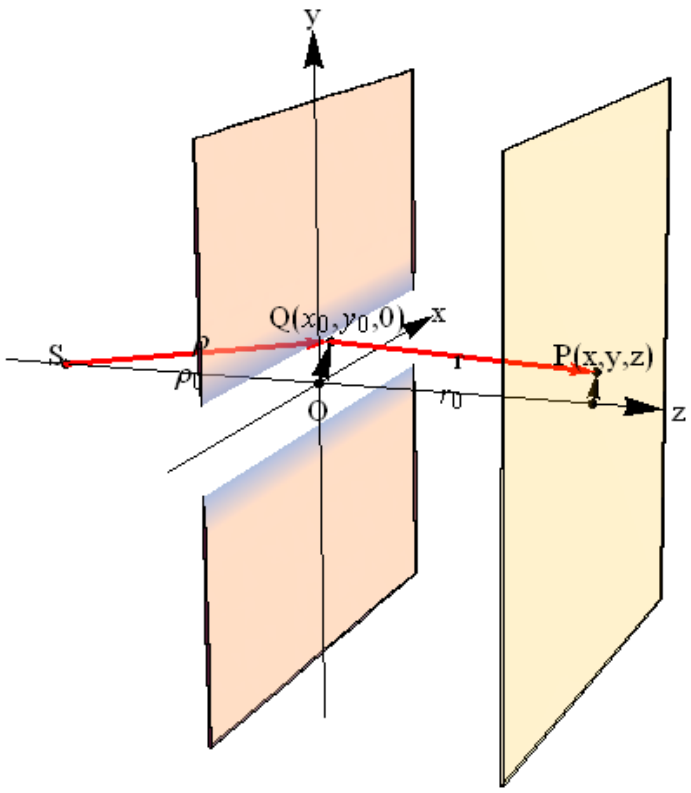


Fig. Cornu plot (enlarged part of the above figure). P_1, P_2, P_3, and P_4 are the local maximum points of the intensity vs y, and Q_1, Q_2, and Q_3 are the local minimum points of the intensity I vs y.

15S.15 Fresnel diffraction with a single slit aperture

We calculate the intensity of the Fresnel diffraction with a single slit aperture by using the Mathematica (Plot3D, ContourPlot). λ and z are fixed. The size of the square ($L = a$)


y
x
z
S
Q(x0,y0,0)
P(x,y,z)
O
ρ
ρ0
r
r0

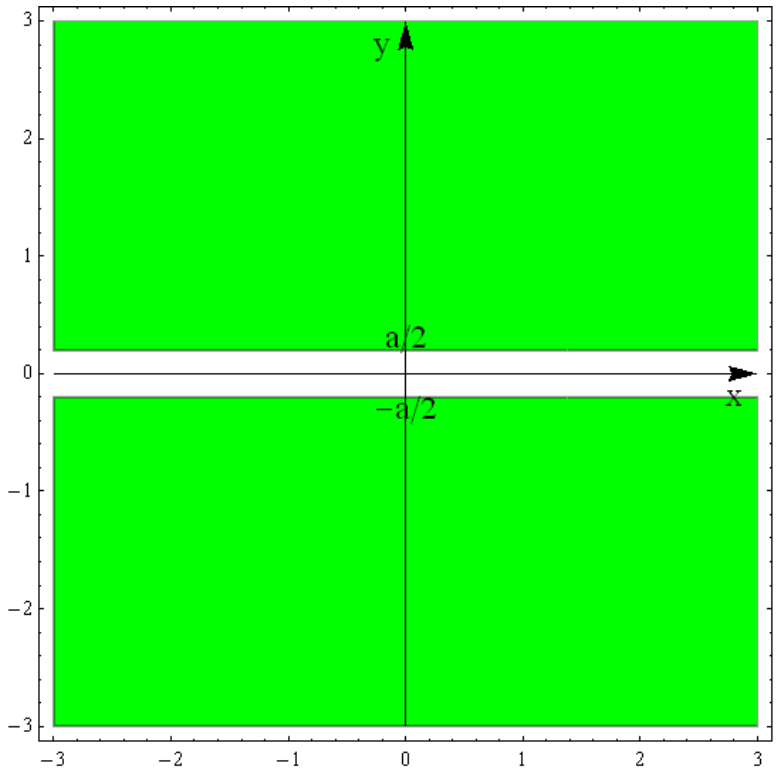


Fig. Single slit aperture. The width of the single slit is *a*.

The electric filed distribution is given by

$$U(x,y)=\frac{e^{ikz}}{2i}\{C(\xi_2)-C(\xi_1)+i[S(\xi_2)-S(\xi_1)]\}\{C(\eta_2)-C(\eta_1)+i[S(\eta_2)-S(\eta_1)]\}$$

The intensity is proportional to $|U(x,y)|^2$, and is given by the form

$$I(x,y)=\frac{I_0}{4}\{[C(\xi_2)-C(\xi_1)]^2+[S(\xi_2)-S(\xi_1)]^2\}\{[C(\eta_2)-C(\eta_1)]^2+[S(\eta_2)-S(\eta_1)]^2\}$$

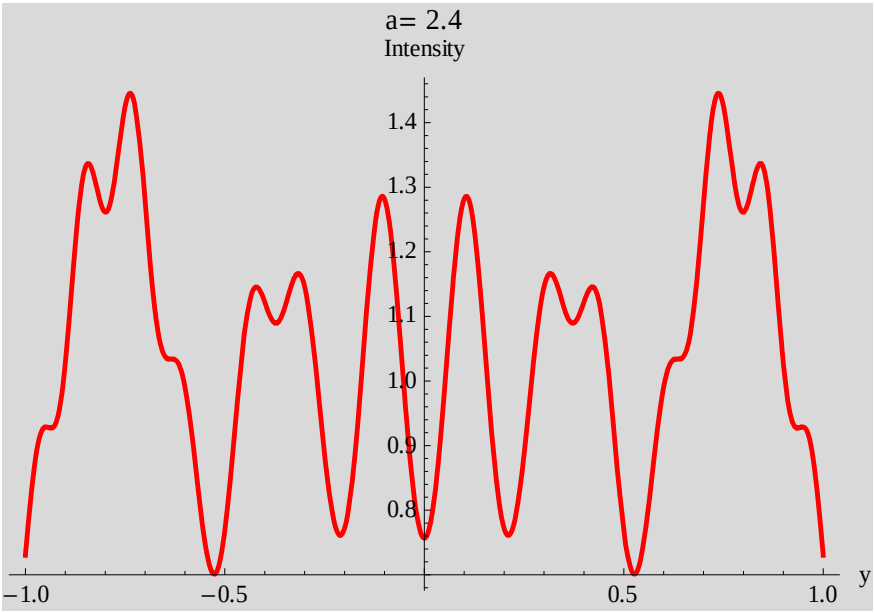
where

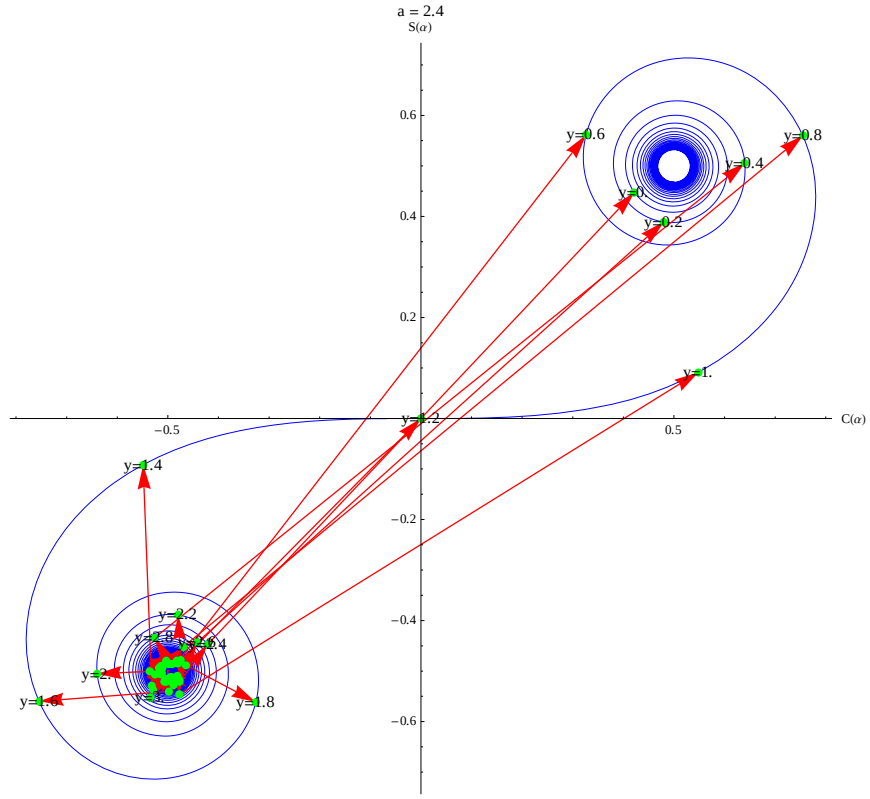
$$\xi_1=-\infty,\qquad \xi_2=\infty,\qquad \eta_1=-\sqrt{\frac{k}{\pi z}}\left(\frac{a}{2}+y\right),\qquad \eta_2=\sqrt{\frac{k}{\pi z}}\left(\frac{a}{2}-y\right)$$

Using these values of x_1, x_2, h_{11}, and h_2, we get

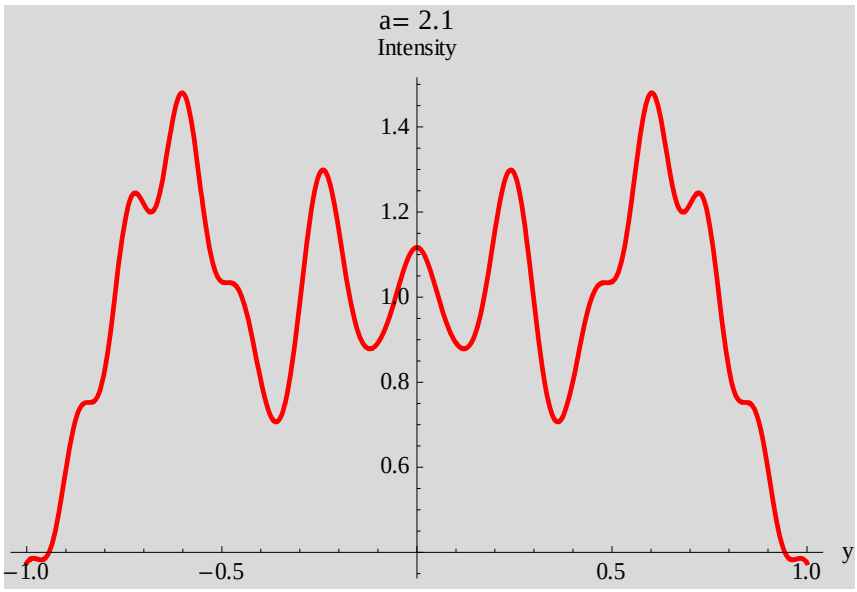
$$U(x,y)=\frac{e^{ikz}}{2i}\sqrt{2}e^{i\frac{\pi}{4}}\{C(\eta_2)+iS(\eta_2)-[C(\eta_1)+iS(\eta_1)]\}$$

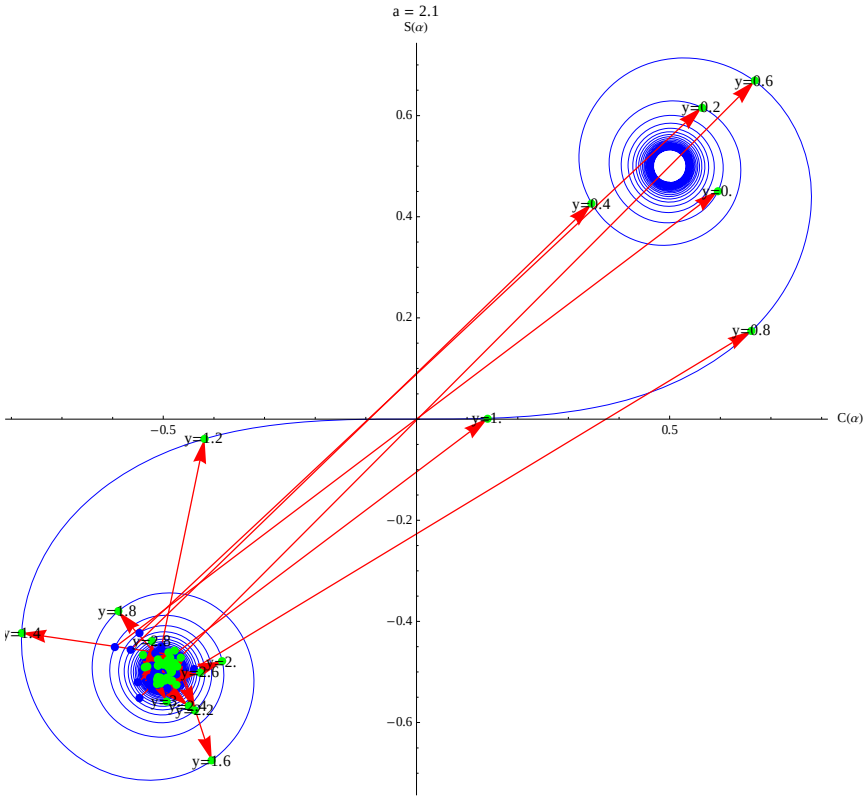
(a) a = 2.4 mm



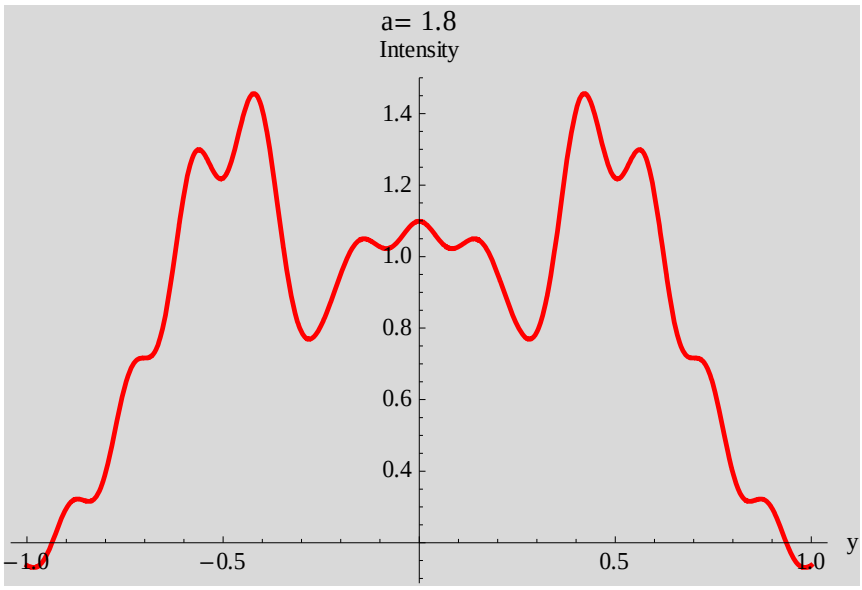


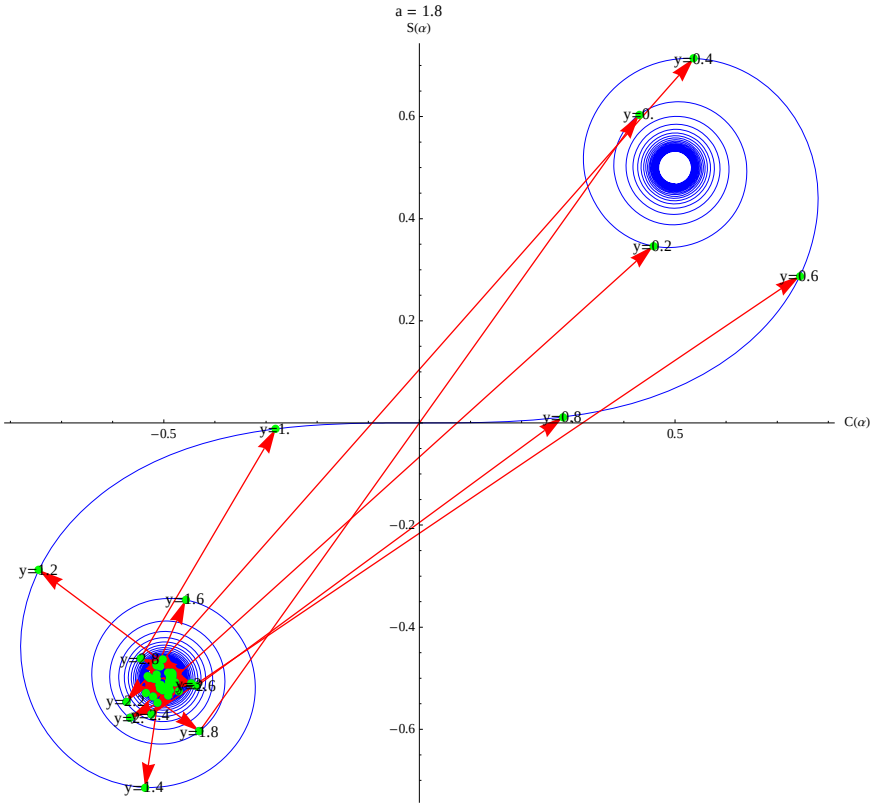
(b) a = 2.1 mm


a= 2.1
Intensity
1.4
1.2
1.0
0.8
0.6
−1.0
−0.5
0.5
1.0
y

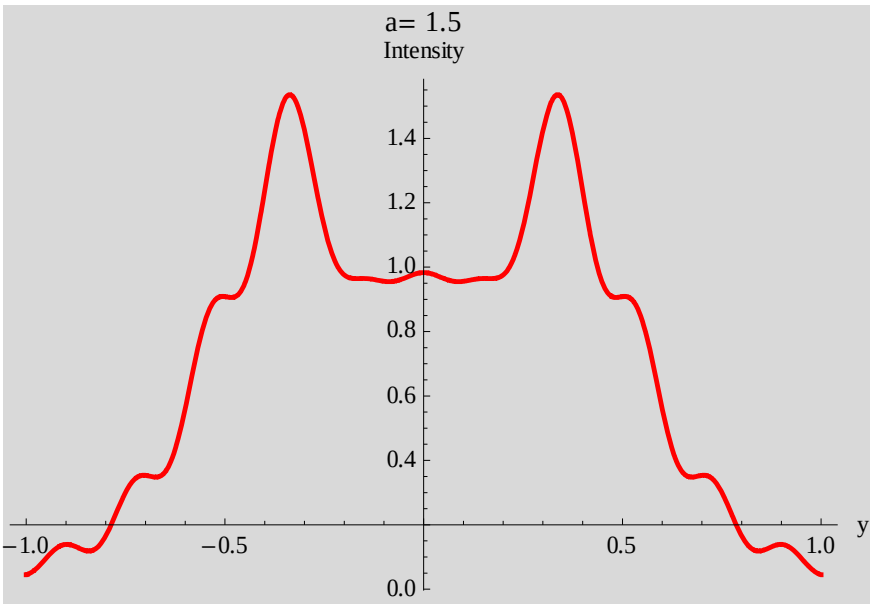

a = 2.1
S(α)
C(α)
y=0.6
y=0.2
y=0.4
y=0.8
y=1.2
y=1.8
y=1.6

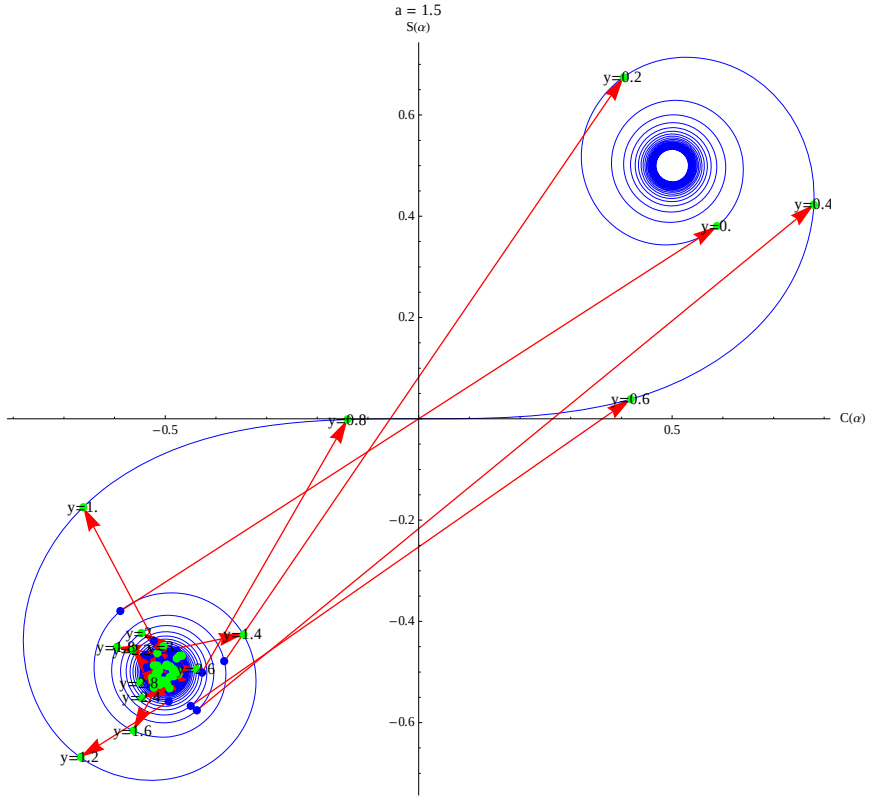
(c) a = 1.8 mm


a= 1.8
Intensity
1.4
1.2
1.0
0.8
0.6
0.4
−1.0
−0.5
0.5
1.0
y

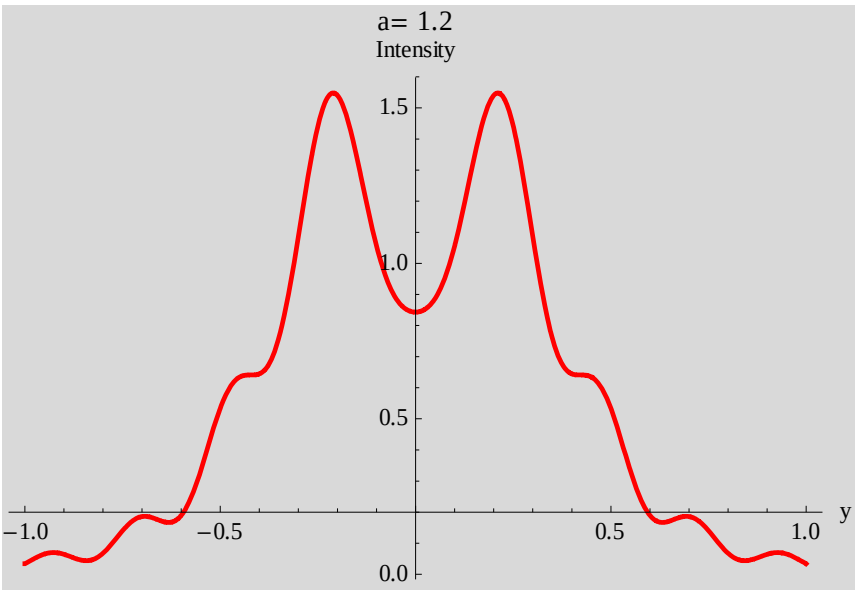

a = 1.8
S(α)
C(α)
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.5
0.5
y=0.4
y=0.
y=0.2
y=0.6
y=0.8
y=1.
y=1.2
y=1.6
y=1.8
y=1.4

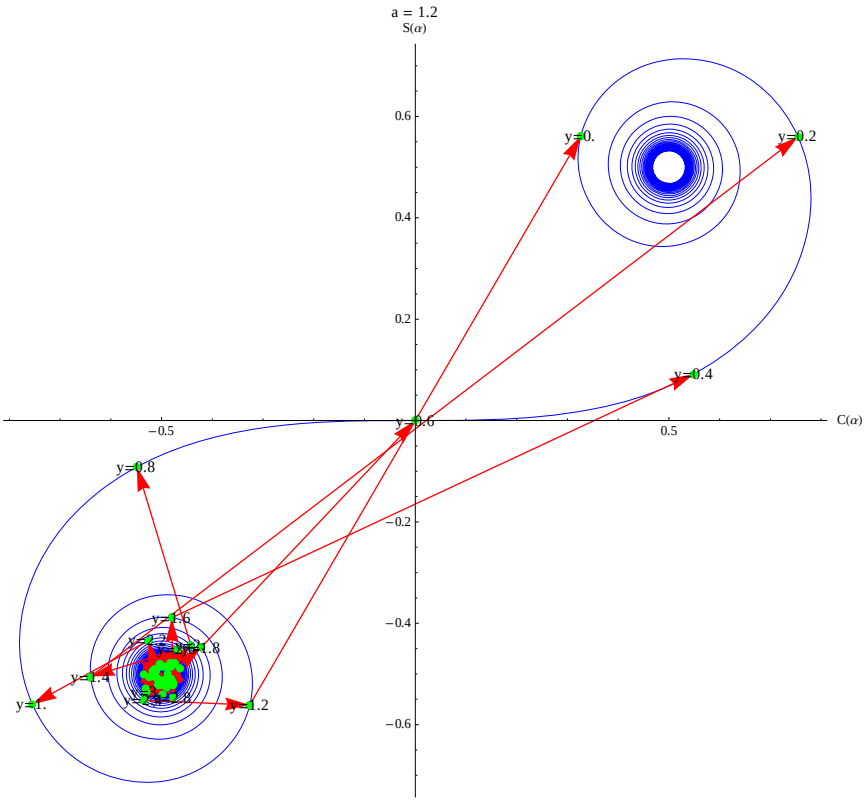
(d) a = 1.5 mm


a= 1.5
Intensity
1.4
1.2
1.0
0.8
0.6
0.4
0.0
−1.0
−0.5
0.5
1.0
y

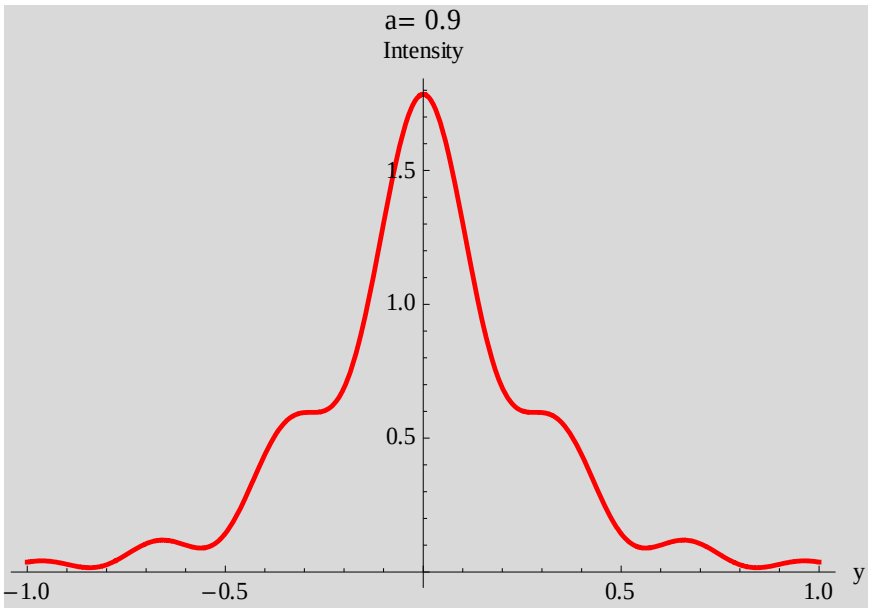

a = 1.5
S(α)
C(α)
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.5
0.5
y=0.2
y=0.4
y=0.6
y=0.8
y=1.
y=1.2
y=1.4
y=1.6

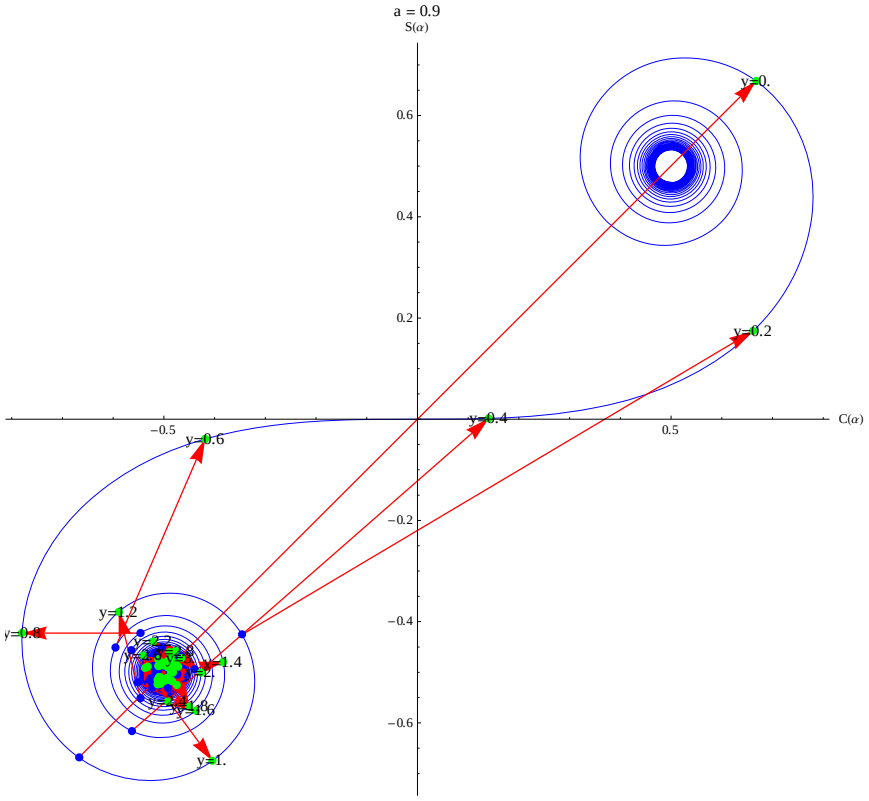
(e) a = 1.2 mm


a= 1.2
Intensity
1.5
1.0
0.5
0.0
−1.0
−0.5
0.5
1.0
y

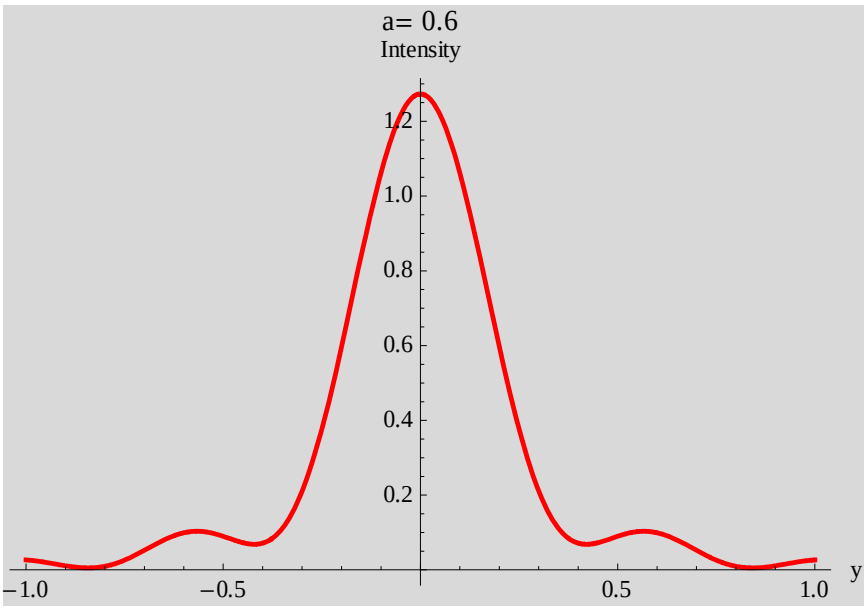

a = 1.2
S(α)
C(α)
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.5
0.5
y=0.
y=0.2
y=0.4
y=0.6
y=0.8
y=1.
y=1.2
y=1.4
y=1.6
y=1.8

(f) a = 0.9 mm


a= 0.9
Intensity
1.5
1.0
0.5
−1.0
−0.5
0.5
1.0
y


a = 0.9
S(α)
C(α)
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.5
0.5
y=0.
y=0.2
y=0.4
y=0.6
y=1.2
y=1.4
y=1.
y=1.6

(g) a = 0.6 mm



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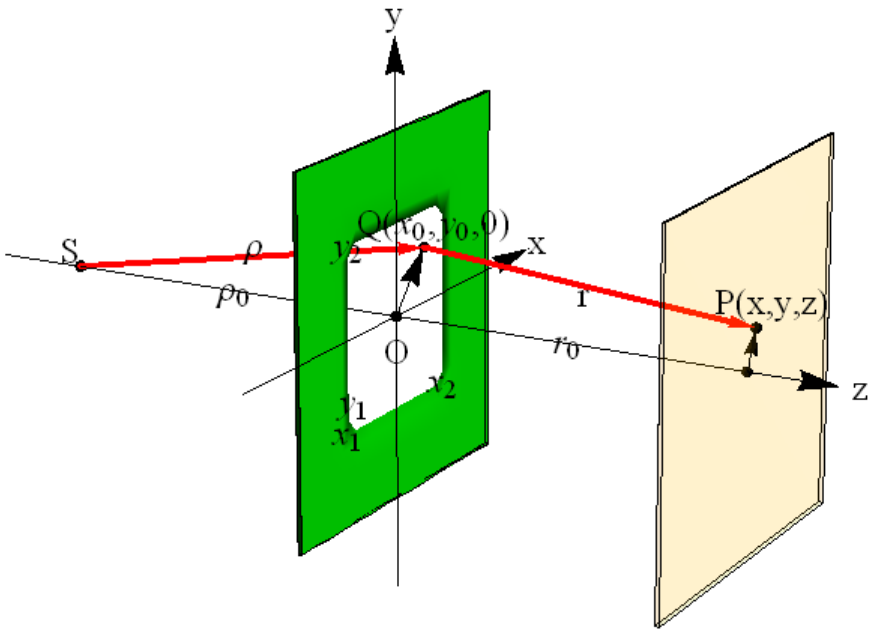
APPENDIX

Appendix. Square aperture in the case of finite distance ρ_0

(Landau and Lifshitz)



$$\rho = \sqrt{\rho_0^{\,2} + x_0^{\,2} + y_0^{\,2}}$$

$$r = \sqrt{r_0^{\,2} + (x - x_0)^2 + (y - y_0)^2}$$

$x_1 \le x \le x_2$, $\quad y_1 \le y \le y_2$ (square aperture)

The total path is given by

$$\rho + r = \rho_0 + r_0 + \frac{x_0^{\,2} + y_0^{\,2}}{2\rho_0} + \frac{(x - x_0)^2 + (y - y_0)^2}{2r_0}$$

$$= \frac{\rho_0 + r_0}{2\rho_0 r_0}\left[\left(x_0 - \frac{r_0 x}{\rho_0 + r_0}\right)^2 + \left(y_0 - \frac{r_0 y}{\rho_0 + r_0}\right)^2\right] + \frac{x^2 + y^2}{2(\rho_0 + r_0)} + (\rho_0 + r_0)$$

$$\exp[ik(\rho_0 + r_0)]\exp[ik\frac{x^2 + y^2}{2(\rho_0 + r_0)}]$$

$$\int_{x_1}^{x_2} \exp[ik(\frac{\rho_0 + r_0}{2\rho_0 r_0}(x_0 - \frac{xr_0}{\rho_0 + r_0})^2]dx_0$$

$$\int_{y_1}^{y_2} \exp[ik(\frac{\rho_0 + r_0}{2\rho_0 r_0})(y_0 - \frac{yr_0}{\rho_0 + r_0})^2]dy_0$$

We note that

$$\frac{\rho_0 + r_0}{2\rho_0 r_0}(x_0 - \frac{r_0 x}{\rho_0 + r_0})^2 = \frac{\left(\frac{\rho_0 + r_0}{2\rho_0 r_0}\right)^2}{\frac{\rho_0 + r_0}{2\rho_0 r_0}}(x_0 - \frac{xr_0}{\rho_0 + r_0})^2$$

$$= \frac{1}{\frac{\rho_0 + r_0}{2\rho_0 r_0}}[\frac{\rho_0 + r_0}{2\rho_0 r_0}x_0 - \frac{\rho_0 + r_0}{2\rho_0 r_0}\frac{r_0 x}{\rho_0 + r_0}]^2$$

$$= \frac{\rho_0 + r_0}{2\rho_0 r_0}(\frac{\rho_0 + r_0}{\rho_0 r_0}x_0 - \frac{x}{\rho_0})^2$$

$$\rho = \sqrt{\rho_0^{\,2} + x_0^{\,2} + y_0^{\,2}}$$

$$r = \sqrt{r_0^{\,2} + (x - x_0)^2 + (y - y_0)^2}$$

$x_1 \le x \le x_2$, $y_1 \le y \le y_2$ (square aperture)

The total path is given by

$$\rho + r = \rho_0 + r_0 + \frac{x_0^{\,2} + y_0^{\,2}}{2\rho_0} + \frac{(x - x_0)^2 + (y - y_0)^2}{2r_0}$$

$$= \frac{\rho_0 + r_0}{2\rho_0 r_0}[(x_0 - \frac{r_0 x}{\rho_0 + r_0})^2 + (y_0 - \frac{r_0 y}{\rho_0 + r_0})^2] + \frac{x^2 + y^2}{2(\rho_0 + r_0)}$$
$$+ (\rho_0 + r_0)$$

$$\exp[ik(\rho_0 + r_0)]\exp[ik\frac{x^2 + y^2}{2(\rho_0 + r_0)}]$$

$$\int_{x_1}^{x_2} \exp[ik(\frac{\rho_0 + r_0}{2\rho_0 r_0}(x_0 - \frac{xr_0}{\rho_0 + r_0})^2]dx_0$$

$$\int_{y_1}^{y_2} \exp[ik(\frac{\rho_0 + r_0}{2\rho_0 r_0})(y_0 - \frac{yr_0}{\rho_0 + r_0})^2]dy_0$$

We note that

$$\frac{\rho_0 + r_0}{2\rho_0 r_0}(x_0 - \frac{r_0 x}{\rho_0 + r_0})^2 = \frac{\left(\frac{\rho_0 + r_0}{2\rho_0 r_0}\right)^2}{\frac{\rho_0 + r_0}{2\rho_0 r_0}}(x_0 - \frac{xr_0}{\rho_0 + r_0})^2$$

$$= \frac{1}{\frac{\rho_0 + r_0}{2\rho_0 r_0}}[\frac{\rho_0 + r_0}{2\rho_0 r_0}x_0 - \frac{\rho_0 + r_0}{2\rho_0 r_0}\frac{r_0 x}{\rho_0 + r_0}]^2$$

$$= \frac{\rho_0 + r_0}{2\rho_0 r_0}(\frac{\rho_0 + r_0}{\rho_0 r_0}x_0 - \frac{x}{\rho_0})^2$$

Then we have

$$I_1 = \int_{x_1}^{x_2} \exp[ik(\frac{\rho_0 + r_0}{2\rho_0 r_0})(x_0 - \frac{r_0 x}{\rho_0 + r_0})^2]dx_0$$

$$= \int_{x_2}^{x_2} \exp\{ik\frac{1}{2}\frac{\rho_0 r_0}{\rho_0 + r_0}(\frac{\rho_0 + r_0}{\rho_0 r_0}x_0 - \frac{x}{\rho_0})^2\}dx_0$$

We put

$$t = \sqrt{\frac{k}{\pi}\frac{\rho_0 r_0}{\rho_0 + r_0}}(\frac{\rho_0 + r_0}{\rho_0 r_0}x_0 - \frac{x}{\rho_0})$$

$$= \sqrt{\frac{k}{\pi}\frac{\rho_0 r_0}{(\rho_0 + r_0)\rho_0^2}}[(1 + \frac{\rho_0}{r_0})x_0 - x]$$

$$= \sqrt{\frac{k}{\pi}\frac{r_0}{(\rho_0 + r_0)\rho_0}}[(1 + \frac{\rho_0}{r_0})x_0 - x]$$

and

$$dt = \sqrt{\frac{kr_0(1+\frac{\rho_0}{r_0})^2}{\pi\rho_0(\rho_0+r_0)}}dx_0 = \sqrt{\frac{k(\rho_0+r_0)}{\pi\rho_0 r_0}}dx_0$$

$$I_1 = \sqrt{\frac{\pi}{k}\frac{\rho_0 r_0}{\rho_0+r_0}}\int_{\xi_1}^{\xi_2}\exp(i\frac{\pi t^2}{2})dt$$

where

$$\xi_1 = \sqrt{\frac{k}{\pi}\frac{r_0}{(\rho_0+r_0)\rho_0}}[(1+\frac{\rho_0}{r_0})x_1 - x]$$

.

$$\xi_2 = \sqrt{\frac{k}{\pi}\frac{r_0}{(\rho_0+r_0)\rho_0}}[(1+\frac{\rho_0}{r_0})x_2 - x]$$

Similarly for the *y* direction. we have

$$I_2 = \sqrt{\frac{\pi}{k}\frac{\rho_0 r_0}{\rho_0+r_0}}\int_{\eta_1}^{\eta_2}\exp(i\frac{\pi t^2}{2})dt$$

$$\eta_1 = \sqrt{\frac{k}{\pi}\frac{r_0}{(\rho_0+r_0)\rho_0}}[(1+\frac{\rho_0}{r_0})y_1 - y]$$

$$\eta_2 = \sqrt{\frac{k}{\pi}\frac{r_0}{(\rho_0+r_0)\rho_0}}[(1+\frac{\rho_0}{r_0})y_2 - y]$$

The resultant electric field at the point P at the coordinate (*x, y,z)* is given by

$$\sqrt{\frac{\pi\rho_0 r_0}{k(\rho_0+r_0)}}\exp[ik(\rho_0+r_0+\frac{x^2+y^2}{2(\rho_0+r_0)}]$$

$$\int_{\xi_1}^{\xi_2}\exp[i\frac{\pi t^2}{2}]dt\int_{\eta_1}^{\eta_2}\exp[i\frac{\pi t^2}{2}]dt$$

where

$$\int_0^{\alpha} \exp[i\frac{\pi t^2}{2}]dt = C(\alpha) + iS(\alpha)$$

$$\int_{\xi_1}^{\xi_2} \exp[i\frac{\pi t^2}{2}]dt = \int_0^{\xi_2} \exp[i\frac{\pi t^2}{2}]dt - \int_0^{\xi_1} \exp[i\frac{\pi t^2}{2}]dt$$
$$= [C(\xi_2) + iS(\xi_2)] - [C(\xi_1) + iS(\xi_1)]$$

The intensity is determined by the square of the field. Thus, we have

$$I = \frac{1}{4} I_0 \,|\, ([C(\xi_2) + iS(\xi_2)] - [C(\xi_1) + iS(\xi_1)])([C(\eta_2) + iS(\eta_2)] - [C(\eta_1) + iS(\eta_1)]\,|^2$$

Then we have

$$I_1 = \int_{x_1}^{x_2} \exp[ik(\frac{\rho_0 + r_0}{2\rho_0 r_0})(x_0 - \frac{r_0 x}{\rho_0 + r_0})^2]dx_0$$
$$= \int_{x_2}^{x_2} \exp\{ik\frac{1}{2}\frac{\rho_0 r_0}{\rho_0 + r_0}(\frac{\rho_0 + r_0}{\rho_0 r_0}x_0 - \frac{x}{\rho_0})^2\}dx_0$$

We put

$$t = \sqrt{\frac{k}{\pi}\frac{\rho_0 r_0}{\rho_0 + r_0}}(\frac{\rho_0 + r_0}{\rho_0 r_0}x_0 - \frac{x}{\rho_0})$$
$$= \sqrt{\frac{k}{\pi}\frac{\rho_0 r_0}{(\rho_0 + r_0)\rho_0^2}}[(1 + \frac{\rho_0}{r_0})x_0 - x]$$
$$= \sqrt{\frac{k}{\pi}\frac{r_0}{(\rho_0 + r_0)\rho_0}}[(1 + \frac{\rho_0}{r_0})x_0 - x]$$

and

$$dt = \sqrt{\frac{kr_0(1 + \frac{\rho_0}{r_0})^2}{\pi\rho_0(\rho_0 + r_0)}}dx_0 = \sqrt{\frac{k(\rho_0 + r_0)}{\pi\rho_0 r_0}}dx_0$$

$$I_1 = \sqrt{\frac{\pi}{k}\frac{\rho_0 r_0}{\rho_0 + r_0}}\int_{\xi_1}^{\xi_2} \exp(i\frac{\pi t^2}{2})dt$$

where

$$\xi_1 = \sqrt{\frac{k}{\pi}\frac{r_0}{(\rho_0 + r_0)\rho_0}}[(1+\frac{\rho_0}{r_0})x_1 - x]$$

.

$$\xi_2 = \sqrt{\frac{k}{\pi}\frac{r_0}{(\rho_0 + r_0)\rho_0}}[(1+\frac{\rho_0}{r_0})x_2 - x]$$

Similarly for the *y* direction. we have

$$I_2 = \sqrt{\frac{\pi}{k}\frac{\rho_0 r_0}{\rho_0 + r_0}}\int_{\eta_1}^{\eta_2} \exp(i\frac{\pi t^2}{2})dt$$

$$\eta_1 = \sqrt{\frac{k}{\pi}\frac{r_0}{(\rho_0 + r_0)\rho_0}}[(1+\frac{\rho_0}{r_0})y_1 - y]$$

$$\eta_2 = \sqrt{\frac{k}{\pi}\frac{r_0}{(\rho_0 + r_0)\rho_0}}[(1+\frac{\rho_0}{r_0})y_2 - y]$$

The resultant electric field at the point P at the coordinate (*x, y,z)* is given by

$$\sqrt{\frac{\pi\rho_0 r_0}{k(\rho_0 + r_0)}}\exp[ik(\rho_0 + r_0 + \frac{x^2 + y^2}{2(\rho_0 + r_0)}]$$

$$\int_{\xi_1}^{\xi_2}\exp[i\frac{\pi t^2}{2}]dt\int_{\eta_1}^{\eta_2}\exp[i\frac{\pi t^2}{2}]dt$$

where

$$\int_0^{\alpha}\exp[i\frac{\pi t^2}{2}]dt = C(\alpha) + iS(\alpha)$$

$$\int_{\xi_1}^{\xi_2} \exp[i\frac{\pi t^2}{2}]dt = \int_{0}^{\xi_2} \exp[i\frac{\pi t^2}{2}]dt - \int_{0}^{\xi_1} \exp[i\frac{\pi t^2}{2}]dt$$
$$= [C(\xi_2) + iS(\xi_2)] - [C(\xi_1) + iS(\xi_1)]$$

The intensity is determined by the square of the field. Thus, we have

$$I = \frac{1}{4} I_0 \,|\, ([C(\xi_2) + iS(\xi_2)] - [C(\xi_1) + iS(\xi_1)])([C(\eta_2) + iS(\eta_2)] - [C(\eta_1) + iS(\eta_1)]\,|^2$$

A.2 Fresnel diffraction intensity for the square aperture: Mathematica

FresnelS[α];
FresnelC[α]
Plot3D
ParametricPlot
DensityPlot
RegionPlot3D
Or[*a*, *b*]

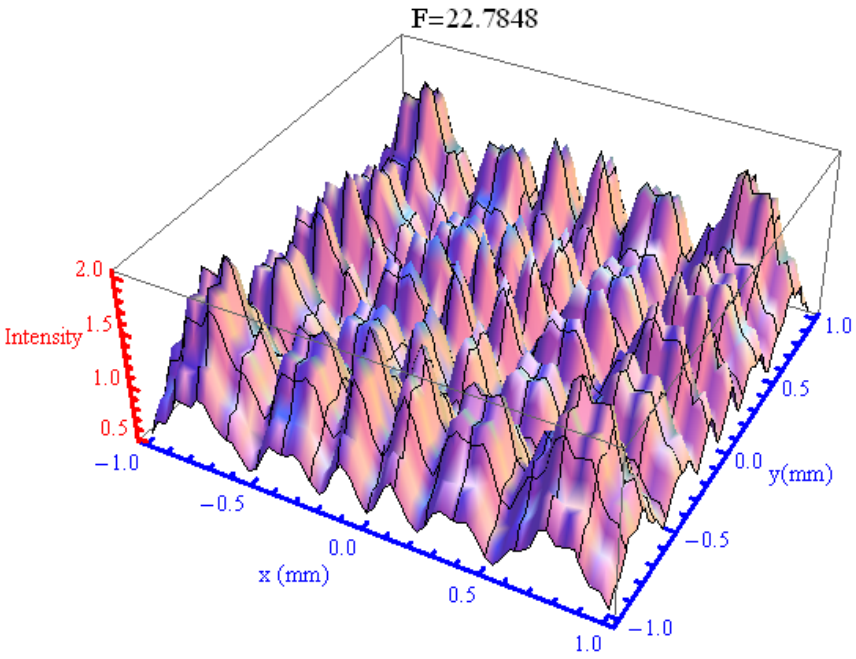
Fresnel diffraction with a square aperture
λ = 632 nm, z = 400 mm, L = 2.4 mm
The unit of x and y are mm.

```
Clear["Gobal`"]; F = L^2/(z λ);

rule1 = {k → (2 π)/(632 × 10^-9), λ → 632 × 10^-9, z → 400 × 10^-3, L → 2.4  10^-3,
    x1 → 10^-3 x, y1 → 10^-3 y};

ξ1 = -Sqrt[k/(π z)] (L/2 + x1) /. rule1; ξ2 = Sqrt[k/(π z)] (L/2 - x1) /. rule1;

η1 = -Sqrt[k/(π z)] (L/2 + y1) /. rule1;

η2 = Sqrt[k/(π z)] (L/2 - y1) /. rule1;

I1[α1_, α2_, β1_, β2_] :=
  1/4 ((FresnelC[α2] - FresnelC[α1])^2 + (FresnelS[α2] - FresnelS[α1])^2)
   ((FresnelC[β2] - FresnelC[β1])^2 + (FresnelS[β2] - FresnelS[β1])^2);
```

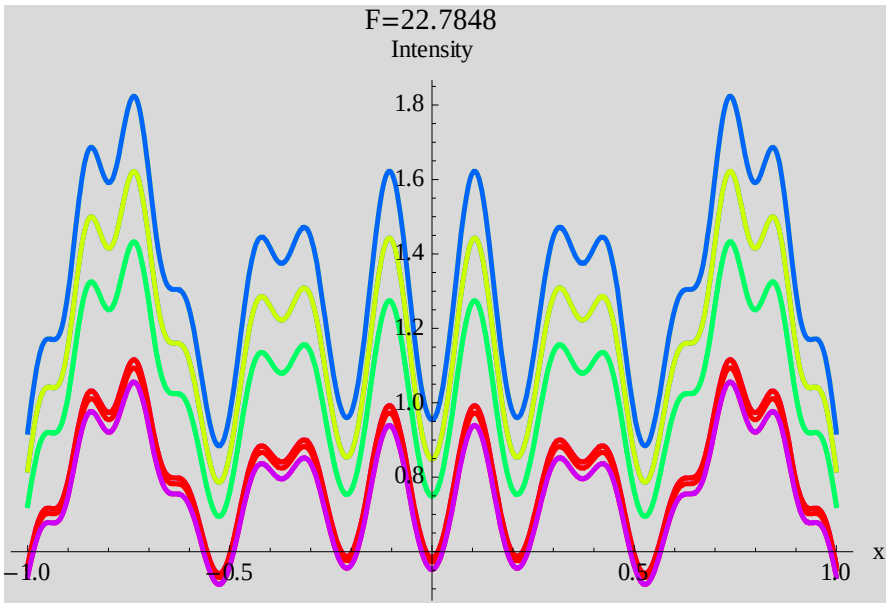
Fresnel number

```
F1 = F /. rule1
22.7848
```

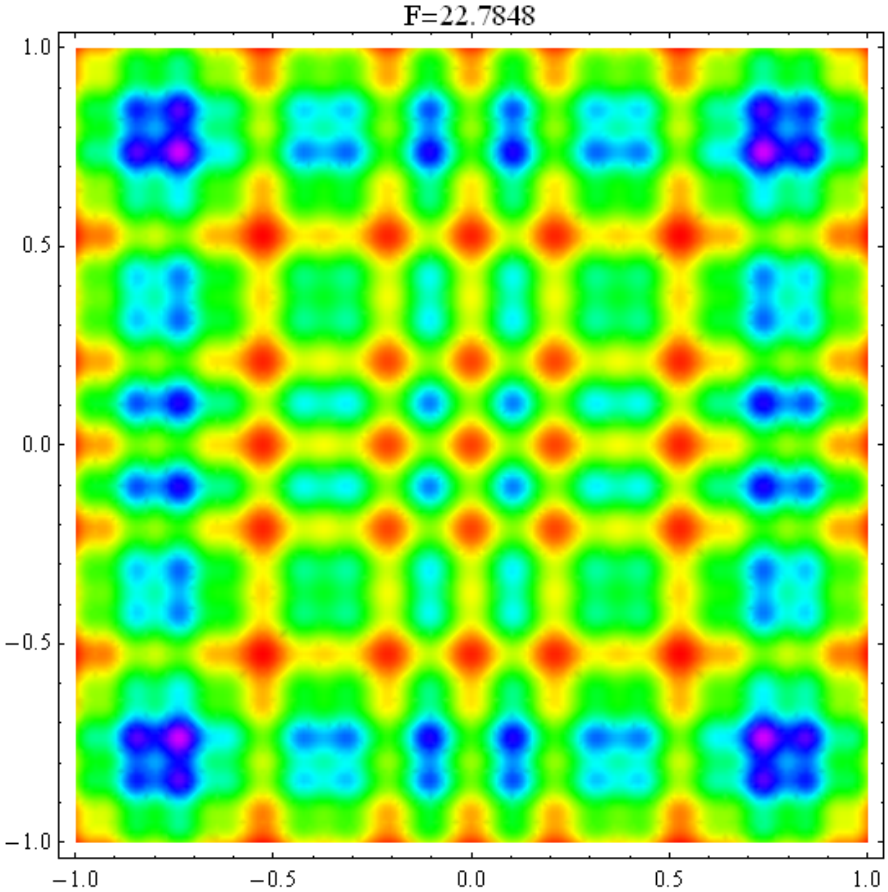
```
Plot3D[I1[ξ1, ξ2, η1, η2], {x, -1, 1}, {y, -1, 1}, PlotRange → All,
 AxesLabel → {"x (mm)", "y(mm)", "Intensity"},
 AxesEdge → {{-1, -1}, {1, -1}, {-1, -1}},
 AxesStyle → {{Blue, Thick}, {Blue, Thick}, {Red, Thick}},
 PlotLabel → "F=" <> ToString[F1]]
```




```
Plot[Evaluate[Table[I1[ξ1, ξ2, η1, η2], {y, -1, 1, 0.2}]], {x, -1, 1},
 PlotRange → All, AxesLabel → {"x", "Intensity"}, Background → LightGray,
 PlotStyle → Table[{Hue[0.2 i], Thick}, {i, 0, 5}],
 PlotLabel → "F=" <> ToString[F1]]
```




```
DensityPlot[I1[ξ1, ξ2, η1, η2], {x, -1, 1}, {y, -1, 1}, PlotRange → All,
 AxesLabel → {"x (mm)", "y(mm)"}, PlotPoints → 40,
 ColorFunction → (Hue[0.8 #] &), PlotLabel → "F=" <> ToString[F1] ]
```



A3. Cornu spiral

```
Clear["Global`*"];

f1 = ParametricPlot[{FresnelC[x], FresnelS[x]}, {x, 0, 10}, PlotStyle → {Blue, Thin},
  PlotRange → All, AxesLabel → {"C(α)", "S(α)"}];
f2 = Graphics[{Black, Table[Point[{FresnelC[α], FresnelS[α]}], {α, 0, 10, 0.1}], Red,
   Table[Point[{FresnelC[α], FresnelS[α]}], {α, 0, 10, 0.5}],
   Table[Text[Style["α=" <> ToString[α], Black, 12], {FresnelC[α], FresnelS[α]}],
    {α, 0, 5, 0.1}]}];
Show[f1, f2, PlotRange → All]
```