# Chapter 34 <br> Maxwell's equation: plane waves <br> Masatsugu Sei Suzuki <br> Department of Physics, SUNY at Binghamton <br> (Date: November 21, 2010) 

Maxwell's equation
Snell's law
Fresnel's equation
Brewster's angle
Skin depth
Wave guides
James Clerk Maxwell (13 June 1831- 5 November 1879) was a Scottish theoretical physicist and mathematician. His most important achievement was formulating classical electromagnetic theory. This united all previously unrelated observations, experiments and equations of electricity, magnetism and even optics into a consistent theory. His set of equations-Maxwell's equations-demonstrated that electricity, magnetism and even light are all manifestations of the same phenomenon, the electromagnetic field. Subsequently, all other classic laws or equations of these disciplines were simplified cases of Maxwell's equations. Maxwell's work in electromagnetism has been called the "second great unification in physics", after the first one carried out by Isaac Newton.

http://en.wikipedia.org/wiki/James_Clerk_Maxwell
The Maxwell's equation in vacuum (SI units)

$$
\nabla \cdot \mathbf{E}=0, \quad \nabla \cdot \mathbf{B}=0, \quad \nabla \times \mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B}=\mu_{0} \varepsilon_{0} \frac{\partial \mathbf{E}}{\partial t}
$$

### 34.1 Wave equations in vacuum

## From these equations, we have

$$
\begin{aligned}
\nabla \times(\nabla \times \mathbf{B}) & =\nabla(\nabla \cdot \mathbf{B})-\nabla^{2} \mathbf{B} \\
& =\mu_{0} \varepsilon_{0} \nabla \times \frac{\partial \mathbf{E}}{\partial t} \\
& =\mu_{0} \varepsilon_{0} \frac{\partial}{\partial t} \nabla \times \mathbf{E}=-\mu_{0} \varepsilon_{0} \frac{\partial^{2} \mathbf{B}}{\partial t^{2}}
\end{aligned}
$$

or

$$
\nabla^{2} \mathbf{B}=\mu_{0} \varepsilon_{0} \frac{\partial^{2} \mathbf{B}}{\partial t^{2}}=\frac{1}{c^{2}} \frac{\partial^{2} \mathbf{B}}{\partial t^{2}}
$$

or

$$
\left(\nabla^{2}-\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}}\right) \mathbf{B}=0, \quad \quad \text { (wave equation) }
$$

where

$$
c=\frac{1}{\sqrt{\varepsilon_{0} \mu_{0}}}
$$

Similarly, we have

$$
\nabla \times(\nabla \times \mathbf{E})=\nabla(\nabla \cdot \mathbf{E})-\nabla^{2} \mathbf{E}=-\frac{\partial}{\partial t}(\nabla \times \mathbf{B})=-\mu_{0} \varepsilon_{0} \frac{\partial^{2}}{\partial t^{2}} \mathbf{E},
$$

or

$$
\nabla^{2} \mathbf{E}=\mu_{0} \varepsilon_{0} \frac{\partial^{2}}{\partial t^{2}} \mathbf{E}=\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}} \mathbf{E} . \quad \text { (wave equation) }
$$

We consider the special case when $\boldsymbol{E}$ or $\boldsymbol{B}$ depend only on $x$. In this case the equation for the field becomes

$$
\frac{\partial^{2}}{\partial t^{2}} f=c^{2} \frac{\partial^{2}}{\partial x^{2}} f,
$$

where by $f$ is understood any component of the vector $\boldsymbol{E}$ or $\boldsymbol{B}$.

$$
\left(\frac{\partial}{\partial t}-c \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}+c \frac{\partial}{\partial x}\right) f=0 .
$$

We introduce new variables

$$
\begin{aligned}
& \xi=t-\frac{x}{c} \\
& \eta=t+\frac{x}{c}
\end{aligned}
$$

So that the equation for $f$ becomes

$$
\frac{\partial^{2} f}{\partial \xi \partial \eta}=0 .
$$

The solution obviously has the form

$$
f=f_{1}(\xi)+f_{2}(\eta),
$$

where $f_{1}$ and $f_{2}$ are arbitrary function.
or

$$
f=f_{1}\left(t-\frac{x}{c}\right)+f_{2}\left(t+\frac{x}{c}\right) .
$$

The function $f_{1}$ represents a plane wave moving in the positive direction along the $x$ axis. The function $f_{2}$ represents a plane wave moving in the negative direction along the $x$ axis.

### 34.2 Method using Fourier transform

We use the Fourier transformation technique.

$$
\begin{aligned}
& F(x, \omega)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} e^{-i \omega t} f(x, t) d t \\
& f(x, t)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} e^{i \omega t} F(x, \omega) d \omega \\
& \frac{\partial^{2}}{\partial x^{2}} f-\frac{1}{c^{2}} f=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} e^{i \omega t}\left[\frac{\partial^{2}}{\partial x^{2}} F(x, \omega)+\frac{\omega^{2}}{c^{2}} F(x, \omega)\right] d \omega=0
\end{aligned}
$$

Then we have

$$
\frac{\partial^{2}}{\partial x^{2}} F(x, \omega)+k^{2} F(x, \omega)=0
$$

where

$$
k=\frac{\omega}{c} .
$$

The solution of this equation is

$$
F(x, \omega)=g(\omega) \frac{e^{ \pm i k x}}{\sqrt{2 \pi}}=g(\omega) \frac{e^{-i \frac{\omega x}{c}}}{\sqrt{2 \pi}}
$$

where $g(\omega)$ is an arbitrary function of $\omega$. Finally we get

$$
f(x, t)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} g(\omega) e^{i \omega\left(t \pm \frac{x}{c}\right)} d \omega
$$

Therefore, $f(x, t)$ is a function of $t \pm \frac{x}{C}$.

### 34.3 Plane wave representation

We suppose that

$$
\begin{aligned}
& \mathbf{E}=\operatorname{Re}\left[\widetilde{\mathbf{E}}_{0} e^{i(\mathbf{k} \cdot \boldsymbol{r}-\omega t)}\right]=\operatorname{Re}\left[\widetilde{\mathbf{E}}_{0} e^{i \theta}\right]=\frac{\widetilde{\mathbf{E}}_{0} e^{i \theta}+\widetilde{\mathbf{E}}_{0}{ }^{*} e^{-i \theta}}{2} \\
& \mathbf{B}=\operatorname{Re}\left[\widetilde{\mathbf{B}}_{0} e^{i(\mathrm{k} \cdot \boldsymbol{r}-\omega t)}\right]=\operatorname{Re}\left[\widetilde{\mathbf{B}}_{0} e^{i \theta}\right]=\frac{\widetilde{\mathbf{B}}_{0} e^{i \theta}+\widetilde{\mathbf{B}}_{0}{ }^{*} e^{-i \theta}}{2}
\end{aligned}
$$

From

$$
\nabla \cdot \mathbf{E}=0, \quad \text { and } \quad \nabla \cdot \mathbf{B}=0
$$

we have

$$
\mathbf{k} \cdot \widetilde{\mathbf{E}}_{0}=0, \quad \text { and } \quad \mathbf{k} \cdot \widetilde{\mathbf{B}}_{0}=0
$$

From

$$
\nabla \times \mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t} .
$$

we have

$$
i\left(\mathbf{k} \times \widetilde{\mathbf{E}}_{0}\right)=i \omega \widetilde{\mathbf{B}}_{0},
$$

or

$$
\left(\mathbf{k} \times \widetilde{\mathbf{E}}_{0}\right)=c k \widetilde{\mathbf{B}}_{0}
$$

or

$$
\left(\hat{\mathbf{k}} \times \widetilde{\mathbf{E}}_{0}\right)=c \widetilde{\mathbf{B}}_{0}
$$

or

$$
\widetilde{\mathbf{B}}_{0}=\frac{1}{c}\left(\hat{\mathbf{k}} \times \widetilde{\mathbf{E}}_{0}\right) .
$$

((Note)) Dispersion relation
From

$$
\nabla^{2} \mathbf{E}=\mu_{0} \varepsilon_{0} \frac{\partial^{2}}{\partial t^{2}} \mathbf{E}=\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}} \mathbf{E}
$$

we have a dispersion relation

$$
\omega=c k=c|\mathbf{k}| .
$$

### 34.4 Energy and momentum in electromagnetic wave

The energy density:

$$
u=\frac{1}{2}\left(\varepsilon_{0} \mathbf{E}^{2}+\frac{1}{\mu_{0}} \mathbf{B}^{2}\right)
$$

The Poynting vector:

$$
\mathbf{S}=\frac{1}{\mu_{0}}(\mathbf{E} \times \mathbf{B})
$$

Calculation of $\underline{\mathbf{E}^{2}}$

$$
\begin{gathered}
\mathbf{E}^{2}=\left(\frac{\widetilde{\mathbf{E}}_{0} e^{i \theta}+\widetilde{\mathbf{E}}_{0}{ }^{*} e^{-i \theta}}{2}\right)^{2}=\frac{1}{4}\left(\widetilde{\mathbf{E}}_{0}{ }^{2} e^{2 i \theta}+\widetilde{\mathbf{E}}_{0}^{* 2} e^{-2 i \theta}+2 \widetilde{\mathbf{E}}_{0} \cdot \widetilde{\mathbf{E}}_{0}^{*}\right) \\
\left\langle\mathbf{E}^{2}\right\rangle=\frac{1}{T} \int_{0}^{T} \mathbf{E}^{2} d t=\frac{1}{2} \widetilde{\mathbf{E}}_{0} \cdot \widetilde{\mathbf{E}}_{0}^{*}=\frac{1}{2}\left|\widetilde{\mathbf{E}}_{0}\right|^{2}
\end{gathered}
$$

where $T=\frac{2 \pi}{\omega}$ and $\frac{1}{T} \int_{0}^{T} e^{ \pm 2 i \theta} d t=0$. Similarly, we have

$$
\left\langle\mathbf{B}^{2}\right\rangle=\frac{1}{T} \int_{0}^{T} \mathbf{B}^{2} d t=\frac{1}{2} \widetilde{\mathbf{B}}_{0} \cdot \widetilde{\mathbf{B}}_{0}^{*}=\frac{1}{2}\left|\widetilde{\mathbf{B}}_{0}\right|^{2}
$$

The time-average of the energy density is given by

$$
\langle u\rangle=\frac{1}{2}\left(\varepsilon_{0}\left\langle\mathbf{E}^{2}\right\rangle+\frac{1}{\mu_{0}}\left\langle\mathbf{B}^{2}\right\rangle\right)=\frac{1}{4}\left(\varepsilon_{0}\left|\widetilde{\mathbf{E}}_{0}\right|^{2}+\frac{1}{\mu_{0}}\left|\widetilde{\mathbf{B}}_{0}\right|^{2}\right)
$$

Here we note

$$
\left|\widetilde{\mathbf{B}}_{0}\right|^{2}=\frac{1}{c^{2}}\left(\hat{\mathbf{k}} \times \widetilde{\mathbf{E}}_{0}\right) \cdot\left(\hat{\mathbf{k}} \times \widetilde{\mathbf{E}}_{0}{ }^{*}\right)=\frac{1}{c^{2}}\left[\widetilde{\mathbf{E}}_{0} \cdot \widetilde{\mathbf{E}}_{0}{ }^{*}-\left(\hat{\mathbf{k}} \cdot \widetilde{\mathbf{E}}_{0}\right)\left(\hat{\mathbf{k}} \cdot \widetilde{\mathbf{E}}_{0}{ }^{*}\right)\right]=\frac{1}{c^{2}}\left|\widetilde{\mathbf{E}}_{0}\right|^{2}
$$

Then we have

$$
\langle u\rangle=\frac{1}{4}\left(\varepsilon_{0}\left|\widetilde{\mathbf{E}}_{0}\right|^{2}+\frac{1}{\mu_{0} c^{2}}\left|\widetilde{\mathbf{E}}_{0}\right|^{2}\right)=\frac{1}{2} \varepsilon_{0}\left|\widetilde{\mathbf{E}}_{0}\right|^{2}
$$

## Calculation of poynting vector $\boldsymbol{S}$

$$
\begin{aligned}
& \mathbf{S}=\frac{1}{\mu_{0}}(\mathbf{E} \times \mathbf{B})=\frac{1}{4 \mu_{0}}\left(\widetilde{\mathbf{E}}_{0} e^{i \theta}+\widetilde{\mathbf{E}}_{0}{ }^{*} e^{-i \theta}\right) \times\left(\widetilde{\mathbf{B}}_{0} e^{i \theta}+\widetilde{\mathbf{B}}_{0}{ }^{*} e^{-i \theta}\right) \\
& \langle\mathbf{S}\rangle=\frac{1}{4 \mu_{0}}\left(\widetilde{\mathbf{E}}_{0} \times \widetilde{\mathbf{B}}_{0}{ }^{*}+\widetilde{\mathbf{E}}_{0}{ }^{*} \times \widetilde{\mathbf{B}}_{0}\right)=\frac{1}{2 \mu_{0}} \operatorname{Re}\left[\widetilde{\mathbf{E}}_{0} \times \widetilde{\mathbf{B}}_{0}{ }^{*}\right]
\end{aligned}
$$

Note that

$$
\widetilde{\mathbf{E}}_{0} \times \widetilde{\mathbf{B}}_{0}^{*}=\frac{1}{c} \widetilde{\mathbf{E}}_{0} \times\left(\hat{\mathbf{k}} \times \widetilde{\mathbf{E}}_{0}^{*}\right)=\hat{\mathbf{k}} \frac{1}{c}\left|\widetilde{\mathbf{E}}_{0}\right|^{2}
$$

Then we have

$$
\begin{aligned}
\langle\mathbf{S}\rangle & =\frac{1}{4 \mu_{0}}\left(\widetilde{\mathbf{E}}_{0} \times \widetilde{\mathbf{B}}_{0}{ }^{*}+\widetilde{\mathbf{E}}_{0}{ }^{*} \times \widetilde{\mathbf{B}}_{0}\right) \\
& =\frac{1}{2 \mu_{0}} \operatorname{Re}\left[\widetilde{\mathbf{E}}_{0} \times \widetilde{\mathbf{B}}_{0}{ }^{*}\right] \\
& =\frac{1}{2 \mu_{0}} \operatorname{Re}\left[\hat{\mathbf{k}} \frac{1}{c}\left|\widetilde{\mathbf{E}}_{0}\right|^{2}\right] \\
& =\frac{1}{2 \mu_{0}} \hat{\mathbf{k}} \frac{1}{c}\left|\widetilde{\mathbf{E}}_{0}\right|^{2}
\end{aligned}
$$

or

$$
\langle\mathbf{S}\rangle=c \frac{1}{2 \mu_{0}} \hat{\mathbf{k}} \frac{1}{c^{2}}\left|\widetilde{\mathbf{E}}_{0}\right|^{2}=c \hat{\mathbf{k}} \frac{\varepsilon_{0}}{2}\left|\widetilde{\mathbf{E}}_{0}\right|^{2}=c \hat{\mathbf{k}}\langle u\rangle
$$


$\langle S\rangle$ is the energy flux (energy per unit area per unit time). We define the intensity $I$ given by

$$
I=\langle S\rangle=c\langle u\rangle=\frac{1}{2} c \varepsilon_{0}\left|\widetilde{\mathbf{E}}_{0}\right|^{2}=\frac{1}{2 \mu_{0} c}\left|\widetilde{\mathbf{E}}_{0}\right|^{2}
$$

We now consider the photon (the velocity is $c$ ) flows. During the time $\Delta t$, the total energy passing through the area $A$ is

$$
E=c \Delta t A\langle u\rangle=A \Delta t S
$$

where the volume is $c \Delta t A$ and the energy density is $\langle u\rangle$. Here we define the momentum density $\boldsymbol{G}$ by

$$
\mathbf{G}=\frac{1}{c^{2}} \mathbf{S}
$$

### 34.5 Reflection and transmission

$$
\nabla \cdot \mathbf{E}=0, \quad \nabla \cdot \mathbf{B}=0, \quad \nabla \times \mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t}, \nabla \times \mathbf{B}=\mu \varepsilon \frac{\partial \mathbf{E}}{\partial t} .
$$

with the velocity $v$ defined by

$$
v=\frac{1}{\sqrt{\varepsilon \mu}}=\frac{c}{n}
$$

where $n$ is the index of refraction;

$$
n=\sqrt{\frac{\varepsilon \mu}{\varepsilon_{0} \mu_{0}}}
$$

For most materials, $\mu=\mu_{0}$


Incident wave

$$
\begin{aligned}
& \widetilde{\mathbf{E}}_{I}=\widetilde{E}_{01} e^{i\left(k_{1}-\omega t\right)} \hat{\chi} \\
& \widetilde{\mathbf{B}}_{I}=\widetilde{B}_{0 I} e^{i\left(k_{1}-\omega t\right)} \hat{y} \\
& \widetilde{B}_{0 I}=\frac{1}{v_{1}} \widetilde{E}_{0 I}
\end{aligned}
$$

## Reflected wave

$$
\begin{aligned}
& \widetilde{\mathbf{E}}_{R}=\widetilde{E}_{0 R} e^{i\left(-k_{1} z-\omega t\right)} \hat{X} \\
& \widetilde{\mathbf{B}}_{R}=-\widetilde{B}_{0 R} e^{i\left(-k_{1} z-\omega t\right)} \hat{y} \\
& \widetilde{B}_{0 R}=\frac{1}{v_{1}} \widetilde{E}_{0 R}
\end{aligned}
$$

Transmitted wave

$$
\begin{aligned}
& \widetilde{\mathbf{E}}_{T}=\widetilde{E}_{0 T} e^{i\left(k_{2}-\omega t\right)} \hat{X} \\
& \widetilde{\mathbf{B}}_{T}=\widetilde{B}_{0 T} e^{i\left(k_{2} z-\omega t\right)} \hat{y} \\
& \widetilde{B}_{0 T}=\frac{1}{v_{2}} \widetilde{E}_{0 T}
\end{aligned}
$$

The dispersion relation:

$$
\omega=v_{1} k_{1}=v_{2} k_{2} .
$$

This condition is required. Otherwise the boundary condition is different at different times.

The boundary condition

$$
\begin{array}{ll}
\widetilde{E}_{0 I}+\widetilde{E}_{0 R}=\widetilde{E}_{0 T} & \text { (tangential) } \\
\frac{1}{\mu_{1}}\left(\widetilde{B}_{0 I}-\widetilde{B}_{0 R}\right)=\frac{1}{\mu_{2}} \widetilde{B}_{0 T} & \text { (tangential) }
\end{array}
$$

or

$$
\frac{1}{\mu_{1}}\left(\frac{1}{v_{1}} \widetilde{E}_{0 I}-\frac{1}{v_{1}} \widetilde{E}_{0 R}\right)=\frac{1}{\mu_{2}} \frac{1}{v_{2}} \widetilde{E}_{0 T}
$$

or

$$
\widetilde{E}_{0 I}-\widetilde{E}_{0 R}=\frac{\mu_{1}}{\mu_{2}} \frac{v_{1}}{v_{2}} \widetilde{E}_{0 T}=\beta \widetilde{E}_{0 T}
$$

where

$$
\beta=\frac{\mu_{1}}{\mu_{2}} \frac{v_{1}}{v_{2}}=\frac{\mu_{1}}{\mu_{2}} \frac{n_{2}}{n_{1}}=\frac{\mu_{1}}{\mu_{2}} \frac{\sqrt{\varepsilon_{2} \mu_{2}}}{\sqrt{\varepsilon_{1} \mu_{1}}}=\frac{\sqrt{\varepsilon_{2} \mu_{1}}}{\sqrt{\varepsilon_{1} \mu_{2}}} .
$$

Then we have

$$
\begin{aligned}
& -\widetilde{E}_{0 R}+\widetilde{E}_{0 T}=\widetilde{E}_{0 I} \\
& \widetilde{E}_{0 R}+\beta \widetilde{E}_{0 T}=\widetilde{E}_{0 I}
\end{aligned}
$$

From these two equations

$$
\begin{aligned}
& \widetilde{E}_{0 T}=\frac{2}{1+\beta} \widetilde{E}_{0 I} \\
& \widetilde{E}_{0 R}=\frac{1-\beta}{1+\beta} \widetilde{E}_{0 I}
\end{aligned}
$$

The intensity is given by

$$
I=\frac{1}{2} v \varepsilon\left|\widetilde{\mathbf{E}}_{0}\right|^{2} .
$$

The reflection coefficient $R$ is defined by

$$
R=\frac{I_{R}}{I_{I}}=\frac{\frac{1}{2} v_{1} \varepsilon_{1}\left|\widetilde{\mathbf{E}}_{0 R}\right|^{2}}{\frac{1}{2} v_{1} \varepsilon_{1}\left|\widetilde{\mathbf{E}}_{0 I}\right|^{2}}=\frac{\left|\widetilde{\mathbf{E}}_{0 R}\right|^{2}}{\left|\widetilde{\mathbf{E}}_{0 I}\right|^{2}}=\left(\frac{1-\beta}{1+\beta}\right)^{2} .
$$

The transmission coefficient $T$ is defined by

$$
T=\frac{I_{T}}{I_{I}}=\frac{\frac{1}{2} v_{2} \varepsilon_{2}\left|\widetilde{\mathbf{E}}_{0 T}\right|^{2}}{\frac{1}{2} v_{1} \varepsilon_{1}\left|\widetilde{\mathbf{E}}_{0 I}\right|^{2}}=\frac{v_{2} \varepsilon_{2}}{v_{1} \varepsilon_{1}} \frac{\left|\widetilde{\mathbf{E}}_{0 T}\right|^{2}}{\left|\widetilde{\mathbf{E}}_{0 I}\right|^{2}}=\beta\left(\frac{2}{1+\beta}\right)^{2} .
$$

Therefore we have

$$
T+R=\left(\frac{1-\beta}{1+\beta}\right)^{2}+\beta\left(\frac{2}{1+\beta}\right)^{2}=1 .
$$

((Note)) The boundary condition
The normal component is continuous

$$
\begin{array}{lll}
D_{1}{ }^{\perp}=D_{2}{ }^{\perp} & \varepsilon_{1} E_{1}{ }^{\perp}=\varepsilon_{2} \\
B_{1}{ }^{\perp}=B_{2}{ }^{\perp} & & B_{1}{ }^{\perp}=B_{2}{ }^{\perp}
\end{array}
$$

The tangential components is continuous

$$
\begin{array}{lll}
E_{1}^{\prime \prime}=E_{2}^{\prime \prime} & E_{1}^{\prime \prime}=E_{2}^{\prime \prime} \\
H_{1}^{\prime \prime}=H_{2}^{\prime \prime} & & \text { or } \\
\mu_{1} & B_{1}^{\prime \prime}=\frac{1}{\mu_{2}} B_{2}^{\prime \prime}
\end{array}
$$

34.6

Reflection and transmission at oblique incidence


Fig. Components of plane wave. An incident plane wave in a medium of index $n_{1}$ results in a reflected wave in $n_{1}$ and a refracted wave in the medium of index $n_{2}$.

$$
\begin{array}{lll}
\widetilde{\mathbf{E}}_{I}=\widetilde{\mathbf{E}}_{0 I} e^{i\left(\mathbf{k}_{I} \cdot \mathbf{r}-\omega t\right)} & \widetilde{\mathbf{E}}_{R}=\widetilde{\mathbf{E}}_{0 R} e^{i\left(\mathbf{k}_{R} \cdot \mathbf{r}-\omega t\right)} & \widetilde{\mathbf{E}}_{T}=\widetilde{\mathbf{E}}_{0 T} e^{i\left(\mathbf{k}_{T} \cdot \mathbf{r}-\omega t\right)} \\
\widetilde{\mathbf{B}}_{I}=\widetilde{\mathbf{B}}_{0 I} e^{i\left(\mathbf{k}_{I} \cdot \mathbf{r}-\omega t\right)} & \widetilde{\mathbf{B}}_{R}=\widetilde{\mathbf{B}}_{0 R} e^{i\left(\mathbf{k}_{R} \cdot \mathbf{r}-\omega t\right)} & \widetilde{\mathbf{B}}_{T}=\widetilde{\mathbf{B}}_{0 T} e^{i\left(\mathbf{k}_{T} \cdot \mathbf{r}-\omega t\right)} \\
\widetilde{\mathbf{B}}_{0 I}=\frac{1}{v_{1}}\left(\hat{\mathbf{k}}_{I} \times \widetilde{\mathbf{E}}_{0 I}\right) & \widetilde{\mathbf{B}}_{0 R}=\frac{1}{v_{1}}\left(\hat{\mathbf{k}}_{R} \times \widetilde{\mathbf{E}}_{0 R}\right) & \widetilde{\mathbf{B}}_{0 T}=\frac{1}{v_{2}}\left(\hat{\mathbf{k}}_{T} \times \widetilde{\mathbf{E}}_{0 T}\right) \\
\theta_{i}=\mathbf{k}_{I} \cdot \mathbf{r} & & \\
\theta_{R}=\mathbf{k}_{R} \cdot \mathbf{r} & & \\
\theta_{T}=\mathbf{k}_{T} \cdot \mathbf{r} & &
\end{array}
$$

Dispersion relation

$$
\omega=k_{I} v_{1}=k_{R} v_{1}=k_{T} v_{2} \quad \text { or } \quad k_{I}=k_{R}=k_{T} \frac{v_{2}}{v_{1}}
$$

Boundary condition at $z=0$

$$
\begin{aligned}
& \widetilde{\mathbf{E}}_{I}^{\prime \prime}+\widetilde{\mathbf{E}}_{R}^{\prime \prime}=\widetilde{\mathbf{E}}_{T}{ }^{\prime \prime} \\
& \widetilde{\mathbf{B}}_{I}{ }^{\perp}+\widetilde{\mathbf{B}}_{R}^{\perp}=\widetilde{\mathbf{B}}_{T}{ }^{\perp} \\
& \varepsilon_{1}\left(\widetilde{\mathbf{E}}_{I}{ }^{\perp}+\widetilde{\mathbf{E}}_{R}{ }^{\perp}\right)=\varepsilon_{2} \widetilde{\mathbf{E}}_{T}{ }^{\prime} \\
& \frac{1}{\mu_{1}}\left(\widetilde{\mathbf{B}}_{I}^{\prime \prime}+\widetilde{\mathbf{B}}^{\prime \prime}\right)=\frac{1}{\mu_{2}} \widetilde{\mathbf{B}}_{T}^{\prime \prime}
\end{aligned}
$$

or

$$
\begin{aligned}
& \widetilde{\mathbf{E}}_{0 I}{ }^{\prime \prime} e^{i \theta_{1}}+\widetilde{\mathbf{E}}_{0 R}{ }^{\prime \prime} e^{i \theta_{R}}=\widetilde{\mathbf{E}}_{0 T}{ }^{\prime \prime} e^{i \theta_{T}} \\
& \widetilde{\mathbf{B}}_{0 I}{ }^{\perp} e^{i \theta_{I}}+\widetilde{\mathbf{B}}_{0 R}{ }^{1} e^{i \theta_{R}}=\widetilde{\mathbf{B}}_{0 T}{ }^{1} e^{i \theta_{T}} \\
& \left.\varepsilon_{1} \widetilde{\mathbf{E}}_{0 I}{ }^{1} e^{i \theta_{I}}+\widetilde{\mathbf{E}}_{0 R}{ }^{1} e^{i \theta_{R}}\right)=\varepsilon_{2} \widetilde{\mathbf{E}}_{0 T}{ }^{1 i e^{i \theta_{T}}} \\
& \frac{1}{\mu_{1}}\left(\widetilde{\mathbf{B}}_{0 I}{ }^{\prime \prime} e^{i \theta_{I}}+\widetilde{\mathbf{B}}_{0 R}{ }^{\prime \prime} e^{i \theta_{R}}\right)=\frac{1}{\mathbf{B}_{0 T}} e^{i \theta_{T}}
\end{aligned}
$$

These boundary conditions must hold at all points on the plane ( $z=0$ ), and for all times. These exponential must be equal.

$$
e^{i \theta_{T}}=e^{i \theta_{R}}=e^{i \theta_{T}} \quad \text { at } Z=0 .
$$

or

$$
\mathbf{k}_{I} \cdot \mathbf{r}=\mathbf{k}_{R} \cdot \mathbf{r}=\mathbf{k}_{T} \cdot \mathbf{r}
$$

for all $x$ and $y$.
Then we have

$$
\begin{aligned}
& \left(\mathbf{k}_{I}\right)_{x}=\left(\mathbf{k}_{R}\right)_{x}=\left(\mathbf{k}_{T}\right)_{x} \\
& \left(\mathbf{k}_{I}\right)_{y}=\left(\mathbf{k}_{R}\right)_{y}=\left(\mathbf{k}_{T}\right)_{y}
\end{aligned}
$$



Suppose that $\left(\mathbf{k}_{I}\right)_{y}=0$, then we have $\left(\mathbf{k}_{R}\right)_{y}=\left(\mathbf{k}_{T}\right)_{y}=0$

$$
\begin{aligned}
& \left(\mathbf{k}_{I}\right)_{x}=k_{I} \sin \theta_{i} \\
& \left(\mathbf{k}_{R}\right)_{x}=k_{R} \sin \theta_{R} \\
& \left(\mathbf{k}_{T}\right)_{x}=k_{T} \sin \theta_{T}
\end{aligned}
$$

First law
The incident, reflected, and transmitted wave vectors form a plane (plane of incidence), which also includes the normal to the surface.

Second law

$$
\theta_{i}=\theta_{R} \quad \text { (Law of reflection) }
$$

since

$$
\begin{aligned}
& k_{I} \sin \theta_{i}=k_{R} \sin \theta_{R} \\
& k_{I}=k_{R}
\end{aligned}
$$

Third law

$$
\frac{\sin \theta_{i}}{\sin \theta_{T}}=\frac{k_{T}}{k_{I}}=\frac{v_{1}}{v_{2}}=\frac{\frac{c}{n_{1}}}{\frac{c}{n_{2}}}=\frac{n_{2}}{n_{1}} \quad \text { (Snell's law of refraction) }
$$

since

$$
k_{I} \sin \theta_{i}=k_{T} \sin \theta_{T}
$$

$$
k_{I}=k_{T} \frac{v_{2}}{v_{1}}
$$

### 34.7 Reflection and transmission for the polarization vector in the plane of incidence

Augustin-Jean Fresnel (10 May 1788-14 July 1827), was a French physicist who contributed significantly to the establishment of the theory of wave optics. Fresnel studied the behavior of light both theoretically and experimentally.

http://en.wikipedia.org/wiki/File:Augustin_Fresnel.jpg

$$
\begin{aligned}
& \widetilde{\mathbf{E}}_{0 I}^{\prime \prime}+\widetilde{\mathbf{E}}_{0 R}{ }^{\prime \prime}=\widetilde{\mathbf{E}}_{0 T}^{\prime \prime} \\
& \widetilde{\mathbf{B}}_{01}{ }^{\perp}+\widetilde{\mathbf{B}}_{0 R}{ }^{\perp}=\widetilde{\mathbf{B}}_{0 T}{ }^{\perp} \\
& \varepsilon_{1}\left(\widetilde{\mathbf{E}}_{0 I}^{\perp}+\widetilde{\mathbf{E}}_{0 R}{ }^{\perp}\right)=\varepsilon_{2} \widetilde{\mathbf{E}}_{0 T}{ }^{\perp} \\
& \frac{1}{\mu_{1}}\left(\widetilde{\mathbf{B}}_{0 I}^{\prime \prime}+\widetilde{\mathbf{B}}_{0 R}{ }^{\prime \prime}\right)=\frac{1}{\mu_{2}} \widetilde{\mathbf{B}}_{0 T}^{\prime \prime}
\end{aligned}
$$

or

$$
\begin{aligned}
& \left(\widetilde{\mathbf{E}}_{0 I}+\widetilde{\mathbf{E}}_{0 R}\right)_{x, y}=\left(\widetilde{\mathbf{E}}_{0 T}\right)_{x, y} \\
& \left(\widetilde{\mathbf{B}}_{0 I}+\widetilde{\mathbf{B}}_{0 R}\right)_{z}=\left(\widetilde{\mathbf{B}}_{0 T}\right)_{z} \\
& \varepsilon_{1}\left(\widetilde{\mathbf{E}}_{0 I}+\widetilde{\mathbf{E}}_{0 R}\right)_{z}=\varepsilon_{2}\left(\widetilde{\mathbf{E}}_{0 T}\right)_{z} \\
& \frac{1}{\mu_{1}}\left(\widetilde{\mathbf{B}}_{0 I}+\widetilde{\mathbf{B}}_{0 R}\right)_{x, y}=\frac{1}{\mu_{2}}\left(\widetilde{\mathbf{B}}_{0 T}\right)_{x, y}
\end{aligned}
$$

where

$$
\begin{aligned}
& \widetilde{\mathbf{B}}_{0 I}=\frac{1}{v_{1}}\left(\hat{\mathbf{k}}_{I} \times \widetilde{\mathbf{E}}_{0 I}\right) \\
& \widetilde{\mathbf{B}}_{0 R}=\frac{1}{v_{1}}\left(\hat{\mathbf{k}}_{R} \times \widetilde{\mathbf{E}}_{0 R}\right) \\
& \widetilde{\mathbf{B}}_{0 T}=\frac{1}{v_{2}}\left(\hat{\mathbf{k}}_{T} \times \widetilde{\mathbf{E}}_{0 T}\right)
\end{aligned}
$$

The polarization of the incident wave is parallel to the plane of incidence. The reflected and transmitted waves are also polarized in this plane.


$$
\begin{aligned}
& \widetilde{\mathbf{E}}_{0 I}=\left(\widetilde{E}_{0 I} \cos \theta_{I}, 0, \widetilde{E}_{0 I} \sin \theta_{I}\right) \\
& \widetilde{\mathbf{E}}_{0 R}=\left(\widetilde{E}_{0 R} \cos \theta_{R}, 0,-\widetilde{E}_{0 R} \sin \theta_{R}\right) \\
& \widetilde{\mathbf{E}}_{0 T}=\left(\widetilde{E}_{0 T} \cos \theta_{T}, 0, \widetilde{E}_{0 T} \sin \theta_{T}\right) \\
& \widetilde{\mathbf{B}}_{0 I}=\left(0, \frac{1}{v_{1}} \widetilde{E}_{0 I}, 0\right) \\
& \widetilde{\mathbf{B}}_{0 R}=\left(0,-\frac{1}{v_{1}} \widetilde{E}_{0 R}, 0\right) \\
& \widetilde{\mathbf{B}}_{0 T}=\left(0, \frac{1}{v_{2}} \widetilde{E}_{0 T}, 0\right)
\end{aligned}
$$

Then the boundary conditions can be rewritten as

$$
\begin{aligned}
& \widetilde{E}_{0 I} \cos \theta_{I}+\widetilde{E}_{0 R} \cos \theta_{R}=\widetilde{E}_{0 T} \cos \theta_{T} \\
& \varepsilon_{1}\left(\widetilde{E}_{0 I} \sin \theta_{I}-\widetilde{E}_{0 R} \sin \theta_{R}\right)=\varepsilon_{2} \widetilde{E}_{0 T} \sin \theta_{T} \\
& \frac{1}{\mu_{1} v_{1}}\left(\widetilde{E}_{0 I}-\widetilde{E}_{0 R}\right)=\frac{1}{\mu_{2} v_{2}} \widetilde{E}_{0 T}
\end{aligned}
$$

or

$$
\begin{aligned}
& \widetilde{E}_{0 I}+\widetilde{E}_{0 R}=\frac{\cos \theta_{T}}{\cos \theta_{I}} \widetilde{E}_{0 T}=\alpha \widetilde{E}_{0 T} \\
& \begin{aligned}
\widetilde{E}_{0 I}-\widetilde{E}_{0 R} & =\frac{\varepsilon_{2}}{\varepsilon_{1}} \frac{\sin \theta_{T}}{\sin \theta_{I}} \widetilde{E}_{0 T} \\
& =\frac{\varepsilon_{2}}{\varepsilon_{1}} \frac{n_{1}}{n_{2}} \widetilde{E}_{0 T}=\frac{\varepsilon_{2}}{\varepsilon_{1}} \frac{\frac{c}{v_{1}}}{c} \widetilde{E}_{0 T} \\
& =\frac{\varepsilon_{2} v_{2}}{\varepsilon_{1} v_{1}} \widetilde{E}_{0 T}=\frac{\frac{1}{\mu_{2} v_{2}^{2}} v_{2}}{\frac{1}{\mu_{1} v_{1}^{2}} v_{1}} \widetilde{E}_{0 T}=\frac{\mu_{1} v_{1}}{\mu_{2} v_{2}} E_{0 T} \\
\widetilde{E}_{0 I}-\widetilde{E}_{0 R} & =\frac{\mu_{1} v_{1}}{\mu_{2} v_{2}} \widetilde{E}_{0 T}=\beta \widetilde{E}_{0 T}
\end{aligned}
\end{aligned}
$$

The second and the third equations are the same. Then we have two independent equations. Here we define

$$
\begin{aligned}
& \alpha=\frac{\cos \theta_{T}}{\cos \theta_{I}}=\frac{\sqrt{1-\left(\frac{n_{1}}{n_{2}}\right)^{2} \sin ^{2} \theta_{I}}}{\cos \theta_{I}} \\
& \beta=\frac{\mu_{1} v_{1}}{\mu_{2} v_{2}}
\end{aligned}
$$

Then we get the Fresnel's equation

$$
\begin{aligned}
& \widetilde{E}_{0 R}=\left(\frac{\alpha-\beta}{\alpha+\beta}\right) \widetilde{E}_{0 I} \\
& \widetilde{E}_{0 T}=\left(\frac{2}{\alpha+\beta}\right) \widetilde{E}_{0 I}
\end{aligned}
$$

(i) $\widetilde{E}_{0 T}$ is always in phase with $\widetilde{E}_{0 I}$.
(ii) $\widetilde{E}_{0 R}$ is in phase with $\widetilde{E}_{0 I}$ for $\alpha<\beta$, while $\widetilde{E}_{0 R}$ is out of phase with $\widetilde{E}_{0 I}$ for $\alpha<\beta$.


Power per unit area striking the interface is

$$
\begin{aligned}
& I_{I}=\frac{1}{2} \varepsilon_{1} v_{1}\left|\widetilde{E}_{0 I}\right|^{2} \cos \theta_{I} \\
& I_{R}=\frac{1}{2} \varepsilon_{1} v_{1}\left|\widetilde{E}_{0 R}\right|^{2} \cos \theta_{R} \\
& I_{T}=\frac{1}{2} \varepsilon_{2} v_{2}\left|\widetilde{E}_{0 T}\right|^{2} \cos \theta_{T}
\end{aligned}
$$

Reflection coefficient:

$$
R_{/ /}=\frac{I_{R}}{I_{I}}=\frac{\left|\widetilde{E}_{0 R}\right|^{2}}{\left|\widetilde{E}_{0 I}\right|^{2}}=\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^{2} .
$$

Transmission coefficient:

$$
T_{/ /}=\frac{I_{T}}{I_{I}}=\frac{\varepsilon_{2} v_{2}}{\varepsilon_{1} v_{1}} \frac{\left|\widetilde{E}_{0 R}\right|^{2}}{\left|\widetilde{E}_{0 I}\right|^{2}} \frac{\cos \theta_{T}}{\cos \theta_{I}}=\alpha \beta\left(\frac{2}{\alpha+\beta}\right)^{2}=\frac{4 \alpha \beta}{(\alpha+\beta)^{2}} .
$$

where // means that the polarization vector is in the plane of incidence.

## 34.8 <br> Reflection and transmission for the polarization vector perpendicular to the plane of incidence



$$
\begin{aligned}
& \left(\widetilde{\mathbf{E}}_{0 I}+\widetilde{\mathbf{E}}_{0 R}\right)_{y}=\left(\widetilde{\mathbf{E}}_{0 T}\right)_{y} \\
& \left(\widetilde{\mathbf{B}}_{0 I}+\widetilde{\mathbf{B}}_{0 R}\right)_{z}=\left(\widetilde{\mathbf{B}}_{0 T}\right)_{z} \\
& \frac{1}{\mu_{1}}\left(\widetilde{\mathbf{B}}_{0 I}+\widetilde{\mathbf{B}}_{0 R}\right)_{x}=\frac{1}{\mu_{2}}\left(\widetilde{\mathbf{B}}_{0 T}\right)_{x}
\end{aligned}
$$

where

$$
\begin{aligned}
& \widetilde{\mathbf{E}}_{0 I}=-v_{1}\left(\hat{\mathbf{k}}_{I} \times \widetilde{\mathbf{B}}_{0 I}\right) \\
& \widetilde{\mathbf{E}}_{0 R}=-v_{1}\left(\hat{\mathbf{k}}_{R} \times \widetilde{\mathbf{B}}_{0 R}\right) \\
& \widetilde{\mathbf{E}}_{0 T}=-v_{2}\left(\hat{\mathbf{k}}_{T} \times \widetilde{\mathbf{B}}_{0 T}\right)
\end{aligned}
$$

or

$$
\begin{aligned}
& \widetilde{\mathbf{E}}_{0 I}=-v_{1}\left(\hat{\mathbf{k}}_{I} \times \widetilde{\mathbf{B}}_{0 I}\right) \\
& \widetilde{\mathbf{E}}_{0 R}=-v_{1}\left(\hat{\mathbf{k}}_{R} \times \widetilde{\mathbf{B}}_{0 R}\right) \\
& \widetilde{\mathbf{E}}_{0 T}=-v_{2}\left(\hat{\mathbf{k}}_{T} \times \widetilde{\mathbf{B}}_{0 T}\right)
\end{aligned}
$$

$$
\widetilde{\mathbf{B}}_{0 I}=\left(-\widetilde{B}_{0 I} \cos \theta_{I}, 0,-\widetilde{B}_{0 I} \sin \theta_{I}\right)
$$

$$
\widetilde{\mathbf{B}}_{0 R}=\left(\widetilde{B}_{0 R} \cos \theta_{R}, 0,-\widetilde{B}_{0 R} \sin \theta_{R}\right)
$$

$$
\widetilde{\mathbf{B}}_{0 T}=\left(-\widetilde{B}_{0 T} \cos \theta_{T}, 0,-\widetilde{B}_{0 T} \sin \theta_{T}\right)
$$

$$
\widetilde{\mathbf{E}}_{0 I}=\left(0, v_{1} \widetilde{B}_{0 I}, 0\right)
$$

$$
\widetilde{\mathbf{E}}_{0 R}=\left(0, v_{1} \widetilde{B}_{0 R}, 0\right)
$$

$$
\widetilde{\mathbf{E}}_{0 T}=\left(0, v_{2} \widetilde{B}_{0 T}, 0\right)
$$

or

$$
\begin{aligned}
& \widetilde{B}_{0 I}=\frac{1}{v_{1}} \hat{E}_{0 I} \\
& \widetilde{B}_{0 R}=\frac{1}{v_{1}} \hat{E}_{0 R} \\
& \widetilde{B}_{0 T}=\frac{1}{v_{2}} \hat{E}_{0 T}
\end{aligned}
$$

The boundary condition

$$
\begin{aligned}
& v_{1}\left(\widetilde{B}_{0 I}+\widetilde{B}_{0 R}\right)=v_{2} \widetilde{B}_{0 T} \\
& \widetilde{B}_{0 I} \sin \theta_{I}+\widetilde{B}_{0 R} \sin \theta_{R}=\widetilde{B}_{0 T} \sin \theta_{T} \\
& \frac{1}{\mu_{1}}\left(-\widetilde{B}_{0 I} \cos \theta_{I}+\widetilde{B}_{0 R} \cos \theta_{R}\right)=\frac{1}{\mu_{2}}\left(-\widetilde{B}_{0 T} \cos \theta_{T}\right)
\end{aligned}
$$

or

$$
\begin{aligned}
& \widetilde{B}_{0 I}+\widetilde{B}_{0 R}=\frac{v_{2}}{v_{1}} \widetilde{B}_{0 T}= \\
& -\widetilde{B}_{0 I}+\widetilde{B}_{0 R}=\frac{-\mu_{1}}{\mu_{2}} \frac{\cos \theta_{T}}{\cos \theta_{R}} \widetilde{B}_{0 T}
\end{aligned}
$$

or

$$
\begin{aligned}
& \hat{E}_{0 I}+\hat{E}_{0 R}=\hat{E}_{0 T} \\
& \hat{E}_{0 I}-\hat{E}_{0 R}=\frac{\mu_{1} v_{1}}{\mu_{2} v_{2}} \frac{\cos \theta_{T}}{\cos \theta_{R}} \hat{E}_{0 T}=\alpha \beta \hat{E}_{0 T}
\end{aligned}
$$

where

$$
\begin{aligned}
& \alpha=\frac{\cos \theta_{T}}{\cos \theta_{I}} \\
& \beta=\frac{\mu_{1} v_{1}}{\mu_{2} v_{2}}
\end{aligned}
$$

Then we get the Fresnel's equation

$$
\begin{aligned}
& \hat{E}_{0 T}=\left(\frac{1}{1+\alpha \beta}\right) \hat{E}_{0 I} \\
& \hat{E}_{0 R}=\left(\frac{1-\alpha \beta}{1+\alpha \beta}\right) \hat{E}_{0 I}
\end{aligned}
$$

for polarization vector perpendicular to the plane of incidence.
Reflection coefficient:

$$
R_{\perp}=\frac{I_{R}}{I_{I}}=\frac{\left|\widetilde{E}_{0 R}\right|^{2}}{\left|\widetilde{E}_{0 I}\right|^{2}}=\left(\frac{1-\alpha \beta}{1+\alpha \beta}\right)^{2},
$$

Transmission coefficient:

$$
T_{\perp}=\frac{I_{T}}{I_{I}}=\frac{\varepsilon_{2} v_{2}}{\varepsilon_{1} v_{1}} \frac{\left|\widetilde{E}_{0 R}\right|^{2}}{\left|\widetilde{E}_{0 I}\right|^{2}} \frac{\cos \theta_{T}}{\cos \theta_{I}}=\alpha \beta\left(\frac{2}{1+\alpha \beta}\right)^{2}=\frac{4 \alpha \beta}{(1+\alpha \beta)^{2}},
$$

where $\perp$ means that the polarization vector perpendicular to the plane of incidence.

### 34.9 Brewster's angle

Sir David Brewster (11 December 1781-10 February 1868) was a Scottish physicist, mathematician, astronomer, inventor, and writer.

http://en.wikipedia.org/wiki/David_Brewster


Fig. Incident ray (unpolarized). Refracted ray (slightly polarized). Reflected ray (polarized). $\theta_{\mathrm{I}}$ is equal to the Brewster's angle $\theta_{\mathrm{B}}: \tan \left(\theta_{B}\right)=\frac{n_{2}}{n_{1}} \cdot \theta_{I}+\theta_{T}=\frac{\pi}{2}$. We use $n_{1}=1$ and $n_{2}=1.65 . \theta_{\mathrm{B}}=58.7816^{\circ}$.

We now consider the case when $\mu_{1}=\mu_{2}$

$$
\begin{aligned}
& \alpha=\frac{\cos \theta_{T}}{\cos \theta_{I}} \\
& \beta=\frac{\mu_{1} v_{1}}{\mu_{2} v_{2}}=\frac{v_{1}}{v_{2}}=\frac{n_{2}}{n_{1}}=\frac{\sin \theta_{I}}{\sin \theta_{T}} \\
& R_{/ /}=\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^{2}=\left(\frac{\sin \left(2 \theta_{T}\right)-\sin \left(2 \theta_{I}\right)}{\sin \left(2 \theta_{T}\right)+\sin \left(2 \theta_{I}\right)}\right)^{2}
\end{aligned}
$$

$$
R_{\perp}=\left(\frac{1-\alpha \beta}{1+\alpha \beta}\right)^{2}=\frac{\sin ^{2}\left(\theta_{T}-\theta_{I}\right)}{\sin ^{2}\left(\theta_{T}+\theta_{I}\right)}
$$

Note that $R_{/ /}=0$ for $\sin \left(2 \theta_{T}\right)=\sin \left(2 \theta_{I}\right)$.

This means that

$$
2 \theta_{T}=2 \theta_{I} \quad \text { or } \quad 2 \theta_{T}=\pi-2 \theta_{I}
$$

or

$$
\theta_{T}+\theta_{I}=\frac{\pi}{2} \quad \text { (Brewster angle). }
$$



Fig. Brewster's angle. $n_{1}=1 . n_{2}=1.65 . \theta_{\mathrm{I}}=\theta_{\mathrm{R}}=\theta_{\mathrm{B}}=58.7816^{\circ} . \theta_{\mathrm{T}}=90^{\circ}-\theta_{1}$.

The condition for the Brewster's angle:
Using the Snell's law, we have

$$
n_{1} \sin \theta_{I}=n_{2} \sin \left(\frac{\pi}{2}-\theta_{I}\right)=n_{2} \cos \theta_{I}
$$

we have a Brewster's angle

$$
\tan \theta_{I}=\frac{n_{2}}{n_{1}}
$$




Fig. $\quad n_{1}=1$ and $n_{2}=1.65$. The Brewster's angle is $\theta_{\mathrm{B}}=58.7816^{\circ} . R_{\perp}=1-T_{\perp}$. $R_{/ /}=1-T_{/ /}$.

### 34.10 Skin effect in metal

Skin effect is the tendency of an alternating electric current (AC) to distribute itself within a conductor so that the current density near the surface of the conductor is greater than that at its core. That is, the electric current tends to flow at the skin of the conductor, at an average depth called the skin depth.

$$
\begin{aligned}
& \nabla \cdot \mathbf{D}=\rho_{f} \\
& \nabla \cdot \mathbf{B}=0 \\
& \nabla \times \mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t} \\
& \nabla \times \mathbf{H}=\frac{\partial}{\partial t} \mathbf{D}+\mathbf{J}_{f} \\
& \mathbf{J}_{f}=\sigma \mathbf{E} \quad \\
& \begin{array}{ll}
\sigma & \text { current density } \\
\rho_{\mathrm{f}} & \text { conductivity } \\
\mathbf{B}=\mu \mathbf{H} & \\
\mathbf{D}=\varepsilon \mathbf{E} &
\end{array} .
\end{aligned}
$$

From the above equations, we have

$$
\nabla \times \mathbf{B}=\mu\left(\varepsilon \frac{\partial}{\partial t} \mathbf{E}+\sigma \mathbf{E}\right)
$$

Continuity equation for free charge;

$$
\nabla \cdot \mathbf{J}_{f}=-\frac{\partial}{\partial t} \rho_{f}
$$

or

$$
\frac{\partial}{\partial t} \rho_{f}=-\sigma \nabla \cdot \mathbf{E}=-\frac{\sigma}{\varepsilon} \rho_{f}
$$

The solution of this equation is

$$
\rho_{f}(t)=\rho_{f}(0) \exp \left(-\frac{\sigma}{\varepsilon} t\right)
$$

For perfect conductors we have $\sigma=\infty$. Thus $\rho_{\mathrm{f}}$ should be zero. Thus the starting Maxwell equations are

$$
\begin{aligned}
& \nabla \cdot \mathbf{E}=0 \\
& \nabla \cdot \mathbf{B}=0 \\
& \nabla \times \mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t} \\
& \nabla \times \mathbf{B}=\mu \varepsilon \frac{\partial}{\partial t} \mathbf{E}+\mu \sigma \mathbf{E} \\
& \nabla \times(\nabla \times \mathbf{E})=\nabla(\nabla \cdot \mathbf{E})-\nabla^{2} \mathbf{E}=-\frac{\partial}{\partial t}(\nabla \times \mathbf{B})
\end{aligned}
$$

or

$$
\nabla^{2} \mathbf{E}=\mu \varepsilon \frac{\partial^{2}}{\partial t^{2}} \mathbf{E}+\mu \sigma \frac{\partial}{\partial t} \mathbf{E}
$$

Similarly

$$
\nabla \times(\nabla \times \mathbf{B})=\nabla(\nabla \cdot \mathbf{B})-\nabla^{2} \mathbf{B}=\nabla \times\left(\mu \varepsilon \frac{\partial}{\partial t} \mathbf{E}+\mu \sigma \mathbf{E}\right)
$$

or

$$
\nabla^{2} \mathbf{B}=\mu \varepsilon \frac{\partial^{2}}{\partial t^{2}} \mathbf{B}+\mu \sigma \frac{\partial}{\partial t} \mathbf{B}
$$

We assume the plane-wave solution,

$$
\begin{aligned}
& \mathbf{E}=\operatorname{Re}\left[\widetilde{E}_{0}(z) \mathbf{e}_{x} e^{-i o t}\right] \\
& \mathbf{B}=\operatorname{Re}\left[\widetilde{\mathbf{B}}_{0}(z) e^{-i \omega t}\right] \\
& \nabla^{2} \mathbf{E}=\mu \varepsilon \frac{\partial^{2}}{\partial t^{2}} \mathbf{E}+\mu \sigma \frac{\partial}{\partial t} \mathbf{E}
\end{aligned}
$$

or

$$
\frac{d^{2}}{d z^{2}} \widetilde{E}_{0}(z)+\widetilde{K}^{2} \widetilde{E}_{0}(z)=0
$$

where

$$
\mu \varepsilon \omega^{2}+i \omega \mu \sigma=\widetilde{K}^{2}
$$

From the relation $\nabla \times \mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t}$, we have

$$
\begin{aligned}
\nabla \times\left[\widetilde{E}_{0}(z) \mathbf{e}_{x} e^{-i \omega t}\right] & =\hat{y} \frac{\partial}{\partial z} \widetilde{E}_{0}(z) e^{-i \omega t} \\
& =-\frac{\partial}{\partial t}\left[\widetilde{\mathbf{B}}_{0}(z) e^{-i \omega t}\right]=i \omega \widetilde{\mathbf{B}}_{0}(z) e^{-i \omega t}
\end{aligned}
$$

or

$$
\widetilde{\mathbf{B}}_{0}(z)=\mathbf{e}_{y} \frac{1}{i \omega} \frac{\partial}{\partial z} \widetilde{E}_{0}(z),
$$

or

$$
\widetilde{\mathbf{H}}_{0}(z)=\mathbf{e}_{y} \frac{1}{i \mu \omega} \frac{\partial}{\partial z} \widetilde{E}_{0}(z),
$$



Fig. Electric field (along the $x$ axis) and magnetic field (along the $y$ axis).

Boundary condition

$$
\widetilde{E}_{0}(z), \frac{\partial}{\partial z} \widetilde{E}_{0}(z) \text { are continuous at the boundary. }
$$

### 34.11 Skin depth

$$
\begin{aligned}
& \widetilde{K}=k_{1}+i \kappa=K_{0} e^{i \phi} \\
& \widetilde{K}^{2}=K_{0}^{2} e^{2 i \phi}=i \mu \sigma \omega+\mu \varepsilon \omega^{2}=\mu \varepsilon \omega^{2}\left(1+\frac{i \sigma}{\varepsilon \omega}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
& K_{0}=\sqrt{\mu \varepsilon} \omega\left(1+\frac{\sigma^{2}}{\varepsilon^{2} \omega^{2}}\right)^{1 / 4} \\
& \tan (\phi)=\frac{\kappa}{k_{1}}
\end{aligned}
$$



$$
\left(k_{1}+i \kappa\right)^{2}=i \mu \sigma \omega+\mu \varepsilon \omega^{2}
$$

or

$$
k_{1}^{2}-\kappa^{2}+2 i k_{1} \kappa=i \mu \sigma \omega+\mu \varepsilon \omega^{2}
$$

Real part:

$$
k_{1}^{2}-\kappa^{2}=\mu \varepsilon \omega^{2}
$$

Imaginary part: $\quad 2 k_{1} \kappa=\mu \sigma \omega$
or

$$
k_{1}^{2} \kappa^{2}=\frac{\mu^{2} \sigma^{2} \omega^{2}}{4}
$$

The solution for $k_{1}$ and $\kappa$ is given as

$$
\begin{aligned}
& k_{1}=\omega \sqrt{\frac{\varepsilon \mu}{2}}\left(1+\sqrt{1+\frac{\sigma^{2}}{\varepsilon^{2} \omega^{2}}}\right)^{1 / 2} \\
& \kappa=\omega \sqrt{\frac{\varepsilon \mu}{2}}\left[-1+\sqrt{1+\frac{\sigma^{2}}{\varepsilon^{2} \omega^{2}}}\right]^{1 / 2}
\end{aligned}
$$

We now assume that

$$
\frac{\sigma}{\varepsilon \omega} » 1
$$

Then we have

$$
\begin{aligned}
& \kappa=\omega \sqrt{\frac{\varepsilon \mu}{2}}\left(-1+\sqrt{1+\frac{\sigma^{2}}{\varepsilon^{2} \omega^{2}}}\right)^{1 / 2} \approx \omega \sqrt{\frac{\varepsilon \mu}{2} \frac{\sigma}{\varepsilon \omega}}=\sqrt{\frac{\mu \sigma \omega}{2}} \\
& k_{1}=\omega \sqrt{\frac{\varepsilon \mu}{2}}\left[1+\sqrt{1+\frac{\sigma^{2}}{\varepsilon^{2} \omega^{2}}}\right]^{1 / 2} \approx \sqrt{\frac{\mu \sigma \omega}{2}} \\
& \tan (\phi)=\frac{\kappa}{k_{1}} \approx 1 \quad \text { or } \quad \phi=\frac{\pi}{4} \\
& \widetilde{K}=k_{1}+i \kappa=\sqrt{\frac{\mu \sigma \omega}{2}} e^{i \pi / 4} .
\end{aligned}
$$

The skin depth $d$ is defined by

$$
d=\frac{1}{\kappa}=\frac{1}{\sqrt{\frac{\mu \sigma \omega}{2}}}=\sqrt{\frac{2}{\mu \sigma \omega}} .
$$

Note that the skin depth $d$ decreases with increasing $\omega$. Here we have

$$
\begin{aligned}
& \mathbf{E}=\operatorname{Re}\left[\widetilde{\mathbf{E}}_{0} e^{i(\widetilde{K}-\omega t)}\right]=\operatorname{Re}\left[\widetilde{\mathbf{E}}_{0} e^{i\left(\left(k_{1}+i \kappa\right) z-\omega t\right)}\right]=e^{-\kappa z} \operatorname{Re}\left[\widetilde{\mathbf{E}}_{0} e^{i\left(k_{1} z-\omega t\right)}\right] \\
& \mathbf{B}=\operatorname{Re}\left[\widetilde{\mathbf{B}}_{0} e^{i(\widetilde{K}-\omega t)}\right]=\operatorname{Re}\left[\widetilde{\mathbf{B}}_{0} e^{i\left(\left(k_{1}+i \kappa\right) z-\omega t\right)}\right]=e^{-\kappa z} \operatorname{Re}\left[\widetilde{\mathbf{B}}_{0} e^{i\left(k_{1} z-\omega t\right)}\right]
\end{aligned}
$$

Both $\boldsymbol{E}$ and $\boldsymbol{B}$ exponentially decay in the metal.


Fig. Skin depth.

### 34.12 Solution of the boundary problem

For $z \leq 0$,

$$
\begin{aligned}
& \widetilde{E}_{1}(z)=E_{11} e^{i k z}+E_{12} e^{-i k z}, \\
& \frac{d \widetilde{E}_{1}(z)}{d z}=i k\left(E_{11} e^{i k z}-E_{12} e^{-i k z}\right)
\end{aligned}
$$

For $z \geq 0$,

$$
\begin{gathered}
\widetilde{E}_{2}(z)=E_{21} e^{i \widetilde{K} z} \\
\frac{d \widetilde{E}_{2}(z)}{d z}=i \widetilde{K} E_{21} e^{i \widetilde{K} z}
\end{gathered}
$$

Boundary condition at $z=0$,

$$
\begin{aligned}
& E_{21}=E_{11}+E_{12} \\
& \frac{1}{i \mu_{1} \omega} i k\left(E_{11}-E_{12}\right)=\frac{1}{i \mu_{2} \omega} i \widetilde{K} E_{21}
\end{aligned}
$$

$$
\begin{equation*}
\frac{k}{\mu_{1}}\left(E_{11}-E_{12}\right)=\frac{\widetilde{K}}{\mu_{2}} E_{21} \tag{2}
\end{equation*}
$$

From Eqs.(1) and (2) we get

$$
\begin{aligned}
& \frac{E_{21}}{E_{11}}=\frac{2 k \mu_{2}}{\widetilde{K} \mu_{1}+k \mu_{2}} \\
& \frac{E_{12}}{E_{11}}=\frac{k \mu_{2}-\mu_{1} \widetilde{K}}{\widetilde{K} \mu_{1}+k \mu_{2}}
\end{aligned}
$$

Then we have

$$
\widetilde{E}_{2}(z)=E_{21} e^{i \widetilde{K} z}=\frac{2 k \mu_{2}}{\widetilde{K} \mu_{1}+k \mu_{2}} e^{i \widetilde{K} z}
$$

When $\mu_{1}=\mu_{2}=\mu_{0}$

$$
\begin{aligned}
& \widetilde{E}_{2}(z)=\frac{2 k}{\widetilde{K}_{1}+k} e^{i \widetilde{K} z} \\
& \frac{E_{21}}{E_{11}}=\frac{2 k}{\widetilde{K}+k} \\
& \frac{E_{12}}{E_{11}}=\frac{k-\widetilde{K}}{\widetilde{K}+k} .
\end{aligned}
$$

The reflection coefficient:

$$
\begin{aligned}
R & =\left|\frac{1-\frac{\widetilde{K}}{k}}{1+\frac{\widetilde{K}}{k}}\right|^{2} \\
& \left.=\left\lvert\, \frac{1-\sqrt{\frac{\varepsilon_{2} \mu_{2}}{2 \varepsilon_{1} \mu_{1}}}\left(1+\sqrt{1+\frac{\sigma^{2}}{\varepsilon_{2}{ }^{2} \omega^{2}}}\right)^{1 / 2}-i\left(\sqrt{\frac{\varepsilon_{2} \mu_{2}}{2 \varepsilon_{1} \mu_{1}}}\left(-1+\sqrt{1+\frac{\sigma^{2}}{\varepsilon_{2}{ }^{2} \omega^{2}}}\right]^{1 / 2}\right.}{1+\sqrt{\frac{\varepsilon_{2} \mu_{2}}{2 \varepsilon_{1} \mu_{1}}}\left(1+\sqrt{1+\frac{\sigma^{2}}{\varepsilon_{2}{ }^{2} \omega^{2}}}\right)^{1 / 2}+i \sqrt{\frac{\varepsilon_{2} \mu_{2}}{2 \varepsilon_{1} \mu_{1}}}\left(-1+\sqrt{1+\frac{\sigma^{2}}{\varepsilon_{2}{ }^{2} \omega^{2}}}\right)^{1 / 2}}\right.\right) \mid
\end{aligned}
$$

where

$$
\begin{aligned}
& k=\omega \sqrt{\varepsilon_{1} \mu_{1}} \\
& \widetilde{K}=k_{1}+i \kappa \\
& k_{1}=\omega \sqrt{\frac{\varepsilon_{2} \mu_{2}}{2}}\left(1+\sqrt{1+\frac{\sigma^{2}}{\varepsilon_{2}{ }^{2} \omega^{2}}}\right)^{1 / 2} \\
& \kappa=\omega \sqrt{\frac{\varepsilon_{2} \mu_{2}}{2}}\left(-1+\sqrt{1+\frac{\sigma^{2}}{\varepsilon_{2}{ }^{2} \omega^{2}}}\right)^{1 / 2}
\end{aligned}
$$

### 34.13 Pressure of radiation

$$
k=\omega \sqrt{\varepsilon_{1} \mu_{1}}
$$

For $\frac{\sigma}{\omega \varepsilon_{0}} \gg 1$

$$
\begin{aligned}
& \kappa=\omega \sqrt{\frac{\varepsilon_{2} \mu_{2}}{2}}\left(-1+\sqrt{1+\frac{\sigma^{2}}{\varepsilon_{2}{ }^{2} \omega^{2}}}\right)^{1 / 2} \approx \omega \sqrt{\frac{\varepsilon_{2} \mu_{2}}{2}} \frac{\sigma}{\varepsilon_{2} \omega}
\end{aligned}=\sqrt{\frac{\mu_{2} \sigma \omega}{2}} .
$$

Then we have

$$
\widetilde{K}=\sqrt{\frac{\mu_{2} \sigma \omega}{2}}(1+i)
$$

and

$$
\begin{aligned}
& \frac{\widetilde{K}}{k}=\sqrt{\frac{\mu_{2} \sigma}{2 \varepsilon_{1} \mu_{1} \omega}}(1+i) \\
& \frac{E_{21}}{E_{11}}=\frac{2 k}{\widetilde{K}+k}=\frac{2}{\frac{\widetilde{K}}{k}+1}=0
\end{aligned}
$$

$$
\frac{E_{12}}{E_{11}}=\frac{k-\widetilde{K}}{\widetilde{K}+k}=\frac{1-\frac{\widetilde{K}}{k}}{1+\frac{\widetilde{K}}{k}}=-1
$$

We now consider the magnetic field

$$
\begin{aligned}
& \widetilde{E}_{1}(z)=E_{11} e^{i k z}+E_{12} e^{-i k z}=E_{11}\left(e^{i k z}-e^{-i k z}\right) \\
& \widetilde{\mathbf{B}}_{1}(z)=\mathbf{e}_{y} \frac{1}{i \omega} \frac{\partial}{\partial z} \widetilde{E}_{1}(z)=\mathbf{e}_{y} \frac{1}{i \omega} E_{11} i k\left(e^{i k z}-e^{-i k z}\right)=\mathbf{e}_{y} \frac{k}{\omega} E_{11}\left(e^{i k z}+e^{-i k z}\right)
\end{aligned}
$$

When $z=0, \widetilde{E}_{1}(z)=0$ and $\widetilde{\mathbf{B}}_{1}(z)=\mathbf{e}_{y} \frac{k}{\omega} 2 E_{11}$
The $y$ component of the magnetic field is discontinuous at $z=0$.


We apply the Ampere's law

$$
\oint B \cdot d l=\mu_{0} I_{e n c},
$$

where $I_{\text {enc }}$ is an induced surface current.

$$
B_{\text {out }} a=\mu_{0} I_{\text {enc }} \text {. }
$$

with

$$
I_{e n c}=K_{b} a,
$$

and

$$
\frac{B_{\text {out }}}{\mu_{0}}=K_{b}
$$

where $K_{\mathrm{b}}$ is the surface induced surface (line) current.
The average magnetic field

$$
\bar{B}=\frac{1}{2}\left(B_{\text {out }}+0\right)=\frac{1}{2} B_{\text {out }} .
$$

The total force $\boldsymbol{F}$ is

$$
\mathbf{F}=b \mathbf{I}_{\text {enc }} \times \overline{\mathbf{B}}=a b\left(\mathbf{K}_{b} \times \overline{\mathbf{B}}\right)=a b \frac{B_{\text {out }}}{\mu_{0}} \frac{1}{2} B_{\text {out }}=\frac{B_{\text {out }}{ }^{2}}{2 \mu_{0}} a b .
$$

The force is directed into the inside of the conductor.

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