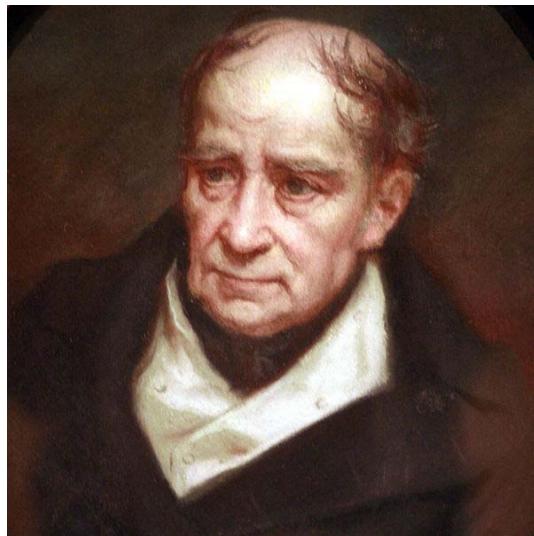


**Chapter 5**  
**Linear second order differential equation:**  
**Series expansion and second solution**  
**(October 1, 2010)**

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**Józef Maria Hoene-Wroński** (French: *Josef Hoëné-Wronski*; 23 August 1776 – 9 August 1853) was a Polish Messianist philosopher who worked in many fields of knowledge, not only as philosopher but also as mathematician, physicist, inventor, lawyer, economist. He was born Hoene but changed his name in 1815.



[http://en.wikipedia.org/wiki/J%C3%BCzef\\_Maria\\_Hoene-Wro%C5%84nski](http://en.wikipedia.org/wiki/J%C3%BCzef_Maria_Hoene-Wro%C5%84nski)

### **5.1 Regular point and regular singular point**

Linear, second order homogeneous differential equation

$$y'' + P(x)y' + Q(x)y = 0.$$

- (a) Ordinary point: (regular point):

*If  $P(x)$  or  $Q(x)$  remain finite at a point  $x = x_0$ . This point is called an ordinary point of the differential equation.*

- (b) Singular point:

*If  $P(x)$  or  $Q(x)$  diverges at a point  $x = x_0$ . This point is called an singular point.*

- (c) Regular singular point:

*If  $P(x)$  or  $Q(x)$  diverges at a point  $x = x_0$ , but  $(x-x_0)P(x)$  and  $(x-x_0)^2Q(x)$  remain finite at  $x = x_0$ , This point is called an regular singular point.*

- (d) Irregular or essential singularities:

*All other singular points are irregular or essential singularities.*

((Example-1))      Bessel differential equation

$$x^2 y'' + xy' + (x^2 - n^2) y = 0,$$

or

$$y'' + \frac{1}{x} y' + \left(1 - \frac{n^2}{x^2}\right) y = 0,$$

$$P(x) = \frac{1}{x}, \quad Q(x) = 1 - \frac{n^2}{x^2},$$

At  $x = 0$ ,  $P(x)$  and  $Q(x)$  diverges.

$x P(x) = 1$  and  $x^2 Q(x) = x^2 - n^2$  remain finite at  $x = 0$ .

$x = 0$  is a regular singular point of the Bessel differential equation.

((Example-2))      Legendre differential equation

$$(1-x^2) y'' - 2xy' + l(l+1)y = 0,$$

or

$$y'' - \frac{2x}{1-x^2} y' + \frac{l(l+1)}{1-x^2} y = 0,$$

$$P(x) = \frac{2x}{(x+1)(x-1)}, \quad Q(x) = -\frac{l(l+1)}{(x+1)(x-1)},$$

At  $x = \pm 1$ ,  $P(x)$  and  $Q(x)$  diverges.

$(x-1)P(x)$  and  $(x-1)^2 Q(x)$  remain finite at  $x = 1$ ,  
 $(x+1)P(x)$  and  $(x+1)^2 Q(x)$  remain finite at  $x = -1$ .

$x = \pm 1$  are regular singular point of the Legendre differential equation.

## 5.2 Change of variable of differential equation I

The analysis of point  $x \rightarrow \infty$  is to set  $x = 1/z$ , substitute into the differential equation, and then let  $z \rightarrow 0$ . We now consider a change of variable in differential equation

$$x = f(z)$$

$$\frac{dy}{dx} = \frac{dz}{dx} \frac{dy}{dz} = \frac{1}{\frac{dx}{dz}} \frac{d}{dz} y = \frac{1}{f'(z)} \frac{d}{dz} y.$$

Similarly, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left[ \frac{1}{f'(z)} \frac{d}{dz} \right] y = \frac{1}{f'(z)} \frac{d}{dz} \left[ \frac{1}{f'(z)} \frac{d}{dz} \right] y.$$

Then we have

$$y''(z) + [P(f(z))f'(z) - \frac{f''(z)}{f'(z)}]y'(z) + Q(f(z))[f'(z)]^2 y(z) = 0.$$

When  $x = f(z) = \frac{1}{z}$ , we get

$$y''(z) + \left( \frac{2z - P(z^{-1})}{z^2} \right) y'(z) + \frac{Q(z^{-1})}{z^4} y(z) = 0.$$

((Example)) Bessel's equation

$$y'' + \frac{1}{x} y' + \left(1 - \frac{n^2}{x^2}\right) y = 0.$$

When  $x = 1/z$ , this equation is transformed as

$$y''(z) + \frac{1}{z} y'(z) + \left(\frac{1-n^2 z^2}{z^4}\right) y(z) = 0,$$

for  $y(x=1/z) = y(z)$ . Since the coefficient of  $y(z)$  diverges  $1/z^4$  as  $z \rightarrow 0$ , the point  $x = \infty$  is an irregular or essential singularity.

It is interesting to calculate such transformation using the Mathematica. To this end, we just introduce the differential operator in the Mathematica.

$$OP := \frac{1}{D[f[z], z]} D[\#, z] &.$$

((Mathematica))

```

Clear["Global`*"];

OP :=  $\frac{1}{D[f[z], z]} D[\# , z] \&$ 

OP[ψ[z]]


$$\frac{\psi'[z]}{f'[z]}$$


Nest[OP, ψ[z], 1] // Simplify


$$\frac{\psi'[z]}{f'[z]}$$


Nest[OP, ψ[z], 2] // Simplify


$$\frac{-\psi'[z] f''[z] + f'[z] \psi''[z]}{f'[z]^3}$$


```

The secod order differential equation

$$\psi''[x] + P[x] \psi'[x] + Q[x] \psi[x] = 0.$$

```

eq1 = Nest[OP, ψ[z], 2] + P[f[z]] Nest[OP, ψ[z], 1] + Q[f[z]] ψ[z] // Simplify

Q[f[z]] ψ[z] + 
$$\frac{P[f[z]] f'[z]^2 \psi'[z] - \psi'[z] f''[z] + f'[z] \psi''[z]}{f'[z]^3}$$


eq2 = Solve[eq1 == 0, ψ''[z]]


$$\left\{ \psi''[z] \rightarrow \frac{-Q[f[z]] \psi[z] f'[z]^3 - P[f[z]] f'[z]^2 \psi'[z] + \psi'[z] f''[z]}{f'[z]} \right\}$$


eq3 = ψ''[z] - (ψ''[z] /. eq2[[1]]) // Simplify

Q[f[z]] ψ[z] f'[z]^2 + P[f[z]] f'[z] ψ'[z] - 
$$\frac{\psi'[z] f''[z]}{f'[z]} + \psi''[z]$$


```

```

Coefficient[eq3, ψ'[z]]
```

$$P[f[z]] f'[z] - \frac{f''[z]}{f'[z]}$$

```

Coefficient[eq3, ψ[z]]
```

$$Q[f[z]] f'[z]^2$$

of x=f[z]=1/z

```

rule1 = {f → (1/#) &} ;
```

```

eq3 == 0 /. rule1 // Simplify
```

$$\frac{Q\left[\frac{1}{z}\right] \psi[z]}{z^4} + \frac{\left(2 z - P\left[\frac{1}{z}\right]\right) \psi'[z]}{z^2} + \psi''[z] == 0$$

Bessel function       $P[x] = \frac{1}{x} \cdot Q[x] = 1 - \frac{n^2}{x^2}$

rule2 =  $\left\{ P \rightarrow \left( \frac{1}{\#} \& \right), Q \rightarrow \left( 1 - \frac{n^2}{\#^2} \& \right) \right\}$

eq3 = 0 /. rule1 /. rule2

$$\frac{(1 - n^2 z^2) \psi[z]}{z^4} + \frac{\psi'[z]}{z} + \psi''[z] = 0$$

### 5.3 Change of variable in differential equation II

We make a more convenient Mathematica program for the change of variable in the differential equation. Using this program we derive several differential equations for  $x = 1/z$ .

- (a) Legendre differential equation;

$$y''(x) - \frac{2x}{1-x^2} y'(x) + \frac{l(l+1)}{1-x^2} y(x) = 0$$

and

$$y''(z) - \frac{2z}{1-z^2} y'(z) - \frac{l(l+1)}{z^2(1-z^2)} y(z) = 0$$

for  $x = 1/z$ .

- (b) Hermite differential equation;

$$y''(x) - 2xy'(x) + 2\alpha y(x) = 0$$

and

$$y''(z) + \frac{2(z^2 + 1)}{z^3} y'(z) + \frac{2\alpha}{z^4} y(z) = 0$$

for  $x = 1/z$ .

((Mathematica))

```
Clear["Global`*"]

vchange[Eq_, ψ_, x_, z_, f_] :=
  Eq /. {D[ψ[x], {x, n_}] :> Nest[(1/D[f, z]) #, ψ[z], n], ψ[x] :> ψ[z], x :> f}

secdifchang[Eq_, ψ_, x_, z_, f_] :=
  Module[{req1, req2, req3, req4}, req1 = vchange[Eq, ψ, x, z, f] // Simplify;
  req2 = Solve[req1, ψ''[z]] // Simplify // Flatten;
  req3 = ψ''[z] - (ψ''[z] /. req2) == 0 // Simplify]

eq1 = y''[x] + P[x] y'[x] + Q[x] y[x] == 0
Q[x] y[x] + P[x] y'[x] + Y''[x] == 0

s1 = secdifchang[eq1, y, x, z, 1/z]
Q[1/z] y[z] + (2 z - P[1/z]) y'[z] + Y''[z] == 0
```

Bessel differential equation

```
rule1 = {P :> (1/# &), Q :> (1 - n^2/#^2 &)}
{P :> (1/z &), Q :> (1 - n^2/z^2 &)}

s2 = s1 /. rule1
(1 - n^2 z^2) Y[z] + Y'[z]/z + Y''[z] == 0
```

### Legendre differential equation

```
rule2 = {P → (-2 # / (1 - #2) &), Q → (# (#+1) / (1 - #2) &)}
```

$$\left\{ P \rightarrow \left( -\frac{2 \# 1}{1 - \# 1^2} \& \right), Q \rightarrow \left( \frac{\# (\# + 1)}{1 - \# 1^2} \& \right) \right\}$$

```
s3 = s1 /. rule2 // FullSimplify
```

$$\frac{\# (\# + 1) Y[z] + 2 z^3 Y'[z]}{z (-1 + z^2)} + z Y''[z] == 0$$

### Laguerre's differential equation

```
rule3 = {P → ((1 - #) / # &), Q → (a / # &)}
```

$$\left\{ P \rightarrow \left( \frac{1 - \# 1}{\# 1} \& \right), Q \rightarrow \left( \frac{a}{\# 1} \& \right) \right\}$$

```
s4 = s1 /. rule3 // FullSimplify
```

$$\frac{a Y[z]}{z} + (1 + z) Y'[z] + z^2 Y''[z] == 0$$

### Chebyshev differential equation

```
rule4 = {P → ((- #) / (1 - #2) &), Q → (n2 / (1 - #2) &)}
```

$$\left\{ P \rightarrow \left( -\frac{\# 1}{1 - \# 1^2} \& \right), Q \rightarrow \left( \frac{n^2}{1 - \# 1^2} \& \right) \right\}$$

```
s5 = s1 /. rule4 // FullSimplify
```

$$\frac{n^2 Y[z] + z (-1 + 2 z^2) Y'[z]}{z (-1 + z^2)} + z Y''[z] == 0$$

### Hermite differential equation

```
rule5 = {P → (- 2 # &), Q → (2 α #0 &)}
```

$$\left\{ P \rightarrow (-2 \# 1 \&), Q \rightarrow (2 \alpha \# 1^0 \&) \right\}$$

```
s6 = s1 /. rule5 // FullSimplify
```

$$\frac{2 \alpha Y[z] + 2 (z + z^3) Y'[z] + z^4 Y''[z]}{z} == 0$$

---

## 5.4 Fuch's theorem (I)

If we are expanding about an ordinary point or at worst about a regular singular point, the series expansion approach will yield at least one solution.

We assume that  $x_0$  is an ordinary point or a regular singular point.

$$y'' + P(x)y' + Q(x)y = 0,$$

with

$$P(x) = \frac{p_0}{x - x_0} + p_1 + p_2(x - x_0) + \dots = \sum_{\lambda=0}^{\infty} p_{\lambda}(x - x_0)^{\lambda-1},$$

$$Q(x) = \frac{q_0}{(x - x_0)^2} + \frac{q_1}{x - x_0} + q_2 + q_3(x - x_0) + \dots = \sum_{\lambda=0}^{\infty} q_{\lambda}(x - x_0)^{\lambda-2}.$$

The solution of the differential equation is assumed to be given by

$$y(x) = (x - x_0)^k \sum_{\lambda=0}^{\infty} y_{\lambda}(x - x_0)^{\lambda},$$

with  $y_0 \neq 0$ .  $k$  and  $y_{\lambda}$ 's must be determined. For simplicity, we put  $\xi = x - x_0$ .

$$y'' + P(\xi)y' + Q(\xi)y = 0,$$

with

$$y(\xi) = \sum_{\lambda=0}^{\infty} y_{\lambda}\xi^{\lambda+k},$$

$$P(\xi) = \sum_{\lambda=0}^{\infty} p_{\lambda}\xi^{\lambda-1},$$

$$Q(\xi) = \sum_{\lambda=0}^{\infty} q_{\lambda}\xi^{\lambda-2}.$$

Substituting these forms into the differential equation can be made using Mathematica.

The indicial equation:

$$k(k-1) + p_0k + q_0 = 0$$

This is quadratic, so there are two roots;  $k_1$  and  $k_2$ . For each root, we find a different set of values for  $y_1, y_2, y_3, \dots$ , i.e., the two solutions of the differential equation.

((Mathematica)) Determination of indicial equation

Series expansion ( indicial equation)

```
Clear["Global`*"]

eq1 = y'''[x] + P[x] y'[x] + Q[x] y[x];
rule1 = {P → (Sum[p[λ] #^λ-1 &, {λ, 0, 4}], Q → (Sum[q[λ] #^λ-2 &, {λ, 0, 4}], y → (Sum[y[λ] #^λ+k &, {λ, 0, 4}])};

eq2 = eq1 /. rule1 // Simplify; eq3 = eq2 x^{-k+3} // Expand;
Table[{λ - 1, Coefficient[eq3, x^λ]}, {λ, 1, 2, 1}] // FullSimplify

{{0, (k (-1 + k + p[0]) + q[0]) y[0]}, 
{1, (k p[1] + q[1]) y[0] + ((1 + k) (k + p[0]) + q[0]) y[1]}}

eq4 = Coefficient[eq3, x^1] /. {y[0] → 1} // Simplify
k^2 + k (-1 + p[0]) + q[0]

eq5 = Coefficient[eq3, x^2] /. {y[0] → 1} // Simplify;
eq6 = Solve[eq5 == 0, y[1]]
{{Y[1] → (-k p[1] - q[1]) / (k + k^2 + p[0] + k p[0] + q[0])}}

eq7 = Coefficient[eq3, x^3] /. {y[0] → 1} /. eq6[[1]] // Simplify;
eq8 = Solve[eq7 == 0, y[2]] // FullSimplify
{{Y[2] → (-k (1+k) (-p[1]^2 + (k+p[0]) p[2]) - k p[2] q[0] + (1+2 k) p[1] q[1] + q[1]^2) / ((1+k) (k+p[0]) + q[0]) - q[2] / ((2+k) (1 + k + p[0]) + q[0])}}
```

```

Example - 1 Bessel differential equation
 $x^2 y'' + xy' + (x^2 - n^2) y = 0$ 
 $P(x) = 1/x; P(0) = 1; Q(x) = 1 - n^2/x^2; Q(0) = -n^2, Q(1) = 0, Q(2) = 1$ 

rule2 = {p[0] → 1, p[1] → 0, p[2] → 0, p[3] → 0, p[4] → 0, q[0] → -n^2, q[1] → 0,
q[2] → 1, q[3] → 0, q[4] → 0}

{p[0] → 1, p[1] → 0, p[2] → 0, p[3] → 0,
p[4] → 0, q[0] → -n^2, q[1] → 0, q[2] → 1, q[3] → 0, q[4] → 0}

Indicial equation

s4 = Solve[eq4 == 0 /. rule2, k]
{{k → -n}, {k → n} }

s5 = Solve[eq5 == 0 /. rule2 /. s4[[2]], y[1]]
{{y[1] → 0} }

s7 = Solve[eq7 == 0 /. rule2 /. {k → n} /. s5[[1]], y[2]]
{{y[2] → -1/(4 (1 + n))}}

```

## 5.5 Second solution

Fuch's theorem (II)

- (a) If the two roots of the indicial equation are equal,  $k_1 = k_2$ , then we can obtain only one solution by series substitution method (Frobenius' method).
- (b) If the two roots differ by a non-integral number ( $k_2 - k_1 = \text{non-integral}$ ), two independent solutions may be obtained.
- (c) If the two roots differ by integer, the larger of the two will yield a solution.

## 5.6 Wronskian

Given a set of functions  $\varphi_\lambda$  ( $\lambda = 1, 2, \dots, n$ ), the criterion for linear dependence is the existence of the form

$$c_1 \varphi_1 + c_2 \varphi_2 + c_3 \varphi_3 + \dots + c_n \varphi_n = 0.$$

Let us assume that  $\varphi_\lambda$ 's are differentiable as needed. Then we have

$$c_1 \varphi_1^{(1)} + c_2 \varphi_2^{(1)} + c_3 \varphi_3^{(1)} + \dots + c_n \varphi_n^{(1)} = 0$$

$$c_1 \varphi_1^{(2)} + c_2 \varphi_2^{(2)} + c_3 \varphi_3^{(2)} + \dots + c_n \varphi_n^{(2)} = 0$$

$$c_1 \varphi_1^{(3)} + c_2 \varphi_2^{(3)} + c_3 \varphi_3^{(3)} + \dots + c_n \varphi_n^{(3)} = 0$$

$$\dots$$

$$c_1 \varphi_1^{(n-1)} + c_2 \varphi_2^{(n-1)} + c_3 \varphi_3^{(n-1)} + \dots + c_n \varphi_n^{(n-1)} = 0$$

If the determinant of the coefficient of the  $c_\lambda$ 's vanishes, then there is a solution  $c_\lambda \neq 0$  (nontrivial solution). This means that Wronskian is equal to zero,

$$W = \begin{vmatrix} \varphi_1 & \varphi_2 & \varphi_3 & \cdot & \cdot & \cdot & \cdot & \varphi_n \\ \varphi_1' & \varphi_2' & \varphi_3' & \cdot & \cdot & \cdot & \cdot & \varphi_n' \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \varphi_3^{(n-1)} & \cdot & \cdot & \cdot & \cdot & \varphi_n^{(n-1)} \end{vmatrix} = 0.$$

If  $\varphi_\lambda$  ( $\lambda = 1, 2, \dots, n$ ) are linearly dependent, then  $W = 0$  over the range of  $x$ .

### ((Mathematica))

Example 9-6-2 (Arfken) We have  $\varphi_1(x) = \exp(x)$ ,  $\varphi_2(x) = \exp(-x)$ , and  $\varphi_3(x) = \cosh[x]$ . Show that these functions are linearly dependent.

```
Clear["Global`*"];

ϕ1[x_] = Exp[x]; ϕ2[x_] = Exp[-x]; ϕ3[x_] = Cosh[x];

Wronskian[{ϕ1[x], ϕ2[x], ϕ3[x]}, x]
```

0

Now we consider a second-order differential equation

$$y'' + P(x)y' + Q(x)y = 0.$$

Suppose that we have two solutions of this differential equation;  $y_1$  and  $y_2$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

If  $y_1$  and  $y_2$  are linearly dependent,  $W=0$ .

If  $y_1$  and  $y_2$  are linearly independent,  $W \neq 0$ .

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix},$$

$$W'(x) = \begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ -Py_1' - Qy_1 & -Py_2' - Qy_2 \end{vmatrix},$$

or

$$W'(x) = \begin{vmatrix} y_1 & y_2 \\ -Py_1' & -Py_2' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 \\ -Qy_1 & -Qy_2 \end{vmatrix} = -P \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} - Q \begin{vmatrix} y_1 & y_2 \\ y_1 & y_2 \end{vmatrix},$$

or

$$W'(x) = \frac{dW(x)}{dx} = -PW(x).$$

Then we have

$$\int_a^x \frac{dW(x)}{W(x)} dx = - \int_a^x P(x) dx,$$

or

$$\ln \frac{W(x)}{W(a)} = - \int_a^x P(x) dx,$$

or

$$W(x) = W(a) \exp\left(- \int_a^x P(x_l) dx_l\right). \quad (\text{Liouville's formula})$$

((Mathematica))

Calculation of Wronskian  $W(x)$  for the typical second-order differential equation.

Wronskian determinant

Hermite differential equation

```
Clear["Global`*"]

Wronskian[y''[x] - 2 x y'[x] + 2 α y[x] == 0, y, x]

$$e^{x^2}$$

```

Bessel differential equation

```
Wronskian[x^2 y'''[x] + x y'[x] + (x^2 - n^2) y[x] == 0, y, x]

$$\frac{1}{x}$$

```

Legendre differential equation

```
Wronskian[(1 - x^2) y'''[x] - 2 x y'[x] + l (l + 1) y[x] == 0, y, x]

$$\frac{1}{-1 + x^2}$$

```

((Note))

We assume that  $y_1$  and  $y_2$  are the solutions of the equation given by

$$y'' + P(x)y' + Q(x)y = 0.$$

The Wronskian given by  $W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ ,

is the same as the Mathematica calculation

$$\text{Wronskian}[y'' + P(x)y' + Q(x)y == 0, y, x],$$

except for a constant.

((Example)) From the Documentation Center of the Wolfram

The Wronskian for a linear equation:

```
In[1]:= Wronskian[y''[x] - x y'[x] == 0, y, x]
Out[1]= 1
```

Except for a constant, the result is the same as for the explicit solution:

```
In[2]:= DSolve[y''[x] - x y'[x] == 0, y, x]
Out[2]= {{y → Function[{x}, AiryAi[x] C[1] + AiryBi[x] C[2]]}}
In[3]:= Wronskian[{AiryAi[x], AiryBi[x]}, x]
Out[3]= 1/π
```

---

## 5.7 Selected problems from Arfken

### 5.7.1 Arfken 9-6-3

Using the Wronskian determinant, show that the set of functions

$$\{1, \frac{x^n}{n!} (n = 1, 2, \dots, N)\}$$

is linearly independent.

((Mathematica))

```
Clear["Global`*"]
wse[n_] := Table[(x^k)/k!, {k, 0, n}]
Table[Wronskian[wse[n], x], {n, 2, 12}] // Simplify
{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
```

### 5.7.2 Arfken 9-6-9

Legendre's differential equation

$$(1-x^2)y''-2xy'+n(n+1)=0,$$

has a regular solution  $P_n(x)$  and an irregular solution  $Q_n(x)$ . Show that the Wronskian of  $P_n$  and  $Q_n$  is given by  $A_n/(1-x^2)$  with  $A_n$  independent of  $x$ .

Arfken 9 - 6 - 9

Method-I      Direct calculation from the definition of Wronskian

```
eq1 = (1 - x2) y''[x] - 2 x y'[x] + n (n + 1) y[x] == 0
n (1 + n) y[x] - 2 x y'[x] + (1 - x2) y''[x] == 0

DSolve[eq1, y[x], x]
{{y[x] → C[1] LegendreP[n, x] + C[2] LegendreQ[n, x]}}

W[x_, n_] :=
Det[{{LegendreP[n, x], LegendreQ[n, x]},
{D[LegendreP[n, x], x], D[LegendreQ[n, x], x]}}]

W[x, 1] // Simplify
1
—
1 - x2

W[x, 2] // Simplify
1
—
1 - x2

W[x, 3] // Simplify
1
—
1 - x2

W[x, 4] // Simplify
1
—
1 - x2
```

Method - II

$$\begin{aligned} \text{eq2} &= W'[x] == \frac{2 x}{1 - x^2} W[x] \\ W'[x] &= \frac{2 x W[x]}{1 - x^2} \end{aligned}$$

```
DSolve[eq2, W[x], x]
{{W[x] → C[1] / (-1 + x2)}}
```

Method - III

```
Wronskian[eq1, y, x]
1
—
-1 + x2
```

## 5.8 Second solution

Now we consider how this leads to a second solution of the differential equation

$$\frac{d}{dx} \left( \frac{y_2}{y_1} \right) = \frac{y_2' y_1 - y_2 y_1'}{y_1^2} = \frac{W(x)}{y_1^2},$$

or

$$W(x) = y_1^2 \frac{d}{dx} \left( \frac{y_2}{y_1} \right) = W(a) \exp \left( - \int_a^x P(x_1) dx_1 \right).$$

Thus we have

$$\frac{d}{dx} \left( \frac{y_2}{y_1} \right) = W(a) \frac{\exp \left( - \int_a^x P(x_1) dx_1 \right)}{[y_1(x)]^2},$$

or

$$\frac{y_2(x)}{y_1(x)} = W(a) \int_b^x \frac{\exp \left( - \int_a^{x_1} P(x_1) dx_1 \right)}{[y_1(x_2)]^2} dx_2 + \frac{y_2(b)}{y_1(b)},$$

or

$$y_2(x) = W(a) y_1(x) \int_b^x \frac{\exp \left( - \int_a^{x_1} P(x_1) dx_1 \right)}{[y_1(x_2)]^2} dx_2 + \frac{y_2(b)}{y_1(b)} y_1(x),$$

or

$$y_2(x) = W(a) y_1(x) \left[ \int_a^x \frac{\exp \left( - \int_a^{x_1} P(x_1) dx_1 \right)}{[y_1(x_2)]^2} dx_2 + \int_b^x \frac{\exp \left( - \int_a^{x_1} P(x_1) dx_1 \right)}{[y_1(x_2)]^2} dx_2 \right] + \frac{y_2(b)}{y_1(b)} y_1(x).$$

We are now interested in the first term, even if the second and third terms are dropped, it leads to nothing new.

We set  $W(a) = 1$ , since  $W(a)$  is a constant.

$$y_2(x) = y_1(x) \int_a^x \frac{\exp\left[-\int_{x_1}^{x_2} P(x_1) dx_1\right]}{\left[y_1(x_2)\right]^2} dx_2.$$

More generally

$$y_2(x) = y_1(x) \int_x^a \frac{\exp\left[-\int_{x_1}^{x_2} P(x_1) dx_1\right]}{\left[y_1(x_2)\right]^2} dx_2.$$

Note: the inclusion of lower limits on the two integrals leads to nothing new; that is, it throws in only overall factors and/or a multiple of the known solution  $y_1(x)$ .

### **5.9 Example-I ((Mathematica))**

```
Clear["Global`*"];
```

Example-1

```
Clear[y];  
  
eq1 = y''[x] - 2 y'[x] + y[x] == 0;  
  
DSolve[eq1, y[x], x]  
{ {y[x] → e^x C[1] + e^x x C[2]} }  
  
y11[x_] := e^x;  
  
y12[x_] := y11[x] ∫_0^x Exp[- ∫_0^{x2} (-2) dx1] dx2  
          / (y11[x2])^2  
  
y12[x] // Simplify  
e^x x
```

Example-2

```
Clear[y];  
  
eq2 = y''[x] + y[x] == 0;  
  
DSolve[eq2, y[x], x]  
{ {y[x] → C[1] Cos[x] + C[2] Sin[x]} }  
  
y21[x_] := Sin[x];  
  
y22[x_] := y21[x] ∫_{π/2}^x 1 / (y21[x2])^2 dx2;  
  
Simplify[y22[x], x > 0]  
-Cos[x]
```

---

**5.10 Example II** Arfken Example 9-6-4 (p.586).  
((Mathematica))

Bessel differential equation with n = 0

$$x^2 y'''[x] + x y'[x] + x^2 y[x] = 0$$

Suppose that we know one solution Bessel function J0[x]

```
Clear["Global`*"];
```

```
Clear[y];
```

```
eq1 = x^2 y'''[x] + x y'[x] + x^2 y[x] == 0;
```

```
DSolve[eq1, y[x], x]
```

```
{ {y[x] \rightarrow BesselJ[0, x] C[1] + BesselY[0, x] C[2]} }
```

```
eq2 = Exp[-Integrate[1/(x2), {x2, 1, x}]] // Simplify[#, x2 > 0] &
```

$$\frac{1}{x^2}$$

```
y1[x_] := BesselJ[0, x];
```

```
eq3 = Series[1/(y1[x2]^2, {x2, 0, 11}] // Normal
```

$$1 + \frac{x^2}{2} + \frac{5 x^4}{32} + \frac{23 x^6}{576} + \frac{677 x^8}{73728} + \frac{7313 x^{10}}{3686400}$$

```
eq4 = Simplify[Integrate[1/x2 (eq3), {x2, 1, x}] // Expand
```

$$-\frac{65705753}{221184000} + \frac{x^2}{4} + \frac{5 x^4}{128} + \frac{23 x^6}{3456} + \frac{677 x^8}{589824} + \frac{7313 x^{10}}{36864000} + \text{Log}[x]$$

$$\alpha = \frac{65705753}{221184000} // N$$

$$0.297064$$

```
\gamma = EulerGamma // N
```

$$0.577216$$

The second solution :

$$\begin{aligned}
 N0[x] &= \frac{2}{\pi} J0[x] (\text{eq4} - \text{Log}[x] + \alpha) \\
 &+ \frac{2}{\pi} J0[x] \text{Log}[x] + \frac{2}{\pi} J0[x] (-\text{Log}[2] + \gamma) \\
 &= \frac{2}{\pi} J0[x] \text{eq4} + (\alpha + \gamma - \text{Log}[2]) \frac{2}{\pi} J0[x]
 \end{aligned}$$

We compare our approximation of  $Y0[x]$  with the second solution  $\text{BesselY}[0, x]$ .

$$\text{eq5} = \frac{2}{\pi} \text{BesselJ}[0, x] (\text{eq4} + \alpha - \text{Log}[x]);$$

$$\text{eq55} = \text{Series}[\text{eq5}, \{x, 0, 9\}] // \text{Normal}$$

$$\frac{x^2}{2 \pi} - \frac{3 x^4}{64 \pi} + \frac{11 x^6}{6912 \pi} - \frac{25 x^8}{884736 \pi}$$

$$N0[x] = \text{eq55} + \frac{2}{\pi} \text{BesselJ}[0, x] (\text{Log}[x] - \text{Log}[2] + \gamma) //$$

Simplify;

```

f1 = Plot[{BesselJ[0, x], BesselY[0, x], N0[x]}, {x, 0.01, 5},
  PlotStyle -> {{Thick, Hue[0]}, {Thick, Hue[0.4]}, {Thick, Hue[0.7]}}, Background -> GrayLevel[0.7]];

g1 = Graphics[{Text[Style["J0", Red, 12], {1, 0.8}], Text[Style["N0", Blue, 12], {2, 0.6}], Text[Style["Y0", Green, 12], {4.5, -0.1}]}];
Show[f1, g1]

```

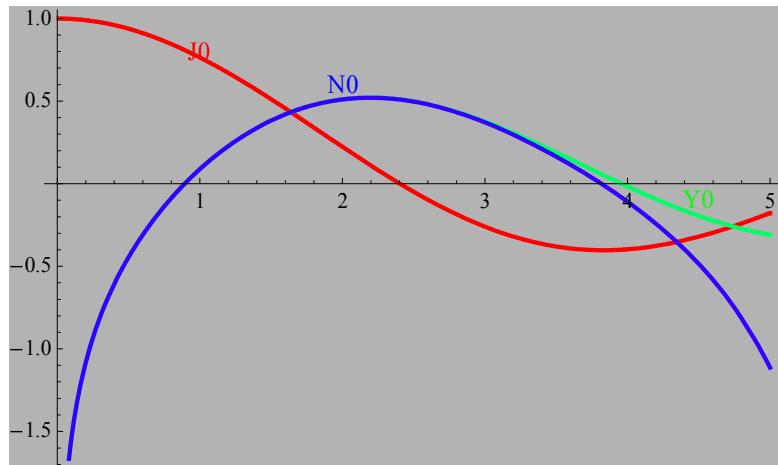


Fig.1 Plot of Bessel functions  $J_0(x)$  (the first kind),  $N_0(x)$  (the second kind), and approximation  $Y_0(x)$  to  $N_0(x)$ .

---

### 5.11 Simple harmonics (quantum mechanics)

We consider the Schrödinger equation for the simple harmonics (quantum mechanics),

$$\langle x | \hat{H} | n \rangle = E \langle x | n \rangle,$$

or

$$\langle x | \frac{1}{2m} \hat{p}^2 + \frac{m\omega_0^2}{2} \hat{x}^2 | n \rangle = E \langle x | n \rangle,$$

or

$$(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{m\omega_0^2}{2} x^2) \varphi_n(x) = E \varphi_n(x),$$

with

$$\varphi_n(x) = \langle x | n \rangle.$$

Here we use  $\xi$  instead of  $x$ ;

$$\xi = \beta x,$$

with

$$\beta = \sqrt{\frac{m\omega_0}{\hbar}}. \quad (\text{unit: cm}^{-1}).$$

Using the relations given by

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} = \beta \frac{\partial}{\partial \xi},$$

and

$$\frac{\partial^2}{\partial x^2} = \beta \frac{\partial}{\partial \xi} (\beta \frac{\partial}{\partial \xi}) = \beta^2 \frac{\partial^2}{\partial \xi^2},$$

we get

$$\left(-\frac{\hbar^2}{2m}\beta^2 \frac{d^2}{d\xi^2} + \frac{m\omega_0^2}{2}\frac{\xi^2}{\beta^2}\right)\langle\xi|n\rangle = E\langle\xi|n\rangle,$$

or

$$\left(\frac{\hbar\omega_0}{2}\frac{d^2}{d\xi^2} + \frac{\hbar\omega_0}{2}\xi^2\right)\langle\xi|n\rangle = E\langle\xi|n\rangle,$$

or

$$\left(\frac{d^2}{d\xi^2} - \xi^2 + 2\varepsilon\right)\varphi_n(\xi) = 0,$$

where

$$\varepsilon = \frac{E}{\hbar\omega_0}$$

and

$$\varphi_n(\xi) = \langle\xi|n\rangle = \frac{1}{\sqrt{\beta}}\varphi_n(x).$$

### 5.12 Differential equation (Series expansion method)

We consider a differential equation given by

$$\left[\frac{d^2}{d\xi^2} - (\xi^2 - 2\varepsilon)\right]\varphi(\xi) = 0.$$

Let us try to predict intuitively the behavior of  $\varphi(\xi)$  for very large  $\xi$ .

$$\left(\frac{d^2}{d\xi^2} - \xi^2\right)\varphi(\xi) = 0.$$

To do this, we consider

$$G_{\pm}(\xi) = e^{\pm\frac{\xi^2}{2}},$$

which satisfies

$$[\frac{d^2}{d\xi^2} - (\xi^2 \pm 1)]G_{\pm}(\xi) = 0.$$

When  $\xi$  approaches infinity,  $\xi^2 \pm 1 \approx \xi^2 \approx \xi^2 - 2\varepsilon$ .

We choose

$$G_-(\xi) = e^{-\frac{\xi^2}{2}}$$

from a physical point of view.

$$\lim_{\xi \rightarrow \infty} G_-(\xi) = 0$$

Now we set

$$\varphi(\xi) = e^{-\frac{\xi^2}{2}} h(\xi)$$

Here  $h(\xi)$  satisfies the differential equation.

$$[\frac{d^2}{d\xi^2} - 2\xi \frac{d}{d\xi} + (2\varepsilon - 1)]h(\xi) = 0$$

$h(\xi)$  should be either even or odd functions from the symmetry of solutions.

### (Proof) The symmetry of solutions.

Suppose that  $h(\xi)$  is a solution of the above differential equation. Then we have

$$[\frac{d^2}{d\xi^2} - 2\xi \frac{d}{d\xi} + (2\varepsilon - 1)]h(\xi) = 0$$

When  $\xi$  is changed into  $-\xi$ , then we get

$$[\frac{d^2}{d\xi^2} - 2\xi \frac{d}{d\xi} + (2\varepsilon - 1)]h(-\xi) = 0 \tag{1}$$

which means that  $h(-\xi)$  is also the solution of the differential equation. Then any solution may be resolved into even and odd parts,

$$h(\xi) = \frac{1}{2}[h(\xi) + h(-\xi)] + \frac{1}{2}[h(\xi) - h(-\xi)].$$

The first bracket on the right gives an even solution, and the second an odd solution.

So we assume that

$$h(\xi) = \xi^p (a_0 + a_2 \xi^2 + a_4 \xi^4 + \dots) = \sum_{m=0}^{\infty} a_{2m} \xi^{2m+p},$$

with  $a_0 \neq 0$ .

$$h'(\xi) = \sum_{m=0}^{\infty} a_{2m} (2m+p) \xi^{2m+p-1},$$

$$h''(\xi) = \sum_{m=0}^{\infty} a_{2m} (2m+p)(2m+p-1) \xi^{2m+p-2},$$

$$\sum_{m=0}^{\infty} a_{2m} (2m+p)(2m+p-1) \xi^{2m+p-2} - 2 \sum_{m=0}^{\infty} a_{2m} (2m+p) \xi^{2m+p},$$

$$+ \sum_{m=0}^{\infty} (2\varepsilon - 1) a_{2m} \xi^{2m+p} = 0,$$

or

$$a_0 p(p-1) \xi^{p-2} + a_2 (p+2)(p+1) \xi^p + [-2a_0 p \xi^p + (2\varepsilon - 1) a_0 \xi^p] + \dots = 0.$$

The coefficient of  $\xi^{p-2}$

$$a_0 p(p-1) = 0.$$

Then  $p = 0$  or  $1$ .

In general, for the co-efficient of  $\xi^{2m+p}$

$$a_{2m} (2\varepsilon - 1 - 4m - 2p) + a_{2m+2} (2m+p+2)(2m+p+1) = 0,$$

or

$$a_{2m+2} = -\frac{2\varepsilon - (4m+2p+1)}{(2m+p+1)(2m+p+2)} a_{2m}, \quad (1)$$

with  $p = 0$  or  $1$ .

We consider what happens when  $\varepsilon$  is not expressed by

$$2\varepsilon = 4m_0 + 2p + 1,$$

where  $m_0$  is some positive integer.

$$\lim_{m \rightarrow \infty} \frac{a_{2m+2}}{a_{2m}} = \lim_{m \rightarrow \infty} \frac{-2\varepsilon + (4m+2p+1)}{(2m+p+1)(2m+p+2)} = \frac{1}{m}.$$

Now we consider the power series of  $e^{\xi^2}$

$$e^{\xi^2} = \sum_{m=0}^{\infty} b_{2m} \xi^{2m},$$

with

$$b_{2m} = \frac{1}{m!}.$$

Thus

$$\lim_{m \rightarrow \infty} \frac{b_{2m+2}}{b_{2m}} \approx \frac{1}{m}.$$

This means that

$$h(\xi) \approx e^{\xi^2}.$$

or

$$\varphi(\xi) = e^{-\frac{\xi^2}{2}} h(\xi) \approx e^{-\frac{\xi^2}{2}} e^{\xi^2} \approx e^{\frac{\xi^2}{2}}.$$

which becomes infinity when  $\xi$  tends to infinity. We must reject this solution. This solution makes no sense physically. The numerator of Eq.(1) goes to zero for a value  $m_0$  of  $m$ .

$$a_{2m} \neq 0 \text{ for } m \leq m_0,$$

and

$$a_{2m} = 0 \text{ for } m > m_0,$$

or

$$4m_0 + 2p - 2\varepsilon + 1 = 0,$$

or

$$\varepsilon = 2m_0 + p + \frac{1}{2} = \frac{E}{\hbar\omega}.$$

If we set  $n = 2m_0 + p$

( $n = \text{even}$  for  $p = 0$  and  $n = \text{odd}$  for  $p = 1$ .)

or

$$E_n = \hbar\omega(n + \frac{1}{2}),$$

and

$$h_n(\xi) = \xi^p (a_0 + a_2 \xi^2 + a_4 \xi^4 + \dots + a_{2m_0} \xi^{2m_0}).$$

Then  $p = 0$  or  $1$ .

We consider the two cases separately.

(a)  $p = 0$ .

The coefficients of  $\xi^0, \xi^2, \xi^4, \xi^6, \xi^8, \dots$

$$\xi^0 \quad (2\varepsilon - 1)a_0 + 2a_2 = 0$$

$$\xi^2 \quad (2\varepsilon - 5)a_2 + 12a_4 = 0$$

$$\xi^4 \quad (2\varepsilon - 9)a_4 + 30a_6 = 0$$

$$\xi^6 \quad (2\varepsilon - 13)a_6 + 56a_8 = 0$$

$$\xi^8 \quad (2\varepsilon - 17)a_8 + 90a_{10} = 0$$

$$\xi^{10} \quad (2\varepsilon - 21)a_{10} + 132a_{12} = 0$$

$$\xi^{12} \quad (2\varepsilon - 25)a_{12} + 182a_{14} = 0$$

$$\xi^{14} \quad (2\varepsilon - 29)a_{14} + 240a_{16} = 0$$

$$\xi^{16} \quad (2\varepsilon - 33)a_{16} + 306a_{18} = 0$$

$$\xi^{18} \quad (2\varepsilon - 37)a_{18} + 380a_{20} = 0$$


---

(b)  $p = 1$

The coefficients of  $\xi^1, \xi^3, \xi^5, \xi^7, \dots$

$$\xi^1 \quad (2\varepsilon - 3)a_0 + 6a_2 = 0$$

$$\xi^3 \quad (2\varepsilon - 7)a_2 + 20a_4 = 0$$

$$\xi^5 \quad (2\varepsilon - 11)a_4 + 42a_6 = 0$$

$$\xi^7 \quad (2\varepsilon - 15)a_6 + 72a_8 = 0$$

$$\xi^9 \quad (2\varepsilon - 19)a_8 + 110a_{10} = 0$$

$$\xi^{11} \quad (2\varepsilon - 23)a_{10} + 156a_{12} = 0$$

$$\xi^{13} \quad (2\varepsilon - 27)a_{12} + 210a_{14} = 0$$

$$\xi^{15} \quad (2\varepsilon - 31)a_{14} + 272a_{16} = 0$$

$$\xi^{17} \quad (2\varepsilon - 35)a_{16} + 342a_{18} = 0$$

$$\xi^{19} \quad (2\varepsilon - 39)a_{18} + 420a_{20} = 0$$

**((Stationary wave function))**

(a) Ground state ( $n = 0$ ):  $\langle \xi | 0 \rangle$ : even parity

$$\varepsilon = 1/2$$

$$m_0 = 0, p = 0.$$

$$a_2 = 0, a_0 \neq 0.$$

$$h(\xi) = a_0$$

$$\varphi_0(\xi) = a_0 e^{-\frac{\xi^2}{2}} \text{ (even function)}$$

Normalization:

$$\int_{-\infty}^{\infty} |\varphi_0(\xi)|^2 d\xi = \int_{-\infty}^{\infty} |a_0|^2 \exp(-\xi^2) d\xi = |a_0|^2 \sqrt{\pi} = 1$$

or

$$\varphi_0(\xi) = \pi^{-1/4} \frac{1}{\sqrt{2^0 0!}} e^{-\frac{\xi^2}{2}}$$

(b)  $n = 1$  state:  $\langle \xi | 1 \rangle$ : odd parity

$$\varepsilon = 3/2$$

$$m_0 = 0, p = 1.$$

$$a_2 = 0, a_0 \neq 0.$$

$$h(\xi) = a_0 \xi$$

$$\varphi_1(\xi) = e^{-\frac{\xi^2}{2}} a_0 \xi$$

$$\int_{-\infty}^{\infty} |\varphi_1(\xi)|^2 d\xi = \int_{-\infty}^{\infty} |a_0|^2 \xi^2 \exp(-\xi^2) d\xi = |a_0|^2 \frac{\sqrt{\pi}}{2} = 1$$

$$\varphi_1(\xi) = \pi^{-1/4} \frac{1}{\sqrt{2^1 1!}} (2\xi) e^{-\frac{\xi^2}{2}}$$

(c)  $n = 2$  state:  $\langle \xi | 2 \rangle$ : even parity

$$\varepsilon = 5/2, \text{ or } E = \hbar \omega_0 (2 + \frac{1}{2}).$$

$$m_0 = 1, p = 0.$$

$$2a_2 + 4a_0 = 0.$$

$$a_2 = -2a_0.$$

$$\varphi_2(\xi) = e^{-\frac{\xi^2}{2}} a_0 (1 - 2\xi^2).$$

$$\int_{-\infty}^{\infty} |\varphi_2(\xi)|^2 d\xi = \int_{-\infty}^{\infty} |a_0|^2 (1 - 2\xi^2)^2 \exp(-\xi^2) d\xi = |a_0|^2 2\sqrt{\pi} = 1.$$

$$\varphi_2(\xi) = \pi^{-1/4} \frac{1}{\sqrt{2^2 2!}} (-2 + 4\xi^2) e^{-\frac{\xi^2}{2}}.$$

(d)  $n = 3$  state:  $\langle \xi | 3 \rangle$ : odd parity

$$\varepsilon = 7/2, \text{ or } E = \hbar\omega_0(3 + \frac{1}{2}).$$

$$m_0 = 1, p = 1.$$

$$6a_2 + 4a_0 = 0.$$

$$\varphi_3(\xi) = e^{-\frac{\xi^2}{2}} a_0 \xi (1 - \frac{2}{3} \xi^2).$$

$$\int_{-\infty}^{\infty} |\varphi_3(\xi)|^2 d\xi = \int_{-\infty}^{\infty} |a_0|^2 \xi^2 (1 - \frac{2}{3} \xi^2)^2 \exp(-\xi^2) d\xi = |a_0|^2 \frac{\sqrt{\pi}}{3} = 1.$$

$$\varphi_3(\xi) = e^{-\frac{\xi^2}{2}} \pi^{-1/4} \frac{1}{\sqrt{2^3 3!}} (-12\xi + 8\xi^3).$$

(e)  $n = 4$  state:  $\langle \xi | 4 \rangle$ : even parity

$$\varepsilon = \frac{9}{2} \text{ or } E = \hbar\omega_0(4 + \frac{1}{2}),$$

$$p = 0, m_0 = 2,$$

$$a_4 = -\frac{1}{12}(2\varepsilon - 5)a_2 = -\frac{1}{3}a_2 = -\frac{1}{3}(-4a_0) = \frac{4}{3}a_0.$$

$$a_2 = -\frac{1}{2}(2\varepsilon - 1)a_0 = -4a_0.$$

$$h_4(\xi) = a_0(1 - 4\xi^2 + \frac{4}{3}\xi^4) = a_0'(12 - 48\xi^2 + 16\xi^4).$$

$$\varphi_4(\xi) = \exp[-\frac{\xi^2}{2}] h_4(\xi).$$

Normalization:

$$\int_{-\infty}^{\infty} \exp[-\xi^2] |a_0|^2 (12 - 48\xi^2 + 16\xi^4)^2 d\xi = |a_0|^2 \sqrt{\pi} 2^4 4! = 1,$$

or

$$|a_0|^2 = \pi^{-1/4} \frac{1}{\sqrt{2^4 4!}}.$$

Thus we have

$$\varphi_4(\xi) = \pi^{-1/4} \frac{1}{\sqrt{2^4 4!}} \exp[-\frac{\xi^2}{2}] H_4(\xi).$$

((Note))

$H_n(\xi)$  is the Hermite polynomial.

$$H_0(\xi) = 1$$

$$H_1(\xi) = 2\xi$$

$$H_2(\xi) = 4\xi^2 - 2$$

$$H_3(\xi) = 8\xi^3 - 12\xi$$

$$H_4(\xi) = 16\xi^4 - 48\xi^2 + 12$$

$$H_5(\xi) = 32\xi^5 - 160\xi^3 + 120\xi$$

$H_n(\xi)$  satisfies the differential equation given by

$$(\frac{d^2}{d\xi^2} - 2\xi \frac{d}{d\xi} + 2n) H_n(\xi) = 0.$$

In Mathematica, the Hermite polynomial is defined as `HermiteH[n, x]`.

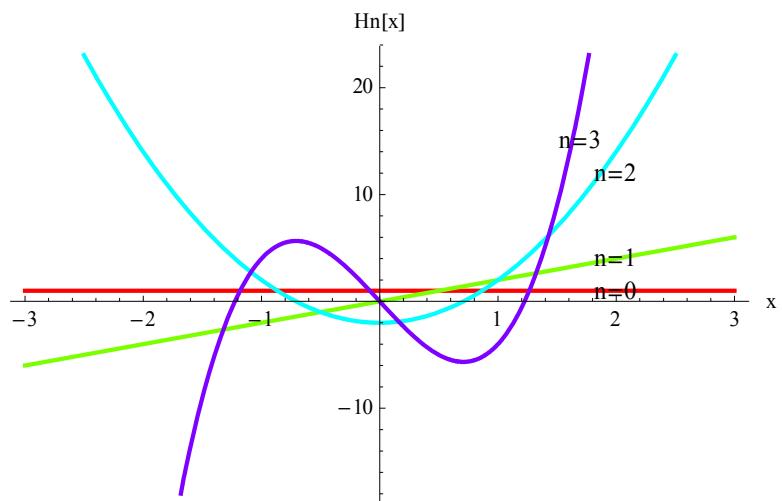


Fig.2 Plot of the Hermite polynomial  $H_n(x)$  as a function of  $x$ .  $n = 0, 1, 2$ , and  $3$ .

### 5.13 Power series by Mathematica

## Hermite differential equation

$$(-1 + 2 \varepsilon) h[\xi] - 2 \xi h'[\xi] + h''[\xi] = 0$$

```

Clear["Global`*"];

Eq1=D[\varphi[\xi],{\xi,2}]-\left(\xi^2-2\varepsilon\right)\varphi[\xi];

rule1=\left\{\varphi\rightarrow\left(\text{Exp}\left[-\frac{\#^2}{2}\right]\,h[\#]&\right)\right\},

\left\{\varphi\rightarrow\left(e^{-\frac{\#1^2}{2}}\,h[\#1]&\right)\right\};

Eq2=Eq1/.rule1//Simplify
e^{-\frac{\xi^2}{2}}\left((-1+2\varepsilon)\,h[\xi]-2\xi\,h'[\xi]+h''[\xi]\right);

e^{-\frac{\xi^2}{2}}\left((-1+2\varepsilon)\,h[\xi]-2\xi\,h'[\xi]+h''[\xi]\right)

Eq3=Eq2 e^{\frac{\xi^2}{2}}//Simplify
(-1+2\varepsilon)\,h[\xi]-2\xi\,h'[\xi]+h''[\xi]

```

$h(\xi)$  is either an even function or an odd function from the symmetry of the differential equation.

Derivation of recurrence relation

$$a[2+2m] = \frac{(1+4m+2p-2\varepsilon)}{(1+2m+p)(2+2m+p)} a[2m]$$

```

f[ξ_] := ξ^p ∑_{k=-1}^1 a[2m + 2k] ξ^{2m+2k};

Eq4 = Eq3 /. h → f // Simplify
ξ^{-4+2m+p} (-ξ^2 (-4m^2 + p - p^2 + 2pξ^2 + (1 - 2ε) ξ^2 + m (2 - 4p + 4ξ^2)) a[2m] +
(6 + 4m^2 + p^2 + 3ξ^2 + 2εξ^2 + 2m (-5 + 2p - 2ξ^2) - p (5 + 2ξ^2)) a[-2+2m] +
ξ^4 (2 + 4m^2 + p^2 - 5ξ^2 + 2εξ^2 + m (6 + 4p - 4ξ^2) + p (3 - 2ξ^2)) a[2+2m])

Eq5 = Eq4 ξ^{6-2m-p} // FullSimplify
ξ^2 (-ξ^2 (2m - 4m^2 + p - 4mp - p^2 + (1 + 4m + 2p - 2ε) ξ^2) a[2m] +
(6 - 10m + 4m^2 - 5p + 4mp + p^2 + (3 - 4m - 2p + 2ε) ξ^2) a[-2+2m] +
ξ^4 (2 + 6m + 4m^2 + 3p + 4mp + p^2 - (5 + 4m + 2p - 2ε) ξ^2) a[2+2m])

list1=Table[{2 n,Coefficient[Eq5,ξ,2 n]}, {n,1,4}]//Simplify//TableForm

2 (6 + 4m^2 - 5p + p^2 + 2m (-5 + 2p)) a[-2+2m]
4 (4m^2 + (-1 + p)p + m (-2 + 4p)) a[2m] + (3 - 4m - 2p + 2ε) a[-2+2m]
6 -(1 + 4m + 2p - 2ε) a[2m] + (2 + 4m^2 + 3p + p^2 + m (6 + 4p)) a[2+2m]
8 -(5 + 4m + 2p - 2ε) a[2+2m]

```

We pick up the appropriate recurrence relation from the above.

```

seq1 = Coefficient[Eq5, ξ, 6] // Simplify
-(1 + 4m + 2p - 2ε) a[2m] + (2 + 4m^2 + 3p + p^2 + m (6 + 4p)) a[2+2m]

Solve[seq1 == 0, a[2+2m]] // Factor
{a[2+2m] → (1 + 4m + 2p - 2ε) a[2m] / ((1 + 2m + p)(2 + 2m + p))}

```

**Indicial equation;  $p(p-1) = 0$ ,**  
 **$p = 0$  (even parity),     $p = 1$  (odd parity).**

```

f0[ξ_] := ξ^p Sum[a[2k] ξ^{2k}, {k, 0, 3}];

Eq6 = Eq3 /. h → f0 // Simplify
ξ^{-2+p}
(p^2 (a[0] + ξ^2 a[2] + ξ^4 a[4] + ξ^6 a[6]) + ξ^2 ((-1 + 2 ε) a[0] + (2 + (-5 + 2 ε) ξ^2) a[2] +
12 ξ^2 a[4] - 9 ξ^4 a[6] + 2 ε ξ^4 a[8] + 30 ξ^6 a[10] - 13 ξ^8 a[12] + 2 ε ξ^8 a[14]) -
p ((1 + 2 ξ^2) a[0] - 3 ξ^2 a[2] + ξ^4 (2 a[2] - 7 a[4]) +
ξ^6 (2 a[4] - 11 a[6]) + 2 ξ^8 a[8])))

Eq7 = Eq6 ξ^{2-p} // FullSimplify
p^2 (a[0] + ξ^2 a[2] + ξ^4 a[4] + ξ^6 a[6]) -
p ((1 + 2 ξ^2) a[0] - 3 ξ^2 a[2] + ξ^4 (2 a[2] - 7 a[4]) + ξ^6 (2 a[4] - 11 a[6]) + 2 ξ^8 a[8]) +
ξ^2 ((-1 + 2 ε) a[0] + (2 + (-5 + 2 ε) ξ^2) a[2] +
ξ^2 (12 + (-9 + 2 ε) ξ^2) a[4] + ξ^4 (30 + (-13 + 2 ε) ξ^2) a[6])

list1 = Table[{2n, Coefficient[Eq7, ξ, 2n]}, {n, 0, 3}] // Simplify // TableForm
0 (-1 + p) p a[0]
2 (-1 - 2 p + 2 ε) a[0] + (2 + 3 p + p^2) a[2]
4 (-5 - 2 p + 2 ε) a[2] + (12 + 7 p + p^2) a[4]
6 (-9 - 2 p + 2 ε) a[4] + (30 + 11 p + p^2) a[6]

Eq8 = Coefficient[Eq7, ξ, 0] // Simplify
(-1 + p) p a[0]

Solve[Eq8 == 0, p]
{{p → 0}, {p → 1}}

```

### 5.14 Simpler method (Mathematica)    Example-1

We show a simple method of solving differential equation by using Mathematica (which is used by Prof. Richard Palmer of Physics Department of Duke University). For simplicity, we solve the differential equation.

$$y''(x) + 9y(x) = 0.$$

The solution of this equation is

$$y = C_1 \cos(3x) + C_2 \sin(3x).$$

((Mathematica))

## Differential equation of simple harmonics

$$y'' + 9 y = 0$$

```
Clear["Global`*"];

eq1 = y''[x] + 9 y[x]==0
9 y[x] + y''[x] == 0

DSolve[eq1, y[x], x]
{ {y[x] \rightarrow C[1] Cos[3 x] + C[2] Sin[3 x]} }
```

We solve this problem using series expansion

```
f[x_] = Sum[a[i] x^i, {i, 0, 10}] + O[x]^11
a[0] + a[1] x + a[2] x^2 + a[3] x^3 + a[4] x^4 + a[5] x^5 +
a[6] x^6 + a[7] x^7 + a[8] x^8 + a[9] x^9 + a[10] x^10 + O[x]^11

eq2 = eq1 /. y \rightarrow f
(9 a[0] + 2 a[2]) + (9 a[1] + 6 a[3]) x + (9 a[2] + 12 a[4]) x^2 +
(9 a[3] + 20 a[5]) x^3 + (9 a[4] + 30 a[6]) x^4 + (9 a[5] + 42 a[7]) x^5 +
(9 a[6] + 56 a[8]) x^6 + (9 a[7] + 72 a[9]) x^7 + (9 a[8] + 90 a[10]) x^8 + O[x]^9 == 0

eq3 = Solve[eq2]

Solve::svrs: Equations may not give solutions for all "solve" variables. >>
{ {a[0] \rightarrow -\frac{44800 a[10]}{729}, a[2] \rightarrow \frac{22400 a[10]}{81},
a[1] \rightarrow \frac{4480 a[9]}{81}, a[3] \rightarrow -\frac{2240 a[9]}{27}, a[4] \rightarrow -\frac{5600 a[10]}{27},
a[5] \rightarrow \frac{112 a[9]}{3}, a[6] \rightarrow \frac{560 a[10]}{9}, a[7] \rightarrow -8 a[9], a[8] \rightarrow -10 a[10]} }
```

```

eq4 = f[x] /. eq3[[1]]

$$-\frac{44800 a[10]}{729} + \frac{4480}{81} a[9] x + \frac{22400}{81} a[10] x^2 - \frac{2240}{27} a[9] x^3 - \frac{5600}{27} a[10] x^4 +$$


$$\frac{112}{3} a[9] x^5 + \frac{560}{9} a[10] x^6 - 8 a[9] x^7 - 10 a[10] x^8 + a[9] x^9 + a[10] x^{10} + O[x]^{11}$$


Collect[eq4, {a[9], a[10]}]

$$\left( \frac{4480 x}{81} - \frac{2240 x^3}{27} + \frac{112 x^5}{3} - 8 x^7 + x^9 \right) a[9] +$$


$$\left( -\frac{44800}{729} + \frac{22400 x^2}{81} - \frac{5600 x^4}{27} + \frac{560 x^6}{9} - 10 x^8 + x^{10} \right) a[10]$$


eq5 = Coefficient[eq4, a[9]] 3  $\left( \frac{81}{4480} \right)$  // Expand

$$3 x - \frac{9 x^3}{2} + \frac{81 x^5}{40} - \frac{243 x^7}{560} + \frac{243 x^9}{4480}$$


eq6 = Coefficient[eq4, a[10]]  $\left( -\frac{729}{44800} \right)$  // Expand

$$1 - \frac{9 x^2}{2} + \frac{27 x^4}{8} - \frac{81 x^6}{80} + \frac{729 x^8}{4480} - \frac{729 x^{10}}{44800}$$


```

Comparison with series expansion of  $\sin(3x)$  and  $\cos(3x)$

```

Series[Sin[3 x], {x, 0, 10}]

$$3 x - \frac{9 x^3}{2} + \frac{81 x^5}{40} - \frac{243 x^7}{560} + \frac{243 x^9}{4480} + O[x]^{11}$$


Series[Cos[3 x], {x, 0, 10}]

$$1 - \frac{9 x^2}{2} + \frac{27 x^4}{8} - \frac{81 x^6}{80} + \frac{729 x^8}{4480} - \frac{729 x^{10}}{44800} + O[x]^{11}$$


```

## 5.15 Solving the differential equation by the simple method: Example-II

Legendre differential equation

$$(1 - x^2) y''[x] - 2x y'[x] + 5(5+1) y[x] = 0$$

At the ordinary point, you can always assume that  $k = 0$  for the indicial equation

Indicial equation,  $k(k-1) + k p_0 + q_0 = 0$ , with  $p_0 = 0, q_0 = 0$

$k = 0$  or  $1$

```

Clear["Global`*"] ;

f[x_] := Sum[a[λ] x^λ, {λ, 0, 12}] + O[x]^13;

eq1 = (1 - x^2) y''[x] - 2 x y'[x] + 5 (5 + 1) y[x] == 0
30 y[x] - 2 x y'[x] + (1 - x^2) y''[x] == 0

eq2 = eq1 /. y → f // Simplify
2 (15 a[0] + a[2]) + (28 a[1] + 6 a[3]) x +
12 (2 a[2] + a[4]) x^2 + (18 a[3] + 20 a[5]) x^3 +
10 (a[4] + 3 a[6]) x^4 + 42 a[7] x^5 + (-12 a[6] + 56 a[8]) x^6 +
(-26 a[7] + 72 a[9]) x^7 + (-42 a[8] + 90 a[10]) x^8 + O[x]^9 == 0

```

**eq3 = Solve[eq2]**

Solve::svrs: Equations may not give solutions for all "solve" variables. >>

$$\left\{ \begin{array}{l} \{a[0] \rightarrow -a[10], a[2] \rightarrow 15 a[10], a[1] \rightarrow \frac{5 a[5]}{21}, a[3] \rightarrow -\frac{10 a[5]}{9}, \\ a[4] \rightarrow -30 a[10], a[6] \rightarrow 10 a[10], a[7] \rightarrow 0, a[8] \rightarrow \frac{15 a[10]}{7}, a[9] \rightarrow 0\} \end{array} \right.$$

**eq4 = f[x] /. eq3[[1]]**

$$-a[10] + \frac{5}{21} a[5] x + 15 a[10] x^2 - \frac{10}{9} a[5] x^3 - 30 a[10] x^4 + a[5] x^5 + 10 a[10] x^6 + \frac{15}{7} a[10] x^8 + a[10] x^{10} + O[x]^{11}$$

```
eq5 = Coefficient[eq4, a[5]]  $\frac{63}{8}$  // Expand
```

$$\frac{15x}{8} - \frac{35x^3}{4} + \frac{63x^5}{8}$$

```
eq6 = Coefficient[eq4, a[10]]  $\frac{8}{15}$  // Expand
```

$$-\frac{8}{15} + 8x^2 - 16x^4 + \frac{16x^6}{3} + \frac{8x^8}{7} + \frac{8x^{10}}{15}$$

The solution can be derived by solving the differential equation directly.

```
eq5 = DSolve[eq1, y[x], x] // Simplify
```

$$\left\{ \left\{ y[x] \rightarrow \frac{1}{8}x(15 - 70x^2 + 63x^4)C[1] + C[2] \left( -\frac{8}{15} + \frac{49x^2}{8} - \frac{63x^4}{8} - \frac{1}{16}x(15 - 70x^2 + 63x^4)(\text{Log}[1-x] - \text{Log}[1+x]) \right) \right\} \right\}$$

```
y1[x_] = y[x] /. eq5[[1]];
```

```
Series[y1[x] /. eq5, {x, 0, 10}] // Collect[#, {C[1], C[2]}] &
```

$$\left\{ \left( \frac{15x}{8} - \frac{35x^3}{4} + \frac{63x^5}{8} \right) C[1] + \left( -\frac{8}{15} + 8x^2 - 16x^4 + \frac{16x^6}{3} + \frac{8x^8}{7} + \frac{8x^{10}}{15} \right) C[2] \right\}$$

## Appendix I Mathematica programs

- Wronskian[{y1, y2, ...}, x], Wronskian[Table[y[n], {n, 1, N}], x]

To give the Wronskian determinant for the functions  $y_1, y_2, \dots$  depending on  $x$ .

- Wronskian[*eqn*, *y*, *x*]

To gives the Wronskian determinant for the basis of the solutions of the linear differential equation *eqn* with dependent variable *y* and independent variable *x*.

## Appendix II: Parity operator

### II.1

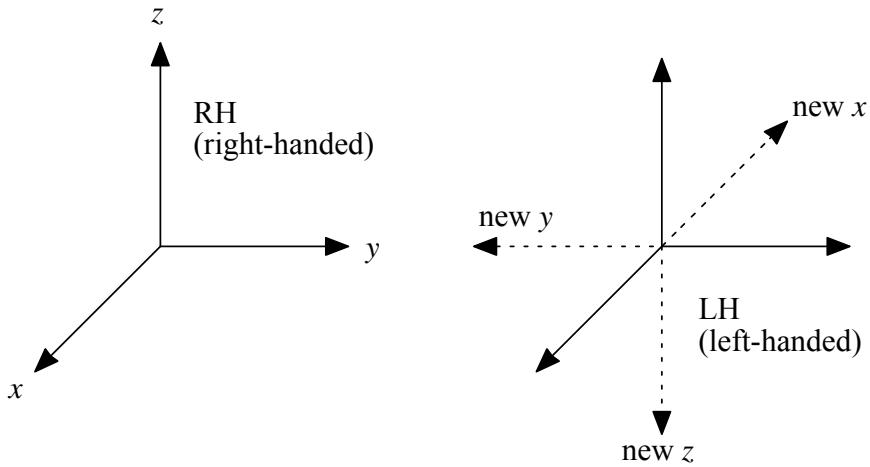


Fig.3

$\hat{\pi}$ : parity operator (unitary operator)

$$|\psi'\rangle = \hat{\pi}|\psi\rangle,$$

or

$$\langle\psi'| = \langle\psi| \hat{\pi}^+.$$

Definition: the average of  $\hat{x}$  in the new state  $|\psi'\rangle$  is opposite to that in the old state  $|\psi\rangle$

$$\langle\psi'|\hat{x}|\psi'\rangle = -\langle\psi|\hat{x}|\psi\rangle,$$

or

$$\langle\psi|\hat{\pi}^+\hat{x}\hat{\pi}|\psi\rangle = -\langle\psi|\hat{x}|\psi\rangle,$$

or

$$\hat{\pi}^+\hat{x}\hat{\pi} = -\hat{x}. \quad (1)$$

The position vector is called a polar vector.

Normalization:

$$\langle\psi'|\psi'\rangle = \langle\psi|\hat{\pi}^+\hat{\pi}|\psi\rangle = \langle\psi|\psi\rangle = 1.$$

or

$$\hat{\pi}^+ \hat{\pi} = \hat{1}. \quad (2)$$

Thus the parity operator is an unitary operator. From Eqs.(1) and (2), we have

$$\hat{x} \hat{\pi} + \hat{\pi} \hat{x} = 0,$$

or

$$\hat{x} \hat{\pi} |x\rangle = -\hat{\pi} \hat{x} |x\rangle = -x \hat{\pi} |x\rangle.$$

Thus

$\hat{\pi}|x\rangle$  is the eigenket of  $\hat{x}$  with the eigenvalue  $(-x)$ .

or

$$\hat{\pi}|x\rangle = |-x\rangle,$$

$$\hat{\pi} \hat{\pi} |x\rangle = \hat{\pi} |-x\rangle = |x\rangle,$$

or

$$\hat{\pi}^2 = \hat{1}.$$

Since  $\hat{\pi}^+ \hat{\pi} = \hat{1}$  and  $\hat{\pi}^2 = \hat{1}$ ,

$$\hat{\pi}^+ \hat{\pi} \hat{\pi} = \hat{\pi},$$

or

$$\hat{\pi}^+ = \hat{\pi}.$$

So the parity operator is a Hermite operator.

$$\begin{aligned}
\hat{\pi}|p\rangle &= \hat{\pi} \int_{-\infty}^{\infty} dx' |x'\rangle \langle x'| p \rangle \\
&= \int_{-\infty}^{\infty} dx' |-x'\rangle \langle x'| p \rangle \\
&= \int_{-\infty}^{\infty} dx' |-x'\rangle \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ipx'}{\hbar}\right) \\
&= \int_{-\infty}^{\infty} dx |x\rangle \frac{1}{\sqrt{2\pi\hbar}} \exp\left(-\frac{ipx}{\hbar}\right) \\
&= \int_{-\infty}^{\infty} dx |x\rangle \langle x| -p \rangle \\
&= |-p\rangle
\end{aligned}$$

Note that  $x' = -x$  and  $dx' = -dx$ ,

$$\hat{\pi}|p\rangle = |-p\rangle,$$

and

$$\hat{p}|p\rangle = p|p\rangle.$$

$$\hat{\pi}\hat{p}|p\rangle = p\hat{\pi}|p\rangle = p|-p\rangle.$$

$$\hat{p}\hat{\pi}|p\rangle = \hat{p}|-p\rangle = -p|-p\rangle.$$

Thus we have

$$\hat{\pi}\hat{p} + \hat{p}\hat{\pi} = 0.$$

Thus the linear momentum is called a polar vector.

## II.2 Eigenvalue problem for the parity operator

We consider the eigenvalue problem for the parity operator.

$$\hat{\pi}|\psi_{\alpha}\rangle = \alpha|\psi_{\alpha}\rangle.$$

$$\hat{\pi}^2|\psi_{\alpha}\rangle = \alpha\hat{\pi}|\psi_{\alpha}\rangle = \alpha\hat{\pi}|\psi_{\alpha}\rangle = \alpha^2|\psi_{\alpha}\rangle = |\psi_{\alpha}\rangle.$$

Thus we have

$$\alpha^2 = 1 \text{ or } \alpha = \pm 1.$$

We define  $|\psi_+\rangle$  and  $|\psi_-\rangle$ , where

$$\hat{\pi}|\psi_{\pm}\rangle = \pm|\psi_{\pm}\rangle.$$

Note that

$$\hat{\pi}|x\rangle = |-x\rangle,$$

or

$$\langle x|\hat{\pi}^+ = \langle x|\hat{\pi} = \langle -x|,$$

$$\langle x|\hat{\pi}|\psi_{\pm}\rangle = \pm\langle x|\psi_{\pm}\rangle,$$

or

$$\langle -x|\psi_{\pm}\rangle = \pm\langle x|\psi_{\pm}\rangle,$$

or

$$\psi_{\pm}(-x) = \pm\psi_{\pm}(x),$$

where

$\psi_+(x)$  is an even function with respect to  $x$ .

$\psi_-(x)$  is an odd function with respect to  $x$ .

### II.3 Commutation relation between the Hamiltonian and parity operator

$$V(-\hat{x}) = V(\hat{x}) : \text{symmetric potential}$$

$$\hat{H} = \frac{1}{2m} \hat{p}^2 + V(\hat{x}),$$

$$\hat{\pi}^+ V(\hat{x}) \hat{\pi} = V(-\hat{x}) = V(\hat{x}),$$

$$\hat{\pi}^+ \hat{p}^2 \hat{\pi} = (-\hat{p})^2 = \hat{p}^2.$$

Thus we have

$$\hat{\pi}^+ \hat{H} \hat{\pi} = \hat{H},$$

or

$$[\hat{\pi}, \hat{H}] = 0.$$

The Hamiltonian  $\hat{H}$  is invariant under parity.

$|\psi_\alpha\rangle$  is the simultaneous eigenket of  $\hat{H}$  and  $\hat{\pi}$ .

$$\hat{H}|\psi_\alpha\rangle = E_\alpha|\psi_\alpha\rangle,$$

and

$$\hat{\pi}|\psi_\alpha\rangle = \alpha|\psi_\alpha\rangle,$$

with  $\alpha = \pm 1$ .

For $\alpha = 1$ , symmetric state	(even parity)
For $\alpha = -1$ , antisymmetric state	(odd parity)

---