Chapter 2
First order differential equation

2.1. Linear equations

2.1.1 Method of solution

We consider a first-order differential equation given by

$$y' + p(x)y = q(x)$$

First solve the problem: $y_1' + p(x)y_1 = 0$

$$y_1(x) = C\exp[-\int^x p(s)ds]$$

((Note)) Separation variable method

$$y_1' + p(x)y_1 = 0 .$$

or

$$\frac{dy_1}{dx} + p(x)y_1 = 0, \qquad \frac{dy_1}{y_1} = -p(x)dx$$

Then we have

$$\int \frac{dy_1}{y_1} = -\int p(x)dx = -\int^x p(s)ds,$$

or

$$\ln(y_1) = -\int^x p(s)ds + const, \qquad y_1(x) = C_1\exp[-\int^x p(s)ds].$$

We assume that $C \rightarrow C(x)$

$$y(x) = C(x)\exp(-\int^x p(s)ds)$$

$$y' = -p(x)C(x)\exp(-\int^{x} p(s)ds) + C'(x)\exp(-\int^{x} p(s)ds)$$

$$= -p(x)y + C'(x)\exp(-\int^{x} p(s)ds)$$

or

$$C'(x)\exp(-\int^{x} p(s)ds) = q(x)$$

or

$$C'(x) = \frac{dC(x)}{dx} = q(x)\exp(\int^{x} p(s)ds)$$

Then we have

$$C(x) = \int^{x} dt q(t) e^{\int^{t} p(s)ds} .$$

Final result is

$$y(x) = C(x)\exp(-\int^{x} p(s)ds) .$$

2.1.2 Example

Solve

$$\frac{dy}{dx} + 2xy = 4x .$$

((**Solution**))

The integrating factor $C(x)$ is obtained as follows.

$$\frac{dy}{dx} + 2xy = 0$$

or

$$\int \frac{1}{y} dy = -\int 2x dx \qquad \text{(separation variable).}$$

Then we have

$$y(x) = C\exp(-x^2) \to C(x)\exp(-x^2)$$

with

$$C'(x) = \frac{dC(x)}{dx} = 4x\exp(x^2)$$

Then we have

$$C(x) = \int 4x\exp(x^2)dx = 2\exp(x^2) + c.$$

The solution is given by

$$y(x) = 2 + c\exp(-x^2).$$

((**Mathematica**))

```
DSolve[y'[x] + 2 x y[x] == 4 x, y[x], x]
```

$$\{\{y[x] \to 2 + e^{-x^2} C[1]\}\}$$

2.2 Solution of the differential equation by Mathematica

We use the Mathematica to solve a differential equation. We use the following command.

DSolve[eqn,*y*,x]
find a formal solution for the ordinary differential equations eqn for the function *y* with the independent variable *x*

1. Write down the differential equation such as

$$eq1 = y'[x] + xy[x] == Exp[-x]$$

With initial condition *y*[0]=1.

2. Using DSolve, we solve the differential equation

$$eq2 = DSolve[\{eq1, y[0] == 1\}, y[x], x]$$

3. Using the conventional technique (this is a very convenient technique)

$y[x\_] = y[x] /. eq2[[1]]$

You can get the solution of $y(x)$.

4. Next we make a plot of $y[x]$ as a function of x. For example,

$$Plot[Evaluate[y[x]], \{x, 0, 10\}, PlotStyle \to \{Hue[0], Thickness[0.01]\}, Background \to LightGray, AxesLabel \to \{"x", "y"\}]$$

5. We can make a plot of the phase space given by $\{y[x], y'[x]\}$. For example,

$$ParametricPlot[Evaluate[\{y[x], y'[x]\}], \{x, 0, 10\}]$$

2.3. Numerical solution by Mathematica

We use the Mathematica to solve a complicated differential equation numerically. We use the following commands.

NDSolve[eqns,y,$\{x, x_{min}, x_{max}\}$]
finds a numerical solution to the ordinary differential equations eqns for the function y with the independent variable x in the range x_{min} and x_{max}

1. Write down the differential equation such as

$$eq1 = y'[x] + xy[x] == Exp[-x]$$

2. Using NDSolve we solve the differential equation with the boundary condition (in this case, $y[0]=1$) for $0 \le x \le x_0$ (=10).

$$eq2 = NDSolve[\{eq1, y[0] == 1\}, y[x], \{x, 0, 10\}]$$

3. Using the conventional technique (this technique is very convenient. You must memorize).

$$y[x\_] = y[x] /. eq2[[1]]$$

You can get the solution of $y(x)$ for $0 \le x \le x_0$ (=10).

4. Next we make a plot of $y[x]$ as a function of x for $0 \le x \le 10$. The command Evaluate is very important for the process of changing the parameters into numerical values

```
Plot[Evaluate[y[x]],{x,0,10},PlotStyle→{Hue[0],Thickness[0.01]},
Background→LightGray,AxesLabel→{"x","y"}]
```

5. We can make a plot of the phase space given by {*y*[*x*],*y*'[*x*]}.

```
ParametricPlot[Evaluate[{y[x],y'[x]}],{x,0,10}]
```

2.4. Exact differential equation

2.4.1 Definition

Here we consider the exact differential equation,

$$d\phi = A(x,y)dx + B(x,y)dy = \frac{\partial \phi}{dx}dx + \frac{\partial \phi}{dy}dy = 0,$$

with

$$A(x,y) = \frac{\partial \phi(x,y)}{\partial x}, \qquad B(x,y) = \frac{\partial \phi(x,y)}{\partial y}.$$

The condition for the exact differential equation is

$$\frac{\partial A(x,y)}{\partial y} = \frac{\partial^2 \phi(x,y)}{\partial y \partial x} = \frac{\partial^2 \phi(x,y)}{\partial x \partial y} = \frac{\partial B(x,y)}{\partial x}.$$

Then we have the solution of the exact differential equation;

ϕ = constant.

2.4.2 Example

Solve

$$x\frac{dy}{dx} + (3x + y) = 0, \quad \text{or} \quad (3x + y)dx + xdy = 0$$

with

$$A(x,y) = 3x + y, \qquad B(x,y) = x.$$

((**Solution**))
Since

$$\frac{\partial A(x,y)}{\partial y} = \frac{\partial B(x,y)}{\partial x} = 1,$$

this differential equation is an exact equation.

$$\phi = \int (3x + y)dx = \frac{3}{2}x^2 + yx + F(y) = c_1.$$

Since $B(x,y) = \frac{\partial \phi}{\partial y} = x$, $F'(y) = 0$ or $F(y) = c_2$.Then we have a final solution

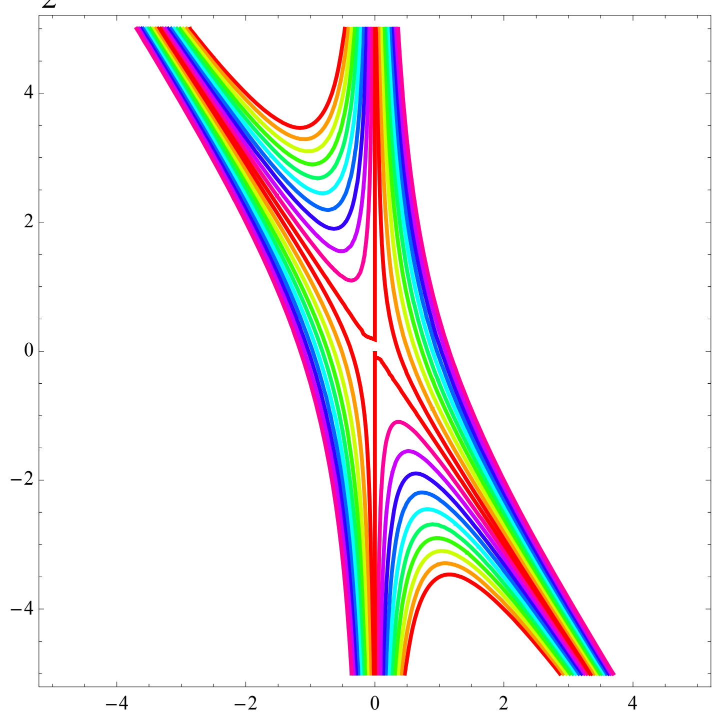
$$\frac{3}{2}x^2 + yx = c_1 - c_2 = c$$

((**Mathematica**))

```
DSolve[x y'[x] + 3 x + y[x] == 0, y[x], x]
```

$$\left\{\left\{y[x] \to -\frac{3x}{2} + \frac{C[1]}{x}\right\}\right\}$$

ContourPlot of $\phi = \frac{3}{2}x^2 + yx = c$, where c = -2 to 2.



2.5. Inexact differential equation

2.5.1 Definition

We now consider a differential equation,

$$P(x,y)dx + Q(x,y)dy = 0\,.$$

If we multiply $\alpha(x,y)$ on this equation,

$$\alpha(x,y)P(x,y)dx + \alpha(x,y)Q(x,y)dy = 0\,,$$

then this becomes exact differential equation.

$$\frac{\partial\phi}{\partial x} = \alpha(x,y)P(x,y)\,, \quad \text{and} \quad \frac{\partial\phi}{\partial y} = \alpha(x,y)Q(x,y)\,,$$

or

$$\begin{aligned}\frac{\partial^2\phi}{\partial y\partial x} &= \frac{\partial}{\partial y}\alpha(x,y)P(x,y)\\ \frac{\partial^2\phi}{\partial x\partial y} &= \frac{\partial}{\partial x}\alpha(x,y)Q(x,y)\end{aligned}\,.$$

Suppose that $\alpha(x,y)$ is a function of only x: $\alpha(x,y) = a(x)$ for simplicity

$$\frac{\partial}{\partial y}\alpha(x)P(x,y) = \frac{\partial}{\partial x}\alpha(x)Q(x,y)\,,$$

$$\alpha(x)\frac{\partial P(x,y)}{\partial y} = \frac{d\alpha(x)}{dx}Q(x,y) + \alpha(x)\frac{\partial Q(x,y)}{\partial x}$$

$$\alpha(x)[\frac{\partial P(x,y)}{\partial y} - \frac{\partial Q(x,y)}{\partial x}] = \frac{d\alpha(x)}{dx}Q(x,y)$$

or

$$\ln[\alpha(x)] = \int\frac{1}{Q(x,y)}[\frac{\partial P(x,y)}{\partial y} - \frac{\partial Q(x,y)}{\partial x}]dx$$

2.5.2 Example

Solve

$$(4x + 3y^2)dx + 2xydy = 0$$

((Solution))

$P(x,y)=4x+3y^2$, $\quad Q(x,y) = 2xy$

$$\frac{\partial P(x,y)}{\partial y} = 6y\,, \qquad \frac{\partial Q(x,y)}{\partial x} = 2y$$

So the differential equation is not exact. We multiply $\alpha(x)$ on both sides of the original equation.

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$$
$$= \alpha(x)(4x + 3y^2)dx + 2\alpha(x)xydy = 0$$

The condition for the exact differential equation is

$$\frac{\partial}{\partial y}[\alpha(x)(4x+3y^2)] = \frac{\partial}{\partial x}[2\alpha(x)xy]$$

or

$$6y\alpha(x) = 2\alpha'(x)xy + 2y\alpha(x)$$

or

$$\int \frac{1}{\alpha(x)}d\alpha(x) = \int \frac{2}{x}dx \qquad \text{(separation variable)}$$

or

$$\alpha(x) = x^2$$

Then we have

$$\frac{\partial \phi}{\partial x} = \alpha(x)(4x+3y^2) = 4x^3 + 3x^2y^2, \qquad \frac{\partial \phi}{\partial y} = \alpha(x)(2xy) = 2x^3y$$

From the first equation, we get

$$\phi(x,y) = x^4 + x^3y^2 + F(y) = c_1$$

From the second equation, we have

$$2x^3y + F'(y) = 2x^3y$$

or

$$F'(y) = 0$$

Then we have

$$\phi(x, y) = x^4 + x^3y^2 = c$$

((**Mathematica**))

```
Clear["Gobal`"]

Clear[y];

eq2 = DSolve[4 x + 3 y[x]^2 + 2 x y[x] y'[x] == 0, y[x], x]
```

$$\left\{\left\{y[x] \to -\sqrt{-x + \frac{C[1]}{x^3}}\right\}, \left\{y[x] \to \sqrt{-x + \frac{C[1]}{x^3}}\right\}\right\}$$

```
y[x_] = y[x] /. eq2[[1]]
```

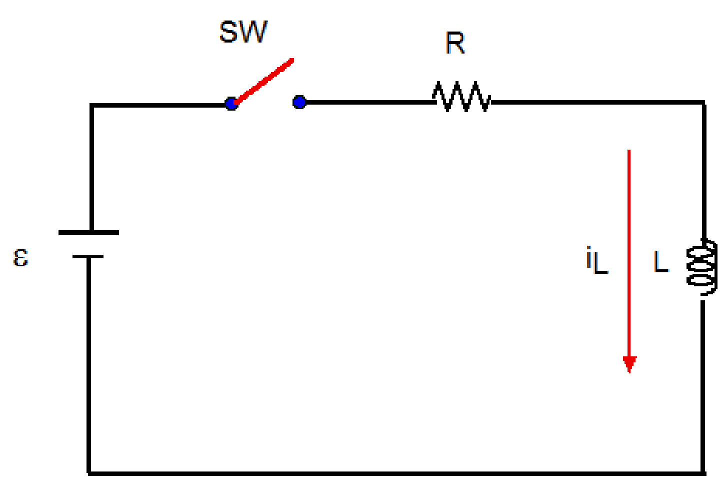
$$-\sqrt{-x + \frac{C[1]}{x^3}}$$

```
x^3 y[x]^2 // Expand
```

$$-x^4 + C[1]$$

2.6 RL circuit

We consider an RL circuit (battery ε-$R-L$ are connected in series);



$$L\frac{dI(t)}{dt}+RI(t)=\varepsilon$$

for t>0, where $I(t = 0) = 0$ as initial condition. We note that $I(t)$ (= $I_L(t)$) is ideal variable for the RL circuit. In other words, $I(t)$ is continuous at $t = 0$. The reason is as follows. The voltage across the inductance L is expressed by

$$V_L(t)=L\frac{dI(t)}{dt}.$$

Then the current flowing through the inductance is given by

$$I(t)=\frac{1}{L}\int V_L(t')dt'.$$

This means that $I(t)$ slowly varies with time t even if $V_L(t)$ drastically changes with time. The current I(t) flowing through the inductance is equal to zero at $t = 0$ and continuously changes with time for t>0.

<u>Step -1</u>.

For $t\to\infty$, $I(t)$ becomes independent of time; $I_0=\frac{\varepsilon}{R}$.

<u>Step-II</u>:

The solution of

$$L\frac{dI_1(t)}{dt}+RI_1(t)=0,$$

is given by

$$I_1(t)=A\exp(-\frac{R}{L}t).$$

Then

$$I(t)=I_1(t)+I_0=A\exp(-\frac{R}{L}t)+\frac{\varepsilon}{R},$$

where A is determined from the initial condition ($I(t = 0) = 0$), $A=-\frac{\varepsilon}{R}$. Therefore we get

$$I(t) = I_0(t) + I_\infty = \varepsilon[1 - \exp(-\frac{R}{L}t)] = I_0[1 - \exp(-\frac{t}{\tau})],$$

where the relaxation time τ is $\tau = \dfrac{L}{R}$.

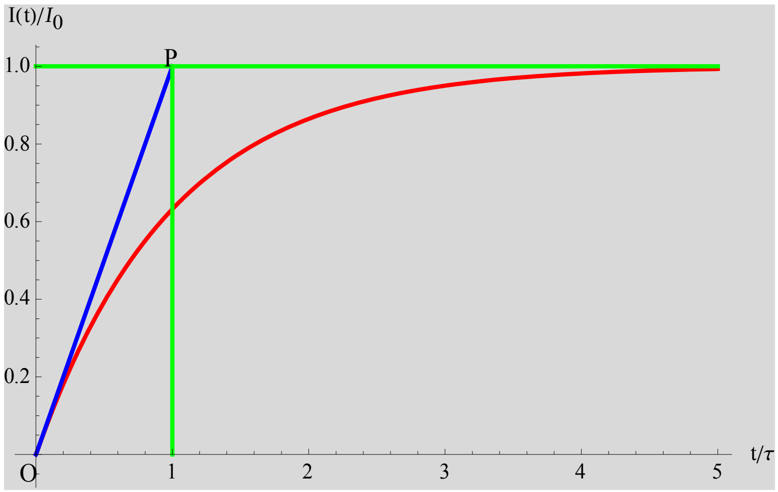


Fig. The blue line is the tangential line at t = 0. At t/τ = 1 (the point P) the tangential line reaches at I/I_0 = 1.

((**Note**)) Another solution

$$L\frac{dI(t)}{dt} + RI(t) = \varepsilon\,. \qquad (1)$$

In the limit of t $t \to \infty$, we have

$$RI_\infty = \varepsilon\,. \qquad (2)$$

Using Eq. (2), Eq.(1) can be rewritten as

$$L\frac{d}{dt}[I(t) - I_\infty] + R[I(t) - I_0] = 0\,.$$

The solution of this equation is given by

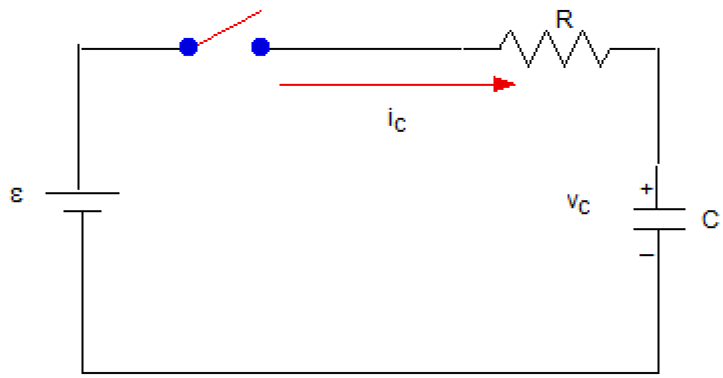
$$I(t) - I_\infty = A\exp(-\frac{R}{L}t)$$

where A is a constant to be determined from the initial condition. When $I(t = 0) = 0$, we have

$$I(t) = I_\infty[1 - \exp(-\frac{R}{L}t)].$$

2.7 RC circuit

We consider an RC circuit (battery ε-$R - C$ are connected in series);



$$V_C(t) + RI_C(t) = \varepsilon,$$

and

$$I_C(t) = C\frac{dV_C(t)}{dt},$$

for t>0, where $V_C(t = 0) = 0$ as initial condition. $I_C(t)$ is the current flowing through the capacitor C. $V_C(t)$ is the voltage across the capacitor C. The differential equation for $V_C(t)$ is given by

$$V_C(t) + RC\frac{dV_C(t)}{dt} = \varepsilon.$$

We note that $V_C(t)$ is ideal variable for the RC circuit. In other words, $V_C(t)$ is continuous at $t = 0$. The reason is that the voltage across the capacitor is given by

$$V_C(t) = \frac{1}{C}\int I_C(t')dt'.$$

This means that $V_C(t)$ slowly varies with time t even if $I_C(t)$ drastically changes with time. The current V_C(t) flowing through the inductance is equal to zero at t = 0 and continuously changes with time for t>0.

<u>Step -1</u>:

For $t \to \infty$, $V_C(t)$ becomes independent of time; ε.

<u>Step-II:</u>

The solution of

$$RC\frac{dV_1(t)}{dt} + V_1(t) = 0\,,$$

is given by

$$V_1(t) = A\exp(-\frac{1}{RC}t)\,.$$

Then

$$V_C(t) = V_1(t) + \varepsilon = A\exp(-\frac{1}{RC}t) + \varepsilon\,.$$

where A is determined from the initial condition ($V_C(t = 0) = 0$), $A = -\varepsilon$. Therefore we get

$$V_C(t) = \varepsilon[1 - \exp(-\frac{t}{\tau})]\,,$$

where the relaxation time τ is $\tau = \frac{1}{RC}$.

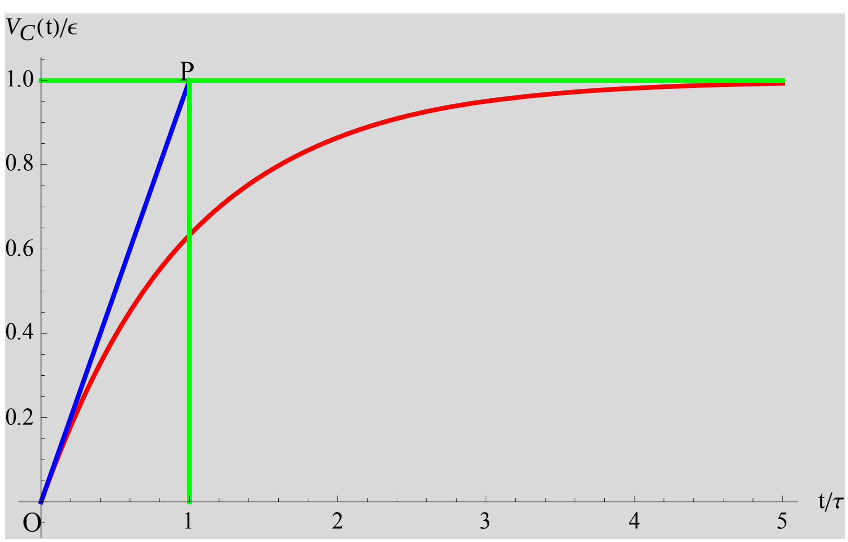


Fig. The blue line is the tangential line at $t = 0$. At $t/\tau = 1$ (the point P) the tangential line reaches at $V_C/\varepsilon = 1$.

((**Note**)) Another solution

$$V_C(t) + RC\frac{dV_C(t)}{dt} = \varepsilon\,. \quad (1)$$

In the limit of t $t \to \infty$, we have

$$V_\infty = \varepsilon\,. \quad (2)$$

Using Eq. (2), Eq.(1) can be rewritten as

$$RC\frac{d}{dt}[V_c(t) - V_\infty] + [V_c(t) - V_0] = 0\,.$$

The solution of this equation is given by

$$V_c(t) - V_\infty = A\exp(-\frac{1}{RC}t)$$

where A is a constant to be determined from the initial condition. When $V_c(t = 0) = 0$, we have

$$V_c(t) = V_\infty[1 - \exp(-\frac{1}{RC}t)]\,.$$

2.8 Example

The solution of

$$\frac{dy}{dx} = -\frac{1}{2yx}(y^2 + \frac{2}{x})$$

is given in the form of $\phi(x, y)$ = constant. We define a field $\boldsymbol{E}$ as

$$\mathbf{E} = (-\frac{\partial\phi}{\partial x}, -\frac{\partial\phi}{\partial y})\,.$$

Note that the differential equation is an exact differential one.

$$(y^2 + \frac{2}{x})dx + 2xy\frac{dy}{dx} = A(x,y)dx + B(x,y)dy = 0$$

where

$$\frac{\partial A(x,y)}{\partial y} = 2y, \qquad \frac{\partial B(x,y)}{\partial x} = 2y.$$

Since

$$\frac{\partial \phi}{\partial x} = A(x,y) = y^2 + \frac{2}{x}, \qquad \frac{\partial \phi}{\partial y} = B(x,y) = 2xy$$

we have

$$\phi(x,y) = xy^2 + 2\ln x + F(y), \quad \frac{\partial \phi(x,y)}{\partial y} = 2xy + F'(y) = 2xy$$

Then we have

$$F'(y) = 0, \qquad F(y) = c_1 \text{ (constant)}$$

Finally we get the form of ϕ as

$$\phi(x,y) = xy^2 + 2\ln x + c_1 = c_2$$

or

$$xy^2 + 2\ln x = c = c_2 - c_1$$

Here we show the solution of the above equation using the Mathematica.
((**Mathematica**))

Solve

$$y'[x] + \frac{1}{2\,y[x]\,x}\left(y[x]^2 + \frac{2}{x}\right) = 0$$

```
Clear["Gobal`"];

Clear[y];

eq1 = DSolve[y'[x] + 1/(2 y[x] x) (y[x]^2 + 2/x) == 0, y[x], x];

y[x_] = y[x] /. eq1[[2]]
```

$$\sqrt{\frac{C[1]}{x} - \frac{2\,\text{Log}[x]}{x}}$$

```
y[x]^2 x + 2 Log[x] // Simplify
```

C[1]

```
Clear[y]

ϕ = y^2 x + 2 Log[x];
f1 = ContourPlot[Evaluate[Table[ϕ == α, {α, -5, 5, 0.5}]],
   {x, 0.1, 5}, {y, -5, 5},
   ContourStyle → Table[{Hue[0.05 i], Thick},
     {i, 0, 20}]];
```

```
eq1 = {-D[ϕ, x], -D[ϕ, y]};

f2 = StreamPlot[eq1, {x, 0.1, 5}, {y, -5, 5}];

Show[f1, f2]
```

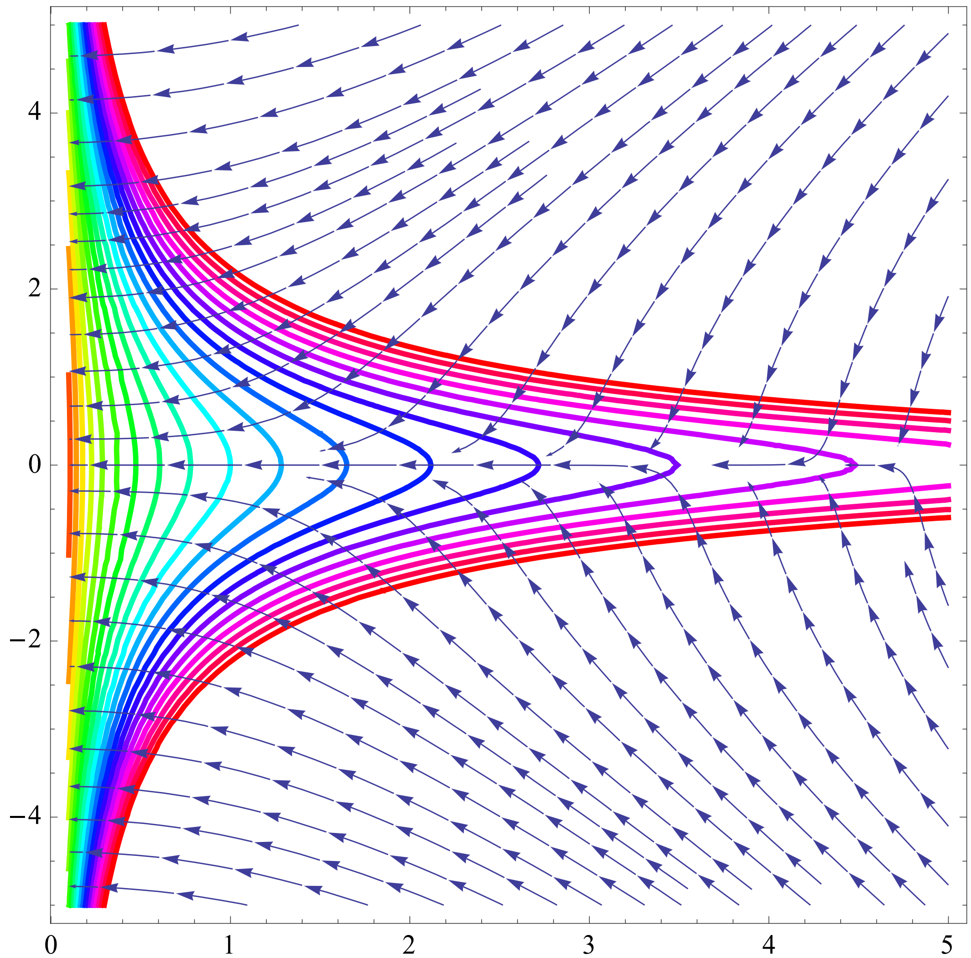


Fig. The contour plot of $\phi(x, y) = xy^2 + 2\ln x$=const, and the field lines of $\mathbf{E} = (-\frac{\partial\phi}{\partial x}, -\frac{\partial\phi}{\partial y})$ in the (x, y) plane.