# Chapter 31 Identical particles <br> Masatsugu Sei Suzuki <br> Department of Physics, SUNY at Binghamton 

(Date November 28, 2010)
Boson
Fermion
Young's Tableaux
Dirac spin exchange operator
Permutaion operator
Symmetrizer
Antisymmetrer
Pauli exclusion principle
Slater determinant
Triplet
Singlet
Spin wave

### 31.1 System of particles 1 and 2

$$
\begin{aligned}
& \left|k^{\prime}\right\rangle,\left|k^{\prime \prime}\right\rangle \\
& \left|\psi_{a}\right\rangle=\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2} \quad\left|\psi_{b}\right\rangle=\left|k^{\prime \prime}\right\rangle_{1}\left|k^{\prime}\right\rangle_{2}
\end{aligned}
$$

Even though the two particles are indistinguishable, mathematically $\left|\psi_{a}\right\rangle$ and $\left|\psi_{b}\right\rangle$ are distinct kets for $\left|k^{\prime}\right\rangle \neq\left|k^{\prime \prime}\right\rangle$. In fact we have $\left\langle\psi_{a} \mid \psi_{b}\right\rangle=0$. Suppose we make a measurement on the two particle system.
$\left|k^{\prime}\right\rangle$ : one particle and $\left|k^{\prime \prime}\right\rangle$ : the other particle.

We do not know a priori whether the state ket is $\left|\psi_{a}\right\rangle=\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2}$ or $\left|\psi_{b}\right\rangle=\left|k^{\prime \prime}\right\rangle_{1}\left|k^{\prime}\right\rangle_{2}$ or for that matter- any linear combination of the two: $c_{a}\left|\psi_{a}\right\rangle+c_{b}\left|\psi_{b}\right\rangle$.

### 31.2 Exchange degeneracy

A specification of the eigenvalue of a complete set of observables does not completely determine the state ket.

Mathematics of permutation symmetry:

$$
\hat{P}_{12}\left|\psi_{a}\right\rangle=\hat{P}_{12}\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2}=\left|k^{\prime \prime}\right\rangle_{1}\left|k^{\prime}\right\rangle_{2}=\left|\psi_{b}\right\rangle .
$$

## Clearly

$$
\hat{P}_{21}=\hat{P}_{12}, \quad \text { and } \hat{P}_{12}^{2}=1
$$

Under $\hat{P}_{12}$, particle 1 having $\left|k^{\prime}\right\rangle$ becomes particle 1 having $\left|k^{\prime \prime}\right\rangle$; particle 2 having $\left|k^{\prime \prime}\right\rangle$ becomes particle 1 having $\left|k^{\prime}\right\rangle$. In other words, it has the effect of interchanging 1 and 2.
((Note))
Matrix element of $\hat{P}_{21}=\hat{P}_{12}$ in terms of $\left|\psi_{a}\right\rangle=\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2}$ and $\left|\psi_{b}\right\rangle=\left|k^{\prime \prime}\right\rangle_{1}\left|k^{\prime}\right\rangle_{2}$

$$
\begin{aligned}
& \hat{P}_{12}\left|\psi_{a}\right\rangle=\hat{P}_{12}\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2}=\left|k^{\prime \prime}\right\rangle_{1}\left|k^{\prime}\right\rangle_{2}=\left|\psi_{b}\right\rangle \\
& \hat{P}_{12}\left|\psi_{b}\right\rangle=\hat{P}_{12}\left|k^{\prime \prime}\right\rangle_{1}\left|k^{\prime}\right\rangle_{2}=\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2}=\left|\psi_{a}\right\rangle
\end{aligned}
$$

Matrix element of $\hat{P}_{21}=\hat{P}_{12}$

$$
\hat{P}_{12}=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)
$$

or

$$
\begin{array}{c|cc} 
& \left|\psi_{a}\right\rangle & \left|\psi_{b}\right\rangle \\
\left\langle\psi_{a}\right| & 0 & 1 \\
\left\langle\psi_{b}\right| & 1 & 0
\end{array}
$$

Eigensystem $\left[\hat{P}_{12}\right]$
$\lambda=1$ (symmetric):

$$
\begin{aligned}
& P_{12}\left|\psi_{\text {symm }}\right\rangle=\left|\psi_{\text {symm }}\right\rangle \\
& \left|\psi_{\text {symm }}\right\rangle=\frac{1}{\sqrt{2}}\left(\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2}+\left|k^{\prime \prime}\right\rangle_{1}\left|k^{\prime}\right\rangle_{2}\right)
\end{aligned}
$$

$\lambda=-1$ (antisymmetric)

$$
\begin{aligned}
& P_{12}\left|\psi_{\text {anti }}\right\rangle=-\left|\psi_{\text {anti }}\right\rangle \\
& \left|\psi_{\text {anti }}\right\rangle=\frac{1}{\sqrt{2}}\left(\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2}-\left|k^{\prime \prime}\right\rangle_{1}\left|k^{\prime}\right\rangle_{2}\right)
\end{aligned}
$$

Our consideration cane be extended to a system made up of many identical particles. A transposition is a permutation which simply exchange the role of two of the particles, without touching others.

$$
\hat{P}_{i j}\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2} \ldots . .\left|k^{i}\right\rangle_{i}\left|k^{i+1}\right\rangle_{i+1} \ldots . .\left|k^{j}\right\rangle_{j} \ldots . .=\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2} \ldots . .\left|k^{j}\right\rangle_{i}\left|k^{i+1}\right\rangle_{i+1} \ldots . .\left|k^{i}\right\rangle_{j} \ldots . .
$$

The transposition operators $\hat{P}_{i j}$ are Hermitian $\left(\hat{P}_{i j}^{+}=\hat{P}_{i j}\right)$

$$
\hat{P}_{i j}{ }^{2}=1
$$

So that $\hat{P}_{i j}$ is Unitary operator. The allowed eigenvalues of $\hat{P}_{i j}$ are $\pm 1$. It is important to note, however, that in general

$$
\left[\hat{P}_{i j}, \hat{P}_{k l}\right] \neq 0
$$

Now we consider a permutation operator $\hat{P}_{123}$ for

$$
\hat{P}_{123}\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2}\left|k^{\prime \prime \prime}\right\rangle_{3}=\left|k^{\prime}\right\rangle_{2}\left|k^{\prime \prime}\right\rangle_{3}\left|k^{\prime \prime \prime}\right\rangle_{1}=\left|k^{\prime \prime}\right\rangle_{1}\left|k^{\prime}\right\rangle_{2}\left|k^{\prime \prime}\right\rangle_{3}
$$

for the system of 3 identical particles $\left(\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2}\left|k^{\prime \prime \prime}\right\rangle_{3}\right)$

$$
P_{123}=\left(\begin{array}{lll}
1 & 2 & 3 \\
2 & 3 & 1
\end{array}\right)
$$

This means replacement of $1 \rightarrow 2,2 \rightarrow 3,3 \rightarrow 1$.

$$
P_{123}=\left(\begin{array}{lll}
1 & 2 & 3 \\
2 & 3 & 1
\end{array}\right)=\left(\begin{array}{lll}
1 & 2 & 3 \\
2 & 1 & 3
\end{array}\right)\left(\begin{array}{lll}
2 & 1 & 3 \\
2 & 3 & 1
\end{array}\right)=P_{12} P_{13}
$$

Quantum mechanically this is not correct. The correct one is

$$
\hat{P}_{123}=\hat{P}_{13} \hat{P}_{12}
$$

Similarly,

$$
P_{123}=\left(\begin{array}{lll}
1 & 2 & 3 \\
2 & 3 & 1
\end{array}\right)=\left(\begin{array}{lll}
1 & 2 & 3 \\
1 & 3 & 2
\end{array}\right)\left(\begin{array}{lll}
1 & 3 & 2 \\
2 & 3 & 1
\end{array}\right)=P_{23} P_{12}
$$

or quantum mechanically

$$
\begin{aligned}
& \hat{P}_{123}=\hat{P}_{12} \hat{P}_{13} \\
& P_{132}=\left(\begin{array}{lll}
1 & 2 & 3 \\
3 & 1 & 2
\end{array}\right)=\left(\begin{array}{lll}
1 & 2 & 3 \\
3 & 2 & 1
\end{array}\right)\left(\begin{array}{lll}
3 & 2 & 1 \\
3 & 1 & 2
\end{array}\right)=P_{13} P_{12}
\end{aligned}
$$

or quantum mechanically

$$
\hat{P}_{132}=\hat{P}_{12} \hat{P}_{13}
$$

## ((Proof))

$$
\hat{P}_{12} \hat{P}_{13}\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2}\left|k^{\prime \prime \prime}\right\rangle_{3}=\hat{P}_{12}\left|k^{\prime \prime \prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2}\left|k^{\prime}\right\rangle_{3}=\left|k^{\prime \prime}\right\rangle_{1}\left|k^{\prime \prime \prime}\right\rangle_{2}\left|k^{\prime}\right\rangle_{3}
$$

and

$$
\hat{P}_{132}\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2}\left|k^{\prime \prime \prime}\right\rangle_{3}=\left|k^{\prime}\right\rangle_{3}\left|k^{\prime \prime}\right\rangle_{1}\left|k^{\prime \prime \prime}\right\rangle_{2}=\left|k^{\prime \prime}\right\rangle_{1}\left|k^{\prime \prime \prime}\right\rangle_{2}\left|k^{\prime}\right\rangle_{3}
$$

Therefore we have $\hat{P}_{132}=\hat{P}_{12} \hat{P}_{13}$.
Any permutation operators can be broken into a product of transposition operators.

$$
\hat{P}_{132}=\hat{P}_{23} \hat{P}_{12}=\hat{P}_{13} \hat{P}_{23}=\hat{P}_{12} \hat{P}_{13}=\hat{P}_{23} \hat{P}_{12} \hat{P}_{23}{ }^{2}=\ldots
$$

The decomposition is not unique. However, for a given permutation, it can be shown that the parity of the number of transposition into which it can be broken down is always the same: it is called the parity of the permutation.

$$
\begin{array}{ll}
\hat{P}_{132}=\hat{P}_{23} \hat{P}_{12}=\hat{P}_{13} \hat{P}_{23}=\hat{P}_{12} \hat{P}_{13}=\hat{P}_{23} \hat{P}_{12} \hat{P}_{23}{ }^{2}=\ldots & \text { even parity } \\
\hat{P}_{123}=\hat{P}_{12} \hat{P}_{23}=\hat{P}_{13} \hat{P}_{12}: & \text { even parity } \\
\hat{P}_{23} & \text { odd parity } \\
\hat{P}_{12} & \text { odd parity } \\
\hat{P}_{31} & \text { odd parity }
\end{array}
$$

((Note))

$$
\hat{P}_{132}=\hat{P}_{321}=\hat{P}_{213}
$$

### 31.3 Symmetrizer and antisymmetrizer

Now we consider the two Hermitian operators

$$
\begin{aligned}
& \hat{S}=\frac{1}{N!} \sum_{\alpha} \hat{P}_{\alpha}: \text { symmetrizer } \\
& \hat{A}=\frac{1}{N!} \sum_{\alpha} \varepsilon_{\alpha} \hat{P}_{\alpha}: \text { antisymmetrizer }
\end{aligned}
$$

where $\varepsilon_{\alpha}=1$ if $\hat{P}_{\alpha}$ is an even permutation and $\varepsilon_{\alpha}=-1$ if $\hat{P}_{\alpha}$ is an odd permutation.

$$
\hat{S}^{+}=\hat{S} \text { and } \hat{A}^{+}=\hat{A} .
$$

## ((Theorem-1))

If $\hat{P}_{\alpha 0}$ is an arbitrary permutation operator, we have

$$
\begin{align*}
& \hat{P}_{\alpha 0} \hat{S}=\hat{S} \hat{P}_{\alpha 0}=\hat{S} \\
& \hat{P}_{\alpha 0} \hat{A}=\hat{A} \hat{P}_{\alpha 0}=\varepsilon_{\alpha 0} A \tag{1}
\end{align*}
$$

## ((Proof))

This is due to the fact that

$$
\hat{P}_{\alpha 0} \hat{P}_{\alpha}=\hat{P}_{\beta}
$$

such that

$$
\varepsilon_{\beta}=\varepsilon_{\alpha_{0}} \varepsilon_{\alpha}
$$

or

$$
\begin{aligned}
& \varepsilon_{\alpha_{0}} \varepsilon_{\beta}=\varepsilon_{\alpha_{0}}{ }^{2} \varepsilon_{\alpha}=\varepsilon_{\alpha} \\
& \hat{P}_{\alpha 0} \hat{S}=\frac{1}{N!} \sum_{\alpha} \hat{P}_{\alpha 0} \hat{P}_{\alpha}=\frac{1}{N!} \sum_{\alpha} \hat{P}_{\beta}=\hat{S} \\
& \hat{P}_{\alpha 0} \hat{A}=\frac{1}{N!} \sum_{\alpha} \varepsilon_{\alpha} \hat{P}_{\alpha 0} \hat{P}_{\alpha}=\frac{1}{N!} \sum_{\alpha} \varepsilon_{\alpha 0} \varepsilon_{\beta} \hat{P}_{\beta}=\varepsilon_{\alpha 0} \hat{A}
\end{aligned}
$$

From Eq.(1) we see the following theorem

## ((Theorem-2))

$$
\hat{S}^{2}=\hat{S}
$$

$$
\hat{A}^{2}=\hat{A}
$$

and

$$
\hat{A} \hat{S}=\hat{S} \hat{A}=0
$$

## ((Proof))

$$
\begin{aligned}
& \hat{S}^{2}=\frac{1}{N!} \sum_{\alpha} \hat{P}_{\alpha} \hat{S}=\frac{1}{N!} \sum_{\alpha} \hat{S}=\hat{S} \\
& \hat{A}^{2}=\frac{1}{N!} \sum_{\alpha} \varepsilon_{\alpha} \hat{P}_{\alpha} \hat{A}=\frac{1}{N!} \sum_{\alpha} \varepsilon_{\alpha}{ }^{2} \hat{A}=\hat{A} \\
& \hat{A} \hat{S}=\hat{A}=\frac{1}{N!} \sum_{\alpha} \varepsilon_{\alpha} \hat{P} \hat{S}=\frac{1}{N!} \hat{S} \sum_{\alpha} \varepsilon_{\alpha}=0
\end{aligned}
$$

since half the $\varepsilon_{\alpha}$ are equal to 1 and half the $\varepsilon_{\alpha}$ equal to $-1 . \hat{S}$ and $\hat{A}$ are therefore projectors. Their action on any ket $|\psi\rangle$ of the state space yields a completely symmetric or completely antisymmetric ket.

$$
\begin{aligned}
& \hat{P}_{\alpha 0} \hat{S}|\psi\rangle=\hat{S}|\psi\rangle \\
& \hat{P}_{\alpha 0} \hat{A}|\psi\rangle=\varepsilon_{\alpha_{0}} \hat{A}|\psi\rangle \\
& \left|\psi_{S}\right\rangle=\hat{S}|\psi\rangle \\
& \left|\psi_{A}\right\rangle=\hat{A}|\psi\rangle
\end{aligned}
$$

((Example))
For $N=3$,

$$
\hat{S}=\frac{1}{6}\left[\hat{1}+\hat{P}_{12}+\hat{P}_{23}+\hat{P}_{31}+\hat{P}_{123}+\hat{P}_{132}\right]
$$

and

$$
\hat{A}==\frac{1}{6}\left[\hat{1}-\hat{P}_{12}-\hat{P}_{23}-\hat{P}_{31}+\hat{P}_{123}+\hat{P}_{132}\right]
$$

where

$$
\begin{aligned}
& \hat{P}_{123}=\hat{P}_{12} \hat{P}_{23}, \quad \hat{P}_{132}=\hat{P}_{12} \hat{P}_{13} \\
& \hat{S}+\hat{A}=\frac{1}{3}\left(\hat{1}+\hat{P}_{123}+\hat{P}_{132}\right) \neq \hat{1}
\end{aligned}
$$

### 31.4 Symmetrization postulate

The system containing $N$ identical particles are either totally symmetrical under the interchange of any pair (boson), or totally antisymmetrical under the interchange of any pair (fermion).

$$
\begin{aligned}
& \hat{P}_{i j}\left|\psi_{N, B}\right\rangle=\left|\psi_{N, B}\right\rangle, \\
& \hat{P}_{i j}\left|\psi_{N, F}\right\rangle=-\left|\psi_{N, F}\right\rangle,
\end{aligned}
$$

where $\left|\psi_{N, B}\right\rangle$ is the eigenket of $N$ identical boson systems and $\left|\psi_{N, F}\right\rangle$ is the eigenket of $N$ identical fermion systems.
((Note)) It is an empirical fact that a mixed symmetry does not occur.
Even more remarkable is that there is a connection between the spin of a particle and the statistics obeyed by it:

Half-integer spin particles are fermion, while integer-spin particles are bosons.

### 31.5 Pauli exclusion principle

Wolfgang Ernst Pauli (April 25, 1900 - December 15, 1958) was an Austrian theoretical physicist and one of the pioneers of quantum physics. In 1945, after being nominated by Albert Einstein, he received the Nobel Prize in Physics for his "decisive contribution through his discovery of a new law of Nature, the exclusion principle or Pauli principle," involving spin theory, underpinning the structure of matter and the whole of chemistry.

http://en.wikipedia.org/wiki/Wolfgang_Pauli
Electron is a fermion. No two electrons can occupy the same state. We discuss the framatic difference between fermions and bosons. Let us consider two particles. Each of which can occupy only two states $\left|k^{\prime}\right\rangle$ and $\left|k^{\prime \prime}\right\rangle$.

For a system of two fermions, we have no choice

$$
\frac{1}{\sqrt{2}}\left(\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2}-\left|k^{\prime \prime}\right\rangle_{1}\left|k^{\prime}\right\rangle_{2}\right) .
$$

For bosons, there are three states.

$$
\begin{aligned}
& \left|k^{\prime}\right\rangle_{1}\left|k^{\prime}\right\rangle_{2} \\
& \left|k^{\prime \prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2} \\
& \frac{1}{\sqrt{2}}\left(\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2}+\left|k^{\prime \prime}\right\rangle_{1}\left|k^{\prime}\right\rangle_{2}\right) .
\end{aligned}
$$

In contrast, for "classical particles" satisfying Maxwell-Boltzmann (M-B) statitics with no restriction on symmetry, we have altogether four independentstates.

$$
\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2},\left|k^{\prime \prime}\right\rangle_{1}\left|k^{\prime}\right\rangle_{2},\left|k^{\prime}\right\rangle_{1}\left|k^{\prime}\right\rangle_{2} \text { and }\left|k^{\prime \prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2}
$$

We see that in the fermion case, it is impossible for both particles to occupy the same state.

### 31.6 Transformation of observables by permutation

For simplicity, we consider a specific case where the two particle state ket is completely specified by the eigenvalues of a single observable $\hat{A}$ for each of the particle.

$$
\hat{A}_{1}\left|a^{\prime}\right\rangle_{1}\left|a^{\prime \prime}\right\rangle_{2}=a^{\prime}\left|a^{\prime}\right\rangle_{1}\left|a^{\prime \prime}\right\rangle_{2}
$$

and

$$
\hat{A}_{2}\left|a^{\prime}\right\rangle_{1}\left|a^{\prime \prime}\right\rangle_{2}=a^{\prime \prime}\left|a^{\prime}\right\rangle_{1}\left|a^{\prime \prime}\right\rangle_{2}
$$

Since

$$
\begin{aligned}
\hat{P}_{12} \hat{A}_{1} \hat{P}_{12}^{-1} \hat{P}_{12}\left|a^{\prime}\right\rangle_{1}\left|a^{\prime \prime}\right\rangle_{2} & =\hat{P}_{12} \hat{A}_{1} \hat{P}_{12}^{-1}\left|a^{\prime \prime}\right\rangle_{1}\left|a^{\prime}\right\rangle_{2} \\
\hat{P}_{12} \hat{A}_{1} \hat{P}_{12}^{-1} \hat{P}_{12}\left|a^{\prime}\right\rangle_{1}\left|a^{\prime \prime}\right\rangle_{2} & =\hat{P}_{12} \hat{A}_{1}\left|a^{\prime}\right\rangle_{1}\left|a^{\prime \prime}\right\rangle_{2} \\
& =a^{\prime} \hat{P}_{12}\left|a^{\prime}\right\rangle_{1}\left|a^{\prime \prime}\right\rangle_{2} \\
& =a^{\prime}\left|a^{\prime \prime}\right\rangle_{1}\left|a^{\prime}\right\rangle_{2}=\hat{A}_{2}\left|a^{\prime \prime}\right\rangle_{1}\left|a^{\prime}\right\rangle_{2}=\hat{A}_{2} \hat{P}_{12}\left|a^{\prime}\right\rangle_{1}\left|a^{\prime \prime}\right\rangle_{1}
\end{aligned}
$$

we obtain

$$
\hat{P}_{12} \hat{A}_{1} \hat{P}_{12}^{-1}=\hat{A}_{2}
$$

Similarly,

$$
\hat{P}_{21} \hat{A}_{2} \hat{P}_{21}^{-1}=\hat{A}_{1}
$$

It follows that $\hat{P}_{12}$ must change the particle label of observables.
There are also observables, such as $\hat{A}_{1}+\hat{B}_{2}, \hat{A}_{1} \hat{B}_{2}$, which involve both indices simultaneously.

$$
\begin{aligned}
& \hat{P}_{12}\left(\hat{A}_{1}+\hat{B}_{2}\right) \hat{P}_{12}^{-1}=\hat{A}_{2}+\hat{B}_{1} \\
& \hat{P}_{12} \hat{A}_{1} \hat{B}_{2} \hat{P}_{12}^{-1}=\hat{P}_{12} \hat{A}_{1} \hat{P}_{12}^{-1} \hat{P}_{12} \hat{B}_{2} \hat{P}_{12}^{-1}=\hat{A}_{2} \hat{B}_{1}
\end{aligned}
$$

These results can be generalized to all observables which can be expressed in terms of observables which can be expressed in terms of observables of the type of $\hat{A}_{1}$ and $\hat{B}_{2}$, to be denoted by $\hat{O}(1,2)$.

$$
\hat{P}_{12} \hat{O}(1,2) \hat{P}_{12}^{-1}=\hat{O}(2,1)
$$

where $\hat{O}(2,1)$ is the observable obtained from $\hat{O}(1,2)$ by exchanging indices 1 and 2 throughout.
$\hat{O}_{s}(1,2)$ is said to be symmetric if

$$
\hat{O}_{s}(1,2)=\hat{O}_{s}(2,1)
$$

or

$$
\left[\hat{O}_{s}(1,2), \hat{P}_{12}\right]=0
$$

Symbolic observables commute with the permutation operator.
In general. the observables $\hat{O}_{s}(1,2,3, \ldots, N)$ which are completely symmetric under exchange of indices $1,2, \ldots, N$ commute with all the transposition operators, and with all the permutation operators

### 31.7 Example

Let us now consider a Hamiltonian of a system of two identical particles.

$$
\hat{H}=\frac{1}{2 m} \hat{\mathbf{p}}_{1}^{2}+\frac{1}{2 m} \hat{\mathbf{p}}_{2}^{2}+V_{\text {pair }}\left(\left|\hat{\mathbf{r}}_{1}-\hat{\mathbf{r}}_{2}\right|\right)+V_{\text {ext }}\left(\hat{\mathbf{r}}_{1}\right)+V_{\text {ext }}\left(\hat{\mathbf{r}}_{2}\right)
$$

Clearly we have

$$
\hat{P}_{12} \hat{H} \hat{P}_{12}^{-1}=\hat{H}
$$

or

$$
\left[\hat{P}_{12}, \hat{H}\right]=0
$$

$\hat{P}_{12}$ is a constant of the motion. Since $\hat{P}_{12}{ }^{2}=1$, the eigenvalue of $\hat{P}_{12}$ allowed are $\pm 1$.

$$
\begin{aligned}
& \hat{H}|\psi\rangle=E|\psi\rangle \\
& \hat{P}_{12}|\psi\rangle=\lambda|\psi\rangle \\
& \hat{P}_{12}{ }^{2}|\psi\rangle=\lambda \hat{P}_{12}|\psi\rangle=\lambda^{2}|\psi\rangle=|\psi\rangle
\end{aligned}
$$

or

$$
\lambda= \pm 1 .
$$

It therefore follows that if the two-particle state ket is symmteric (antisymmettric) to start with, it remains so at all times.
(i) $\quad N=2$ case

We can define the symmetrizer and antisymmetrtizer as follows.

$$
\begin{aligned}
& \hat{S}=\frac{1}{2}\left(1+\hat{P}_{12}\right) \quad \hat{A}=\frac{1}{2}\left(1-\hat{P}_{12}\right) \\
& \hat{S}+\hat{A}=\hat{1} \\
& \hat{S}^{2}=\frac{1}{2}\left(1+\hat{P}_{12}\right) \frac{1}{2}\left(1+\hat{P}_{12}\right)=\frac{1}{4}\left(1+2 \hat{P}_{12}+1\right)=\frac{1}{1}\left(1+\hat{P}_{12}\right)=\hat{S} \\
& \hat{A}^{2}=\frac{1}{2}\left(1-\hat{P}_{12}\right) \frac{1}{2}\left(1-\hat{P}_{12}\right)=\frac{1}{4}\left(1-2 \hat{P}_{12}+1\right)=\frac{1}{2}\left(1-\hat{P}_{12}\right)=\hat{A} \\
& \left|\psi_{S}\right\rangle=\frac{1}{\sqrt{2}}\left(\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2}+\left|k^{\prime \prime}\right\rangle_{1}\left|k^{\prime}\right\rangle_{2}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& \left|\psi_{A}\right\rangle=\frac{1}{\sqrt{2}}\left(\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2}-\left|k^{\prime \prime}\right\rangle_{1}\left|k^{\prime}\right\rangle_{2}\right) \\
& \hat{S}\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2}=\frac{1}{2}\left(1+\hat{P}_{12}\right)\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2}=\frac{1}{2}\left(\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2}+\left|k^{\prime \prime}\right\rangle_{1}\left|k^{\prime}\right\rangle_{2}\right) \\
& \hat{A}\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2}=\frac{1}{2}\left(1-\hat{P}_{12}\right)\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2}=\frac{1}{2}\left(\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2}-\left|k^{\prime \prime}\right\rangle_{1}\left|k^{\prime}\right\rangle_{2}\right)
\end{aligned}
$$

(ii) $\quad N=3$ Cases

$$
\begin{aligned}
& \hat{S}=\frac{1}{6}\left(1+\hat{P}_{12}+\hat{P}_{23}+\hat{P}_{31}+\hat{P}_{123}+\hat{P}_{132}\right) \\
& \hat{A}=\frac{1}{6}\left(1-\hat{P}_{12}-\hat{P}_{23}-\hat{P}_{31}+\hat{P}_{123}+\hat{P}_{132}\right) \\
& \hat{S}+\hat{A}=\frac{1}{3}\left(1+\hat{P}_{123}+\hat{P}_{132}\right) \neq 1
\end{aligned}
$$

$$
\hat{A}\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2}\left|k^{\prime \prime \prime}\right\rangle_{3}=\frac{1}{3!}\left|\begin{array}{lll}
\left.k^{\prime}\right\rangle_{1} & \left|k^{\prime \prime}\right\rangle_{1} & \left|k^{\prime \prime \prime}\right\rangle_{1} \\
\left|k^{\prime}\right\rangle_{2} & \left|k^{\prime \prime}\right\rangle_{2} & \left|k^{\prime \prime \prime}\right\rangle_{2} \\
\left|k^{\prime}\right\rangle_{3} & \left|k^{\prime \prime}\right\rangle_{3} & \left|k^{\prime \prime \prime}\right\rangle_{3}
\end{array}\right|
$$

## Slater determinant

$\hat{A}\left|k^{\prime}\right\rangle_{1}\left|k^{\prime \prime}\right\rangle_{2}\left|k^{\prime \prime \prime}\right\rangle_{3}$ is zero if two of individual states coincide. We obtain Pauli's exclusion principle.

### 31.8 Method developed by Tomonaga

Sin-Itiro Tomonaga or Shin'ichirō Tomonaga (Tomonaga Shin'ichirō, March 31, 1906 - July 8, 1979) was a Japanese physicist, influential in the development of quantum electrodynamics, work for which he was jointly awarded the Nobel Prize in Physics in 1965 along with Richard Feynman and Julian Schwinger.

http://en.wikipedia.org/wiki/Sin-Itiro_Tomonaga

We now consider a system consisting of many spins.

$$
\begin{aligned}
& \hat{\mathbf{S}}=\hat{\mathbf{S}}_{1}+\hat{\mathbf{S}}_{2}+\hat{\mathbf{S}}_{3}+\ldots+\hat{\mathbf{S}}_{N} \\
& \hat{\mathbf{S}} \cdot \hat{\mathbf{S}}=\left(\hat{\mathbf{S}}_{1}+\hat{\mathbf{S}}_{2}+\hat{\mathbf{S}}_{3}+\ldots+\hat{\mathbf{S}}_{N}\right) \cdot\left(\hat{\mathbf{S}}_{1}+\hat{\mathbf{S}}_{2}+\hat{\mathbf{S}}_{3}+\ldots+\hat{\mathbf{S}}_{N}\right)
\end{aligned}
$$

or

$$
\hat{\mathbf{S}}^{2}=\sum_{n=1}^{N} \hat{\mathbf{S}}_{n}{ }^{2}+2 \sum_{n<n^{\prime}}\left(\hat{\mathbf{S}}_{n} \cdot \hat{\mathbf{S}}_{n^{\prime}}\right)=\frac{\hbar^{2}}{4} \sum_{n=1}^{N} \hat{\boldsymbol{\sigma}}_{n}{ }^{2}+\frac{1}{2} \hbar^{2} \sum_{n<n^{\prime}}\left(\hat{\boldsymbol{\sigma}}_{n} \cdot \hat{\boldsymbol{\sigma}}_{n^{\prime}}\right)
$$

or

$$
\frac{\hat{\mathbf{S}}^{2}}{\hbar^{2}}=\frac{3 N}{4}+\frac{1}{2} \sum_{n<n^{\prime}}\left(\hat{\boldsymbol{\sigma}}_{n} \cdot \hat{\boldsymbol{\sigma}}_{n^{\prime}}\right)
$$

Here we define an operator

$$
\hat{O}=\frac{2}{N(N-1)} \sum_{n<n^{\prime}} \hat{P}_{n n^{\prime}}
$$

$\hat{O}$ is Hermitian and $[\hat{P}, \hat{O}]=0$. We assume that

$$
\begin{aligned}
& \hat{P}_{n n^{\prime}}=\frac{1}{2}\left(1+\hat{\boldsymbol{\sigma}}_{n} \cdot \hat{\boldsymbol{\sigma}}_{n^{\prime}}\right) \quad \text { (Dirac exchange interaction) } \\
& \hat{O}=\frac{2}{N(N-1)} \sum_{n<n^{\prime}} \frac{1}{2}\left(1+\hat{\boldsymbol{\sigma}}_{n} \cdot \hat{\boldsymbol{\sigma}}_{n^{\prime}}\right)=\frac{2}{N(N-1)}\left[\frac{1}{2} \frac{N(N-1)}{2}+\frac{1}{2} \sum_{n<n^{\prime}} \hat{\boldsymbol{\sigma}}_{n} \cdot \hat{\boldsymbol{\sigma}}_{n^{\prime}}\right]
\end{aligned}
$$

or

$$
\hat{O}=\frac{1}{2}\left[1+\frac{2}{N(N-1)} \sum_{n<n^{\prime}} \hat{\boldsymbol{\sigma}}_{n} \cdot \hat{\boldsymbol{\sigma}}_{n^{\prime}}\right]
$$

Using the relation,

$$
\frac{\hat{\mathbf{S}}^{2}}{\hbar^{2}}=\frac{3 N}{4}+\frac{1}{2} \sum_{n<n^{\prime}}\left(\hat{\boldsymbol{\sigma}}_{n} \cdot \hat{\boldsymbol{\sigma}}_{n^{\prime}}\right)
$$

we get

$$
\hat{O}=\frac{1}{2}\left[1+\frac{2}{N(N-1)}\left(\frac{2}{\hbar^{2}} \hat{\mathbf{S}}^{2}-\frac{3 N}{2}\right)\right]=\frac{1}{2}\left[\frac{N-4}{(N-1)}+\frac{4}{N(N-1)} \frac{1}{\hbar^{2}} \hat{\mathbf{S}}^{2}\right]
$$

$\left[\hat{\mathbf{S}}^{2}, \hat{O}\right]=0$. When the eigenvalue of $\hat{\mathbf{S}}^{2}$ is given by $\hbar^{2} S(S+1)$, the eigenvalue of $\hat{O}$ is equal to

$$
\chi=\frac{1}{2}\left[\frac{N-4}{(N-1)}+\frac{4 S(S+1)}{N(N-1)}\right]
$$

The eigenvalue of $\hat{O}(\chi)$ specifies the symmetry.
(i) $\operatorname{For} N=2$,

$$
\begin{aligned}
& \mathrm{D}_{1 / 2} \times \mathrm{D}_{1 / 2}=\mathrm{D}_{1}+\mathrm{D}_{0} \\
& \chi=\frac{1}{2}[-2+2 S(S+1)]=-1+S(S+1)
\end{aligned}
$$

When $S=1, \chi=1$ (symmetric).
When $S=0, \chi=-1$ (anti-symmetric).
(ii) $\operatorname{For} N=3$,

$$
\begin{aligned}
& \mathrm{D}_{1 / 2} \times \mathrm{D}_{1 / 2} \times \mathrm{D}_{1 / 2}=\mathrm{D}_{3 / 2}+2 \mathrm{D}_{1 / 2} \\
& \chi=\frac{1}{2}\left[-\frac{1}{2}+\frac{2}{3} S(S+1)\right]
\end{aligned}
$$

When $S=3 / 2, \chi=1$ (symmetric).
When $S=1 / 2, \chi=0$.
(iii) $\operatorname{For} N=4$,

$$
\begin{aligned}
& \mathrm{D}_{1 / 2} \times \mathrm{D}_{1 / 2} \times \mathrm{D}_{1 / 2} \times \mathrm{D}_{1 / 2}=\mathrm{D}_{2}+3 \mathrm{D}_{1}+2 \mathrm{D}_{0} \\
& \chi=\frac{S(S+1)}{6}
\end{aligned}
$$

For $S=2, \chi=1$ (symmetric).
For $S=1, \chi=1 / 3$.
For $S=0, \chi=0$.

### 31.9 Two spin 1/2 particles

$$
\mathrm{D}_{1 / 2} \times \mathrm{D}_{1 / 2}=\mathrm{D}_{1}+\mathrm{D}_{0}
$$

(i) $\quad j=1$ (spin triplet): symmetric states

$$
\begin{aligned}
& |j=1, m=1\rangle=|++\rangle \\
& |1,0\rangle=\frac{1}{\sqrt{2}}(|+-\rangle+|-+\rangle) \\
& |1,-1\rangle=|--\rangle
\end{aligned}
$$

(ii) $\quad j=0$ (singlet): anti-symmetric state

$$
|j=0, m=0\rangle=\frac{1}{\sqrt{2}}[(|+-\rangle-|-+\rangle]
$$

### 31.10 Young's tableau-I

The spin state of an individual electron is to be represented by a box. A single box represents a doublet

symmetric tableau (spin triplet)

antisymmetric tableau (spin singlet)
((Rule))
We do not consider

| 2 | 1 |
| :--- | :--- |

because when we put boxes horizontally, symmetry is understood. So we deduce an important rule. Double counting is avoided if we require that the number (label) not decrease going from the left to the right. Similarly, to eliminate the unwanted symmetry states, we require the number (label) to increase as we go down.

## General rule

In drawing Young tabeleau, going from left to right the number cannot decrease; going down the number must increase.

### 31.11 Three electrons with spin $1 / 2$

$$
\mathrm{D}_{1 / 2} \times \mathrm{D}_{1 / 2} \times \mathrm{D}_{1 / 2}=\left(\mathrm{D}_{1}+\mathrm{D}_{0}\right) \times \mathrm{D}_{1 / 2}=\mathrm{D}_{3 / 2}+\mathrm{D}_{1 / 2}+\mathrm{D}_{1 / 2}
$$

(i) $j=3 / 2$

$$
\begin{aligned}
& \left|j=\frac{3}{2}, m=\frac{3}{2}\right\rangle=|+++\rangle \\
& \left|\frac{3}{2}, \frac{1}{2}\right\rangle=\frac{1}{\sqrt{3}}[|++-\rangle+|+-+\rangle+|-++\rangle] \\
& \left|\frac{3}{2},-\frac{1}{2}\right\rangle=\frac{1}{\sqrt{3}}[|+--\rangle+|-+-\rangle+|--+\rangle] \\
& \left|\frac{3}{2},-\frac{3}{2}\right\rangle=|---\rangle
\end{aligned}
$$

(ii) $j=1 / 2$

$$
\begin{aligned}
& \left.\left|j=\frac{1}{2}, m=\frac{1}{2}\right\rangle=\frac{1}{\sqrt{6}}[-|-++\rangle+2|++-\rangle-\mid+-+]\right\rangle \\
& \left|\frac{1}{2},-\frac{1}{2}\right\rangle=\frac{1}{\sqrt{6}}[|+--\rangle+|-+-\rangle-2|--+\rangle]
\end{aligned}
$$

(iii) $j=1 / 2$

$$
\begin{aligned}
& \left|j=\frac{1}{2}, m=\frac{1}{2}\right\rangle=\frac{1}{\sqrt{2}}[|+-+\rangle-|-++\rangle] \\
& \left|\frac{1}{2},-\frac{1}{2}\right\rangle=\frac{1}{\sqrt{2}}[|+--\rangle-|-+-\rangle],
\end{aligned}
$$

### 31.12 Young's tableaux II



$$
j=3 / 2, m=3 / 2,1 / 2,-1 / 2,-3 / 2
$$



What about the totally antisymmetric states? We may try vertical tableau like

| 1 | 1 |
| :--- | :--- |
| 1 | 2 |
| 1 | 2 |
| 1 | 2 | : forbidden state

But these are illegal, because the numbers must increase as we go down.
$j=1 / 2$


### 31.13 Note

We define a mixed symmetry tableau. The mixed state is orthogonal to the symmetric state and anti-symmetric state.

(a)

We consider a mixed state,


$$
\begin{equation*}
\left|\psi_{1}\right\rangle=|+--\rangle+|--+\rangle=\left(|+\rangle_{1}|-\rangle_{3}+|-\rangle_{1}|+\rangle_{3}\right)|-\rangle_{2} \tag{1}
\end{equation*}
$$

satisfies symmetry under $1 \leftrightarrow 3$, but it is neither symmetric nor anti-symmetric with respect to $2 \leftrightarrow 3$ (or $1 \leftrightarrow 2$ ).


$$
\begin{equation*}
\left|\psi_{2}\right\rangle=|--+\rangle+|-+-\rangle=\left(|-\rangle_{2}|+\rangle_{3}+|+\rangle_{3}|-\rangle_{2}\right)|-\rangle_{1} \tag{2}
\end{equation*}
$$

satisfies symmetry under $2 \leftrightarrow 3$, but it is neither symmetric nor anti-symmetric with respect to $1 \leftrightarrow 2$ (or $1 \leftrightarrow 3$ ).

Subtraction: Eq.(1) - Eq.(2):


$$
\begin{equation*}
\left|\psi_{1}\right\rangle-\left|\psi_{2}\right\rangle=|+--\rangle-|-+-\rangle \tag{3}
\end{equation*}
$$

This satisfies anti-symmetry under $1 \leftrightarrow 2$, but no longer have the original symmetry under $1 \leftrightarrow 2$.

This corresponds to

$$
\left|\frac{1}{2},-\frac{1}{2}\right\rangle=\frac{1}{\sqrt{2}}[|+--\rangle-|-+-\rangle]
$$

which is obtained from the Clebsch-Gordan coefficient.
(b)


$$
\begin{equation*}
\left|\psi_{3}\right\rangle=|+--\rangle-|--+\rangle=\left(|+\rangle_{1}|-\rangle_{3}-|-\rangle_{1}|+\rangle_{3}|-\rangle_{2}\right. \tag{4}
\end{equation*}
$$

This satisfies anti-symmetric under $1 \leftrightarrow 3$.


$$
\begin{equation*}
\left|\psi_{4}\right\rangle=|-+-\rangle-|--+\rangle=\left(|+\rangle_{2}|-\rangle_{3}-|-\rangle_{2}|+\rangle_{3}\right)|-\rangle_{1} \tag{5}
\end{equation*}
$$

This satisfies anti-symmetric under $2 \leftrightarrow 3$. Addition: Eq.(4) + Eq.(5):

$$
\begin{align*}
& \left|\psi_{3}\right\rangle+\left|\psi_{4}\right\rangle=|+--\rangle+|-+-\rangle-2|--+\rangle  \tag{6}\\
& \left.\left|\frac{1}{2},-\frac{1}{2}\right\rangle=\frac{1}{\sqrt{6}}[|+--\rangle+|-+-\rangle-2 \mid--+]\right\rangle
\end{align*}
$$

which is obtained from the Clebsch-Gordan coefficient.
(c)


$$
\begin{equation*}
\left|\psi_{5}\right\rangle=|+-+\rangle+|++-\rangle=\left(|+\rangle_{2}|-\rangle_{3}+|-\rangle_{2}|+\rangle_{3}\right)|+\rangle_{1} . \tag{7}
\end{equation*}
$$

This satisfies symmetric under $2 \leftrightarrow 3$

$$
\begin{equation*}
\left|\psi_{6}\right\rangle=|++-\rangle+|-++\rangle=\left(|+\rangle_{1}|-\rangle_{3}+|-\rangle_{1}|+\rangle_{3}\right)|+\rangle_{2} . \tag{8}
\end{equation*}
$$

This satisfies symmetric under $1 \leftrightarrow 3$.
Eq.(7) - Eq.(8)


$$
\begin{equation*}
\left|\psi_{5}\right\rangle-\left|\psi_{6}\right\rangle=|+-+\rangle-|-++\rangle . \tag{9}
\end{equation*}
$$

This satisfies anti-symmetric under $1 \leftrightarrow 2$.

$$
\left|j=\frac{1}{2}, m=\frac{1}{2}\right\rangle=\frac{1}{\sqrt{2}}[|+-+\rangle-|-++\rangle]
$$

(d)

$$
\begin{equation*}
\left|\psi_{7}\right\rangle=|++-\rangle-|+-+\rangle=\left(|+\rangle_{2}|-\rangle_{3}-|-\rangle_{2}|+\rangle_{3}\right)|+\rangle_{1} . \tag{10}
\end{equation*}
$$

This satisfies anti-symmetric under $2 \leftrightarrow 3$


$$
\begin{equation*}
\left|\psi_{8}\right\rangle=|++-\rangle-|-++\rangle=\left(|+\rangle_{1}|-\rangle_{3}-|-\rangle_{1}|+\rangle_{3}\right)|+\rangle_{2} . \tag{11}
\end{equation*}
$$

This satisfies anti-symmetric under $1 \leftrightarrow 3$.
Subtraction: Eq.(10) - Eq.(11)


$$
\begin{equation*}
-\left|\psi_{7}\right\rangle+\left|\psi_{8}\right\rangle=|+-+\rangle-|-++\rangle . \tag{12}
\end{equation*}
$$

This satisfies antisymmetic under $1 \leftrightarrow 2$.
Addition: Eq.(10) + Eq.(11)


$$
\left|\psi_{7}\right\rangle+\left|\psi_{8}\right\rangle=2|++-\rangle-|+-+\rangle-|-++\rangle
$$

or

$$
\left|j=\frac{1}{2}, m=\frac{1}{2}\right\rangle=\frac{1}{\sqrt{6}}[-|-++\rangle+2|++-\rangle-|+-+\rangle]
$$

### 31.144 electrons with spin $1 / 2$

$$
\begin{aligned}
\mathrm{D}_{1 / 2} \times \mathrm{D}_{1 / 2} \times \mathrm{D}_{1 / 2} \times \mathrm{D}_{1 / 2} & =\left(\mathrm{D}_{3 / 2}+\mathrm{D}_{1 / 2}+\mathrm{D}_{1 / 2}\right) \times \mathrm{D}_{1 / 2} \\
& =\left(\mathrm{D}_{2}+\mathrm{D}_{1}\right)+\left(\mathrm{D}_{1}+\mathrm{D}_{0}\right)+\left(\mathrm{D}_{1}+\mathrm{D}_{0}\right)
\end{aligned}
$$

(i) $j=2$

$$
\begin{aligned}
& |j=2, m=2\rangle=|++++\rangle \\
& |2,1\rangle=\frac{1}{2}[|+++-\rangle+|++-+\rangle+|+-++\rangle+|-+++\rangle] \\
& |2,0\rangle=\frac{1}{\sqrt{6}}[|-++-\rangle+|++--\rangle+|+-+-\rangle+|+--+\rangle+|-+-+\rangle+|--++\rangle] \\
& |2,-1\rangle=\frac{1}{2}[|+---\rangle+|-+--\rangle+|--+-\rangle+|---+\rangle] \\
& |2,-2\rangle=|----\rangle
\end{aligned}
$$

(ii) $j=1$

$$
\begin{aligned}
& \left.|j=1, m=1\rangle=-\frac{1}{2 \sqrt{3}}[|-+++\rangle+|++-+\rangle+|+-++\rangle]+\frac{\sqrt{3}}{2}|+++-\rangle\right] \\
& |1,0\rangle=\frac{1}{\sqrt{6}}[|-++-\rangle+|++--\rangle+|+-+-\rangle]-\frac{1}{\sqrt{6}}[|+--+\rangle+|--++\rangle+|-+-+\rangle] \\
& \left.|1,-1\rangle=\frac{1}{2 \sqrt{3}}[|+---\rangle+|-+--\rangle+|--+-\rangle]-\frac{\sqrt{3}}{2}|---+\rangle\right]
\end{aligned}
$$

(iii) $j=1$
(iv) $j=1$

$$
\begin{aligned}
& |j=1, m=1\rangle=\frac{1}{\sqrt{2}}[|+-++\rangle-|-+++\rangle] \\
& |1,0\rangle=\frac{1}{2}[|+-+-\rangle-|-++-\rangle+|+--+\rangle-|-+-+\rangle] \\
& |1,-1\rangle=\frac{1}{\sqrt{2}}[|+---\rangle-|-+--\rangle]
\end{aligned}
$$

(v) $j=0$

$$
|j=0, m=0\rangle=\frac{1}{2}[|+-+-\rangle-|-++-\rangle-|+--+\rangle+|-+-+\rangle]
$$

### 31.15 Young's tableaux

$j=2$ symmetric state

$j=1$ mixed state

$j=0$

| 1 | 1 |
| :---: | :---: |
| 2 | 2 |
| $m=0$ |  |

31.16 Simplified model for spin $\mathbf{1 / 2}$

Now we introduce a simple way to build a Young diagram.
(a) Two spin 1/2 particles

$2 \times 2=3+1$
$\mathrm{D}_{1 / 2} \times \mathrm{D}_{1 / 2}=\mathrm{D}_{1}+\mathrm{D}_{0}$
(b) Three spin $1 / 2$ particles

$3 \times 2=4+2$
$\mathrm{D}_{1} \times \mathrm{D}_{1 / 2}=\mathrm{D}_{3 / 2}+\mathrm{D}_{1 / 2}$


$$
\mathrm{D}_{0} \times \mathrm{D}_{1 / 2}=\mathrm{D}_{1 / 2}
$$

((Note))

(c) Four particles with $1 / 2$


$$
2 \times 2 \times 2 \times 2=16
$$


$\oplus$

$\mathrm{D}_{3 / 2} \times \mathrm{D}_{1 / 2}=\mathrm{D}_{2}+\mathrm{D}_{1}$

$\oplus$

$\mathrm{D}_{1 / 2} \times \mathrm{D}_{1 / 2}=\mathrm{D}_{1}+\mathrm{D}_{0}$
(d) 5 spin $1 / 2$ particles

$2 \times 2 \times 2 \times 2 \times 2=32$

$\mathrm{D}_{2} \times \mathrm{D}_{1 / 2}=\mathrm{D}_{5 / 2}+\mathrm{D}_{3 / 2}$

$\mathrm{D}_{1} \times \mathrm{D}_{1 / 2}=\mathrm{D}_{3 / 2}+\mathrm{D}_{1 / 2}$

$\mathrm{D}_{0} \times \mathrm{D}_{1 / 2}=\mathrm{D}_{1 / 2}$

### 31.17 Particles with $I=1 ; m=1,0,-1$ ( $p$ electrons)

The labels 1,2 , and 3 may stand for the magnetic quantum number of $p$-orbitals ( $l=1$ particle).

| $\square$ |
| :---: |
| $\mathbf{1}$, |
| $m=1$ |
| 2, |
| $m=0$ |$\underset{m=-1}{3}$

### 31.18 Two particles with spin $1: 3 \times 3=9$ states

For $j=1$

$\mathrm{D}_{1}, \mathrm{D}_{1} \quad \mathrm{D}_{2}, \mathrm{D}_{0} \quad \mathrm{D}_{1}$

The horizontal tableau has six states: the tableau is to be broken down into $j=-2$ (multiplicity 5) and $j=0$ (multiplicity 1 ); both of which are symmetric.

The vertical tableau corresponds to an antisymmetric $j=1$ state.
Concretely,
Symmetric

$$
\begin{array}{r}
\begin{array}{r}
\square \\
\hline
\end{array}, \begin{array}{l|l|}
\hline 1 & 1 \\
\hline
\end{array}, \begin{array}{|l|l|}
\hline 1 & 2 \\
\hline
\end{array}, \begin{array}{|l|l|}
\hline 1 & 3 \\
\hline
\end{array}, \begin{array}{|l|l|l|l|}
\hline 2 & 2 \\
\hline
\end{array}, \begin{array}{|l|l|}
\hline 2 & 3 \\
\hline & 3 \\
\hline
\end{array}, \\
\begin{array}{l}
6 \text { states } \\
m=-2
\end{array}(j=2 \text { and } 0)
\end{array}
$$

Antisymmetric\begin{tabular}{|c|}
\hline 1 <br>
\hline 2 <br>
\hline

,$\underset{m}{\mid 1},$

\hline 2 <br>
\hline
\end{tabular},$\underset{m=-1}{ } 3$

$m=-1$

### 31.19 The three particles with $I=1$

$3 \times 3 \times 3=27$ states

$: 6 \times 3=7+3+5+3$
$\mathrm{D}_{2}, \mathrm{D}_{0} \mathrm{x} \quad \mathrm{D}_{1} \quad \mathrm{D}_{3}, \mathrm{D}_{1} \quad \mathrm{D}_{2}, \mathrm{D}_{1}$


$$
: 3 \times 3=8+1
$$

D1x D1 $\quad D_{2}, D_{1} \quad D_{0}$

Note:


As for $\square$ with eight possibilities altogether, the argument is more involved, but we note that this 8 cannot be broken into $7+1$ because 7 is totally symmetric, while 1 is totally antisymmetric when we know that 8 is of mixed symmetry. So the only possibility is $8=5+3-$ in other words $j=2$ and $j=1$.

Finally, therefore


$$
D_{1} \times D_{1} \times D_{1}=D_{3}+2 D_{2}+3 D_{1}+D_{0}
$$

or


In terms of angular momentum states, we have

$$
\begin{array}{ll}
j=3(7 \text { states }) & \text { once } \quad \text { (totally symmetric) } \\
j=2(5 \text { states })) & \text { twice } \quad \text { (both mixed symmetry) } \\
j=1(3 \text { states }) & \text { three times (one totally symmetric, two mixed symmetry) } \\
j=0(1 \text { state }) & \text { once } \quad \text { (totally antisymmetric) } .
\end{array}
$$

| 1 | 1 | 1 |
| :--- | :--- | :--- |
| $\begin{array}{l}2\end{array}$, |  |  |, | 1 | 1 | 2 |
| :--- | :--- | :--- | :--- |
| $m=2$ |  |  |,$\quad$| 1 | 1 | 3 |
| :--- | :--- | :--- | :--- |
| $m=1$ |  |  |


| 1 | 2 | 2 |
| :--- | :--- | :--- |, | 1 | 2 | 3 |
| :--- | :--- | :--- | :--- |$\quad$| 1 | 3 | 3 | 3 |
| :--- | :--- | :--- | :--- |


| 2 | 2 | 2 |
| :--- | :--- | :--- |
| $m=0$ |  |  |, | 2 | 2 | 3 |
| :--- | :--- | :--- | :--- |
| $m=-1$ |  |  |, | 2 | 3 | 3 |
| :--- | :--- | :--- |
| $m=-2$ |  |  |

\[

\]

| 1 |
| :--- |
| 2 |
| 3 |

$m=0$


$m=0$
$m=-1$
$m=-1$
$m=-2$

### 31.20 Four particles with $\boldsymbol{I}=\mathbf{1}$ (Landau) $(p)^{4}$


$\mathrm{D}_{1}, \mathrm{D}_{2} \quad \mathrm{D}_{1} \quad \mathrm{D}_{1}, \mathrm{D}_{2}, \mathrm{D}_{3}, \quad \mathrm{D}_{0}, \mathrm{D}_{2} \quad \mathrm{D}_{1}$

((Note))
$\oplus$
 is forbidden.

| 1 | 1 | 1 | 1 |
| :--- | :--- | :--- | :--- |


| 1 1 1 2 <br> $m=3$    |
| :--- |
| m |


| 1 | 1 | 1 | 3 |
| :--- | :--- | :--- | :--- | | m=2 |
| :--- | l


| 1 | 1 | 2 | 2 |
| :--- | :--- | :--- | :--- |


| 1 | 1 | 2 | 3 |
| :--- | :--- | :--- | :--- | | m=1 |
| :--- |


| 1 | 1 | 3 | 3 |
| :--- | :--- | :--- | :--- |

$m=2$
$m=1$
$m=0$

| 1 | 2 | 2 | 2 |
| :--- | :--- | :--- | :--- |


| 1 | 2 | 2 | 3 |
| :--- | :--- | :--- | :--- |


| 1 | 2 | 3 | 3 |
| :--- | :--- | :--- | :--- |

$m=1$
$m=0$
$m=-1$

| 1 | 3 | 3 | 3 |
| :--- | :--- | :--- | :--- |


| 2 | 2 | 2 | 2 |
| :--- | :--- | :--- | :--- |


| 2 | 2 | 2 | 3 |
| :--- | :--- | :--- | :--- |

$m=-2$
$m=0$
$m=-1$

| 2 | 2 | 3 | 3 |
| :--- | :--- | :--- | :--- |
| $m=-2$ |  |  |  |


| 2 | 3 | 3 | 3 |
| :--- | :--- | :--- | :--- |

$m=-3$

| 3 | 3 | 3 | 3 |
| :--- | :--- | :--- | :--- | | m $=-4$ |
| :--- |

### 31.21 Two spin states

Hyperfine splitting in hydrogen
The hydrogen atom consists of an electron sitting in the neighborhood of the proton. There are four states for the ground state of the hydrogen atom.


1
$S_{1}$


2
$S_{2}$









For any state, the state can be described by the linear combination of these four states.

We use the following formula to set up the eigenkets of two spins with $\operatorname{spin} S=1 / 2$.

$$
\begin{aligned}
& \hat{\sigma}_{z}|+\rangle=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)\binom{1}{0}=\binom{1}{0}=|+\rangle \\
& \hat{\sigma}_{z}|-\rangle=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)\binom{0}{1}=-\binom{0}{1}=-|-\rangle \\
& \hat{\sigma}_{x}|+\rangle=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)\binom{1}{0}=\binom{0}{1}=|-\rangle \\
& \hat{\sigma}_{x}|-\rangle=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)\binom{0}{1}=\binom{1}{0}=|+\rangle \\
& \hat{\sigma}_{y}|+\rangle=\left(\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right)\binom{1}{0}=\binom{0}{i}=i|-\rangle \\
& \hat{\sigma}_{y}|-\rangle=\left(\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right)\binom{0}{1}=\binom{-i}{0}=-i|+\rangle
\end{aligned}
$$

We now consider the two spin operators: $\hat{\sigma}_{1}$ and $\hat{\sigma}_{2}$. There are four states:

The spin operator $\hat{\sigma}_{1}$ and $\hat{\sigma}_{2}$ work on the first spin state and the second spin state, respectively.

$$
\begin{aligned}
& \hat{\sigma}_{1 y}|-+\rangle=-i|++\rangle \\
& \hat{\sigma}_{1 y}|++\rangle=i|-+\rangle \\
& \hat{\sigma}_{1 y}|+-\rangle=i|--\rangle \\
& \hat{\sigma}_{1 y}|--\rangle=-i|+-\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
& \hat{\sigma}_{2 x}|-+\rangle=|--\rangle \\
& \hat{\sigma}_{2 x}|++\rangle=|+-\rangle
\end{aligned}
$$

$$
\begin{aligned}
& \hat{\sigma}_{2 x}|+-\rangle=|++\rangle \\
& \hat{\sigma}_{2 x}|--\rangle=|-+\rangle
\end{aligned}
$$

In the most general case we could have more complex things.

$$
\begin{aligned}
& \hat{\sigma}_{1 x} \hat{\sigma}_{2 z}|++\rangle=\hat{\sigma}_{1 x}\left(\hat{\sigma}_{2 z}|++\rangle\right)=\hat{\sigma}_{1 x}|++\rangle=|-+\rangle \\
& \hat{\sigma}_{1 x} \hat{\sigma}_{2 z}|+-\rangle=\hat{\sigma}_{1 x}\left(\hat{\sigma}_{2 z}|+-\rangle=-\hat{\sigma}_{1 x}|+-\rangle=-|--\rangle\right. \\
& \hat{\sigma}_{1 x} \hat{\sigma}_{2 z}|-+\rangle=\hat{\sigma}_{1 x}\left(\hat{\sigma}_{2 z}|-+\rangle\right)=\hat{\sigma}_{1 x}|-+\rangle=|++\rangle \\
& \hat{\sigma}_{1 x} \hat{\sigma}_{2 z}|--\rangle=\hat{\sigma}_{1 x}\left(\hat{\sigma}_{2 z}|--\rangle\right)=-\hat{\sigma}_{1 x}|--\rangle=-|+-\rangle
\end{aligned}
$$

### 34.22 Dirac spin exchange operator

$$
\begin{array}{ll}
\hat{A}=\hat{\sigma}_{1} \cdot \hat{\sigma}_{2}=\hat{\sigma}_{1 x} \cdot \hat{\sigma}_{2 x}+\hat{\sigma}_{1 y} \cdot \hat{\sigma}_{2 y}+\hat{\sigma}_{1 z} \cdot \hat{\sigma}_{2 z} \\
\hat{\sigma}_{1 x} \cdot \hat{\sigma}_{2 x}|++\rangle=|--\rangle, & \hat{\sigma}_{1 y} \cdot \hat{\sigma}_{2 y}|++\rangle=-|--\rangle \\
\hat{\sigma}_{1 z} \cdot \hat{\sigma}_{2 z}|++\rangle=|++\rangle, & \hat{\sigma}_{1 x} \cdot \hat{\sigma}_{2 x}|+-\rangle=|-+\rangle \\
\hat{\sigma}_{1 y} \cdot \hat{\sigma}_{2 y}|+-\rangle=|-+\rangle, & \hat{\sigma}_{1 z} \cdot \hat{\sigma}_{2 z}|+-\rangle=-|+-\rangle \\
\hat{\sigma}_{1 x} \cdot \hat{\sigma}_{2 x}|-+\rangle=|+-\rangle, & \hat{\sigma}_{1 y} \cdot \hat{\sigma}_{2 y}|-+\rangle=|+-\rangle \\
\hat{\sigma}_{1 z} \cdot \hat{\sigma}_{2 z}|-+\rangle=-|-+\rangle, & \hat{\sigma}_{1 x} \cdot \hat{\sigma}_{2 x}|--\rangle=|++\rangle \\
\hat{\sigma}_{1 y} \cdot \hat{\sigma}_{2 y}|--\rangle=-|++\rangle, & \hat{\sigma}_{1 z} \cdot \hat{\sigma}_{2 z}|--\rangle=|--\rangle \\
\hat{A}|++\rangle=|++\rangle=2|++\rangle-|++\rangle \\
\hat{A}|--\rangle=|--\rangle=2|--\rangle-|--\rangle \\
\hat{A}|+-\rangle=2|-+\rangle-|+-\rangle & \\
\hat{A}|-+\rangle=2|+-\rangle-|-+\rangle
\end{array}
$$

Now we introduce a new operator $\hat{P}_{12}$, which has the following properties
$\hat{P}_{12}$ is the exchange operator.
When $\hat{P}_{12}$ operates on the state $|\psi\rangle=|\alpha\rangle_{1}|\beta\rangle_{2}$, we have

$$
\begin{aligned}
& \hat{P}_{12}|\psi\rangle=\hat{P}_{12}|\alpha\rangle_{1}|\beta\rangle_{2}=|\alpha\rangle_{2}|\beta\rangle_{1}=|\beta\rangle_{1}|\alpha\rangle_{2} \\
& \hat{P}_{12}|++\rangle=|++\rangle \\
& \hat{P}_{12}|+-\rangle=|-+\rangle \\
& \hat{P}_{12}|-+\rangle=|+-\rangle \\
& \hat{P}_{12}|--\rangle=|--\rangle
\end{aligned}
$$

$\hat{P}_{12}$ is related to $\hat{A}$ as

$$
\hat{A}=2 \hat{P}_{12}-\hat{1}
$$

or

$$
\hat{P}_{12}=\frac{1}{2}(1+\hat{A})=\frac{1}{2}\left(1+\hat{\boldsymbol{\sigma}}_{1} \cdot \hat{\sigma}_{2}\right)
$$

This operator is called the Dirac's spin exchange operator

### 31.23 Total spin angular momentum

$\hat{\mathbf{S}}_{1}$ and $\hat{\mathbf{S}}_{2}$ are commute.
$\hat{\mathbf{S}}=\frac{\hbar}{2}\left(\hat{\boldsymbol{\sigma}}_{1}+\hat{\boldsymbol{\sigma}}_{2}\right)$

$$
\hat{S}_{z}=\frac{\hbar}{2}\left(\hat{\sigma}_{1 z}+\hat{\sigma}_{2 z}\right)
$$

$$
\begin{aligned}
\hat{\mathbf{S}}^{2} & =\frac{\hbar^{2}}{4}\left(\hat{\boldsymbol{\sigma}}_{1}+\hat{\boldsymbol{\sigma}}_{2}\right) \cdot\left(\hat{\boldsymbol{\sigma}}_{1}+\hat{\boldsymbol{\sigma}}_{2}\right)=\frac{\hbar^{2}}{4}\left(\hat{\boldsymbol{\sigma}}_{1}^{2}+\hat{\boldsymbol{\sigma}}_{2}^{2}+2 \hat{\boldsymbol{\sigma}}_{1} \cdot \hat{\boldsymbol{\sigma}}_{2}\right) \\
& =\frac{\hbar^{2}}{2}\left(3+\hat{\boldsymbol{\sigma}}_{1} \cdot \hat{\boldsymbol{\sigma}}_{2}\right)
\end{aligned}
$$

Note that

$$
\hat{P}_{12}=\frac{1}{2}\left(1+\hat{\boldsymbol{\sigma}}_{1} \cdot \hat{\boldsymbol{\sigma}}_{2}\right)
$$

or

$$
\hat{\mathbf{S}}^{2}=\frac{\hbar^{2}}{2}\left(3+\hat{\boldsymbol{\sigma}}_{1} \cdot \hat{\boldsymbol{\sigma}}_{2}\right)=\frac{\hbar^{2}}{2}\left(2+1+\hat{\boldsymbol{\sigma}}_{1} \cdot \hat{\boldsymbol{\sigma}}_{2}\right)=\hbar^{2}\left(1+\hat{P}_{12}\right)
$$

We also see that

$$
\left[\hat{\mathbf{S}}^{2}, \hat{S}_{z}\right]=\hat{0}
$$

We can have simultaneous eigenkets of $\hat{\mathbf{S}}^{2}$ and $\hat{S}_{z}$. Here we use the basis of
which corresponds to the state $|j=1, m=1\rangle$

$$
\begin{aligned}
& \hat{\mathbf{S}}^{2}|--\rangle=\hbar^{2}\left(1+\hat{P}_{12}\right)|--\rangle=2 \hbar^{2}|--\rangle \\
& \hat{S}_{z}|--\rangle=\frac{\hbar}{2}\left(\hat{\sigma}_{1 z}+\hat{\sigma}_{2 z}\right)|--\rangle=-\hbar|--\rangle
\end{aligned}
$$

which corresponds to the state $|j=1, m=-1\rangle$

$$
\begin{aligned}
& \hat{\mathbf{S}}^{2}|+-\rangle=\hbar^{2}\left(1+\hat{P}_{12}\right)|+-\rangle=\hbar^{2}(|+-\rangle+|-+\rangle) \\
& \hat{S}_{z}|+-\rangle=\frac{\hbar}{2}\left(\hat{\sigma}_{1 z}+\hat{\sigma}_{2 z}\right)|+-\rangle=0
\end{aligned}
$$

$$
\begin{aligned}
& \hat{\mathbf{S}}^{2}|-+\rangle=\hbar^{2}\left(1+\hat{P}_{12}\right)|-+\rangle=\hbar^{2}(|-+\rangle+|+-\rangle) \\
& \hat{S}_{z}|-+\rangle=\frac{\hbar}{2}\left(\hat{\sigma}_{1 z}+\hat{\sigma}_{2 z}\right)|-+\rangle=0
\end{aligned}
$$

We now consider the matrix elements of $\hat{S}^{2}$ under the subspace of $|+-\rangle$ and $|-+\rangle$.

$$
\begin{aligned}
& \hat{\mathbf{S}}^{2}=\hbar^{2}\left(\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right) \\
& \hat{S}^{2}\left|\psi_{S}\right\rangle=2 \hbar^{2}\left|\psi_{S}\right\rangle \\
& \hat{S}^{2}\left|\psi_{A}\right\rangle=0
\end{aligned}
$$

where

$$
\begin{aligned}
& \left|\psi_{S}\right\rangle=\frac{1}{\sqrt{2}}(|+-\rangle+|-+\rangle) \\
& \left|\psi_{A}\right\rangle=\frac{1}{\sqrt{2}}(|+-\rangle-|-+\rangle)
\end{aligned}
$$

Note that

$$
\begin{aligned}
& \hat{S}_{z}\left|\psi_{S}\right\rangle=0 \\
& \hat{S}_{z}\left|\psi_{A}\right\rangle=0
\end{aligned}
$$

Thus $\left|\psi_{S}\right\rangle$ is the eigenket of $\hat{\mathbf{S}}^{2}$ with $2 \hbar^{2}$ and of $\hat{S}_{z}$ with $\hbar .\left|\psi_{A}\right\rangle$ is the eigenket of $\hat{\mathbf{S}}^{2}$ with 0 and of $\hat{S}_{z}$ with 0 .

Eigenket
Energy eigenvalue
((Triplet))
((Singlet))

$$
\left|\psi_{A}\right\rangle=\frac{1}{\sqrt{2}}(|+-\rangle-|-+\rangle)
$$

### 31.24 Exchage interaction



Fig. The interaction between the magnetic moment of electron $\left(\mu_{\mathrm{e}} \sigma_{\mathrm{e}}\right)$ and the magnetic moment of proton ( $\mu_{\mathrm{p}} \sigma_{\mathrm{p}}$ ), where $\mu_{\mathrm{e}}<0$ and $\mu_{\mathrm{p}}>0$.

We now consider the spin Hamiltonian between the electron and proton,

$$
\hat{H}=E_{0}+J \hat{\boldsymbol{\sigma}}_{1} \cdot \hat{\boldsymbol{\sigma}}_{2}
$$

(for convenience we assume $E_{0}=0$ ).

$$
\hat{H}|++\rangle=J|++\rangle
$$



$$
\hat{H}|--\rangle=J|--\rangle
$$



$$
\begin{aligned}
& \hat{H}|+-\rangle=J(2|-+\rangle-|+-\rangle) \\
& \hat{H}|-+\rangle=J(2|+-\rangle-|-+\rangle) \\
& \hat{H}=\left(\begin{array}{cc}
-J & 2 J \\
2 J & -J
\end{array}\right)=J\left(\begin{array}{cc}
-1 & 2 \\
2 & -1
\end{array}\right)
\end{aligned}
$$

((Mathematica))

$$
\begin{aligned}
& \text { Clear["Global`*"]; } \\
& \text { H = J }\left(\begin{array}{cc}
-1 & 2 \\
2 & -1
\end{array}\right) ; \\
& \text { eq1 = Eigensystem [H] / / Simplify } \\
& \{\{-3 \mathrm{~J}, \mathrm{~J}\},\{\{-1,1\},\{1,1\}\}\}
\end{aligned}
$$

$$
E=-3 J \text { (antisymmetric state })
$$

$$
\psi A=-\operatorname{Normalize}[\operatorname{eq1}[[2,1]]]
$$

$$
\left\{\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right\}
$$

$\mathrm{E}=\mathrm{J}$ (symmetric states)

$$
\begin{aligned}
& \psi S=\operatorname{Normalize}[\operatorname{eq1}[[2,2]]] \\
& \left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}
\end{aligned}
$$

For $E=-3 J$ (anti-symmetric state)

$$
\left|\psi_{A}\right\rangle=\binom{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}}
$$

For $E=J$ (symmetric state)

$$
\left|\psi_{S}\right\rangle=\binom{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}
$$

In summary we have eigenkets and energy eigenvalues of the system.
Eigenket
Energy eigenvalue
((Triplet))
((Singlet))

$$
\left|\psi_{A}\right\rangle=\frac{1}{\sqrt{2}}(|+-\rangle-|-+\rangle) \quad E_{0}-3 \mathrm{~J}
$$



Fig. Energy diagram.

### 31.25 Zeeman splitting

$$
\hat{H}=J \hat{\sigma}_{1} \cdot \hat{\sigma}_{2}-\mu_{e} \hat{\sigma}_{1} \cdot \mathbf{B}-\mu_{p} \hat{\sigma}_{2} \cdot \mathbf{B}
$$

Magnetic moment of electron: $\frac{2 \mu_{e}}{\hbar} \mathbf{S}_{1}=\mu_{e} \boldsymbol{\sigma}\left(\mu_{\mathrm{e}}<0\right)$.
Magnetic moment of proton: $\frac{2 \mu_{p}}{\hbar} \mathbf{S}_{2}=\mu_{p} \boldsymbol{\sigma}\left(\mu_{\mathrm{p}}>0\right)$
((Note))

$$
\mu_{\mathrm{e}}=-9284.76377 \times 10^{-27} \mathrm{~J} / \mathrm{T}, \quad \mu_{\mathrm{p}}=14.10606662 \times 10^{-27} \mathrm{~J} / \mathrm{T} \text { (NIST) }
$$

When the magnetic field $\boldsymbol{B}$ is applied along the $z$ axis,

$$
\hat{H}=\hat{H}_{0}+\hat{H}_{1}
$$

with

$$
\begin{aligned}
& \hat{H}_{0}=J \hat{\boldsymbol{\sigma}}_{1} \cdot \hat{\boldsymbol{\sigma}}_{2} \\
& \hat{H}_{1}=-\left(\mu_{1} \hat{\sigma}_{1 z}+\mu_{2} \hat{\sigma}_{2 z}\right) B
\end{aligned}
$$

We calculate

$$
\begin{aligned}
& \hat{H}_{1}|++\rangle=-\left(\mu_{1} \hat{\sigma}_{1 z}+\mu_{2} \hat{\sigma}_{2 z}\right) B|++\rangle=-\left(\mu_{1}+\mu_{2}\right) B|++\rangle \\
& \hat{H}_{1}|--\rangle=-\left(\mu_{1} \hat{\sigma}_{1 z}+\mu_{2} \hat{\sigma}_{2 z}\right) B|--\rangle=\left(\mu_{1}+\mu_{2}\right) B|--\rangle \\
& \hat{H}_{1}|+-\rangle=-\left(\mu_{1} \hat{\sigma}_{1 z}+\mu_{2} \hat{\sigma}_{2 z}\right) B|+-\rangle=-\left(\mu_{1}-\mu_{2}\right) B|+-\rangle \\
& \hat{H}_{1}|-+\rangle=-\left(\mu_{1} \hat{\sigma}_{1 z}+\mu_{2} \hat{\sigma}_{2 z}\right) B|-+\rangle=\left(\mu_{1}-\mu_{2}\right) B|-+\rangle \mid-+\langle \\
& \hat{H}_{0}|++\rangle=J|++\rangle \\
& \hat{H}_{0}|--\rangle=J|--\rangle \\
& \hat{H}_{0}|+-\rangle=J(2|-+\rangle-|+-\rangle)
\end{aligned}
$$

$$
\hat{H}_{0}|-+\rangle=J(2|+-\rangle-|-+\rangle)
$$

Thus

$$
\hat{H}|++\rangle=\left[J-\left(\mu_{1}+\mu_{2}\right) B\right]|++\rangle
$$



$$
\hat{H}|--\rangle=\left[J+\left(\mu_{1}+\mu_{2}\right) B\right]|--\rangle
$$



$$
\begin{aligned}
& \hat{H}|+-\rangle=J(2|-+\rangle-|+-\rangle)-\left(\mu_{1}-\mu_{2}\right) B|+-\rangle \\
& \hat{H}|-+\rangle=J(2|+-\rangle-|-+\rangle)+\left(\mu_{1}-\mu_{2}\right) B|+-\rangle \\
& \hat{H}=\left(\begin{array}{cc}
-J-\left(\mu_{1}-\mu_{2}\right) B & 2 J \\
2 J & -J+\left(\mu_{1}-\mu_{2}\right) B
\end{array}\right)
\end{aligned}
$$

## ((Mathematica))

```
Clear["Global`*"];
M =( (-J-(\mu1 - \mu2) B 
eq1 = Eigensystem[M] // Simplify
{{-J-\sqrt{}{4\mp@subsup{J}{}{2}+\mp@subsup{B}{}{2}(\mu1-\mu2\mp@subsup{)}{}{2}},-J+\sqrt{}{4\mp@subsup{J}{}{2}+\mp@subsup{B}{}{2}(\mu\mathbf{1-\mu2)}}\mp@subsup{}{}{2}}}
    {{\frac{-\sqrt{}{4\mp@subsup{J}{}{2}+\mp@subsup{B}{}{2}(\mu1-\mu2\mp@subsup{)}{}{2}}+\mathbf{B}(-\mu\mathbf{1}+\mu2)}{2J},\mathbf{1}},{\frac{\sqrt{}{4\mp@subsup{J}{}{2}+\mp@subsup{B}{}{2}(\mu1-\mu2\mp@subsup{)}{}{2}}+\mathbf{B}(-\mu1+\mu2)}{2J},1}}}
E4 = eq1[[1, 1]];
E3 = eq1[[1, 2]];
E2 = (J - B ( }\mu\mathbf{1}+\mu\mathbf{2}))
E1 = (J + B ( }\mu\mathbf{1}+\mu\mathbf{2}))
rule1 = { < 2 ->-1000 \mu1, J -> 1, B }->1\mp@subsup{0}{}{4}\textrm{B}1,\mu1->1\mp@subsup{0}{}{-7}}
E11 = E1 //. rule1 // Simplify; E22 = E2 // . rule1 // Simplify;
E33 = E3 //. rule1 // Simplify;
E44 = E4 //. rule1 // Simplify;
f1 = Plot[Evaluate[{E11, E22, E33, E44}], {B1, 0, 5},
    PlotStyle }->\mathrm{ Table[{Thick, Hue[0.08 i]}, {i, 0, 5}], Background }->\mathrm{ LightGray,
    AxesLabel }->{"B (T)", "E"}]
f2 = Graphics[{Text[Style["E4", Black, 12], {3, -5}],
    Text[Style["E2", Black, 12], {3, -2}], Text[Style["E3", Black, 12], {3, 2.3}],
    Text[Style["E:", Black, 12], {3, 4}], Text[Style["J = 1", Black, 15], {1, 5}]}];
Show[f1, f2]
```



Figure caption
Zeeman splitting of the ground state of hydrogen.
$B \neq 0$ :the levels are denoted as $E_{1}, E_{3}, E_{2}$, and $E_{4}$ from the top to the bottom.
$B=0$ : there are two levels. One level is $E_{4}$, and another level is degenerate $\left(E_{1}, E_{3}\right.$, $E_{2)}$.

### 31.26 Zeeman splitting of the ground state of hydrogen.

We consider the hydrogen atom consisting of an electron with spin $S_{1}=\hbar / 2$ sitting in the neighborhood of the proton with spin $S_{2}=\hbar / 2$. There are four states for the ground state of hydrogen atom. Any components of $S_{1}$ commutes with any component of $S_{2}$.
[S.B. Crampton, D. Kleppner, and N.F. Ramsey, Phys. Rev. Lett. 11, 338 (1963)].

$$
\begin{aligned}
& \mu_{1}=\mu_{\mathrm{e}}(<0), \mu_{2}=\mu_{\mathrm{p}}(>0) . \\
& \left|\mu_{\mathrm{e}}\right| \approx 1000 \mu_{\mathrm{p}} \\
& \mu_{\mathrm{e}}+\mu_{\mathrm{p}}=-\mu \\
& -\mu_{\mathrm{e}}+\mu_{\mathrm{p}}=\mu
\end{aligned}
$$

( $\mu$ and $\mu^{\prime}$ are positive).

$$
\begin{aligned}
& E_{1}=J+\mu B \\
& E_{2}=J-\mu B \\
& E_{3}=J\left(-1+2 \sqrt{1+\frac{\mu^{\prime 2} B^{2}}{4 J^{2}}}\right) \\
& E_{4}=-J\left(1+2 \sqrt{1+\frac{\mu^{\prime 2} B^{2}}{4 J^{2}}}\right)
\end{aligned}
$$

For $B=0$, there is one transition line observed $(=1.420405751 \mathrm{GHz})$ For $B \neq 0$, six lines are observed.

$$
\begin{array}{lll}
E_{1}-E_{3}, & E_{1}-E_{2}, & E_{1}-E_{4} \\
E_{3}-E_{2}, & E_{3}-E_{4}, & E_{2}-E_{4}
\end{array}
$$

### 31.27 Problems

((Shaum 14-10))
Two spin $1 / 2$ particles are described by an unperturbed Hamiltonian

$$
\hat{H}_{0}=-A\left(\hat{\sigma}_{1 z}+\hat{\sigma}_{2 z}\right) .
$$

We add the perturbation

$$
\hat{H}_{1}=\varepsilon\left(\hat{\sigma}_{1 x} \hat{\sigma}_{2 x}+\hat{\sigma}_{1 y} \hat{\sigma}_{2 y}\right)
$$

with
$\varepsilon \ll A(A>0)$.
(a) Find eigenvalues and eigenfunctions of $\hat{H}_{0}$.
(b) Calculate (exactly) the energy levels and eigenfunctions of $\hat{H}_{0}+\hat{H}_{1}$.
(c) Calculate the first-order corrections to the energy levels of $\hat{H}_{0}$.

$$
\begin{aligned}
& \hat{P}_{12}=\frac{1}{2}\left(\hat{l}+\hat{\boldsymbol{\sigma}}_{1} \cdot \hat{\boldsymbol{\sigma}}_{2}\right) \\
& \hat{\boldsymbol{\sigma}}_{1} \cdot \hat{\boldsymbol{\sigma}}_{2}=2 \hat{P}_{12}-\hat{1}
\end{aligned}
$$

or

$$
\begin{aligned}
& \hat{H}_{1}=\varepsilon\left(\hat{\sigma}_{1 x} \hat{\sigma}_{2 x}+\hat{\sigma}_{1 y} \hat{\sigma}_{2 y}\right)=\varepsilon\left(2 \hat{P}_{12}-\hat{1}-\hat{\sigma}_{1 z} \hat{\sigma}_{2 z}\right) \\
& \hat{H}_{0}|++\rangle=-2 A|++\rangle \\
& \hat{H}_{0}|+-\rangle=0 \\
& \hat{H}_{0}|-+\rangle=0 \\
& \hat{H}_{0}|--\rangle=2 A|--\rangle
\end{aligned}
$$

Eigenkets Energy eigenvalues
(b)

$$
\begin{aligned}
& \hat{H}_{1}|++\rangle=\varepsilon\left(2 \hat{P}_{12}-\hat{1}-\hat{\sigma}_{1 z} \hat{\sigma}_{2 z}\right)|++\rangle=0 \\
& \hat{H}_{1}|+-\rangle=\varepsilon\left(2 \hat{P}_{12}-\hat{1}-\hat{\sigma}_{1 z} \hat{\sigma}_{2 z}\right)|+-\rangle=2 \varepsilon|-+\rangle
\end{aligned}
$$

$$
\begin{aligned}
& \hat{H}_{1}|-+\rangle=\varepsilon\left(2 \hat{P}_{12}-\hat{1}-\hat{\sigma}_{1 z} \hat{\sigma}_{2 z}\right)|-+\rangle=2 \varepsilon|+-\rangle \\
& \hat{H}_{1}|--\rangle=\varepsilon\left(2 \hat{P}_{12}-\hat{1}-\hat{\sigma}_{1 z} \hat{\sigma}_{2 z}\right)|--\rangle
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
& \hat{H}|++\rangle=\left(\hat{H}_{0}+\hat{H}_{1}\right)|++\rangle=-2 A|++\rangle \\
& \hat{H}|+-\rangle=2 \varepsilon|-+\rangle \\
& \hat{H}|-+\rangle=2 \varepsilon|+-\rangle \\
& \hat{H}|--\rangle=\left(\hat{H}_{0}+\hat{H}_{1}\right)|--\rangle=2 A|--\rangle
\end{aligned}
$$

We consider the subsystem $(|+-\rangle$ and $|-+\rangle)$

$$
\hat{H}=\left(\begin{array}{cc}
0 & 2 \varepsilon \\
2 \varepsilon & 0
\end{array}\right)
$$

((Mathematica))
Clear["Global`*"];
$H=\left(\begin{array}{cc}0 & 2 \epsilon \\ 2 \epsilon & 0\end{array}\right) ;$
Eigensystem [H] // Simplify
$\{\{-2 \in, 2 \in\},\{\{-1,1\},\{1,1\}\}\}$

For $E=2 \varepsilon$, (symmetric state)

$$
\left|\psi_{S}\right\rangle=\frac{1}{\sqrt{2}}(|+-\rangle+|-+\rangle)
$$

For $E=-2 \varepsilon$, (anti-symmetric state)

$$
\left|\psi_{A}\right\rangle=\frac{1}{\sqrt{2}}(|+-\rangle-|-+\rangle)
$$

(c) We use the perturbation theory

For the energy level $E=2 A$ (non-degenerate case)

$$
E_{2 A}=2 A+\langle--| \hat{H}_{1}|--\rangle=2 A
$$

For the energy level $E=-2 A$ (non-degenerate case)

$$
E_{-2 A}=-2 A+\langle++| \hat{H}_{1}|++\rangle=-2 A
$$



### 31.28 Spin wave

$$
\hat{H}=-\frac{J}{2} \sum_{n} \hat{\boldsymbol{\sigma}}_{n} \cdot \hat{\boldsymbol{\sigma}}_{n+1}
$$

With this Hamiltonian we have a complete description of the ferromagnet.

where $\hat{\boldsymbol{\sigma}}_{n} \cdot \hat{\boldsymbol{\sigma}}_{n+1}$ interchanges the spins of the $n$-th and ( $n+1$ )-th electrons.

For the ground state all spins are up $(|+\rangle$, so if you exchange a particular pair of spins, one can get back the original state. The ground state is a stationary state: $-J / 2$ for each pair of spins. That is, the energy of the system in the ground state is $-J / 2$ per spin.

It is convenient to measure the energies with respect to the ground state. Our new Hamiltonian is

$$
\hat{H}=-J \sum_{n}\left(\hat{P}_{n, n+1}-1\right)
$$



With this Hamiltonian, the energy of the ground state is zero. Here we define the state $\left|x_{n}\right\rangle$ where all the spins except for the one on the spin at $x_{\mathrm{n}}$.

$$
\begin{aligned}
\hat{H}\left|x_{5}\right\rangle & =-J \sum_{n}\left(\hat{P}_{n, n+1}-1\right)\left|x_{5}\right\rangle \\
& =-J\left(\hat{P}_{5,6}-1\right)\left|x_{5}\right\rangle-J\left(\hat{P}_{4,5}-1\right)\left|x_{5}\right\rangle \\
& =-J\left(\left|x_{6}\right\rangle-2\left|x_{5}\right\rangle+\left|x_{4}\right\rangle\right)
\end{aligned}
$$

where

$$
\hat{P}_{45}\left|x_{5}\right\rangle=\left|x_{4}\right\rangle, \hat{P}_{56}\left|x_{5}\right\rangle=\left|x_{6}\right\rangle, \hat{P}_{78}\left|x_{5}\right\rangle=\left|x_{5}\right\rangle, \text { and } \quad \hat{P}_{34}\left|x_{5}\right\rangle=\left|x_{5}\right\rangle .
$$

Similarly,

$$
\hat{H}\left|x_{n}\right\rangle=-J\left(\left|x_{n+1}\right\rangle-2\left|x_{n}\right\rangle+\left|x_{n-1}\right\rangle\right.
$$

$$
\hat{H}\left|x_{n+1}\right\rangle=-J\left(\left|x_{n+2}\right\rangle-2\left|x_{n+1}\right\rangle+\left|x_{n}\right\rangle\right.
$$

Here we consider

$$
|\psi\rangle=\sum_{n} C_{n}\left|x_{n}\right\rangle
$$

Eigenvalue problem

$$
\hat{H}|\psi\rangle=E|\psi\rangle
$$

or

$$
\sum_{n} C_{n} \hat{H}\left|x_{n}\right\rangle=E \sum_{n} C_{n}\left|x_{n}\right\rangle
$$

or

$$
\sum_{n} C_{n} \hat{H}\left|x_{n}\right\rangle=E \sum_{n} C_{n}\left|x_{n}\right\rangle
$$

or

$$
\sum_{n}(-J)\left(C_{n}\left|x_{n+1}\right\rangle-2 C_{n}\left|x_{n}\right\rangle+C_{n}\left|x_{n-1}\right\rangle=E \sum_{n} C_{n}\left|x_{n}\right\rangle\right.
$$

or

$$
\sum_{n}(-J)\left(C_{n-1}-2 C_{n}+C_{n+1}\right)\left|x_{n}\right\rangle=E \sum_{n} C_{n}\left|x_{n}\right\rangle
$$

or

$$
(-J)\left(C_{n-1}-2 C_{n}+C_{n+1}\right)=E C_{n}
$$

Let us take as a trial function

$$
\begin{aligned}
& C_{n}=e^{i k x_{n}} \\
& (-J)\left(e^{i k\left(x_{n}-b\right)}-2 e^{i k x_{n}}+e^{i k\left(x_{n}+b\right)}\right)=E e^{i k x_{n}} \\
& E=2 J[1-\cos (k b)] \quad \text { (energy dispersion) }
\end{aligned}
$$

The difference energy solutions corresponds to "waves" of down spin-called "spin waves. For $k b \ll 1, E$ is approximated by

$$
E=2 A \frac{k^{2} b^{2}}{2}=A k^{2} b^{2}
$$

## REFERENCES

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