

Chapter 43 Gauge transformation in quantum mechanics; Aharonov-Bohm effect

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David Joseph Bohm (20 December 1917 – 27 October 1992) was an American-born British quantum physicist who made contributions in the fields of theoretical physics, philosophy and neuropsychology, and to the Manhattan Project.



http://en.wikipedia.org/wiki/David_Bohm

Yakir Aharonov (born 1932 in Haifa, Israel) is an Israeli physicist specializing in Quantum Physics. He is a Professor of Theoretical Physics and the James J. Farley Professor of Natural Philosophy at Chapman University in California. He is also a distinguished professor in Perimeter Institute. He also serves as a professor emeritus at Tel Aviv University in Israel. He is president of the Iyar, The Israeli Institute for Advanced Research. His research interests are nonlocal and topological effects in quantum mechanics, quantum field theories and interpretations of quantum mechanics. In 1959, he and David Bohm proposed the Aharonov-Bohm Effect for which he co-received the 1998 Wolf Prize.

http://en.wikipedia.org/wiki/Yakir_Aharonov

Gauge transformation
 Aharonov-Bohm effect
 Young's double slits experiment
 Vector potential

43.1. Gauge transformations in electromagnetism

We start with the Maxwell's equations,

$$\begin{cases} \nabla \cdot \mathbf{E} = 4\pi\rho \\ \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \end{cases}$$

where

\mathbf{B} : magnetic field
 \mathbf{E} : electric field
 ρ : charge density
 \mathbf{J} : current density

The Lorentz force is defined as

$$\mathbf{F} = q[\mathbf{E} + \frac{1}{c}(\mathbf{v} \times \mathbf{B})].$$

The Lorentz force is expressed in terms of fields \mathbf{E} and \mathbf{B} , which is invariant under the gauge transformation (gauge independent). The magnetic field \mathbf{B} and electric field \mathbf{E} can be expressed by

$$\mathbf{B} = \nabla \times \mathbf{A},$$

$$\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \phi,$$

where \mathbf{A} is a vector potential and ϕ is a scalar potential. When \mathbf{E} and \mathbf{B} are given, ϕ and \mathbf{A} are not uniquely determined. If we have a set of possible values for the vector potential \mathbf{A} and the scalar potential ϕ , we obtain other potentials \mathbf{A}' and ϕ' which describes the same electromagnetic field by the gauge transformation,

$$\mathbf{A}' = \mathbf{A} + \nabla \chi,$$

$$\phi' = \phi - \frac{1}{c} \frac{\partial \chi}{\partial t},$$

where χ is an arbitrary function of \mathbf{r} . We note that \mathbf{B} and \mathbf{E} are gauge-invariant;

$$\mathbf{B}' = \nabla \times \mathbf{A}' = \nabla \times (\mathbf{A} + \nabla \chi) = \nabla \times \mathbf{A} = \mathbf{B}$$

$$\mathbf{E}' = -\frac{1}{c} \frac{\partial \mathbf{A}'}{\partial t} - \nabla \phi' = -\frac{1}{c} \frac{\partial (\mathbf{A} + \nabla \chi)}{\partial t} - \nabla \left(\phi - \frac{1}{c} \frac{\partial \chi}{\partial t} \right) = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \phi = \mathbf{E}$$

43.2 Canonical momentum and mechanical momentum

We now consider the Lagrangian which is defined by

$$L = \frac{1}{2} m \mathbf{v}^2 - q \left(\phi - \frac{1}{c} \mathbf{v} \cdot \mathbf{A} \right),$$

where m and q are the mass and charge of the particle. The Canonical momentum is defined as

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = m \mathbf{v} + \frac{q}{c} \mathbf{A}.$$

The mechanical momentum $\boldsymbol{\pi}$ is given by

$$\boldsymbol{\pi} = m \mathbf{v} = \mathbf{p} - \frac{q}{c} \mathbf{A}.$$

Then the Hamiltonian is obtained as

$$H = \mathbf{p} \cdot \mathbf{v} - L = \left(m \mathbf{v} + \frac{q}{c} \mathbf{A} \right) \cdot \mathbf{v} - L = \frac{1}{2} m \mathbf{v}^2 + q \phi = \frac{1}{2m} \left(\mathbf{p} - \frac{q}{c} \mathbf{A} \right)^2 + q \phi.$$

The Hamiltonian formalism uses \mathbf{A} and ϕ , and not \mathbf{E} and \mathbf{B} , directly. The result is that the description of the particle depends on the gauge chosen.

43.3 Analogy from Classical mechanics

The Newton's second law indicates that the position and the velocity take on, at every point, values independent of the gauge. Consequently,

$$\mathbf{r}' = \mathbf{r} \quad \text{and} \quad \mathbf{v}' = \mathbf{v},$$

or

$$\boldsymbol{\pi}' = \boldsymbol{\pi},$$

Since $\boldsymbol{\pi} = m\mathbf{v} = \mathbf{p} - \frac{q}{c} \mathbf{A}$, we have

$$\mathbf{p}' - \frac{q}{c} \mathbf{A}' = \mathbf{p} - \frac{q}{c} \mathbf{A},$$

or

$$\mathbf{p}' = \mathbf{p} + \frac{q}{c} (\mathbf{A}' - \mathbf{A}) = \mathbf{p} + \frac{q}{c} \nabla \chi.$$

In the Hamilton formalism, the value at each instant of the dynamical variables describing a given motion depends on the gauge chosen.

43.4 Gauge invariance in quantum mechanics

In quantum mechanics, we describe the states in the old gauge and the new gauge as $|\psi\rangle$ and $|\psi'\rangle$. The analogue of the relation in the classical mechanics is thus given by the relations between average values.

$$\langle \psi' | \hat{\mathbf{r}} | \psi' \rangle = \langle \psi | \hat{\mathbf{r}} | \psi \rangle \text{ (gauge independent)}$$

$$\langle \psi' | \hat{\boldsymbol{\pi}} | \psi' \rangle = \langle \psi | \hat{\boldsymbol{\pi}} | \psi \rangle \text{ (gauge independent)}$$

or equivalently

$$\langle \psi' | \hat{\mathbf{p}} - \frac{q}{c} \mathbf{A}' | \psi' \rangle = \langle \psi | \hat{\mathbf{p}} - \frac{q}{c} \mathbf{A} | \psi \rangle$$

(we will discuss the proof later).

We now seek a unitary operator \hat{U} which enables one to go from $|\psi\rangle$ to $|\psi'\rangle$.

$$|\psi'\rangle = \hat{U} |\psi\rangle$$

From the condition $\langle \psi' | \psi' \rangle = \langle \psi | \psi \rangle$, we have

$$\hat{U}^\dagger \hat{U} = \hat{U} \hat{U}^\dagger = \hat{1}$$

From the condition, $\langle \psi' | \hat{\mathbf{r}} | \psi' \rangle = \langle \psi | \hat{\mathbf{r}} | \psi \rangle$

$$\hat{U}^\dagger \hat{\mathbf{r}} \hat{U} = \hat{\mathbf{r}}$$

or

$$[\hat{\mathbf{r}}, \hat{U}] = 0 = i\hbar \frac{\partial \hat{U}}{\partial \hat{\mathbf{p}}}$$

\hat{U} is independent of $\hat{\mathbf{p}}$. We also get

$$\hat{U}^+ (\hat{\mathbf{p}} - \frac{q}{c} \mathbf{A}') \hat{U} = \hat{\mathbf{p}} - \frac{q}{c} \mathbf{A}$$

or

$$\hat{U}^+ (\hat{\mathbf{p}} - \frac{q}{c} \mathbf{A} - \frac{q}{c} \nabla \chi) \hat{U} = \hat{\mathbf{p}} - \frac{q}{c} \mathbf{A}$$

or

$$\begin{aligned} \hat{U}^+ \hat{\mathbf{p}} \hat{U} &= \frac{q}{c} \hat{U}^+ \mathbf{A} \hat{U} + \frac{q}{c} \hat{U}^+ \nabla \chi \hat{U} + \hat{\mathbf{p}} - \frac{q}{c} \mathbf{A} \\ &= \frac{q}{c} \mathbf{A} + \frac{q}{c} \nabla \chi + \hat{\mathbf{p}} - \frac{q}{c} \mathbf{A} \\ &= \frac{q}{c} \nabla \chi + \hat{\mathbf{p}} \end{aligned}$$

Note that $[\hat{\mathbf{r}}, \hat{U}] = 0$, and \mathbf{A} is a function of $\hat{\mathbf{r}}$.

((Note))

Here we show that

$$\langle \psi' | \hat{\pi} | \psi' \rangle = \langle \psi' | \hat{\pi} | \psi \rangle$$

is equivalent to

$$\langle \psi' | \hat{\mathbf{p}} - \frac{q}{c} \mathbf{A}' | \psi' \rangle = \langle \psi' | \hat{\mathbf{p}} - \frac{q}{c} \mathbf{A} | \psi \rangle,$$

where $\hat{\pi} = \hat{\mathbf{p}} - \frac{q}{c} \mathbf{A}$ and $\mathbf{A}' = \mathbf{A} + \frac{q}{c} \nabla \chi$.

((Proof))

$$\begin{aligned}
\langle \psi' | \hat{\mathbf{p}} - \frac{q}{c} \mathbf{A}' | \psi' \rangle &= \langle \psi | \hat{\mathbf{p}} + \frac{q}{c} \nabla \chi | \psi \rangle - \langle \psi | \hat{U}^\dagger \frac{q}{c} \mathbf{A}' \hat{U} | \psi \rangle \\
&= \langle \psi | \hat{\mathbf{p}} + \frac{q}{c} \nabla \chi | \psi \rangle - \langle \psi | \frac{q}{c} \mathbf{A}' | \psi \rangle \\
&= \langle \psi | \hat{\mathbf{p}} + \frac{q}{c} \nabla \chi - \frac{q}{c} \mathbf{A}' | \psi \rangle \\
&= \langle \psi | \hat{\mathbf{p}} - \frac{q}{c} (\mathbf{A}' - \nabla \chi) | \psi \rangle \\
&= \langle \psi | \hat{\mathbf{p}} - \frac{q}{c} \mathbf{A} | \psi \rangle
\end{aligned}$$

where $[\hat{U}, \mathbf{A}'] = \hat{0}$, since \mathbf{A}' is a function of $\hat{\mathbf{r}}$.

43.5 Expression of the unitary operator

We assert that \hat{U}

$$\hat{U} = \exp\left[\frac{iq}{\hbar c} \chi(\hat{\mathbf{r}})\right], \quad \hat{U}^\dagger = \exp\left[-\frac{iq}{\hbar c} \chi(\hat{\mathbf{r}})\right].$$

Then we get

$$\begin{aligned}
\hat{U}^\dagger \hat{\mathbf{p}} \hat{U} &= \hat{U}^\dagger [\hat{\mathbf{p}}, \hat{U}] + \hat{\mathbf{p}} \\
&= -\hat{U}^\dagger i\hbar \frac{\partial}{\partial \hat{\mathbf{r}}} \hat{U} + \hat{\mathbf{p}} \\
&= -\hat{U}^\dagger \hat{U} i\hbar \frac{iq}{\hbar c} \nabla \chi + \hat{\mathbf{p}} \\
&= \hat{\mathbf{p}} + \frac{q}{c} \nabla \chi
\end{aligned}$$

which coincides with the expression described above.

((Note)) We use the notation such that

$$\frac{\partial}{\partial \hat{\mathbf{r}}} \hat{U} = \frac{iq}{\hbar c} \hat{U} \frac{\partial \chi(\hat{\mathbf{r}})}{\partial \hat{\mathbf{r}}} = \frac{iq}{\hbar c} \hat{U} (\nabla \chi).$$

So we get the gauge transformation for the wave function;

$$|\psi'\rangle = \exp\left[\frac{iq}{\hbar c} \chi(\hat{\mathbf{r}})\right] |\psi\rangle.$$

or

$$\langle \mathbf{r} | \psi' \rangle = \exp\left[\frac{iq}{\hbar c} \chi(\mathbf{r})\right] \langle \mathbf{r} | \psi \rangle$$

The phase factor of the wave function depends on the choice of the form of χ in the gauge transformation. potential

43.6 Hamiltonian under the gauge transformation

We consider the Schrödinger equation given by

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle .$$

and

$$i\hbar \frac{\partial}{\partial t} |\psi'\rangle = \hat{H}' |\psi'\rangle ,$$

or

$$i\hbar \frac{\partial}{\partial t} \hat{U} |\psi\rangle = \hat{H}' \hat{U} |\psi\rangle ,$$

or

$$i\hbar \left[\frac{\partial \hat{U}}{\partial t} |\psi\rangle + \hat{U} \frac{\partial}{\partial t} |\psi\rangle \right] = \hat{H}' \hat{U} |\psi\rangle .$$

Since $\frac{\partial \hat{U}}{\partial t} = \frac{iq}{\hbar c} \frac{\partial \chi}{\partial t} \hat{U}$, we get

$$-\frac{q}{c} \frac{\partial \chi}{\partial t} \hat{U} |\psi\rangle + \hat{U} i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H}' \hat{U} |\psi\rangle ,$$

or

$$-\frac{q}{c} \frac{\partial \chi}{\partial t} \hat{U} |\psi\rangle + \hat{U} \hat{H} |\psi\rangle = \hat{H}' \hat{U} |\psi\rangle ,$$

or

$$\hat{H}' \hat{U} = -\frac{q}{c} \frac{\partial \chi}{\partial t} \hat{U} + \hat{U} \hat{H} .$$

Thus we have

$$\hat{H}' = -\frac{q}{c} \frac{\partial \chi}{\partial t} + \hat{U} \hat{H} \hat{U}^+.$$

Note that

$$\hat{U}^+ \hat{\mathbf{p}} \hat{U} = \hat{\mathbf{p}} + \frac{q}{c} \nabla \chi,$$

or

$$\hat{\mathbf{p}} \hat{U} = \hat{U} \left(\hat{\mathbf{p}} + \frac{q}{c} \nabla \chi \right),$$

or

$$\hat{\mathbf{p}} \hat{U} = \hat{U} \hat{\mathbf{p}} + \frac{q}{c} \hat{U} \nabla \chi.$$

We also note that

$$\begin{aligned} \hat{U} \hat{\mathbf{p}} \hat{U}^+ &= \hat{U} [\hat{\mathbf{p}}, \hat{U}^+] + \hat{\mathbf{p}} \\ &= -\hat{U} i \hbar \frac{\partial}{\partial \mathbf{r}} \hat{U}^+ + \hat{\mathbf{p}} \\ &= \hat{U} \hat{U}^+ i \hbar \frac{iq}{\hbar c} \nabla \chi + \hat{\mathbf{p}} \\ &= \hat{\mathbf{p}} - \frac{q}{c} \nabla \chi \end{aligned}$$

or

$$\hat{U} \hat{\mathbf{p}} \hat{U}^+ = \left(\hat{\mathbf{p}} - \frac{q}{c} \nabla \chi \right).$$

From the these relations, we get

$$\hat{U} \left(\hat{\mathbf{p}} - \frac{q}{c} \mathbf{A} \right) \hat{U}^+ = \left(\hat{\mathbf{p}} - \frac{q}{c} \mathbf{A} - \frac{q}{c} \nabla \chi \right) = \left(\hat{\mathbf{p}} - \frac{q}{c} \mathbf{A}' \right),$$

and

$$\hat{U}(\hat{\mathbf{p}} - \frac{q}{c} \mathbf{A})^2 \hat{U}^+ = (\hat{\mathbf{p}} - \frac{q}{c} \mathbf{A}')^2.$$

Then we have

$$\hat{H}' = -\frac{q}{c} \frac{\partial \chi}{\partial t} + \hat{U} \left[\frac{1}{2m} (\hat{\mathbf{p}} - \frac{q}{c} \mathbf{A})^2 + q\phi \right] \hat{U}^+$$

or

$$\hat{H}' = -\frac{q}{c} \frac{\partial \chi}{\partial t} + \frac{1}{2m} (\hat{\mathbf{p}} - \frac{q}{c} \mathbf{A}')^2 + q\phi = \frac{1}{2m} (\hat{\mathbf{p}} - \frac{q}{c} \mathbf{A}')^2 + q\phi'.$$

Therefore the Schrödinger equation can be written in the same way in any gauge chosen.

43.7 Invariance of physical predictions under a gauge transformation

The current density is invariant under the gauge transformation.

$$\mathbf{J} = \frac{1}{m} \text{Re}[\langle \psi | \hat{\mathbf{p}} - \frac{q}{c} \mathbf{A} | \psi \rangle]$$

$$\langle \psi' | \hat{\mathbf{p}} - \frac{q}{c} \mathbf{A}' | \psi' \rangle = \langle \psi | \hat{U}^+ (\hat{\mathbf{p}} - \frac{q}{c} \mathbf{A}') \hat{U} | \psi \rangle$$

$$\hat{U}^+ (\hat{\mathbf{p}} - \frac{q}{c} \mathbf{A}') \hat{U} = \hat{\mathbf{p}} + \frac{q}{c} \nabla \chi - \frac{q}{c} \mathbf{A}' = \hat{\mathbf{p}} - \frac{q}{c} \mathbf{A}$$

$$\mathbf{J}' = \frac{1}{m} \text{Re}[\langle \psi' | \hat{\mathbf{p}} - \frac{q}{c} \mathbf{A}' | \psi' \rangle] = \frac{1}{m} \text{Re}[\langle \psi | \hat{\mathbf{p}} - \frac{q}{c} \mathbf{A} | \psi \rangle] = \mathbf{J}$$

Note: after the gauge transformation, $\mathbf{A} \rightarrow \mathbf{A}'$ in the current density operator. This is identified from the form of Hamiltonian.

$$\hat{H}' = \frac{1}{2m} (\hat{\mathbf{p}} - \frac{q}{c} \mathbf{A}')^2 + q\phi'.$$

We note that the density is gauge invariant,

$$\rho' = |\langle \mathbf{r} | \psi' \rangle|^2 = \rho = |\langle \mathbf{r} | \psi \rangle|^2.$$

43.8 Aharonov-Bohm effect

In the best known version, electrons are aimed so as to pass through two regions that are free of electromagnetic field, but which are separated from each other by a long

cylindrical solenoid (which contains magnetic field line), arriving at a detector screen behind. At no stage do the electrons encounter any non-zero field \mathbf{B} .

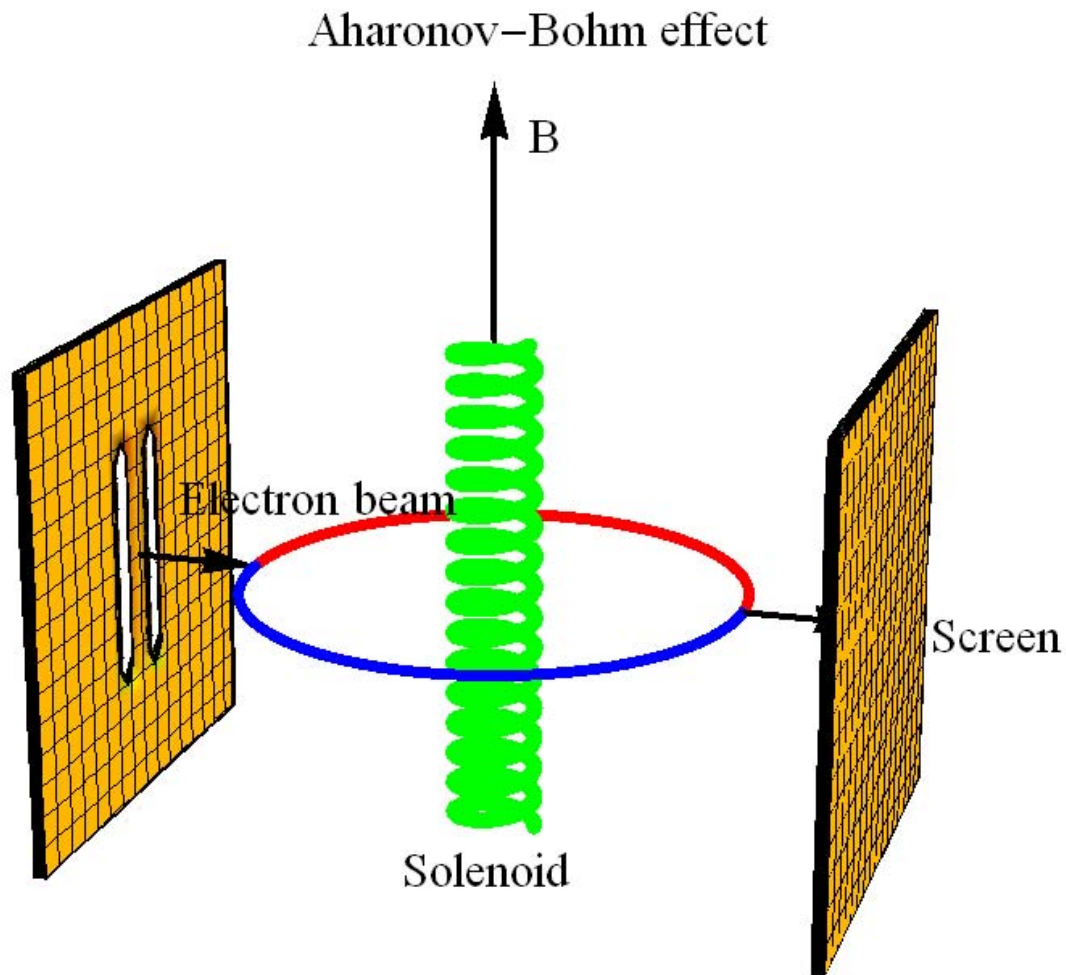


Fig. Schematic diagram of the Aharonov-Bohm experiment. Electron beams are split into two paths that go to either a collection of lines of magnetic flux (achieved by means of a long solenoid). The beams are brought together at a screen, and the resulting quantum interference pattern depends upon the magnetic flux strength—despite the fact that the electrons only encounter a zero magnetic field. Path denoted by red (counterclockwise). Path denoted by blue (clockwise)

We assume that $q = -e$ ($e > 0$). In the space when $\mathbf{B} = 0$, we have

$$\mathbf{B} = \nabla \times \mathbf{A} = 0,$$

or

$$\mathbf{A} = \nabla \chi ,$$

or

$$\chi(\mathbf{r}) = \int_{\mathbf{r}_0}^{\mathbf{r}} d\mathbf{r} \cdot \mathbf{A}(\mathbf{r}) ,$$

where \mathbf{r}_0 is an arbitrary initial point in the field region. We now consider the gauge transformation such that

$$\mathbf{A}' = \mathbf{A} + \nabla(-\chi) = 0 .$$

The new wavefunction $\psi'(\mathbf{r})$ can be written as

$$\psi'(\mathbf{r}) = \exp\left(\frac{ie\chi}{\hbar c}\right)\psi(\mathbf{r}) .$$

The Schrödinger equation for $\psi'(\mathbf{r})$ is

$$-\frac{\hbar^2}{2m}\nabla^2\psi' = i\hbar\frac{\partial}{\partial t}\psi' ,$$

where ψ' is the field-free wave function and the new Hamiltonian is that of free particle;

$$\hat{H}' = \frac{1}{2m}\hat{\mathbf{p}}^2 .$$

Then we have

$$\psi = \psi' \exp\left(-\frac{ie\chi}{\hbar c}\right) = \psi' \exp\left[-\frac{ie}{\hbar c} \int_{\mathbf{r}_0}^{\mathbf{r}} d\mathbf{r} \cdot \mathbf{A}(\mathbf{r})\right] ,$$

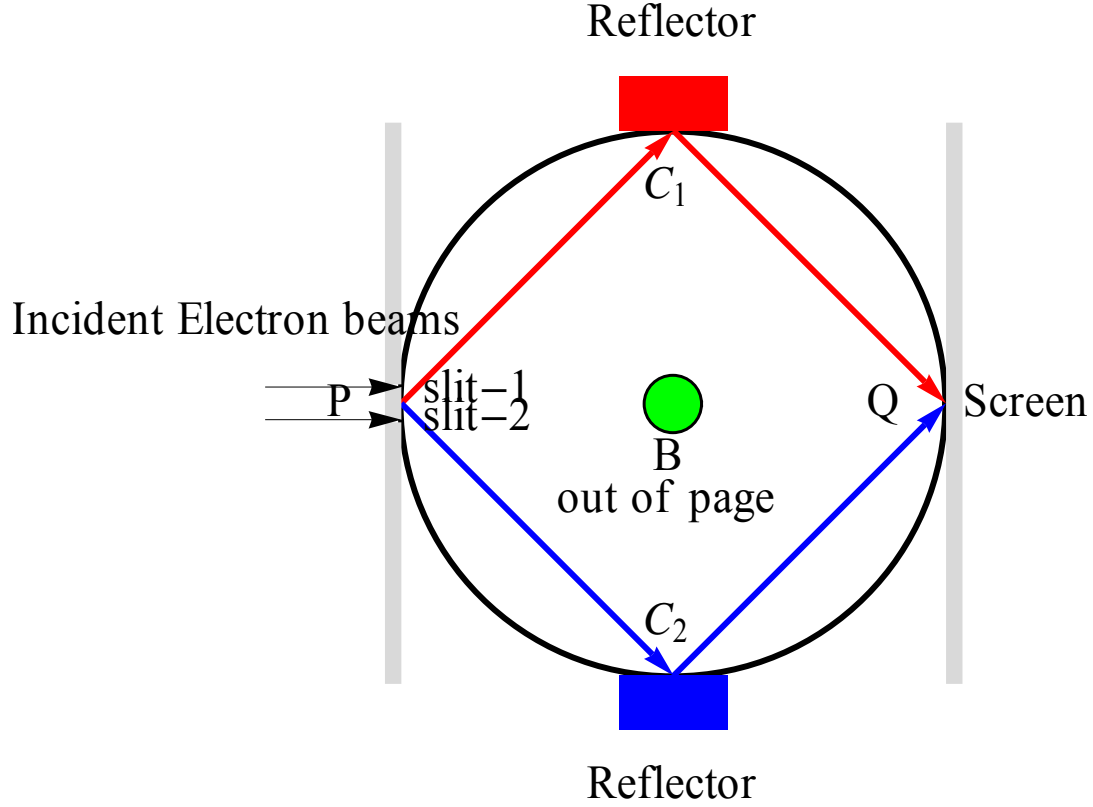


Fig. Schematic diagram of the Aharonov-Bohm experiment. Incident electron beams go into the two narrow slits (one beam denoted by blue arrow, and the other beam denoted by red arrow). The diffraction pattern is observed on the screen. The reflector plays a role of mirror for the optical experiment.

Let ψ_{1B} be the wave function when only slit 1 is open.

$$\psi_{1,B}(\mathbf{r}) = \psi_{1,0}(\mathbf{r}) \exp\left[-\frac{ie}{\hbar c} \int_{Path1} d\mathbf{r} \cdot \mathbf{A}(\mathbf{r})\right], \quad (1)$$

The line integral runs from the source through slit 1 to \mathbf{r} (screen). Similarly, for the wave function when only slit 2 is open, we have

$$\psi_{2,B}(\mathbf{r}) = \psi_{2,0}(\mathbf{r}) \exp\left[-\frac{ie}{\hbar c} \int_{Path2} d\mathbf{r} \cdot \mathbf{A}(\mathbf{r})\right], \quad (2)$$

The line integral runs from the source through slit 2 to \mathbf{r} (screen). Superimposing Eqs.(1) and (2), we obtain

$$\psi_B(\mathbf{r}) = \psi_{1,0}(\mathbf{r}) \exp\left[-\frac{ie}{\hbar c} \int_{Path1} d\mathbf{r} \cdot \mathbf{A}(\mathbf{r})\right] + \psi_{2,0}(\mathbf{r}) \exp\left[-\frac{ie}{\hbar c} \int_{Path2} d\mathbf{r} \cdot \mathbf{A}(\mathbf{r})\right]$$

The relative phase of the two terms is

$$\begin{aligned}\int_{Path1} d\mathbf{r} \cdot \mathbf{A}(\mathbf{r}) - \int_{Path2} d\mathbf{r} \cdot \mathbf{A}(\mathbf{r}) &= \oint d\mathbf{r} \cdot \mathbf{A}(\mathbf{r}) = \int d\mathbf{a} \cdot (\nabla \times \mathbf{A}) \\ &= \int d\mathbf{a} \cdot \mathbf{B} = \Phi_B\end{aligned}$$

by using the Stokes' theorem. Φ_B is the magnetic flux. Then we have

$$\psi_B(\mathbf{r}) = \exp\left[-\frac{ie}{\hbar c} \int_{Path2} d\mathbf{r} \cdot \mathbf{A}(\mathbf{r})\right] \psi_{1,0}(\mathbf{r}) \exp\left(-\frac{ie}{\hbar c} \Phi_B\right) + \psi_{2,0}(\mathbf{r}),$$

where the relative phase now is expressed in terms of the flux of the magnetic field through the closed path.

When

$$\frac{e}{\hbar c} \Phi_B = \frac{e}{\hbar c} \int_{\text{Closed path}} d\mathbf{a} \cdot \mathbf{B} = 2\pi n \quad (n = 0, 1, 2, 3, \dots).$$

The pattern will be the same as without the magnetic field present.

When

$$\frac{e}{\hbar c} \Phi_B = \frac{e}{\hbar c} \int_{\text{Closed path}} d\mathbf{a} \cdot \mathbf{B} = 2\pi\left(n + \frac{1}{2}\right),$$

or

$$\Phi_B = \frac{2\pi\hbar c}{e} \left(n + \frac{1}{2}\right) = 2\Phi_0 \left(n + \frac{1}{2}\right),$$

the position of the minimum and the maximum in the pattern will be interchanged. Φ_0 is the magnetic flux quanta and is given by

$$\Phi_0 = \frac{2\pi\hbar c}{2e} = 2.067833667 \times 10^{-7} \text{ Gauss cm}^2 = 2.067833667 \times 10^{-15} \text{ T m}^2.$$

((Note))

$$\psi_{1,0}(\mathbf{r}) \approx e^{ikr_1}, \quad \psi_{2,0}(\mathbf{r}) \approx e^{ikr_2}$$

The condition for constructive interference in the presence of a magnetic field is

$$kr_1 - \frac{e}{\hbar c} \Phi_B - kr_2 = 2\pi\ell$$

where ℓ is integers.

$$r_1 - r_2 = \frac{1}{k} \left(\frac{e}{\hbar c} \Phi_B + 2\pi\ell \right).$$

The positions of the interference maxima are shifted due to the variation in Φ_B , although the electron does not penetrate into the region of finite magnetic field.

(i) When $\frac{e}{\hbar c} \Phi_B = 2\pi n$

$$r_1 - r_2 = \frac{1}{k} 2\pi(n + \ell).$$

The pattern is the same as without B .

(ii) When $\frac{e}{\hbar c} \Phi_B = 2\pi(n + \frac{1}{2})$,

$$r_1 - r_2 = \frac{1}{k} 2\pi(n + \ell + \frac{1}{2}).$$

The pattern is different from that without B .

43.9 Young's double slit experiment for the Aharonov-Bohm effect

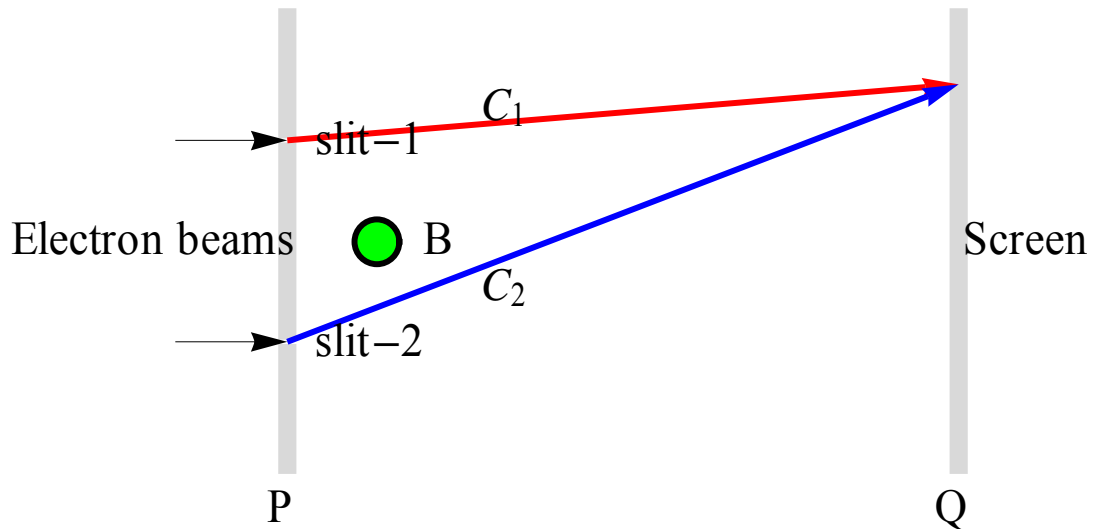


Fig. Young's double slit experiment with the electron beam source. The magnetic field is applied just behind the slits. There is no magnetic field around the paths C_1 and C_2 .

$$\phi_B = \frac{e}{\hbar c} \Phi_B = \frac{eBA}{\hbar c}.$$

$$|r_2 - r_1| = d \sin \theta.$$

The phase difference;

$$\phi_0 = \frac{2\pi}{\lambda} |r_2 - r_1| = \frac{2\pi}{\lambda} d \sin \theta \approx \frac{2\pi}{\lambda} d \theta$$

Since $y = D \tan \theta \approx D \theta$, we have

$$\phi_0 = \frac{2\pi}{\lambda} \frac{d}{D} y.$$

The intensity I

$$I = |\psi_B(\mathbf{r})|^2 = [1 + e^{i(\phi_0 - \phi_B)}][1 + e^{-i(\phi_0 - \phi_B)}] = 4 \sin^2\left(\frac{\phi_0 - \phi_B}{2}\right).$$

The intensity is described by

$$I = 4 \sin^2\left(\frac{\phi_0 - \phi_B}{2}\right) = 4 \sin^2\left(\frac{\frac{2\pi}{\lambda} \frac{d}{D} y - \frac{eBA}{\hbar c}}{2}\right).$$

When the effect of the width of the slit a is taken into account, the intensity is modified as

$$I = 4 \sin^2\left(\frac{\phi_0 - \phi_B}{2}\right) = 4 \sin^2\left(\frac{\frac{2\pi}{\lambda} \frac{d}{D} y - \frac{eBA}{\hbar c}}{2}\right) \frac{\sin^2 \beta}{\beta^2},$$

where

$$\beta = \frac{\pi a}{\lambda} \sin \theta \approx \frac{\pi a}{\lambda} \theta = \frac{\pi a}{\lambda} \frac{y}{D}.$$

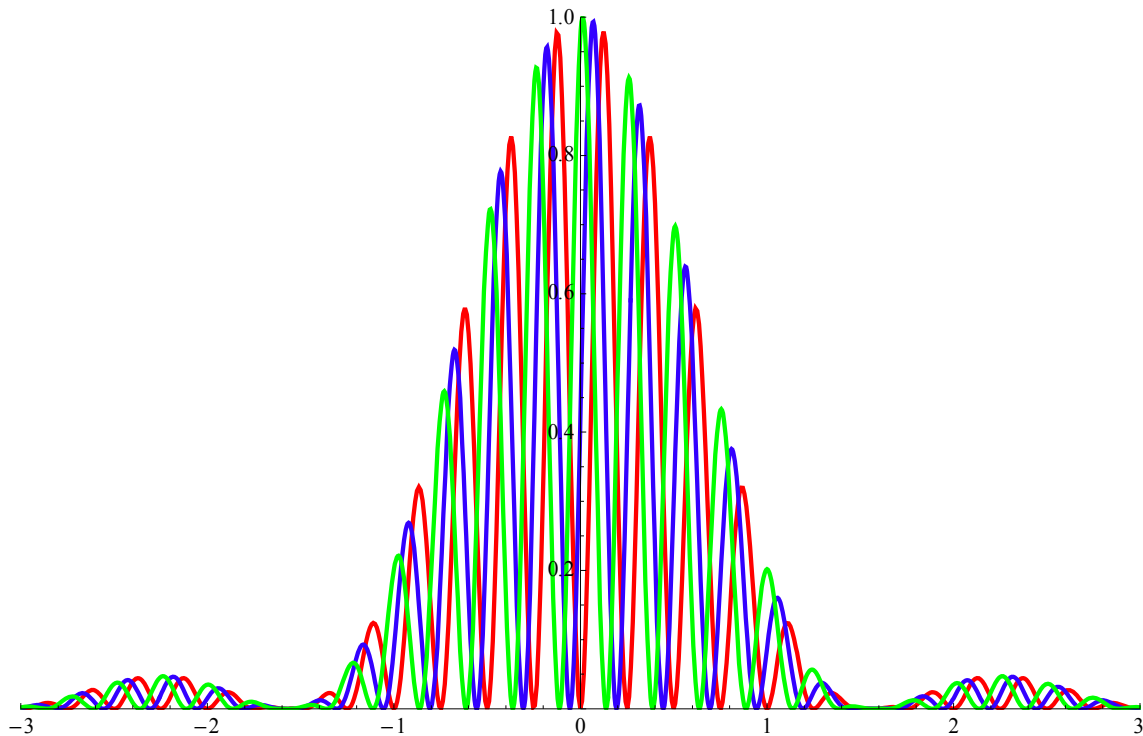


Fig. Young's double slits diffraction with Aharonov-Bohm effect. The diffraction pattern changes with the magnetic field. red ($B = 0$). Blue (B =intermediate value). Green (B = stronger field).

43.10. The observation of Aharonov-Bohm effect by Akira Tonomura

Summary of the article (by A. Tonomura)

[http://physicaplus.org.il/zope/home/en/1224031001/Tonomura_en]

- (i) A toroidal ferromagnet (permalloy) instead of a straight solenoid, which has inevitable leakage fluxes from both ends of the solenoid. An ideal geometry with no flux leakage can be achieved by the finite system of a toroidal magnetic field.
- (ii) The toroidal ferromagnet is covered with a superconducting niobium layer to completely confine the magnetic field.
- (iii) An electron wave is incident to a tiny toroidal sample fabricated using lithography techniques.
- (iv) The relative phase shift between two waves passing through the hole and around the toroid is measured as an interferogram. A relative phase shift of π is produced, indicating the existence of the AB effect even when the magnetic fields are confined within the superconductor and shielded from the electron wave. An electron wave must be physically influenced by the vector potentials. Therefore, it can be concluded that electron waves passing through the field-free regions inside and outside the toroidal magnet are phase-shifted by π , although the waves never touch the magnetic fields.

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