# Chapter 32 The heat conduction: Green's function <br> Masatsugu Sei Suzuki <br> Department of Physics, SUNY at Binghamton <br> (Date: November 24, 2010) 

### 32.1 Thermal conductivity

Fourier's law

$$
\mathbf{J}_{u}=-K \nabla T,
$$

describes the energy flux density in terms of the thermal conductivity $K$ and the temperature gradient. This forms assumes that there is a net transport of energy, but not of particles. The equation of continuity for the energy density is

$$
C \frac{\partial T}{\partial t}+\nabla \cdot \mathbf{J}_{u}=0
$$

where $C$ is the heat capacity per unit volume. We combine these two equations to obtain the heat conduction

$$
C \frac{\partial T}{\partial t}-K \nabla^{2} T=0
$$

or

$$
\frac{\partial T}{\partial t}-D \nabla^{2} T=0
$$

where $D=K / C$ is called the thermal diffusivity. This equation describes the time-dependent diffusion for the temperature.

### 32.2 Green's function

We want to solve the heat equation in one dimension,

$$
\left(\frac{\partial}{\partial t}-D \frac{\partial^{2}}{\partial x^{2}}\right) T(x, t)=Q(x, t)
$$

where $Q(x, t)$ is a heat source. First we start to find the Green's function which is defined by

$$
\left(\frac{\partial}{\partial t}-D \frac{\partial^{2}}{\partial x^{2}}\right) G(x, t)=-\delta(x) \delta(t)
$$

Using the Fourier transform, we have

$$
\begin{aligned}
& G(x, t)=\frac{1}{2 \pi} \iint G(k, \omega) e^{i(k x-\omega t)} d k d \omega \\
& \delta(x) \delta(t)=\frac{1}{(2 \pi)^{2}} \iint e^{i(k x-\omega t)} d k d \omega
\end{aligned}
$$

The Fourier transform of the above equation:

$$
\left(-i \omega+D k^{2}\right) G(k, \omega)=-\frac{1}{2 \pi}
$$

or

$$
G(k, \omega)=-\frac{i}{2 \pi} \frac{1}{\omega+i D k^{2}} .
$$

Then the inverse Fourier transform is obtained as

$$
\begin{aligned}
G(x, t) & =\frac{1}{2 \pi} \iint G(k, \omega) e^{i(k x-\omega t)} d k d \omega \\
& =-\frac{i}{(2 \pi)^{2}} \iint \frac{1}{\omega+i D k^{2}} e^{i(k x-\omega t)} d k d \omega \\
& =-\frac{i}{(2 \pi)^{2}} \int_{-\infty}^{\infty} d k e^{i k x} \int_{-\infty}^{\infty} d \omega \frac{e^{-i \omega t}}{\omega+i D k^{2}}
\end{aligned}
$$

We now calculate the integral

$$
I=\int_{-\infty}^{\infty} d \omega \frac{e^{-i \omega t}}{\omega+i D k^{2}}
$$


(i) $t>0$

For $t>0$, we use the contour $C_{1}$ in lower half plane (the complex plane). The contour integral along the path $\Gamma_{1}$ (radius $R=\infty$ ) is zero according to the Jordan's lemma. Note that the contour $C_{1}$ is the clock-wise, and that there is a simple pole at

$$
z=-i D k^{2}
$$

inside the contour $\mathrm{C}_{1}$. Then we have

$$
\int_{-\infty}^{\infty} d \omega \frac{e^{-i \omega t}}{\omega+i D k^{2}}+\int_{\Gamma_{1}} d z \frac{e^{-i z t}}{z+i D k^{2}}=\oint_{C_{1}} d z \frac{e^{-i z t}}{z+i D k^{2}}=-2 \pi i \operatorname{Res}\left(z=-i D k^{2}\right)
$$

for $t>0$. Since

$$
I=\int_{-\infty}^{\infty} d \omega \frac{e^{-i \omega t}}{\omega+i D k^{2}}=\oint_{C_{1}} d z \frac{e^{-i z t}}{z+i D k^{2}}==-2 \pi i \exp \left(-D k^{2} t\right)
$$

(ii) $t<0$

For $t<0$, we use the contour $C_{2}$ in the upper half plane (the complex plane). The contour integral along the path $\Gamma_{2}$ (radius $R=\infty$ ) is zero according to the Jordan's lemma. Note that the contour $\mathrm{C}_{2}$ is the clock-wise, and that there is no pole.

$$
\int_{-\infty}^{\infty} d \omega \frac{e^{-i \omega t}}{\omega+\frac{i \mu k^{2}}{c}}+\int_{\Gamma_{1}} d z \frac{e^{-i z t}}{z+\frac{i \mu k^{2}}{c}}=\oint_{C_{1}} d z \frac{e^{-i z t}}{z+\frac{i \mu k^{2}}{c}}=0
$$

or

$$
I=\int_{-\infty}^{\infty} d \omega \frac{e^{-i \omega t}}{\omega+\frac{i \mu k^{2}}{c}}=0
$$

Then we have

$$
I=\int_{-\infty}^{\infty} d \omega \frac{e^{-i \omega t}}{\omega+i D k^{2}}=(-2 \pi i) e^{-D k^{2} t} \Theta(t)
$$

The Green's function is obtained as

$$
\begin{aligned}
G(x, t) & =-\frac{1}{2 \pi} \Theta(t) \int_{-\infty}^{\infty} d k \exp \left[i k x-D k^{2} t\right] \\
& =-\frac{1}{\sqrt{4 \pi D t}} \exp \left(-\frac{x^{2}}{4 D t}\right) \Theta(t)
\end{aligned}
$$

((Mathematica))

$$
\begin{aligned}
& \int_{-\infty}^{\infty} \operatorname{Exp}\left[-D k^{2} t+\dot{i} k x\right] d \mathrm{~d} / / \\
& \text { Simplify }[\#,\{x>0, \quad t>0, \quad D>0\}] \& \\
& \frac{e^{-\frac{x^{2}}{4 D t}} \sqrt{\pi}}{\sqrt{D t}}
\end{aligned}
$$

### 32.3 General solution

$$
\left(\frac{\partial}{\partial t}-D \frac{\partial^{2}}{\partial x^{2}}\right) T(x, t)=Q(x, t)
$$

where $Q(x, t)$ is a heat source. Green's function satisfies

$$
\left(\frac{\partial}{\partial t}-D \frac{\partial^{2}}{\partial x^{2}}\right) G\left(x-x^{\prime}, t-t^{\prime}\right)=-\delta\left(x-x^{\prime}\right) \delta\left(t-t^{\prime}\right)
$$

The solution of the differential equation for the heat conduction is

$$
T(x, t)=-\iint d x^{\prime} d t^{\prime} G\left(x-x^{\prime}, t-t^{\prime}\right) Q\left(x^{\prime}, t^{\prime}\right)
$$

where

$$
, G\left(x-x^{\prime}, t-t^{\prime}\right)=-\frac{1}{\sqrt{4 \pi D\left(t-t^{\prime}\right)}} \exp \left[-\frac{\left(x-x^{\prime}\right)^{2}}{4 D\left(t-t^{\prime}\right)}\right] \Theta\left(t-t^{\prime}\right)
$$

or

$$
T(x, t)=\int_{-\infty}^{\infty} d x^{\prime} \int_{0}^{t} d t^{\prime} \frac{1}{\sqrt{4 \pi D\left(t-t^{\prime}\right)}} \exp \left[-\frac{\left(x-x^{\prime}\right)^{2}}{4 D\left(t-t^{\prime}\right)}\right] Q\left(x^{\prime}, t^{\prime}\right)
$$

((Example)) Dirac comb
We assume that

$$
\begin{aligned}
Q(x, t) & =\delta(x) f(t) \\
T(x, t) & =\int_{-\infty}^{\infty} d x^{\prime} \int_{0}^{t} d t^{\prime} \frac{1}{\sqrt{4 \pi D\left(t-t^{\prime}\right)}} \exp \left[-\frac{\left(x-x^{\prime}\right)^{2}}{4 D\left(t-t^{\prime}\right)}\right] \delta\left(x^{\prime}\right) f\left(t^{\prime}\right) \\
& =\int_{0}^{t} d t^{\prime} \frac{1}{\sqrt{4 \pi D\left(t-t^{\prime}\right)}} \exp \left[-\frac{x^{2}}{4 D\left(t-t^{\prime}\right)}\right] f\left(t^{\prime}\right)
\end{aligned}
$$

We further assume that $f(t)$ is described by a Dirac comb given by

$$
f(t)=\sum_{n} \delta\left(t-n T_{0}\right)
$$



Fig. Dirac comb with the heat pulses at $x=0$ applied periodically ( $T_{0}$ is a period time).

Then we have

$$
\begin{aligned}
T(x, t) & =\int_{0}^{t} d t^{\prime} \frac{1}{\sqrt{4 \pi D\left(t-t^{\prime}\right)}} \exp \left[-\frac{x^{2}}{4 D\left(t-t^{\prime}\right)}\right] f\left(t^{\prime}\right) \\
& =\frac{1}{\sqrt{4 \pi D t}} \exp \left(-\frac{x^{2}}{4 D t}\right)+\frac{2}{\sqrt{4 \pi D}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{t-n T_{0}}} \exp \left[-\frac{x^{2}}{4 D\left(t-n T_{0}\right)}\right] \Theta\left(t-n T_{0}\right)
\end{aligned}
$$



Fig. $\quad$ Simulation. Plot of $T(x, t)$ vs $t$ with $x$ as a parameter $(\mathrm{x}=1-10) . D=1 . T_{0}=5$.

### 32.3 Initial condition

We consider the solution of the differential equation given by

$$
\left(\frac{\partial}{\partial t}-D \frac{\partial^{2}}{\partial x^{2}}\right) T(x, t)=0
$$

with the initial condition given by

$$
T(x, t=0)=T_{0}(x)
$$

which is an initial temperature distribution at $t=0$. We assume that the solution is given by

$$
T(x, t)=\Theta(t) u(x, t) .
$$

Since there is no heat source, this solution satisfies

$$
\left(\frac{\partial}{\partial t}-D \frac{\partial^{2}}{\partial x^{2}}\right) T(x, t)=0
$$

or

$$
\Theta(t)\left[\frac{\partial}{\partial t} u(x, t)-D \frac{\partial^{2}}{\partial x^{2}} u(x, t)\right]=-\delta(t) u(x, t)=-\delta(t) T_{0}(x),
$$

or

$$
\frac{\partial}{\partial t} u(x, t)-D \frac{\partial^{2}}{\partial x^{2}} u(x, t)=-\delta(t) T_{0}(x)
$$

where

$$
\Theta^{\prime}(t)=\delta(t)
$$

The Green's function satisfies

$$
\frac{\partial}{\partial t} G(x, t)-D \frac{\partial^{2}}{\partial x^{2}} G(x, t)=-\delta(x) \delta(t) .
$$

The form of $G(x, t)$ is given by

$$
G(x, t)=-\frac{1}{\sqrt{4 \pi D t}} \exp \left(-\frac{x^{2}}{4 D t}\right) \Theta(t)
$$

The solutions of $u(x, t)$ and $T(x, t)$ are obtained as

$$
\begin{aligned}
u(x, t) & =-\iint d \tau d \xi G(x-\xi, t-\tau) \delta(\tau) T_{0}(\xi) \\
& =-\int_{-\infty}^{\infty} d \xi G(x-\xi, t) T_{0}(\xi)
\end{aligned}
$$

and

$$
T(x, t)=-\Theta(t) \int_{-\infty}^{\infty} d \xi G(x-\xi, t) T_{0}(\xi)
$$

## ((Example)) Dirac delta function

$$
T_{0}(\xi)=\delta(\xi)
$$

Then we have

$$
\begin{aligned}
T(x, t) & =-\Theta(t) \int_{-\infty}^{\infty} d \xi G(x-\xi, t) \delta(\xi) \\
& =\frac{1}{\sqrt{4 \pi D t}} \exp \left(-\frac{x^{2}}{4 D t}\right) \Theta(t)
\end{aligned}
$$



Fig. $\quad t$ dependence of $T(x, t)$ with each $x . D=1$.


Fig. $\quad T(x, t)$ has a maximum $\left(=\frac{1}{\sqrt{2 e \pi} x}\right)$ at $t=\frac{x^{2}}{2 D}$ as a function of $t$ where $x$ is fixed as a parameter. $D=1$. This figure is the same as the above figure.

The derivative of $T(x, t)$ with respect to $t$ is obtained as

$$
\frac{\partial T(x, t)}{\partial t}=\frac{D\left(x^{2}-2 D t\right) \exp \left(-\frac{x^{2}}{4 D t}\right)}{8 \sqrt{\pi}(D t)^{5 / 2}} .
$$

which implies that $T(x, t)$ has a maximum $\left(=\frac{1}{\sqrt{2 e \pi x}}\right)$ as a function of $t$ at

$$
t=\frac{x^{2}}{2 D}
$$

The pulse spreads out with increasing time. The mean square value of $x$ is given by

$$
<x^{2}>=\frac{\int x^{2} T(x, t) d x}{\int T(x, t) d x}=2 D t
$$

The root mean square value is

$$
x_{r m s}=\sqrt{\left\langle x^{2}\right\rangle}=\sqrt{2 D t}
$$

### 32.4 Brownian motion

The width of the distribution increases as $\sqrt{t}$, which is a general characteristic of diffusion and random walk problem in one dimension. The connection with Brownian motion or the random walk problem follows if we let $t_{0}$ be the duration of each step of a random walk.

$$
t=N t_{0},
$$

where $N$ is the number of steps. It follows that

$$
x_{r m s}(t)=\sqrt{2 D t_{0}} \sqrt{N}
$$

So that the rms displacement is proportional to the square of the number of steps. This is the result of the Brownian motion, the random motion of suspensions of small particles in liquids (Kittel and Kroemer).

### 32.5 Experiment

In the sophomore laboratory (Physics 227, Binghamton University) we have an experiment of "Thermal Wave." The purpose of this experiment is to measure the thermal diffusivity $(D)$ of a brass rod.


Fig. 1 Diagram of apparatus (from the report by Corrine Blum (Fall, 2009, Sophomore Laboratory).

Figure 1 shows a diagram of the apparatus used. There are three temperature sensors located on the brass rod. The separation distance between adjacent sensors is $\Delta x(=10 \mathrm{~cm})$. A heat source at the one of the edge, is applied for the time $T$ on and the same time $T$ off, in a square wave for one hour, where $T_{1}=5$ and 10 minutes (two trials). The temperature data from each sensor are recorded as a function of time using the computer. Figures 2 and 3 show the temperature data measured by the three sensors, as a function of time t , where $T=5$ minutes for the trial 1 (Fig.2) and the trial (Fig.3). As shown in Figs. 2 and 3, the heat propagates from the sensor 1 to the sensor 2 and from the sensor 2 to the sensor 3 in the same finite time $\Delta t$. The diffusion diffusivity D can be estimates as

$$
D=\frac{(\Delta x)^{2}}{2 \Delta t}
$$

The thermal diffusivity is found to be $3.63 \times 10^{-5} \mathrm{~m}^{2} / \mathrm{s}$ for the trial 1 and $3.33 \times 10^{-5} \mathrm{~m}^{2} / \mathrm{s}$ for the trial 2. Note that the theoretical value of D for the brass rod is $3.376 \times 10^{-5} \mathrm{~m}^{2} / \mathrm{s}$.


Fig. 2 Temperatures measured by three sensors as a function of time (trial 1 ). $T=5$ minutes.


Fig. 3 Temperatures measured by three sensors as a function of time (trial 2). $T=10$ minutes.

## REFERENCES

C. Kittel and H. Kroemer, Thermal Physics, second edition (W.H. Freeman and Company, New York, 1980).
Walter Appel, Mathematics for Physics \& Physicists (Princeton University Press, Princeton and Oxford, 2007).
Corrine Blum Report of Thermal Wave (Fall, 2009, Phys.227, Binghamton University).
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