Chapter 38
van der Pol equation
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Balthasar van der Pol (Utrecht, 27 January 1889 - Wassenaar, 6 October 1959) was a Dutch physicist. van der Pol studied physics in Utrecht, and in 1920 he was awarded his doctorate (PhD). He studied experimental physics with John Ambrose Fleming and Sir J. J. Thomson in England. He joined Philips Research Labs in 1921, where he worked until his retirement in 1949.

(http://en.wikipedia.org/wiki/Balthasar_van_der_Pol)

### 38.1 Types of van der Pol equation

The van der Pol oscillator was originally discovere by Balthasar van der Pol. Van der Pol found stable oscillations, now known as limit cycles, in electrical circuits employing vacuum tubes. When these circuits are driven near the limit cycle they become entrained, i.e. the driving signal pulls the current along with it. Van der Pol and his colleague van der Mark reported in the September 1927 issue of Nature that at certain drive frequencies an irregular noise was heard. This irregular noise was always heard near the natural entrainment frequencies. This was one of the first discovered instances of deterministic chaos. The van der Pol equation has a long history of being used in both the physical and biological sciences.

The van der Pol equation arises in the study of circuits containing vacuum tubes and is given by

$$
\begin{equation*}
\ddot{x}-\varepsilon\left(1-\dot{x}^{2}\right) \dot{x}+x=0 \quad(\varepsilon>0) . \tag{1}
\end{equation*}
$$

This differential equation is called as van der Pol equation (type-I), or Rayleigh's equation. We assume that $\varepsilon \geq 0$. If $\varepsilon=0$, the equation reduces to the equation of simple harmonic motion.

Here we introduce a new variable $y$ such that

$$
y=\sqrt{3} \dot{x} \quad \text { or } \quad \dot{x}=\frac{y}{\sqrt{3}}
$$

Taking a derivative of Eq.(1) with respect to $t$, we get

$$
\dddot{x}-\varepsilon \ddot{x}\left(1-\dot{x}^{2}\right)-\varepsilon \dot{x}(-2 \ddot{x} \ddot{x})+\dot{x}=0,
$$

or

$$
\ddot{y}-\varepsilon \dot{y}\left(1-\frac{y^{2}}{3}\right)+\frac{2}{3} \varepsilon y^{2} \dot{y}+y=0 .
$$

Then Eq.(1) can be written as

$$
\begin{equation*}
\ddot{y}-\varepsilon\left(1-y^{2}\right) \dot{y}+y=0 \quad \text { (van der Pol equation, type-II) } \tag{2}
\end{equation*}
$$

This differential equation is also called as van der Pol equation. Here we call Eq.(1) as a van der Pol equation (type-I) and Eq.(2) as a van der Pol equation (type-II). We discuss mainly the nonlinear nature of Eq.(1). We also show the nonlinear nature of Eq.(2) later.

### 38.2 Nonlinear characteristic of van der Pol equation type-I; Rayleigh's equation

Now we consider the physical meaning of Eq.(1).

$$
\ddot{x}-\varepsilon\left(1-\dot{x}^{2}\right) \dot{x}+x=0,
$$

or

$$
\ddot{x}+K_{v}+x=0
$$

with

$$
K_{v}=-\varepsilon \dot{x}\left(1-\dot{x}^{2}\right)=-\varepsilon v\left(1-v^{2}\right)
$$

The term $K_{\mathrm{v}}$ should be regarded as friction or resistance, and this is the case when the coefficient is positive. However, if $K_{v}$ is negative, then we have the case of "negative resistance."


Fig. $\quad K_{\mathrm{v}}$ vs v with $\varepsilon$ changed as a parameter. Negative resistance occurs in the region of $0<v<1$.

When $\dot{x} \approx 0$, then we have

$$
K_{v}=-\varepsilon v,
$$

which is negative resistance. The differential equation is given by

$$
\ddot{x}-\varepsilon \dot{X}+x=0 .
$$

The solution is

$$
x=A \exp (p t)+B \exp (q t)
$$

with $p$ and $q$ given by

$$
p, q=\frac{\varepsilon \pm \sqrt{\varepsilon^{2}-4}}{2}
$$

This means that $x$ diverges as $t \rightarrow \infty$. When $|\dot{x}|$ is very large,, then we have

$$
F_{v}=\varepsilon v^{3}
$$

which is positive resistance.

### 38.3 Limiting cycle

We consider the phase plane of $v$ vs $x$ for small $\varepsilon$ (see the case for $\varepsilon=0.1$ ). Depending on the initial condition (in our case $v_{0}$ is given as a parameter and $x_{0}=0$ ). When $v_{0}=5$, the rapid motion occurring at early time gradually decays and approaches a closed circle with radius $r$. The closed curve showing a periodic motion in the limit of $t$ $\rightarrow \infty$, is called a limit cycle. When $v_{0}=0$ (for example), the motion undergoes a gradual growth and approaches the limit cycle. In the case of small $\varepsilon$, the limit cycle is close to a circle. From the numerical calculations, we find that the radius of the circle is $r$ in the phase space. What is the value of $r$ ? In the periodic motion, the average of the work done by the friction force over a period $T$ ( $=2 \pi$ in the present case) should be equal to zero.

$$
\Delta W=\int_{0}^{2 \pi} F_{v} d x=\int_{0}^{2 \pi} F_{v} v d t=0
$$

In the limit of $\varepsilon \rightarrow 0$, the limit cycle is approximated by a circle denoted by

$$
x=a \sin t
$$

Then we have

$$
\Delta W=\pi a^{2}\left(1-\frac{3 a^{2}}{4}\right)=0
$$

leading to the value of $a$ as

$$
a=\frac{2 \sqrt{3}}{3}=1.1547
$$

The radius of the limit cycle is 1.1547 .

### 38.4 Phase plane: $(x, v)$

We consider the van der Pol equation given by

$$
\begin{equation*}
\ddot{x}+k(v)+x=0, \tag{3}
\end{equation*}
$$

where

$$
k(v)=-\varepsilon v\left(1-v^{2}\right),
$$

with $\quad v=\dot{x}$. Equation (1) may be rewritten as

$$
\frac{d v}{d t}+k(v)+x=0 .
$$

or

$$
\begin{equation*}
\frac{d v}{d x}=\frac{d v / d t}{d x / d t}=-\frac{x+k(v)}{v} \tag{4}
\end{equation*}
$$

The slope of the trajectory on the ( $x, v$ ) plane (the phase plane) is defined at every point by Eq.(4) and may be determined graphically very easily. This method is called the Lienard construction. Here we define the null-isocline denoted by the curve K

$$
x=-k(v)=\varepsilon v\left(1-v^{2}\right) .
$$

where $\frac{d v}{d t}=0$

### 38.5 Liénard construction



Fig. Lienard construction. . $\varepsilon=0.5$. The red line is a null-isocline $K$ : $x=-k(v)=\varepsilon v\left(1-v^{2}\right)$ (red line). $\mathrm{P}\left(x_{0}, v_{0}\right): x_{0}=0.3 . v_{0}=0.2$. C is the center of curvature of the trajectory S at P .

The slope of the trajectory on the ( $x, v$ ) phase (the phase plane) is defined at every point by

$$
\frac{d v}{d x}=-\frac{x+k(v)}{v}
$$

and may be determined graphically very easily. The curve K is defined by

$$
\text { Curve K; } \quad x=-k(v) .
$$

## ((Tangential line PS))

To find the tangent to the trajectory at any point $\mathrm{P}(x, y)$
(i) Draw a horizontal line to meet the curve K at the point Q .
(ii) Drop a vertical line from the point Q to cut the $x$-axis at N .
(iii) Construct the normal to PN at the point P and the tangent to the curve K at the point Q . Find their intersection G . The line QG is the tangential line of the curve K at the point Q .
(iv) Then S , drawn normal to NP , is the tangent to the trajectory at the point P . The line GPS is perpendicular to the line NP.

## ((Curvature C))

(i) Join GN and let it cut PQ at F.
(vi) Then C lies on PN vertically below F. C is the center of the circle with the radius CP . The line PS is the tangential line at the point P for the circle.

The proof follows immediately from the fact that

$$
\begin{aligned}
& \mathrm{PQ}=x+k(v), \quad \mathrm{QN}=v, \\
& \tan \theta=\frac{Q N}{P Q}=\frac{v}{x+k(v)}
\end{aligned}
$$

or

$$
\frac{d v}{d x}=-\frac{1}{\tan \theta}=-\frac{x+k(v)}{v}
$$

The argument may be extended to provide a construction, not just for the slope, but for the curvature, so that the trajectory may be synthesized as a succession of the circular arcs. In particular, on the null-isoclines $x=-k(v)$ and $v=0$, we have $\frac{d v}{d x}=0$, and $\frac{d v}{d x}=\infty$, respectively.

Base on this construction, the phase space of the van der Pol equation can be drawn.


Fig. Lienard diagram. $K$ (red line). $\mathrm{P}\left(x_{0}, v_{0}\right)$, where the values of $x_{0}$ and $v_{0}$ are changed as a parameter.


Fig. $\quad \mathrm{C}$ is the intersection of PN and $\mathrm{P}^{\prime} \mathrm{N}^{\prime}$ in the limit as $\mathrm{P}^{\prime}$ is moved towards $\mathrm{P}\left(\mathrm{P} \rightarrow \mathrm{P}^{\prime}\right)$. FC and GC' are vertical lines.

### 38.6 General rule for Lienard construction

There are some rules for the Lienard construction diagram. The curve $S$ is the line of the phase space, which we are looking for.
(i) When the curve S cuts the $x$ axis, it does so vertically, and the center of curvature C is the intersection of the curve K with the $x$ axis.

(ii) When the curve S cuts the curve K , it does so horizontally, and the center of curvature C lies on the $x$ axis vertically below.

(iii) The center of curvature also lies on the $x$ axis when the corresponding point on the curve K has a vertical tangent.


### 38.7 Numerical calculations: time dependence

We solve this differential equation with each value of $\varepsilon$ by using the Mathematica (NDSolve). We assume that

$$
x_{0}=0,
$$

and $v_{0}$ is changed as a parameter, $v_{0}=-5,-4,-3,, 4$, and 5 . We also show the FFT calculation.
(1) van der Pol oscillation with $\varepsilon=0.01 . t=0-100$.



FFT spectrum (the intensity vs $n$ ). $\varepsilon=0.01 . T=150 . \omega_{0}=2 \pi / 150 . \omega=n \omega_{0}$. There is a sharp peak at $n=25$, which means $\omega=25(2 \pi / 150)=\pi / 3$. The period $T$ is evaluated as 6 sec.
(2) van der pol oscillation with $\varepsilon=0.1 . t=0-100$.


Fig. Limit cycles of the van der Pol oscillator. Two transient trajectories approaching the limit cycle from the inside and from the outside are also shown.


FFT spectrum. $\varepsilon=0.1 . T=150 . \omega_{0}=2 \pi / 150 . \omega=n \omega_{0}$. The peaks appear at $n=25$ and 72.
(3) van der pol oscillation with $\varepsilon=0.5 . t=0-100$.




FFT spectrum. $\varepsilon=0.5 . T=150 . \omega_{0}=2 \pi / 150 . \omega=n \omega_{0}$. The peaks appear at $n=25,72$, 118 , and 165.
(4) van der Pol oscillation with $\varepsilon=1 . t=0-100$.



FFT spectrum. $\varepsilon=1.0 . T=150 . \omega_{0}=2 \pi / 150 . \omega=n \omega_{0}$. The peaks appear at $n=23.5,68$, $113,157,204,248,293$, and so on.
(5) van der pol oscillation with $\varepsilon=5 \cdot t=0-100$.



FFT spectrum. $\varepsilon=5.0 . T=150 . \omega_{0}=2 \pi / 150 . \omega=n \omega_{0}$
(5) van der pol oscillation with $\varepsilon=10 . t=0-100$.




FFT spectrum. $\varepsilon=10.0 . T=150 . \omega_{0}=2 \pi / 150 . \omega=n \omega_{0}$
(7) van der Pol oscillation with $\varepsilon=30 . t=0-100$.


The period is almost equal to $T=50 \mathrm{sec}$ for $\varepsilon==30$. The ratio $T / \varepsilon=50 / 30=1.667$, which is very close to the prediction ( $=1.6137$ ) in the limit of $\varepsilon \rightarrow \infty$.

### 38.8 FFT spectrum

The period $T$ for each $\varepsilon$ can be determined from the FFT spectrum. Period T increases with increasing $\varepsilon$.


FFT spectrum. $\varepsilon=1$ (red). $\varepsilon=3$ (green). $\varepsilon=10$ (blue). $T=150 . \omega_{0}=2 \pi / 150 . \omega=n \omega_{0}$.


Fig. Period $T$ vs $\varepsilon$, which is obtained from the FFT spectrum. Since the data points are obtained from our calculation of the FFT spectrum, there is some uncertainty in the positions of the data points. For $\varepsilon=15, \mathrm{~T}=21.4 \mathrm{sec}(T=1.43 \varepsilon)$. In the limit of large $\varepsilon$, it is predicted that $T=1.6137 \varepsilon$.


Fig. The trajectory (in the limit of large $\varepsilon$ ) in the phase plane ( $x, v$ ) falls well on the curve K on the lines $\mathrm{P}_{1} \mathrm{P}_{2}$ and $\mathrm{P}_{3}$ and $\mathrm{P}_{4}$.

We can evaluate the period $T$ of the limit cycle with $\varepsilon \rightarrow \infty$. The limit cycle will simply consist of portions of the characteristic curve ( $\mathrm{P}_{1} \mathrm{P}_{2}$ and $\mathrm{P}_{3} \mathrm{P}_{4}$ ) plus vertical lines $\left(\mathrm{P}_{2} \mathrm{P}_{3}\right.$ and $\left.\mathrm{P}_{4} \mathrm{P}_{1}\right)$. To this end, we need to calculate a line integral over the limit cycle. The period $T$ is given by

$$
T=\oint \frac{d x}{v}=\varepsilon \oint \frac{d \xi}{v},
$$

where

$$
\xi=\frac{x}{\varepsilon}, \quad v=\frac{d x}{d t}=\varepsilon \frac{d \xi}{d t}
$$

Since the straight vertical line portions make no contribution to $T(\mathrm{~d} x=0)$. we have, with the curve K;

$$
\xi=v\left(1-v^{2}\right),
$$

along the curved paths on the contour. Then we have

$$
T=2 \varepsilon \int_{v_{3}}^{v_{4}} \frac{d\left(v-v^{3}\right)}{v}=2 \varepsilon \int_{v_{3}}^{v_{4}} \frac{\left(1-3 v^{2}\right) d v}{v}=\left.2 \varepsilon\left(\ln v-\frac{3}{2} v^{2}\right)\right|_{v_{3}} ^{v_{4}}=1.6137 \varepsilon
$$

where the point $P_{3}$ is located at $\left(\xi_{3}=-\frac{2 \sqrt{3}}{9}=-0.3849, v_{3}=1.1547\right)$ and the point $\mathrm{P}_{4}$ is located at $\left(\xi_{4}=\frac{2 \sqrt{3}}{9}=0.3849, v_{4}=\frac{1}{\sqrt{3}}=0.57735\right)$.
((Mathematica))

Clear["Global`*"];

$$
\begin{aligned}
& \mathbf{K} 1=\mathbf{v}\left(\mathbf{1}-\mathbf{v}^{\mathbf{2}}\right) ; \\
& \mathbf{e q 1}=\text { Solve }[\mathbf{D}[\mathbf{K} \mathbf{1}, \mathrm{v}]=\mathbf{0}, \mathrm{v}] \\
& \left\{\left\{\mathbf{v} \rightarrow-\frac{1}{\sqrt{3}}\right\},\left\{\mathbf{v} \rightarrow \frac{1}{\sqrt{3}}\right\}\right\}
\end{aligned}
$$

v4 = v / . eq1 [ [2] ]

$$
\frac{1}{\sqrt{3}}
$$

$$
\xi 4=\mathrm{K} 1 / . \operatorname{eq1}[[2]] / / \mathrm{N}
$$

$$
0.3849
$$

$$
\xi 3=-\xi 4
$$

$$
-0.3849
$$

$$
\mathrm{eq} 2=\mathrm{K} 1==\xi 3 ;
$$

$$
\text { eq3 }=\text { FindRoot }[e q 2,\{v, 0,2\}] / / N
$$

$$
\{v \rightarrow 1.1547\}
$$

v3 = v / . eq3

$$
1.1547
$$

$$
\text { I1 }=2 \int_{\mathrm{v} 3}^{\mathrm{v} 4} \frac{1-3 v^{2}}{v} d v / / \text { Simplify }
$$

$$
1.61371
$$

Here we consider the solution of the van der Pol equation (type-II)

$$
\ddot{x}-\varepsilon\left(1-x^{2}\right) \dot{x}+x=0
$$

We solve this differential equation with each value of $\varepsilon$ by using the Mathematica (NDSolve),

$$
\begin{aligned}
& \dot{x}=v \\
& \dot{v}-\varepsilon\left(1-x^{2}\right) v+x=0 .
\end{aligned}
$$

We assume that

$$
x_{0}=0,
$$

and $v_{0}$ is changed as a parameter, $v_{0}=-5,-4,-3,, 4$, and 5 . We also show the FFT calculation.
(1) van der pol oscillation with $\varepsilon=0.01 . t=0-100$.




FFT spectrum (the intensity vs $n$ ). $\varepsilon=0.01 . T=150 . \omega_{0}=2 \pi / 150 . \omega=n \omega_{0}$. There is a sharp peak at $n=25$, which means $\omega=25(2 \pi / 150)=\pi / 3$. The period $T$ is evaluated as 6 sec.
(2) van der pol oscillation with $\varepsilon=0.1 . t=0-100$.



FFT spectrum. $\varepsilon=0.1 . T=150 . \omega_{0}=2 \pi / 150 . \omega=n \omega_{0}$. The peaks appear at $\mathrm{n}=25$ and 72.
(3) van der pol oscillation with $\varepsilon=0.5 . t=0-100$.



FFT spectrum. $\varepsilon=0.5 . T=150 . \omega_{0}=2 \pi / 150 . \omega=n \omega_{0}$. The peaks appear at $\mathrm{n}=25,72$, 118 , and 165.
(4) van der pol oscillation with $\varepsilon=1 . t=0-100$.


FFT spectrum. $\varepsilon=1.0 . T=150 . \omega_{0}=2 \pi / 150 . \omega=n \omega_{0}$. The peaks appear at $\mathrm{n}=23.5,68$, $113,157,204,248,293$, and so on.
(5) van der pol oscillation with $\varepsilon=5 . t=0-100$.



FFT spectrum. $\varepsilon=5.0 . T=150 . \omega_{0}=2 \pi / 150 . \omega=n \omega_{0}$
(5) van der pol oscillation with $\varepsilon=10 . t=0-100$.



FFT spectrum. $\varepsilon=10.0 . T=150 . \omega_{0}=2 \pi / 150 . \omega=n \omega_{0}$
(7) van der pol oscillation with $\varepsilon=20 . t=0-100$.

(8) van der pol oscillation with $\varepsilon=30 . t=0-100$.


### 38.10 Limit cycle

We consider the phase plane of $v$ vs $x$ for small $\varepsilon$ (see the case for $\varepsilon=0.1$ ). Depending on the initial condition (in our case $v_{0}$ is given as a parameter and $x_{0}=0$ ). When $v_{0}=5$, the rapid motion occurring at early time gradually decays and approaches a
closed circle with radius 2 . The closed curve showing a periodic motion in the limit of $t$ $\rightarrow \infty$, is called a limit cycle. When $v_{0}=0$, the motion undergoes a gradual growth and approaches the limit cycle. In the case of small $\varepsilon$, the limit cycle is close to a circle. In a periodic motion, the friction is given by

$$
F=\varepsilon\left(1-x^{2}\right) \dot{x}
$$

The work $\Delta W$ during a period $T$ ( $=2 \pi$ in the present case) is equal to zero for the periodic motion;

$$
\Delta W=\int_{0}^{2 \pi} F d x=\int_{0}^{2 \pi} \varepsilon\left(1-x^{2}\right) \dot{x}^{2} d t
$$

Suppose that the limit cycle is approximated by a circle;

$$
x(t)=a \sin t
$$

with a period $2 \pi$. Then $\Delta W$ can be evaluated as

$$
\Delta W=\frac{\pi}{4} a^{2}\left(4-a^{2}\right)
$$

Then $a=2$. This is an amplitude of periodic motion.

### 38.11 Comment

The limit cycle for the van der Pol equation (type-I) exhibits a circle of radius $(2 \sqrt{3} / 3)$ in the limit of large $\varepsilon$, while the limit cycle for the van der Pol equation (typeII) exhibits a circle of radius 2 . The reason for this is as follows. We note that

$$
\begin{array}{ll}
\ddot{x}-\varepsilon\left(1-\dot{x}^{2}\right) \dot{x}+x=0 & \text { (van der Pol equation, type-I) } \\
\ddot{y}-\varepsilon\left(1-y^{2}\right) \dot{y}+y=0 & (\text { van der Pol equation type-II) }
\end{array}
$$

where

$$
\dot{x}=\frac{y}{\sqrt{3}}
$$

Suppose that we assume the limit cycle for the type-I, is described by

$$
x=a \sin (t)
$$

with $a=\frac{2 \sqrt{3}}{3}$. Then we get

$$
y=\sqrt{3} \dot{x}=\sqrt{3} a \cos (t)=2 \cos (t) .
$$

In other words, the radius of the limit cycle is 2 for the type-II.

## REFERENCE

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