Chapter 25
Translation operator

Masatsugu Sei Suzuki
Department of Physics, SUNY at Bimghamton
(Date: November 22, 2010)

25.1 Definition of the translation operator

Here we discuss the transportation operator

$\hat{T}(a)$: translation operator (unitary operator)

$$|\psi'\rangle = \hat{T}(a)|\psi\rangle$$

or

$$\langle\psi'| = \langle\psi'|\hat{T}^{+}(a).$$

(i) Analogy from classical mechanics for *x*

The average value of $\hat{x}$ in the new state $|\psi'\rangle$ is equal to the average value of $\hat{x}$ in the new state $|\psi\rangle$ plus the *x*-displacement *a*.

$$\langle\psi'|\hat{x}|\psi'\rangle = \langle\psi|\hat{x}+a|\psi\rangle$$

or

$$\langle\psi|\hat{T}^{+}(a)\hat{x}\hat{T}(a)|\psi\rangle = \langle\psi|\hat{x}+a|\psi\rangle$$

or

$$\hat{T}^{+}(a)\hat{x}\hat{T}(a) = \hat{x}+a\hat{1}, \qquad (1)$$

Normalization condition:

$$\langle\psi'|\psi'\rangle = \langle\psi|\hat{T}^{+}(a)\hat{T}(a)|\psi\rangle = \langle\psi|\psi\rangle$$

or

$$\hat{T}^{+}(a)\hat{T}(a) = \hat{1} \qquad (2)$$

((Unitary operator))

From Eqs.(1) and (2), we have

$$\hat{x}\hat{T}(a) = \hat{T}(a)(\hat{x} + a) = \hat{T}(a)\hat{x} + a\hat{T}(a)$$

((Commutation relation))

$$[\hat{x}, \hat{T}(a)] = a\hat{T}(a),$$

$$\hat{x}\hat{T}(a)|x\rangle = \hat{T}(a)\hat{x}|x\rangle + a\hat{T}(a)|x\rangle = (x + a)\hat{T}(a)|x\rangle.$$

Thus $\hat{T}(a)|x\rangle$ is the eigenket of $\hat{x}$ with the eigenvalue (x+a).

or

$$\hat{T}(a)|x\rangle = |x + a\rangle$$

$$\hat{T}^{+}(a)\hat{T}(a)|x\rangle = \hat{T}^{+}(a)|x + a\rangle = |x\rangle$$

When x is replaced by x-a

$$|x - a\rangle = \hat{T}^{+}(a)|x\rangle$$

or

$$\langle x - a| = \langle x|\hat{T}(a).$$

Note that

$$\langle x|\psi'\rangle = \langle x|\hat{T}(a)|\psi\rangle = \langle x - a|\psi\rangle = \psi(x - a).$$

(ii) Analogy from the classical mechanics for *p*

The average value of $\hat{p}$ in the new state $|\psi'\rangle$ is equal to the average value of $\hat{p}$ in the new state $|\psi\rangle$.

$$\langle\psi'|\hat{p}|\psi'\rangle = \langle\psi|\hat{p}|\psi\rangle,$$

or

$$\langle\psi|\hat{T}^{+}(a)\,\hat{p}\hat{T}(a)|\psi\rangle = \langle\psi|\hat{p}|\psi\rangle$$

$$\hat{T}^{+}(a)\hat{p}\hat{T}(a) = \hat{p}$$

So we have the commutation relation

$$[\hat{T}(a), \hat{p}] = 0 .$$

From the above commutation relation, we have

$$\hat{p}\hat{T}(a)|p\rangle = \hat{T}(a)\hat{p}|p\rangle = p\hat{T}(a)|p\rangle .$$

Thus $\hat{T}(a)|p\rangle$ is the eigenket of $\hat{p}$ associated with the eigenvalue *p*.

25.2 Infinitesimal translation operator

We now define the infinitesimal translation operator by

$$\hat{T}(dx) = \hat{1} - \frac{i}{\hbar}\hat{G}dx$$

where $\hat{G}$ is called a generator of translation. The dimension of $\hat{G}$ is that of the linear momentum.

The operator $\hat{T}(dx)$ satisfies the relations:

$$\hat{T}^{+}(dx)\hat{T}(dx) = \hat{1} , \tag{1}$$

$$\hat{T}^{+}(dx)\hat{x}\hat{T}(dx) = \hat{x} + dx ,$$

or

$$\hat{x}\hat{T}(dx) - \hat{T}(dx)\hat{x} = dx\hat{T}(dx) , \tag{2}$$

and

$$[\hat{T}(dx), \hat{p}] = 0 , \tag{3}$$

Using the relation (1), we get

$$(\hat{1} - \frac{i}{\hbar}\hat{G}dx)^{+}(\hat{1} - \frac{i}{\hbar}\hat{G}dx) = \hat{1} ,$$

or

$$(\hat{1}+\frac{i}{\hbar}\hat{G}^{+}dx)(\hat{1}-\frac{i}{\hbar}\hat{G}dx)=\hat{1}+\frac{i}{\hbar}(\hat{G}^{+}-\hat{G})dx+O[(dx)^{2}]=\hat{1},$$

or

$$\hat{G}^{+}=\hat{G}.$$

The operator $\hat{G}$ is a Hermitian operator. Using the relation (2), we get

$$\hat{x}(\hat{1}-\frac{i}{\hbar}\hat{G}dx)-(\hat{1}-\frac{i}{\hbar}\hat{G}dx)\hat{x}=dx(\hat{1}-\frac{i}{\hbar}\hat{G}dx)=dx\hat{1}+O(dx)^{2}$$

or

$$-\frac{i}{\hbar}[\hat{x},\hat{G}]dx=dx\hat{1}$$

or

$$[\hat{x},\hat{G}]=i\hbar\hat{1}.$$

Using the relation (3), we get

$$[\hat{1}-\frac{i}{\hbar}\hat{G}dx,\hat{p}]=0$$

Then we have

$$[\hat{G},\hat{p}]=0$$

From these two commutation relations, we conclude that

$$\hat{G}=\hat{p}.$$

and

$$\hat{T}(dx)=\hat{1}-\frac{i}{\hbar}\hat{p}dx$$

We see that the position operator and the momentum operator $\hat{p}$ obeys the commutation relation

$$[\hat{x},\hat{p}]=i\hbar\hat{1}.$$

which leads to the Heisenberg's principle of uncertainty.

25.3 Momentum operator $\hat{p}$ in the position basis.

$$\hat{T}(\delta x)|\psi\rangle = \hat{T}(\delta x)\int dx'|x'\rangle\langle x'|\psi\rangle = \int dx'|x'+\delta x\rangle\langle x'|\psi\rangle =$$
$$= \int dx'|x'\rangle\langle x'-\delta x|\psi\rangle = \int dx'|x'\rangle\psi(x'-\delta x)$$

We apply the Taylor expansion:

$$\psi(x'-\delta x) = \psi(x') - \delta x\frac{\partial}{\partial x'}\psi(x')$$

Substitution:

$$\begin{aligned}
\hat{T}(\delta x)|\psi\rangle &= \int dx'|x'\rangle\psi(x'-\delta x) \\
&= \int dx'|x'\rangle[\psi(x') - \delta x\frac{\partial}{\partial x'}\psi(x')] \\
&= \int dx'|x'\rangle[\langle x'|\psi\rangle - \delta x\frac{\partial}{\partial x'}\langle x'|\psi\rangle] \\
&= |\psi\rangle - \delta x\int dx'|x'\rangle\frac{\partial}{\partial x'}\langle x'|\psi\rangle \\
&= (\hat{1} - \frac{i}{\hbar}\hat{p}\delta x)|\psi\rangle
\end{aligned}$$

Thus we have

$$\hat{p}|\psi\rangle = \frac{\hbar}{i}\int dx'|x'\rangle\frac{\partial}{\partial x'}\langle x'|\psi\rangle$$

$$\begin{aligned}
\langle x|\hat{p}|\psi\rangle &= \frac{\hbar}{i}\int dx'\langle x|x'\rangle\frac{\partial}{\partial x'}\langle x'|\psi\rangle \\
&= \frac{\hbar}{i}\int dx'\delta(x-x')\frac{\partial}{\partial x'}\langle x'|\psi\rangle \\
&= \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|\psi\rangle
\end{aligned}$$

We obtain a very important formula

$$\langle x|\hat{p}|\psi\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|\psi\rangle$$

$$\langle\psi|\hat{p}|\psi\rangle = \int dx\langle\psi|x\rangle\frac{\hbar}{i}\langle x|\hat{p}|\psi\rangle$$
$$= \int dx\langle\psi|x\rangle\frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|\psi\rangle$$
$$= \frac{\hbar}{i}\int dx\langle x|\psi\rangle^{*}\frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|\psi\rangle$$

These results suggest that in position space the momentum operator takes the form

$$\hat{p} \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial x}.$$

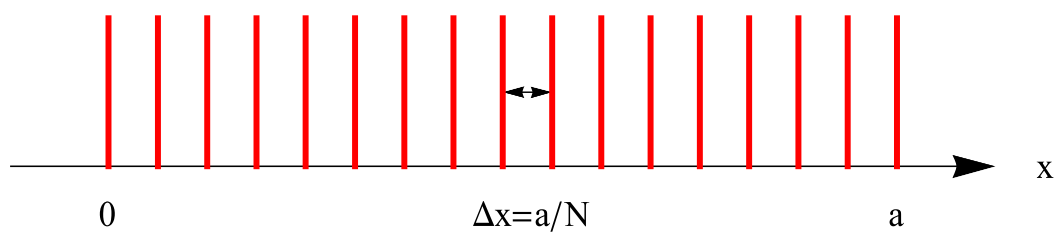
25.4 The finite translation operator

What is the operator $\hat{T}(a)$ corresponding to a finite translation *a*? We find it by the following procedure. We divide the interval into *N* parts of size d*x* = *a*/*N*. As *N*→∞, *a*/*N* becomes infinitesimal.

$$\hat{T}(dx) = \hat{1} - \frac{i}{\hbar}\hat{p}(\frac{a}{N}).$$

Since a translation by *a* equals *N* translations by *a*/*N*, we have

$$\hat{T}(a) = \underset{N\rightarrow\infty}{Lim}[\hat{1} - \frac{i}{\hbar}\hat{p}(\frac{a}{N})]^{N} = \exp(-\frac{i}{\hbar}\hat{p}a)$$



Here we use the formula

$$\lim_{N\rightarrow\infty}(1+\frac{1}{N})^{N} = e, \qquad \lim_{N\rightarrow\infty}(1-\frac{1}{N})^{N} = e^{-1}$$

$$\lim_{N\rightarrow\infty}[(1-\frac{ax}{N})^{\frac{N}{ax}}]^{ax} = \lim_{N\rightarrow\infty}(1-\frac{ax}{N})^{N} = (e^{-1})^{ax} = e^{-ax}$$

In summary, we have

$$\hat{T}(a) = \exp(-\frac{i}{\hbar}\hat{p}a) .$$

25.5 Discussion

It is interesting to calculate

$$\hat{T}^{+}(a)\hat{x}\hat{T}(a) = e^{\frac{i}{\hbar}\hat{p}a}\hat{x}e^{-\frac{i}{\hbar}\hat{p}a}$$

by using the Baker-Hausdorff theorem:

$$\exp(\hat{A}x)\hat{B}\exp(-\hat{A}x) = \hat{B} + \frac{x}{1!}[\hat{A},\hat{B}] + \frac{x^2}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{x^3}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + ...$$

When $x = 1$, we have

$$\exp(\hat{A})\hat{B}\exp(-\hat{A}) = \hat{B} + \frac{1}{1!}[\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{1}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + ...$$

Then we have

$$\hat{T}^{+}(a)\hat{x}\hat{T}(a) = e^{\frac{i}{\hbar}\hat{p}a}\hat{x}e^{-\frac{i}{\hbar}\hat{p}a} = \hat{x} + [\frac{i}{\hbar}\hat{p}a,\hat{x}] = \hat{x} + \frac{i}{\hbar}a[\hat{p},\hat{x}] = \hat{x} + \frac{i}{\hbar}a\frac{\hbar}{i} = \hat{x} + a\hat{1} .$$

So we confirmed that the relation

$$\hat{T}^{+}(a)\hat{x}\hat{T}(a) = \hat{x} + a\hat{1} ,$$

holds for any finite translation operator.

25.6 Invariance of Hamiltonian under the translation

Now we consider the condition for the invariance of Hamiltonian $\hat{H}$ under the translation.

The average value of $\hat{H}$ in the new state $|\psi'\rangle$ is equal to the average value of $\hat{H}$ in the new state $|\psi\rangle$.

$$\langle\psi'|\hat{H}|\psi'\rangle = \langle\psi|\hat{H}|\psi\rangle ,$$

or

$$\hat{T}^{+}(dx)\hat{H}\hat{T}(dx) = \hat{H} , \qquad \text{or} \qquad \hat{H}\hat{T}(dx) = \hat{T}(dx)\hat{H} ,$$

or

$$\hat{H}(\hat{1}-\frac{i}{\hbar}\hat{p}dx)=(\hat{1}-\frac{i}{\hbar}\hat{p}dx)\hat{H}$$

Then we have

$$[\hat{H},\hat{p}]=0\,.$$