One dimensional barrier problems
Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
(Date: January 13, 2012)

1. Overview on the boundary conditions for wave functions

(a) Behavior of a stationary wave function $\varphi(x)$

$$\frac{d^2}{dx^2} \varphi(x) + \frac{2m}{\hbar^2} (E - V) \varphi(x) = 0$$

where $V(x) = V$ in certain regions of space.

(i) $E > V$

$$E - V = \frac{\hbar^2}{2m} k^2 \quad (k > 0)$$

$$\frac{d^2}{dx^2} \varphi(x) + k^2 \varphi(x) = 0$$

$$\varphi(x) = Ae^{ikx} + A'e^{-ikx}$$

where $A$ and $A'$ are complex constants.

(ii) $E < V$

$$-E + V = \frac{\hbar^2}{2m} \kappa^2 \quad (\kappa > 0)$$

$$\frac{d^2}{dx^2} \varphi(x) - \kappa^2 \varphi(x) = 0$$

$$\varphi(x) = Be^{i\kappa x} + B'e^{-i\kappa x}$$

where $B$ and $B'$ are complex constants.

(iii) $E = V$

$$\varphi(x)$$ is a linear function of $x$. 

1
(b) Behavior of $\phi(x)$ at a potential energy discontinuity

When the potential $V(x)$ is discontinuous at $x = x_1$,

(i) $\phi(x)$ and $\frac{d\phi(x)}{dx}$ are continuous at $x = x_1$.

(ii) $\frac{d^2\phi(x)}{dx^2}$ is discontinuous at $x = x_1$, if $V(x)$ remains bounded.

(Note) $V(x) = a\delta(x)$; unbounded function whose integral remains finite. In the case $\phi(x)$ remains continuous but $\frac{d\phi(x)}{dx}$ does not.

2. 1D step barrier potential

(a) Case when $E > V_0$ (partial reflection)

The wave numbers:

$$k_1^2 = \frac{2mE}{h^2} \quad \text{for the region I}$$

$$k_2^2 = \frac{2m(E - V_0)}{h^2} \quad \text{for the region II}$$

The wave functions:
\[ \phi_i(x) = A_i e^{ik_i x} + A_i^* e^{-ik_i x} \] for the region I

\[ \phi_{II}(x) = A_2 e^{ik_2 x} + A_2^* e^{-ik_2 x} \] for the region II

Suppose that \( A_i' = 0 \) (the wave propagates along the positive \( x \) axis in the region II).

From the condition that \( \phi_i(x = 0) = \phi_{II}(x = 0) \),

\[ A_i + A_i' = A_2. \]

From the matching condition that \( \frac{d\phi_i}{dx}(x = 0) = \frac{d\phi_{II}}{dx}(x = 0) \),

\[ k_1(A_i - A_i') = k_2 A_2 \]

Then we have

\[ \frac{A_i'}{A_i} = \frac{k_1 - k_2}{k_1 + k_2}, \quad \frac{A_2}{A_i} = \frac{2k_1}{k_1 + k_2} \]

\( R \): reflection coefficient
\( T \): transmission coefficient

\[ R = \frac{J_{II}}{J_i} = \frac{\hbar k_2}{m \hbar k_1} \left| A_2 \right|^2 = \left| A_i \right|^2 \frac{\left( k_1 - k_2 \right)^2}{\left( k_1 + k_2 \right)^2} = 1 - \frac{4k_1 k_2}{(k + k_{21})^2}. \]

\[ T = \frac{J_{II}}{J_i} = \frac{\hbar k_2}{m \hbar k_1} \left| A_2 \right|^2 = \frac{k_2}{k_1} \left| A_i \right|^2 = \frac{4k_1 k_2}{(k + k_{21})^2}. \]

Thus we have the relation

\[ R + T = 1 \]

((Note))

Contrary to the prediction of classical mechanics, the incident particle has a non-zero probability of turning back.

((Mathematica))
**Fig.** Plot of $R$ and $T$ as a function of $E_0$. $m = 1$. $\hbar = 1$. $V_0 = 1$. $E_0 > 1$.

(b) Case when $E < V_0$: $R = 1$. $T = 0$ (complete reflection)

The wave numbers:

$$k_1^2 = \frac{2mE}{\hbar^2}$$

for the region I

$$p_2^2 = \frac{2m(V_0 - E)}{\hbar^2}$$

for the region II

The wave functions:
\[ \phi_I(x) = A_1 e^{ik_1 x} + A_1' e^{-ik_1 x} \quad \text{for the region I} \]
\[ \phi_{II}(x) = B_2 e^{ik_2 x} + B_2' e^{-ik_2 x} \quad \text{for the region II} \]

Suppose that \( B_2 = 0 \) (the wave propagates along the positive \( x \) axis in the region II).
From the condition that \( \phi_I(x = 0) = \phi_{II}(x = 0) \),
\[ A_1 + A_1' = B_2' \]

From the matching condition that \( \frac{d\phi_I}{dx}(x = 0) = \frac{d\phi_{II}}{dx}(x = 0) \),
\[ ik_1 (A_1 - A_1') = -\rho_2 B_2' \]

Then we get
\[ \frac{A_1'}{A_1} = \frac{k_1 - i\rho_2}{k_1 + i\rho_2}, \quad \frac{B_2'}{A_1} = \frac{2k_1}{k_1 + i\rho_2} \]

The reflection coefficient \( R \) is obtained as
\[ R = \frac{J_1'}{J_1} = \frac{\frac{hk_1}{m} |A_1|^2}{\frac{hk_1}{m} |A_1|^2} = \left|\frac{A_1'}{A_1}\right|^2 = \left|\frac{k_1 - i\rho_2}{k_1 + i\rho_2}\right|^2 = 1 \]

Since \( T + R = 1 \), \( T = 0 \). Since the wave function \( \phi_{II}(x) \) is a real wave function, we always have \( J_{II} = 0 \).

We consider the phase shift in the wave functions from the conditions,
\[ A_1 + A_1' = B_2' \]
\[ A_1 - A_1' = \frac{i\rho_2}{k_1} B_2' \]

When \( \frac{\rho_2}{k_1} = \tan \phi \),
\[ A_1 = \frac{1}{2} (1 + i\frac{\rho_2}{k_1}) B_2' = \frac{1}{2} (1 + i \tan \phi) B_2' = \frac{B_2'}{2 \cos \phi} e^{i\phi} = \frac{I}{2} e^{i\phi} \]

5
and

\[ A_1' = \frac{1}{2} (1 - \frac{i \rho_2}{k_1}) B_2' = \frac{1}{2} (1 - i \tan \phi) B_2' = \frac{B_2'}{2 \cos \phi} e^{-i \phi} = \frac{I}{2} e^{-i \phi} \]

where

\[ \frac{B_2'}{2 \cos \phi} = \frac{I}{2} \]

Then we get

\[ \varphi_1(x) = \frac{I}{2} (e^{i \phi} e^{i \rho_2 x} + e^{-i \phi} e^{-i \rho_2 x}) = I \cos(k_1 x + \phi) \]

\[ \varphi_{II}(x) = I \cos \phi e^{-\rho_2 x} \]

where

\[ \frac{\rho_2}{k_1} = \tan \phi = \sqrt{\frac{V_0 - E}{E}} \]

((Note)) In the case of \( V_0 \to \infty \)

Then we have \( \rho_2 \to \infty \),

or

\[ \frac{\rho_2}{k_1} = \tan \phi \to \infty \]

\[ \phi = \frac{\pi}{2} \]

\[ \varphi_1(x) = I \cos(k_1 x + \frac{\pi}{2}) = -I \sin(k_1 x) \]

and

\[ \varphi_{II}(x) = 0 \]

At \( x = 0 \),

\[ \varphi_1(x = 0) = 0 \]
\[ \varphi_n(x = 0) = 0. \]

So it remains continuous. How about the matching condition?

\[ \frac{d\varphi_L(x)}{dx} = -ik_1 \cos(k_1 x), \text{ which is } -ik_1 \text{ at } x = 0. \]

\[ \frac{d\varphi_R(x)}{dx} = 0 \]

Therefore the derivative is no longer continuous. This is due to the fact that the potential jump is infinite at \( x = 0 \).

((Mathematica))
\[
\begin{align*}
\psi(x) & \quad E (\text{eV}) = 1 \\
\psi(x) & \quad V (\text{eV}) = 1.05 \\
\psi(x) & \quad E (\text{eV}) = 1 \\
\psi(x) & \quad V (\text{eV}) = 1.45 \\
\psi(x) & \quad E (\text{eV}) = 1 \\
\psi(x) & \quad V (\text{eV}) = 1.85 \\
\psi(x) & \quad E (\text{eV}) = 1 \\
\psi(x) & \quad V (\text{eV}) = 2.25 \\
\psi(x) & \quad E (\text{eV}) = 1 \\
\psi(x) & \quad V (\text{eV}) = 2.65 \\
\psi(x) & \quad E (\text{eV}) = 1 \\
\psi(x) & \quad V (\text{eV}) = 3.05 \\
\psi(x) & \quad E (\text{eV}) = 1 \\
\psi(x) & \quad V (\text{eV}) = 3.45 \\
\psi(x) & \quad E (\text{eV}) = 1 \\
\psi(x) & \quad V (\text{eV}) = 3.85
\end{align*}
\]
3. One dimensional square barrier

(a) Fabry-Perot for $E > V_0$

The wave numbers:

$$k_1^2 = \frac{2mE}{\hbar^2} \quad \text{for the region I and III}$$

$$k_2^2 = \frac{2m(-V_0 + E)}{\hbar^2} \quad \text{for the region II}$$

The wave functions:

$$\varphi_i(x) = A_i e^{ik_1x} + A_i' e^{-ik_1x} \quad \text{for the region I}$$

$$\varphi_{II}(x) = A_2 e^{ik_2x} + A_2' e^{-ik_2x} \quad \text{for the region II}$$

$$\varphi_{III}(x) = A_3 e^{ik_1x} + A_3' e^{-ik_1x} \quad \text{for the region III}$$

Let us choose $A_i' = 0$ (neglecting particles coming from $x = +\infty$).

The matching condition at $x = 0$ and $x = l$.

(i) \[ \varphi_i(x = 0) = \varphi_{II}(x = 0) \]
leading to

\[ A_i + A_i' = A_2 + A_2' \]  \hspace{1cm} (1)

(ii) \[ \varphi_{ii}(x = l) = \varphi_{iii}(x = l) \]

leading to

\[ A_2 e^{ik_1} + A_2' e^{-ik_2} = A_2' e^{ik_1} \]  \hspace{1cm} (2)

(iii)

\[ \frac{d\varphi_i(x = 0)}{dx} = \frac{d\varphi_{ii}(x = 0)}{dx} \]

leading to

\[ A_2 ik_1 - A_2' ik_1 = A_2 ik_2 - A_2' ik_2 \]  \hspace{1cm} (3)

(iv)

\[ \frac{d\varphi_{ii}(x = l)}{dx} = \frac{d\varphi_{iii}(x = l)}{dx} \]

leading to

\[ A_2 ik_2 e^{ik_1} - A_2' ik_2 e^{-ik_2} = A_2' ik_2 e^{ik_1} \]  \hspace{1cm} (4)

From Eqs.(1)-(4), we have

\[ A_i = A_2 e^{ik_1} \left[ \cos(k_2 l) - i \left( \frac{k_1^2 + k_2^2}{2k_1 k_2} \right) \sin(k_2 l) \right] \]

\[ A_i' = iA_3 e^{ik_1} \left( \frac{k_2^2 - k_1^2}{2k_1 k_2} \right) \sin(k_2 l) \]

\[ A_2 = \frac{A_1}{2k_2} e^{ik_2} \left[ (k_1 + k_2) \right] \]
\[ A_2 = \frac{A_1}{2k_2} e^{i(k_1 + k_2)}(-k_1 + k_2) \]

Probability current density:
\[ J_1 = \frac{\hbar k_1}{m} \left( |A_1|^2 - |A_2|^2 \right) \]
\[ J_{II} = \frac{\hbar k_2}{m} \left( A_2^* A_2 - |A_2|^2 \right) \]
\[ J_{III} = \frac{\hbar k_1}{m} |A_3|^2 \]

Reflection co-efficient:
\[ R = \left| \frac{A_1}{A_1} \right|^2 = \frac{(k_1^2 - k_2^2)^2 \sin^2(k_2l)}{4k_1^2 k_2^2 + (k_1^2 - k_2^2)^2 \sin^2(k_2l)} \]

Transmission co-efficient:
\[ T = \left| \frac{A_3}{A_1} \right|^2 = \frac{4k_1^2 k_2^2}{4k_1^2 k_2^2 + (k_1^2 - k_2^2)^2 \sin^2(k_2l)} \]
\[ = \frac{4E(E - V_0)}{4E(E - V_0) + V_0^2 \sin^2(k_2l)} \]

for \( E > V_0 \). \( k_2l = l \sqrt{\frac{2m(-V_0 + E)}{\hbar^2}} \)

When \( k_2l = n\pi \) (\( n \): integer), \( T = 1. (\text{Fabry-Perot}) \)

((Mathematica)) Fabry-Perot
Fig. Plot of $T$ as a function of $x = k_2 l$. The parameter $E/V_0$ is changed as 2, 2.5, 3, 3.5, 4, 4.5 and 5.

((Note))

We consider the case of $k_2 l = 2n \pi$, or $k_2 l = n \pi$ (resonance condition).

$T = 1$, $R = 0$

$A'_1 = 0$
\[ A_1 = \cos(n\pi)e^{ik_1x} A_3 = (-1)^n e^{ik_1x} A_3 \]

\[ A_2 = \frac{A_3}{2k_2}(k_1 + k_2)e^{ik_1x} (-1)^n = \frac{1}{2}(1 + \frac{k_1}{k_2})A_i \]

\[ A_2' = \frac{A_1}{2k_2}(-k_1 + k_2)e^{ik_1x} (-1)^n = \frac{1}{2}(1 - \frac{k_1}{k_2})A_i \]

Then we have

\[ \varphi_i(x) = A_i e^{ik_1x} \]

\[ \varphi_{II}(x) = A_2 e^{ik_1x} + A_2'e^{-ik_1x} = \frac{1}{2}(1 + \frac{k_1}{k_2})A_i e^{ik_2x} + \frac{1}{2}(1 - \frac{k_1}{k_2})A_i e^{-ik_2x} \]

\[ = A_i [\cos(k_2x) + \frac{k_1}{k_2} 2i \sin(k_2x)] \]

(resonance scattering)

\[ \varphi_{III}(x) = A_3 e^{ik_1x} = (-1)^n e^{-ik_1x} A_4 e^{ik_1x} \]

(b) Tunneling effect for \( E < V_0 \)
Fig. A beam of particles represented by a plane wave is incident on a potential barrier. Most particles are reflected, but some are transmitted by quantum mechanical tunneling.

We set

\[ k_2 = -i\rho_2 \quad \text{or} \quad \rho_2 = ik_2 \]

where

\[ \rho_2^2 = \frac{2m}{\hbar^2} (V_0 - E) \quad \text{for the region II} \]

The wave functions:

\[ \varphi_I(x) = A_1 e^{ik_1x} + A_1^* e^{-ik_1x} \quad \text{for the region I} \]

\[ \varphi_{II}(x) = A_2 e^{i\rho_2 x} + A_2^* e^{-i\rho_2 x} \quad \text{for the region II} \]

\[ \varphi_{III}(x) = A_3 e^{ik_3 x} \quad \text{for the region III} \]

where

\[ k_1^2 = \frac{2m}{\hbar^2} E \quad \text{for the regions I and III} \]

Boundary conditions: continuity of the wave function and its derivative with respect to \( x \) at \( x = 0 \) and \( x = l \).

\[ A_1 + A_1' = A_2 + A_2' \]

\[ A_2 e^{i\rho_2 l} + A_2^* e^{-i\rho_2 l} = A_3 e^{ik_3 l} \]

\[ ik_1(A_1 - A_1^*) = \rho_2 (A_2 - A_2^*) \]

\[ \rho_2 (A_3 e^{i\rho_3 l} - A_3^* e^{-i\rho_3 l}) = A_3 i k_3 e^{ik_3 l} \]

When \( A_3 \) is a fixed parameter, we get \( A_1, A_1', A_2, \) and \( A_2' \) as

\[ A_1 = \frac{A_3}{4k_1 \rho_2} e^{(ik_1 - i\rho_2 l)} \left[ -i(e^{2i\rho_3 l} - 1)k_1^2 + 2(e^{2i\rho_3 l} + 1)k_1 \rho_2 + i(e^{2i\rho_3 l} - 1)\rho_2^2 \right] \]
\[
A_1' = - \frac{A_3}{4k_1\rho_2} e^{i(k_1-\rho_2)\xi} (e^{2\rho_2\xi} - 1)(k_1 + i\rho_2)(ik_1 + \rho_2)
\]

\[
A_2 = \frac{A_3}{2\rho_2} e^{i(k_1-\rho_2)\xi} (ik_1 + \rho_2)
\]

\[
A_2' = \frac{A_3}{2\rho_2} e^{i(k_1+\rho_2)\xi} (-ik_1 + \rho_2)
\]

Noting that
\[
\cosh(\pm i\theta) = \cosh \theta \quad \text{and} \quad \sinh(\pm i\theta) = \mp i \sinh \theta,
\]
\[
cosh(2x) = 1 + 2\sinh^2(x), \quad \cosh^2(x) - \sinh^2(x) = 1.
\]

we have
\[
A_1 = -A_3 e^{ik_1l} \left[ \cosh(\rho_2 l) + \frac{i}{2} \left( \frac{k_1^2 - \rho_2^2}{k_1 \rho_2} \right) \sinh(\rho_2 l) \right]
\]

\[
A_1' = -iA_3 e^{ik_1l} \left( \frac{k_1^2 + \rho_2^2}{2k_1 \rho_2} \right) \sinh(\rho_2 l)
\]

The wave functions in the regions I, II, and II are obtained as
\[
\psi_I = \frac{A_3}{k_1 \rho_2} e^{ik_1l} \left[ k_1 \rho_2 e^{ik_1x} \frac{\cosh(\rho_2 l)}{\cosh(k_1 l)} - (ik_1^2 \cos(k_1 l) + \rho_2^2 \sin(k_1 l)) \sinh(\rho_2 l) \right]
\]

\[
\psi_{II} = \frac{A_3}{\rho_2} e^{ik_1l} \left[ \rho_2 \cosh(\rho_2 (l - x)) - ik_1 \sinh(\rho_2 (l - x)) \right]
\]

\[
\psi_{II} = A_3 e^{ik_1x}
\]

where \( A_3 \) is fixed parameter. A typical example is shown below.
Fig.  Real part of the wave functions in the tunneling through a barrier (denoted by blue lines).

\[
J_I = \frac{\hbar k}{m} \left( |A_1|^2 - |A_1'|^2 \right)
\]

\[
J_{II} = \frac{-i\hbar}{m} \rho_2 \left( A_2 A_2^* - A_2^* A_2' \right)
\]

\[
J_{III} = \frac{\hbar k}{m} |A_3|^2
\]

where

\[ J_I = J_{II} = J_{III} \]

The transmission coefficient is given by
where $E < V_0$ and $\rho_2 l = l \sqrt{\frac{2m}{\hbar^2}} (V_0 - E)$. The reflection coefficient is given by

$$R = \left| \frac{A_i}{A_i} \right|^2 = \frac{(k_1^2 + \rho_2^2) \sinh^2(\rho_2 l)}{4k_1^2 \rho_2^2 \cosh^2(\rho_2 l) + (k_1^2 - \rho_2^2)^2 \sinh^2(\rho_2 l)}$$

Then we get

$$T + R = 1$$

We make a plot of $T$ as a function of $x = \rho_2 l$, where $E/V_0 = 0.9$.

Fig. Plot of $T$ as a function of $X = \rho_2 l$. $E/V_0 = 0.9$. 

17
When $\rho_2 l >> 1$,

$$\sinh(\rho_2 l) = \frac{e^{\rho_2 l} - e^{-\rho_2 l}}{2} = \frac{e^{\rho_2 l}}{2}$$

Then we have

$$T = \frac{16E(V_0 - E)}{V_0^2} e^{-2\rho_2 l}$$

The particle has a non-zero probability of crossing the potential barrier.

**((Parameter $\rho_2$ for electron))**

$$\rho_2 = \frac{\sqrt{2m}}{\hbar^2 (V_0 - E)}$$

or

$$\frac{1}{\rho_2} = \frac{1.95192}{\sqrt{(V_0 - E) [eV]}} \text{Å}$$

When $V_0 = 2 \text{ eV}$, $l = 1 \text{ Å}$, and $E = 1 \text{ eV}$, we have $1/\rho_2 = 1.96 \text{ Å}$.

or

$$T = 0.78$$

The electron must then have a considerable probability of crossing the barrier.

**((Parameter $\rho_2$ for proton))**

$$\frac{1}{\rho_2} = \sqrt{\frac{2M}{\hbar^2 (V_0 - E)}}$$

or

$$\frac{1}{\rho_2} = \frac{4.5552 \times 10^{-2}}{\sqrt{(V_0 - E) [eV]}} \text{Å}.$$
When \( V_0 = 2 \text{ eV}, l = 1 \, \text{Å}, \) and \( E = 1 \text{ eV}, \) we have \( \frac{1}{\rho^2} = 4.5552 \times 10^{-2} \, \text{Å}. \)

or

\[ T = 3.94 \times 10^{-19}, \]

which is negligibly small.

4. One dimensional square-well potential: Ramsuer effect

The wave numbers:

\[ k_2^2 = \frac{2m}{h^2}(E + V_0) \]

\[ k_1^2 = \frac{2m}{h^2}E \]

\[ A_1 = A_2 e^{ik_2l}[\cos(k_2l) - i \left( \frac{k_1^2 + k_2^2}{2k_1k_2} \right) \sin(k_2l)] \]

\[ A_1' = iA_3 e^{ik_2l} \left( \frac{k_2^2 - k_1^2}{2k_1k_2} \right) \sin(k_2l) \]

Reflection co-efficient
\[ R = \frac{A_2}{A_1} = \frac{(k_1^2 - k_2^2)^2 \sin^2(k_2l)}{4k_1^2k_2^2 + (k_1^2 - k_2^2)^2 \sin^2(k_2l)} \]

Transmission co-efficient

\[ T = \frac{A_2}{A_1} = \frac{4k_1^2k_2^2}{4k_1^2k_2^2 + (k_1^2 - k_2^2)^2 \sin^2(k_2l)} \]

5. Semi infinite well potential

For 0 < x < a (region I)

\[ H \phi_I(x) = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \phi_I(x) = E \phi_I(x) = \frac{\hbar^2}{2m} k^2 \phi_I(x) \]

The solution of this equation is

\[ \phi_I(x) = A \sin(kx) + A_i \cos(kx) \]

where

\[ k = \sqrt{\frac{2mE}{\hbar^2}} \]
Using the boundary condition:

\[ \phi_i(x = 0) = 0 \]

we have

\[ A_1 = 0 \text{ and } A \neq 0. \]

Then we get

\[ \phi_i(x) = A \sin(kx) \]

For \( x > a \) (region-II)

\[ ( - \frac{h^2}{2m} \frac{d^2}{dx^2} + V_0 ) \phi_{ii}(x) = E \phi_{ii}(x) \]

or

\[ \frac{d^2}{dx^2} \phi_{ii}(x) - \kappa^2 \phi_{ii}(x) = 0 \]

where

\[ \kappa = \sqrt{ \frac{2m}{h^2} (V_0 - E) }. \]

The solution of \( \phi_{ii}(x) \) is given by

\[ \phi_{ii}(x) = Be^{-x(x-a)} \]

The condition for the continuity of \( \phi(x) \) and \( \frac{d\phi(x)}{dx} \) at \( x = a \)

\[ A \sin(ka) = B \]

\[ kA \cos(ka) = -B \kappa \]

From these two equations we have
\[
\frac{1}{ka} \tan(ka) = -\frac{1}{\kappa a} \quad \text{or} \quad ka = -\kappa a \tan(ka)
\]

with
\[
(ka)^2 + (ka)^2 = \frac{2m}{\hbar} a^2 V_0
\]

For simplicity, we assume that
\[
x = ka, \quad y = \kappa a, \quad R = \sqrt{\frac{2m}{\hbar^2} a^2 V_0}
\]

Then we need to solve the equations by using graphs,
\[
x = -y \tan(x), \quad x^2 + y^2 = R^2
\]

where \(x>0\) and \(y>0\).

Fig. The intersections of the curve \(y = -\frac{x}{\tan x}\) and the circle \((x^2+y^2 = R^2)\). The radius \(R\) is changed as a parameter. Note that \(y = -\frac{x}{\tan x}\) changes the sign
from negative to positive at $x = \pi/2, 5\pi/2, 7\pi/2, \ldots$. When $\pi/2 < R < 3\pi/2$, there are two intersections, leading to the two energy levels. When $5\pi/2 < R < 7\pi/2$, there are three intersections, leading to the three energy levels.