

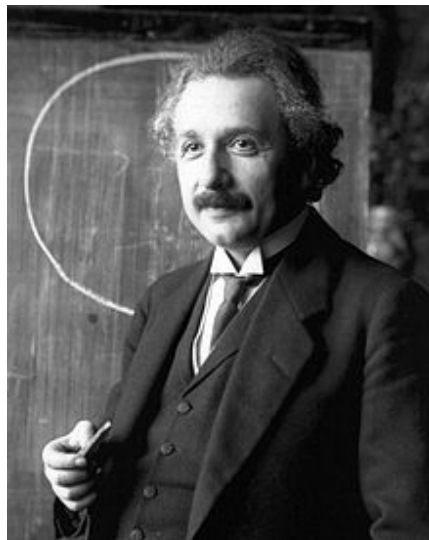
Minkowski space-time diagram in the special relativity

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Albert Einstein (14 March 1879 – 18 April 1955) was a German-born theoretical physicist who developed the theory of general relativity, effecting a revolution in physics. For this achievement, Einstein is often regarded as the father of modern physics. He received the 1921 Nobel Prize in Physics "for his services to theoretical physics, and especially for his discovery of the law of the photoelectric effect". The latter was pivotal in establishing quantum theory within physics.



http://en.wikipedia.org/wiki/Albert_Einstein

Hermann Minkowski (June 22, 1864 – January 12, 1909) was a German mathematician of Lithuanian Jewish descent, who created and developed the geometry of numbers and who used geometrical methods to solve difficult problems in number theory, mathematical physics, and the theory of relativity.



http://en.wikipedia.org/wiki/Hermann_Minkowski

1. **Minkowski space-time diagram (or simply spacetime)**

The space time diagram was first introduced by Hermann Minkowski. He was one of Einstein's teacher at ETH, The Federal Institute of Technology at Zurich, in the late 1890's. The idea of the space diagram came from the paper of Minkowski at 1908. Roger Penrose says that the special relativity was not yet complete, despite the wonderful physical insight of Einstein and the profound contributions of Lorentz and Poincare until Minkowski provided his fundamental and revolutionary viewpoint; spacetime.

A **Minkowski** spacetime diagram is a geometric representation of motions in spacetime. The vertical axis is usually plotted as the time axis. Any point in spacetime is called a world point, and a series of worldpoints representing the motion of some object is called a world line. If ct is used for the time axis, then light on a Minkowski diagram will always travel at 45° to the time axis (either right or left at 45° in a one spatial direction Minkowski diagram, along the surfaces of a 45° cone - called the light cone - in a dual spatial dimension Minkowski diagram).

Any individual event is uniquely represented by some point P. The description of this event is described in the S frame by the coordinates (x, ct) and in the S' frame by the coordinates (x', ct') . If the origins of S and S' are chosen so as to coincide at $ct = ct' = 0$, then the relation between (x, ct) and (x', ct') is contained in the Lorentz transformations. The world line of a light signal starting out at $x = 0, ct = 0$, is a bisector of the angle between the axes. This holds good in both the S and S' frames.

((Lorentz transformation))

$$ct' = \frac{ct - \frac{v}{c}x}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad x' = \frac{x - \frac{v}{c}ct}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$ct = \frac{ct' + \frac{v}{c}x'}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad x = \frac{x' + \frac{v}{c}ct'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Note that

$$x^2 - c^2t^2 = x'^2 - c^2t'^2 \quad (\text{Lorentz invariant})$$

When $ct' = 0$, we get

$$ct = \frac{v}{c}x = (\tan \theta)x$$

where

$$\tan \theta = \frac{v}{c}$$

is the slope of the line $ct' = 0$ (or the x' -line).

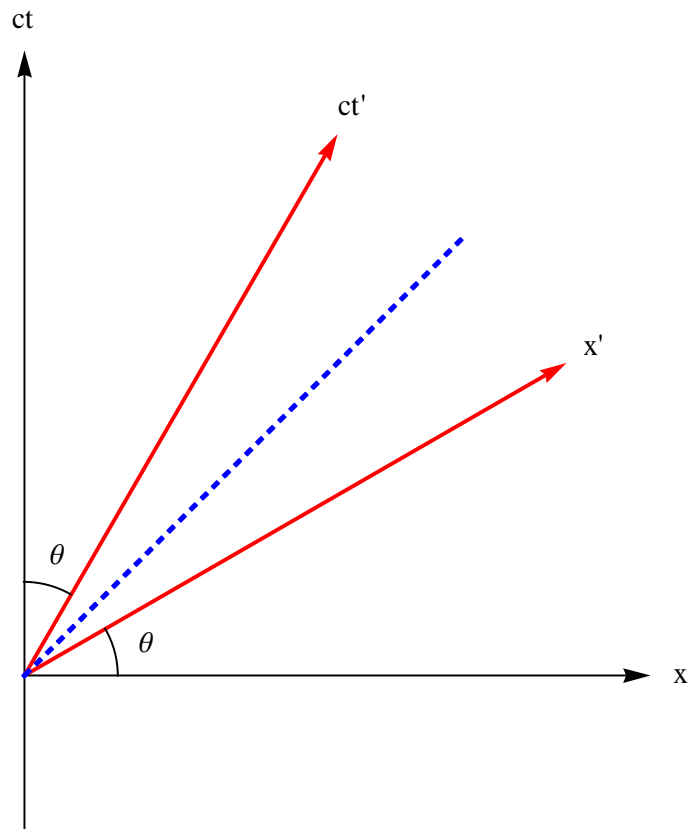


Fig.1 Lorentz transformation. $\tan \theta = \frac{v}{c}$. The dashed line (blue) denotes $ct = x$ for the light.

When $x' = 0$, we get

$$x = \frac{v}{c} ct$$

or

$$ct = \cot \theta x = \tan\left(\frac{\pi}{2} - \theta\right)x$$

$\tan\left(\frac{\pi}{2} - \theta\right)$ is the slope of the line $x' = 0$ (or the ct' -line).

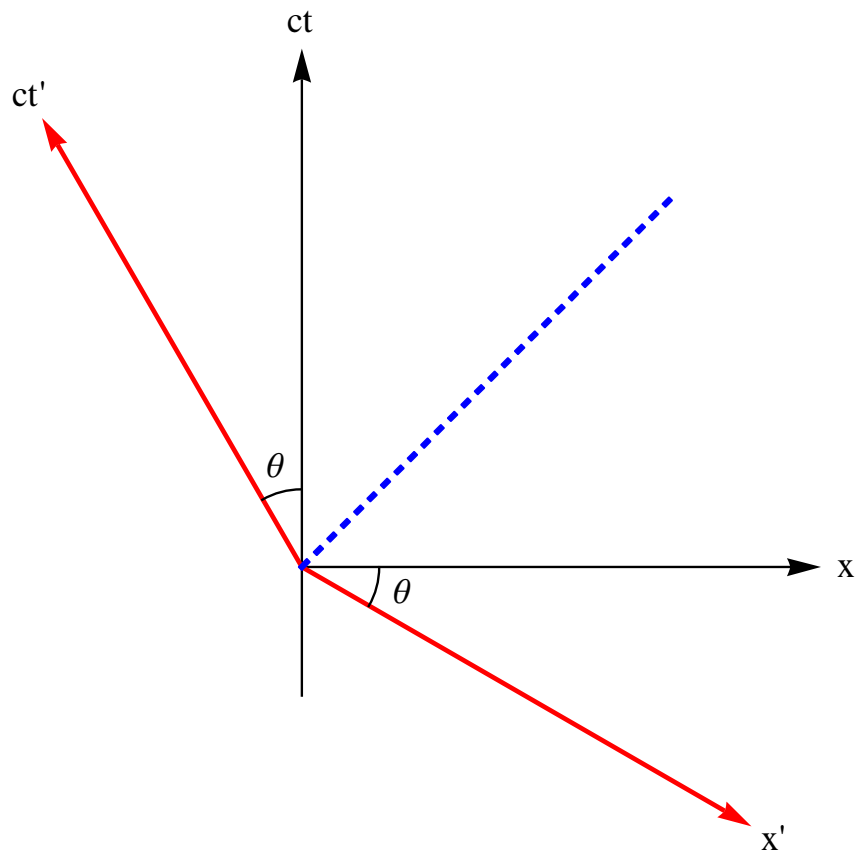


Fig.2 The Lorentz transformation. $\tan \theta = \frac{v}{c}$, with $v < 0$.

2. Definition

(a) Event

A point defines a particular place at a particular instant in time.

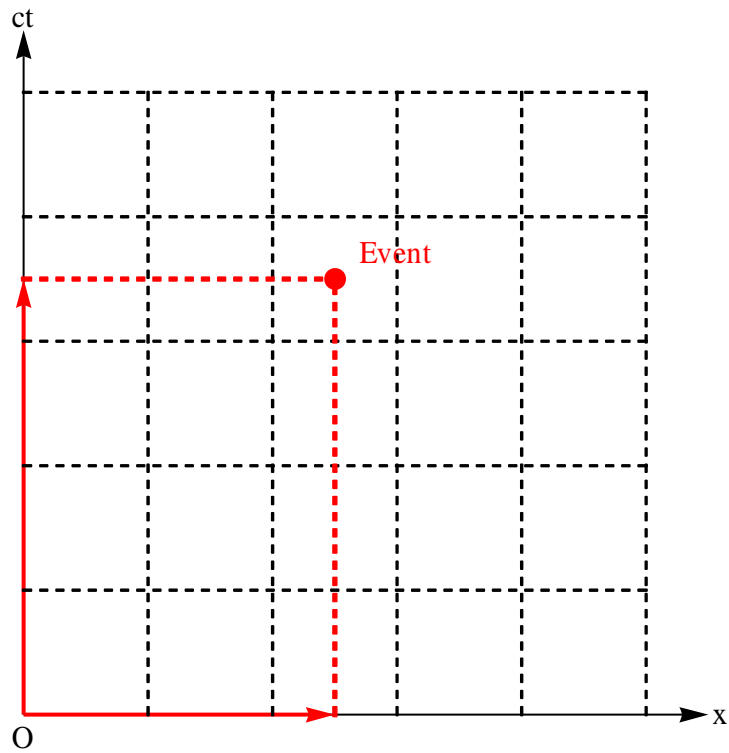


Fig.3 The definition of event in the Minkowski spacetime

(b) World line

A path through the space-time diagram is called a world line. The world line for the acceleration motion is described by a curve, but not a straight line. The angle between the tangential line of the world line at the point (x, ct) and ct -axis is always smaller than 45° .

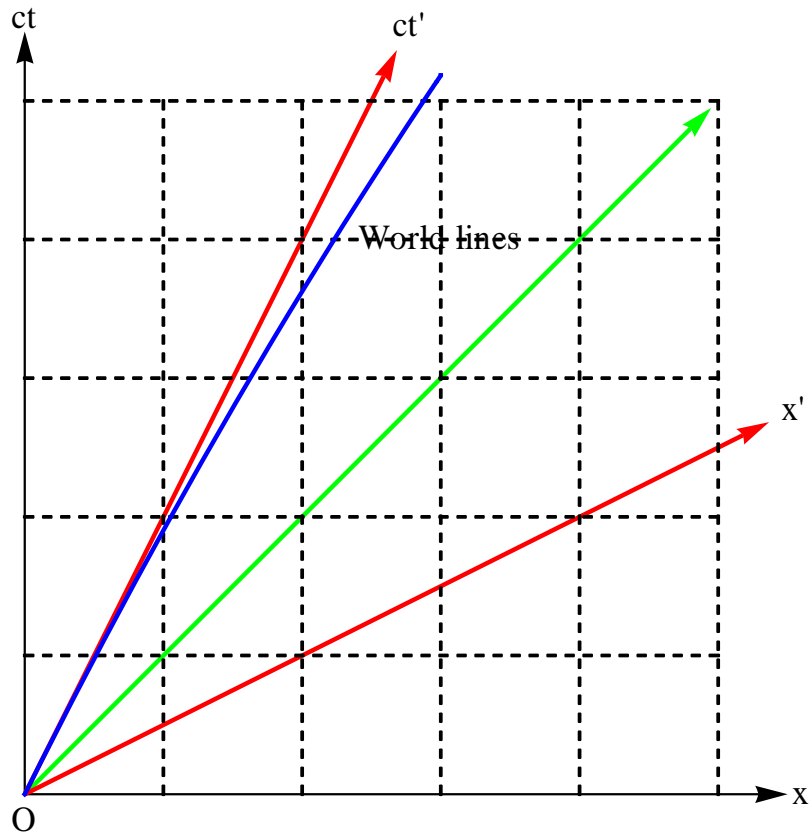


Fig.4 World lines. the blue line (world line of the acceleration motion). The red line (world line of constant motion). The green line (world line of light). We assume $v = c/2$, in this figure.

(c) Time-like, space-like regions

We consider a point at (x, ct) . Under the Lorentz transformation, this point is described by (x', ct') .

$$s^2 = x'^2 - (ct')^2 = x^2 - (ct)^2 \quad (\text{Lorentz invariant}).$$

Suppose that

$$ct > |x| \quad (\text{time-like})$$

which leads to

$$s^2 < 0$$

Then we get

$$(ct')^2 = x'^2 + (-s^2) > 0,$$

Then the inequality $ct' > 0$ is still maintained. Even if this transformation repeated, the conclusion is the same.

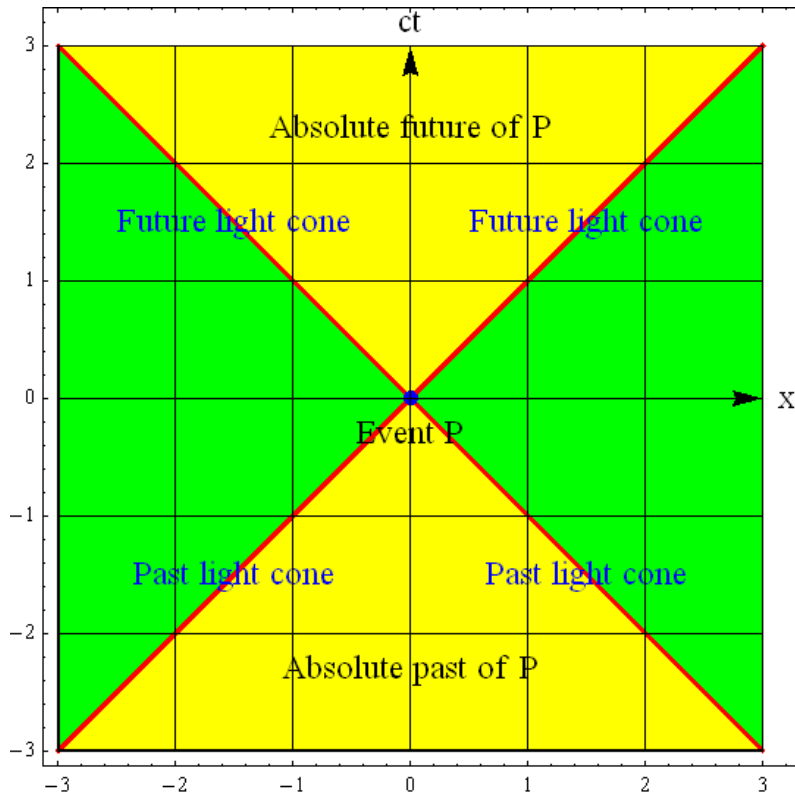


Fig.5 The past and future light cones of event P. $|x|=ct$ (light-like).

(d) Calibration of clocks

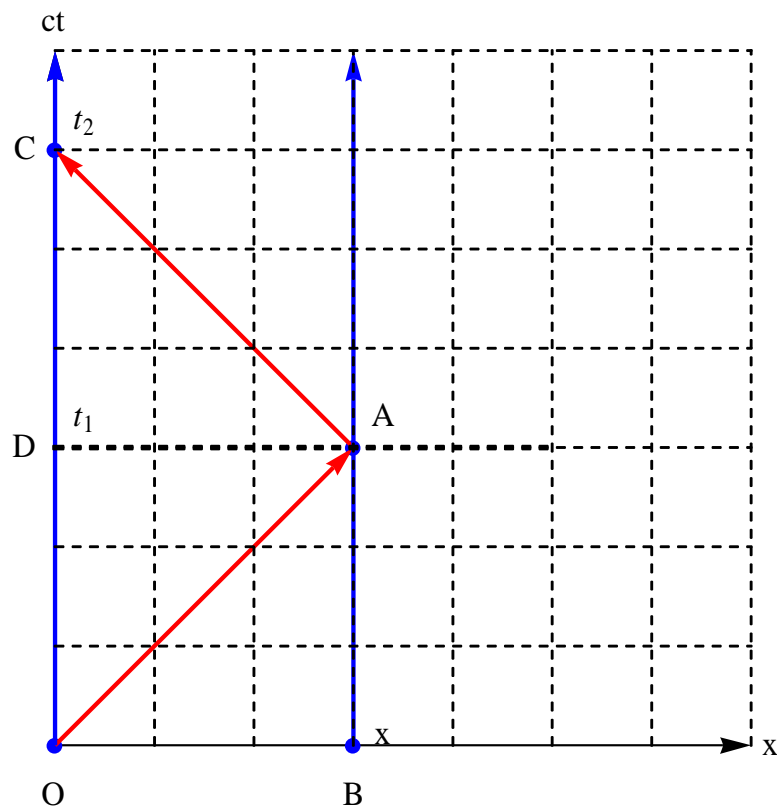


Fig.6 Calibration of clocks. The red line denotes the path of light.

$$(OB)_S = (OD)_S = (DC)_S$$

with

$$(OB)_S = x, \quad (OD)_S = ct_1, \quad (DC)_S = c(t_2 - t_1)$$

3. Lorentz contraction

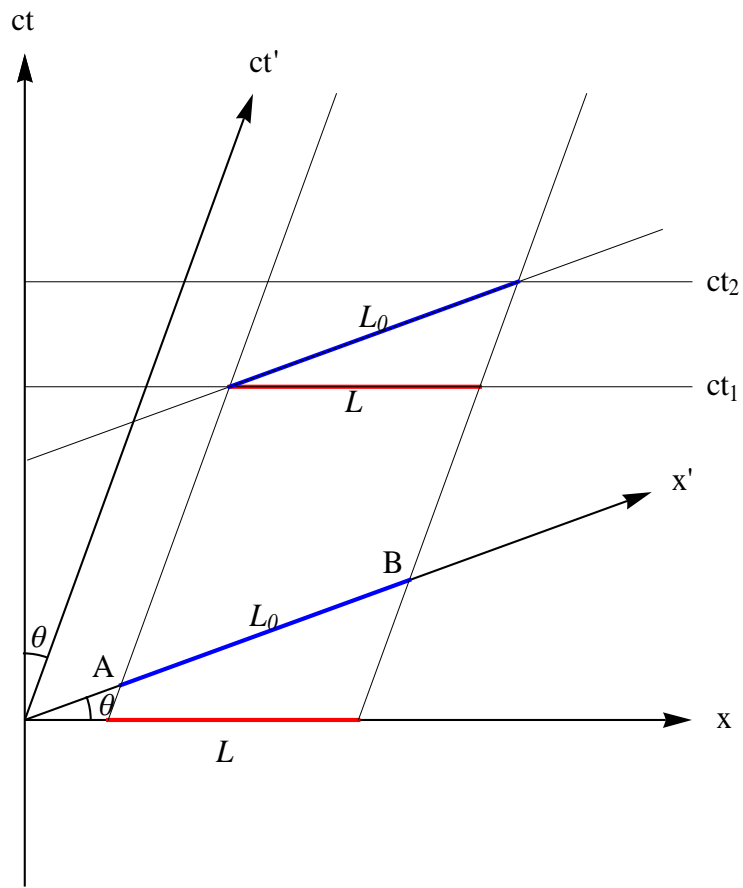


Fig.7 Lorentz contraction

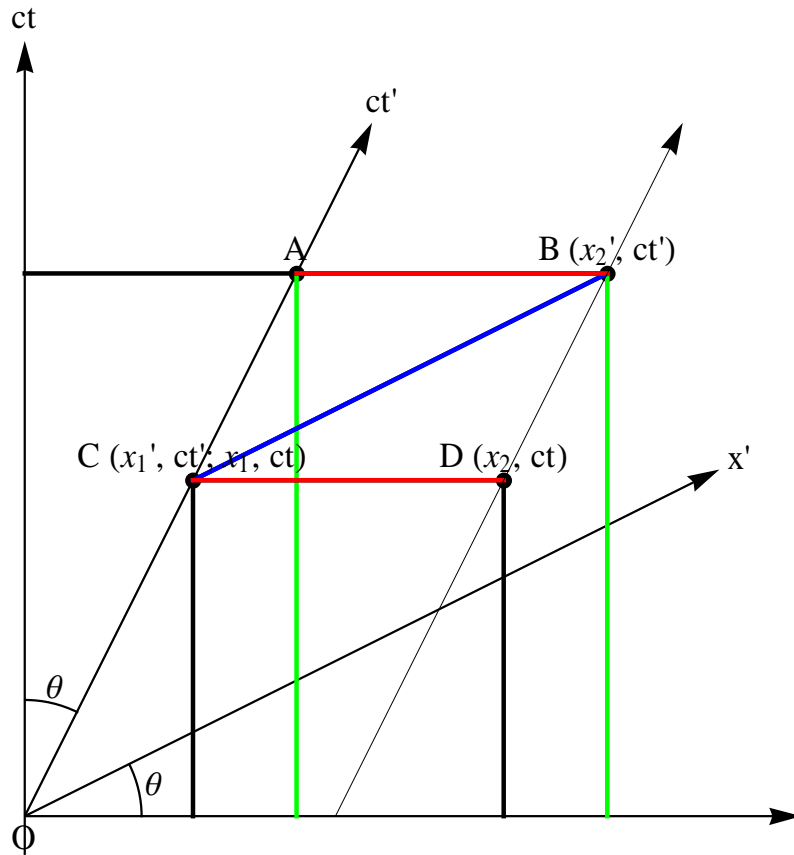


Fig.8 Minkowski spacetime diagram for the Lorentz contraction

Suppose that the rod (the length L_0) is fixed in the frame S' . The points C and B are located at the edges of the rod. The world line of C (x'_1, ct'_1 in the S' frame; x_1, ct_1 in the S frame) passes through the point A. The world line of B passes through the point D (x_2, ct_2 in the S frame). Note that the x' coordinates of these points C and B are x'_1 and x'_2 , where

$$x'_2 - x'_1 = L_0.$$

The Lorentz transformation is given by

$$x'_1 = \frac{x_1 - \frac{v}{c}ct_1}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad x'_2 = \frac{x_2 - \frac{v}{c}ct_2}{\sqrt{1 - \frac{v^2}{c^2}}},$$

or

$$x_1 = \sqrt{1 - \frac{v^2}{c^2}} x_1' + \frac{v}{c} ct_1, \quad x_2 = \sqrt{1 - \frac{v^2}{c^2}} x_2' + \frac{v}{c} ct_2$$

When $t_1 = t_2 = t$ (the point C and D), the distance CD in the S frame is obtained as

$$x_2 - x_1 = \sqrt{1 - \frac{v^2}{c^2}} (x_2' - x_1') = \sqrt{1 - \frac{v^2}{c^2}} L_0 < L_0$$

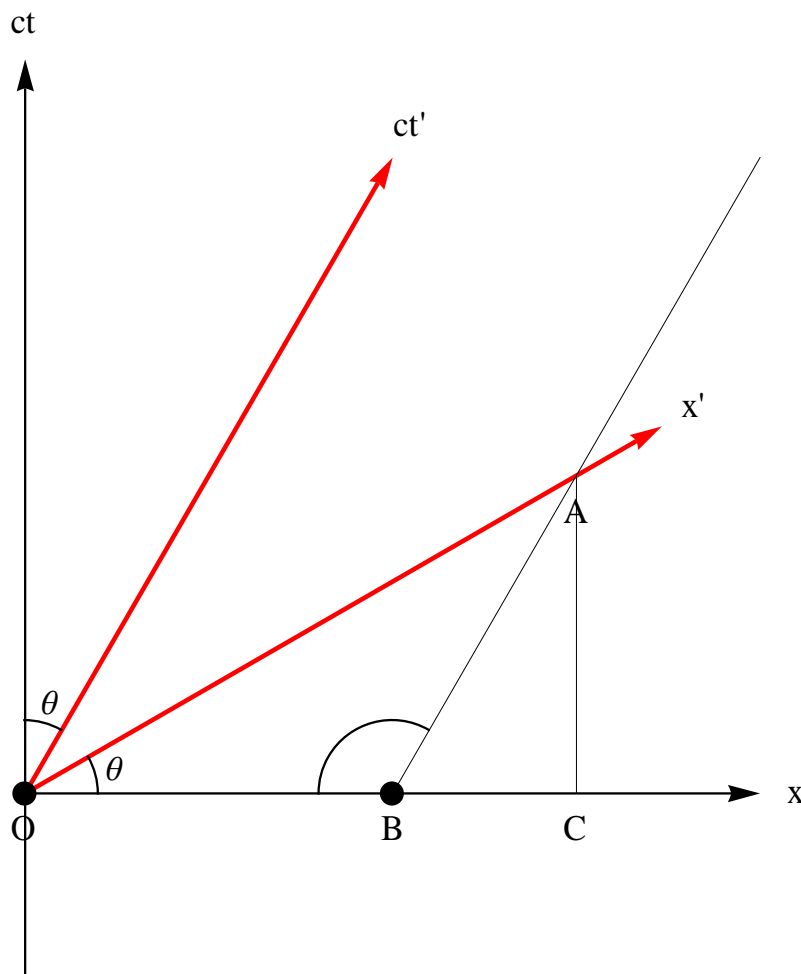


Fig.9 Minkowski space time diagram. $(OA)_{S'}$ = length of the rod in the S' frame. $(OB)_S$ = the length of the rod measured in the S frame. The length \overline{OB} in this figure corresponds to the length $(OB)_{S'}$, which is different from the length $(OB)_S$ measured in the S frame. $(OB)_S = k (OB)_{S'}$ (see the scaling factor k will be discussed in detail later).

4. Time dilation from the Lorentz transformation

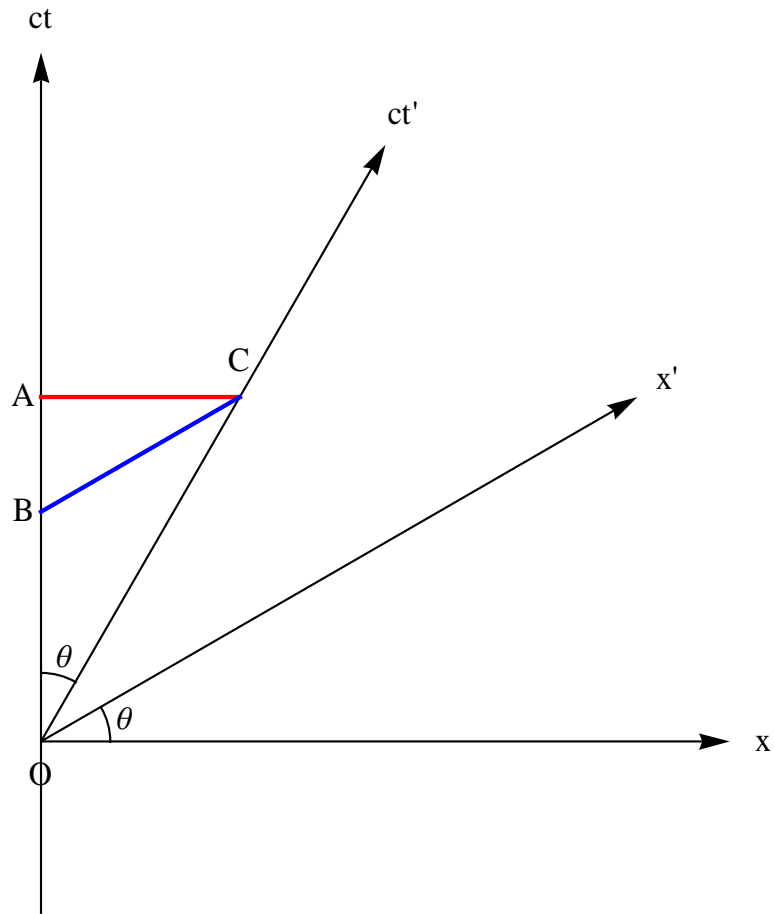


Fig.10 Minkowski space-time diagram for the time dilation. The event C is located at $(ct', x'=0)$ in the S' frame. $(OC)_{S'} = ct'$. $(OA)_S = ct$. The length \overline{OA} in this figure corresponds to the length $(OA)_{S'}$, which is different from the length $(OA)_S$ measured in the S frame. $(OA)_S = k (OA)_{S'}$ (the scaling factor k will be discussed in detail later).

We use the Lorentz transformation for the time dilation,

$$ct' = \frac{ct - \frac{v}{c}x}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad x' = \frac{x - \frac{v}{c}ct}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$ct = \frac{ct' + \frac{v}{c}x'}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad x = \frac{x' + \frac{v}{c}ct'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

When $x' = 0$ (the point C)

$$t = \frac{t'}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{\tau}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (\text{time dilation})$$

where $t' = \tau$. More generally, we have

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{\Delta \tau}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where $\Delta \tau = \Delta t'$. The time τ is called the proper time and is measured at the same x' in the S' frame.

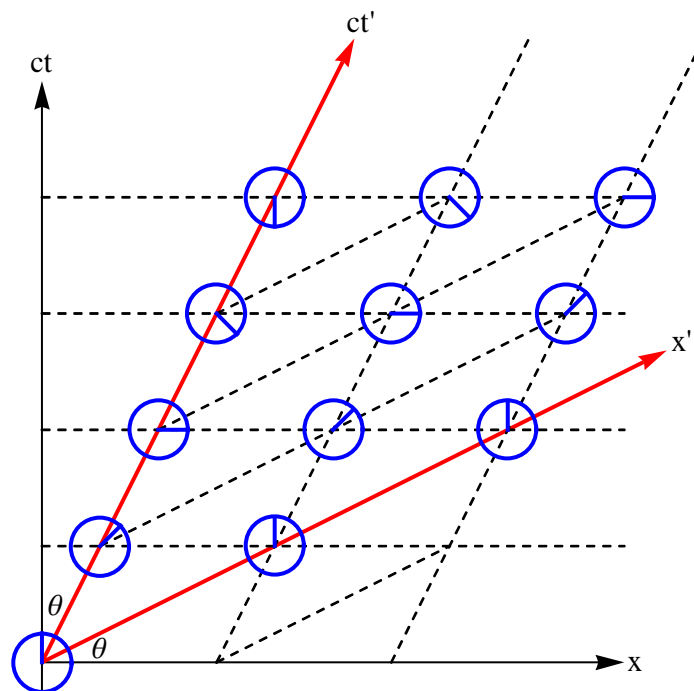


Fig.11 Simultaneity. From the definition, the clocks in the S' frame on the same ct' shows the same time even for different x' . When these clocks are seen from the S frame,

even at the same time ct , the time indication of the clocks is different for the different x .

5. Hyperbola in the S and S' frames

We discuss the unit of length in the Minkowski space-time diagram in the S (x, ct) and S' frames (x', ct').

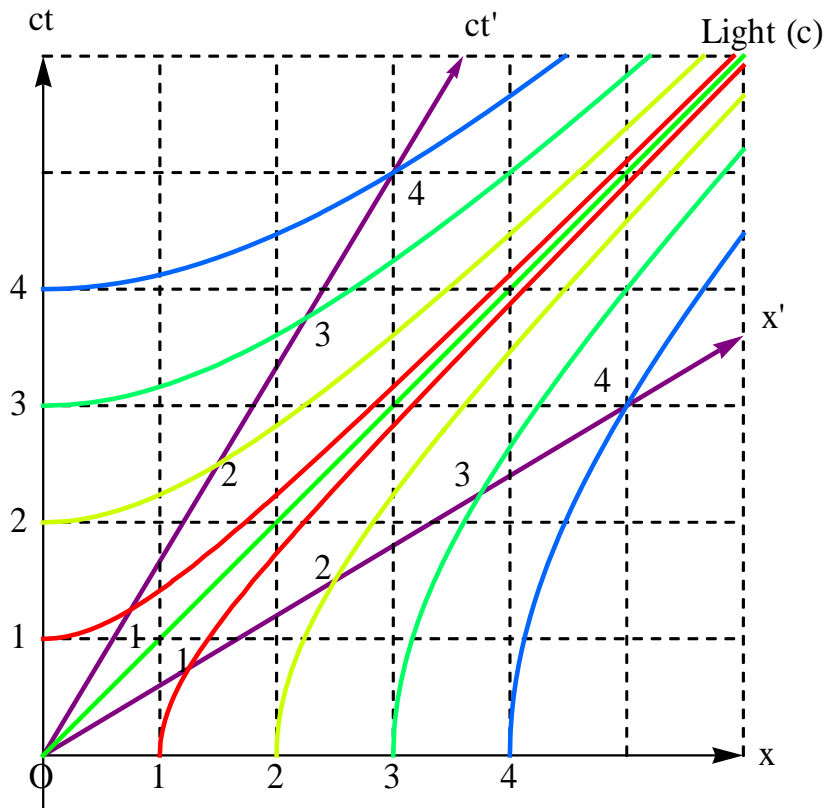


Fig.12 Minkowski space-time diagram, showing two different coordinate systems and a series of hyperbola to define unit distance along x for each system. Plot of the hyperbola $x^2 - y^2 = \sigma^2 = x'^2 - y'^2$ and $x^2 - y^2 = -\sigma^2 = x'^2 - y'^2$, where $\sigma = 1, 2, 3,$ and 4 . $\tan\theta = v/c = 3/5$. $y = ct$. $y' = ct'$.

We consider the equation

$$x^2 - y^2 = \sigma^2 = x'^2 - y'^2, \quad x^2 - y^2 = -\sigma^2 = x'^2 - y'^2$$

where $y = ct$ and $y' = ct'$, and σ is independent of the frames. We make a plot of this equation when σ is changed as a parameter. The curve shows a hyperbola in the (x, y) plane. This curve

should intersect at $x = \sigma$ in the (x, y) plane and at $x' = \sigma$ in the (x', y') plane. However the unit distance (σ) in the (x, y) plane is clearly different from the unit distance (σ) in the (x', y') plane since the coordinate (x', y') is oblique and the coordinate (x, y) is normal.

6. Light (null) cones

This shows spacetime diagram in three dimensions (x, y, ct) . The lightlines form a light cone. The circles centered at ct -axis define simultaneity planes

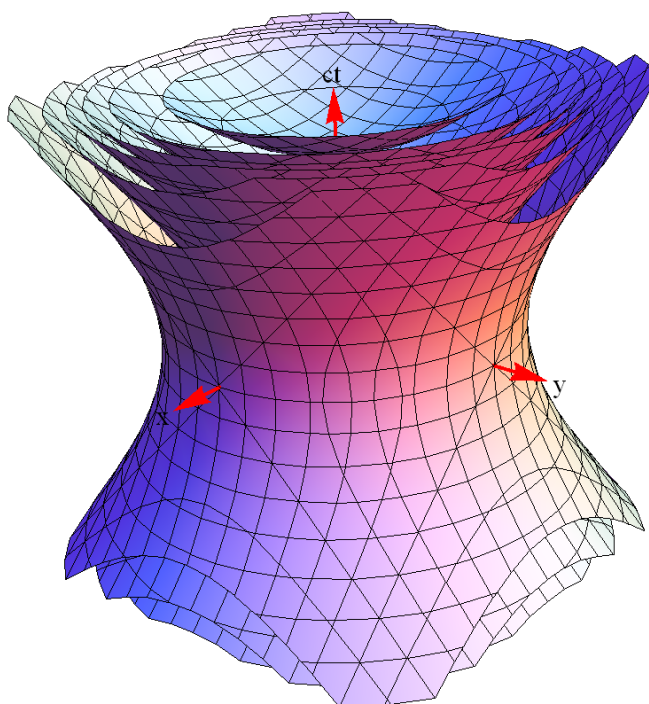


Fig.13 Hyperbolic geometry. Space-time diagram of a two-dimensional world, showing the light cone. ContourPlot of $x^2 + y^2 - (ct)^2 = -\sigma^2$ and $x^2 + y^2 - (ct)^2 = \sigma^2$, where σ is changed as a parameter.

The world line of light pulses play special roles in physics because of the equality of the speed of light for all inertial observers. A light pulse that starts at the origin of the reference frame of S at $t = 0$ reaches the position (x, y, z) at time t where

$$ct = \sqrt{x^2 + y^2 + z^2}$$

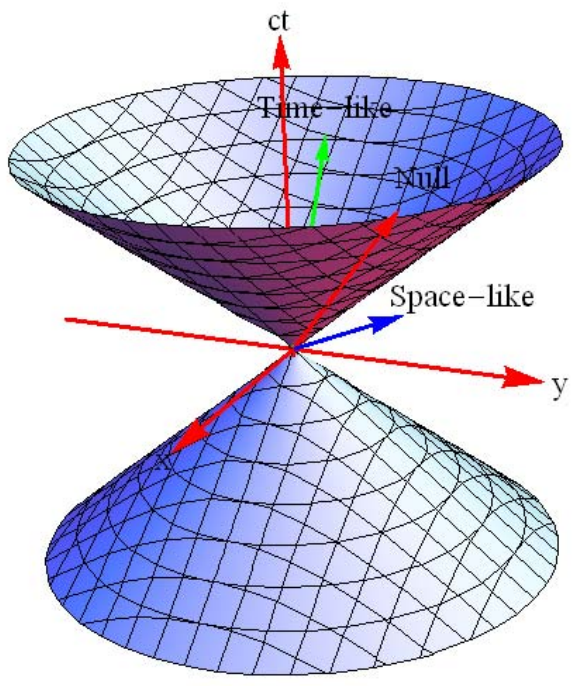


Fig.14

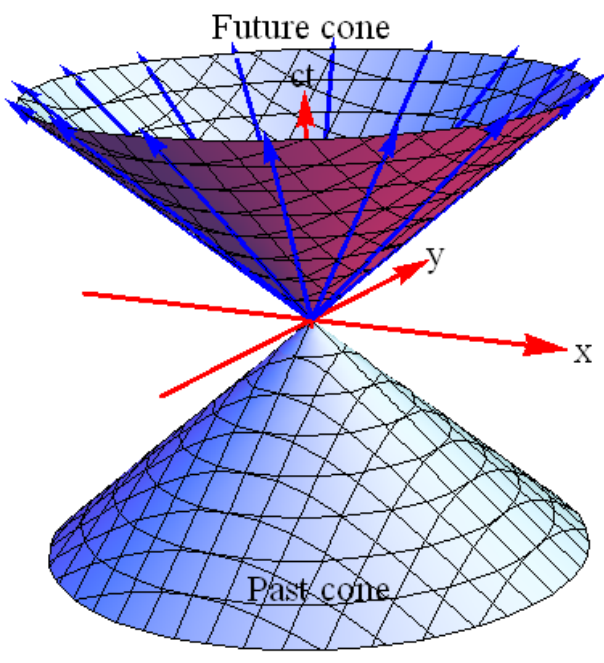


Fig.15

Spacetime diagram of a 2D world, showing the light cone. This figure is a 2D representation (x, y, ct) of a 4D dimensional space. Null cone (light cone). The past null cone refers to light imploding on the event at the origin in same as that the future cone refers to light originating at the event at the origin. The future and the past inside the light cone are accessible. But outside the light cone they are not accessible by any object staring at the origin.

Past:	$x^2 + y^2 < c^2t^2,$	$t < 0$
Future	$x^2 + y^2 < c^2t^2,$	$t > 0$
Light cone	$x^2 + y^2 = c^2t^2$	

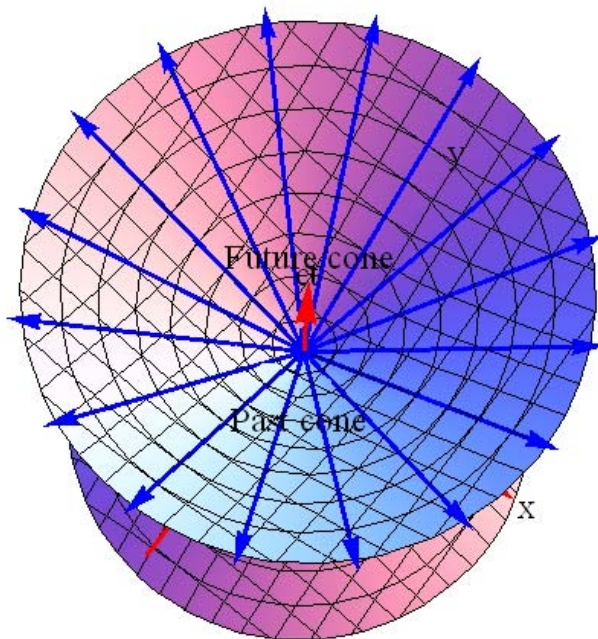
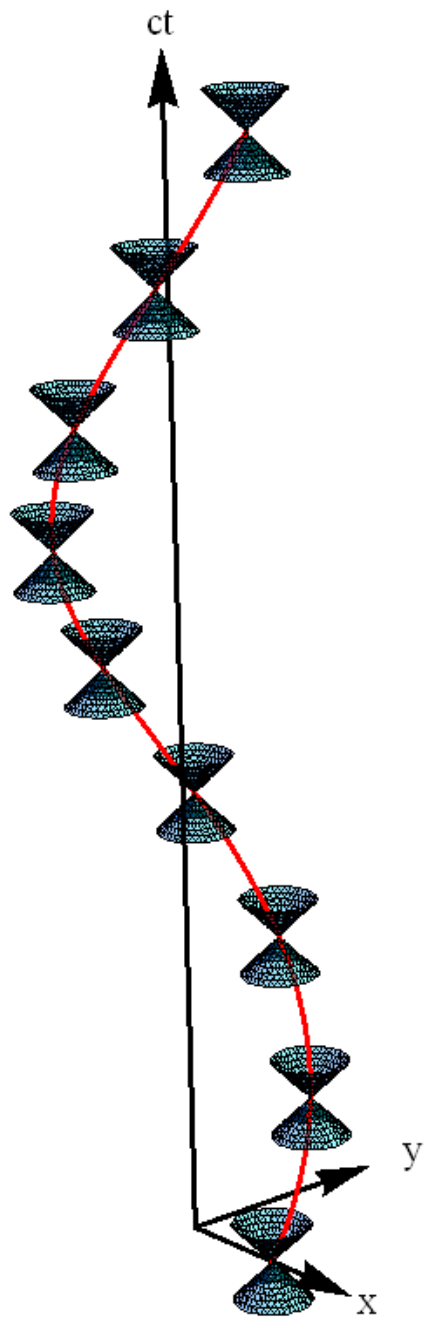


Fig.16

The pase light cone and the future light cone.



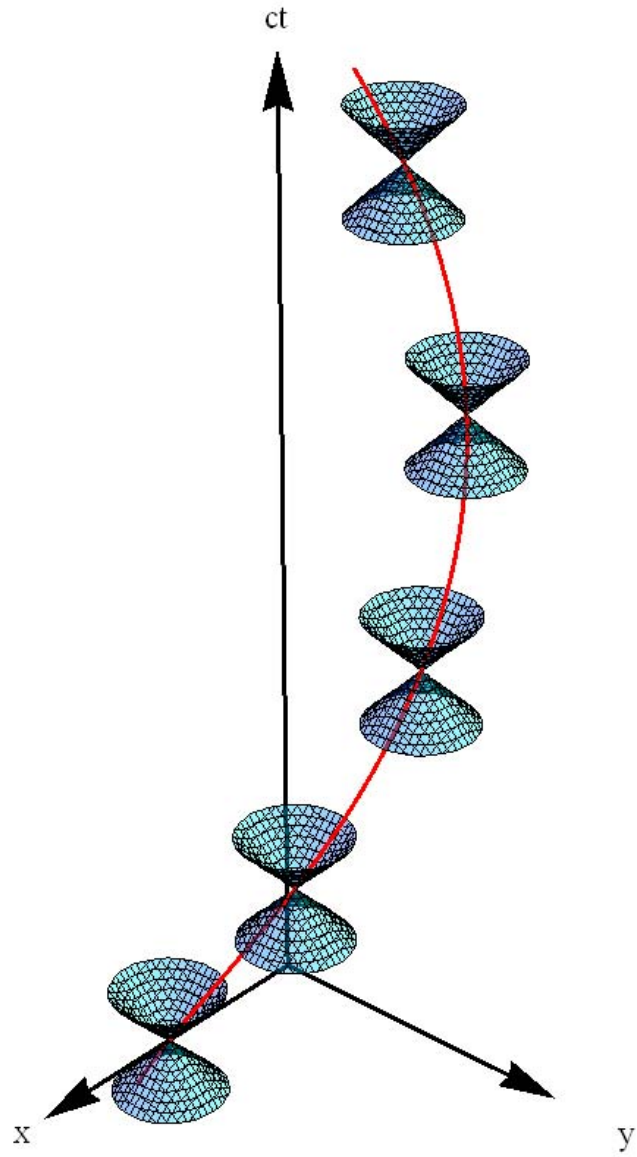


Fig.17 Minkowski space is flat, and its null cones (light cone) are uniformly arranged, depicted here as all being parallel. The curve in the spacetime is a smooth world

line, with tangent always future-time like. The world lines stay inside the light cones. At every origin a tangent line can be drawn; it will have an angle with the time direction less than that of the light cone with the time direction.

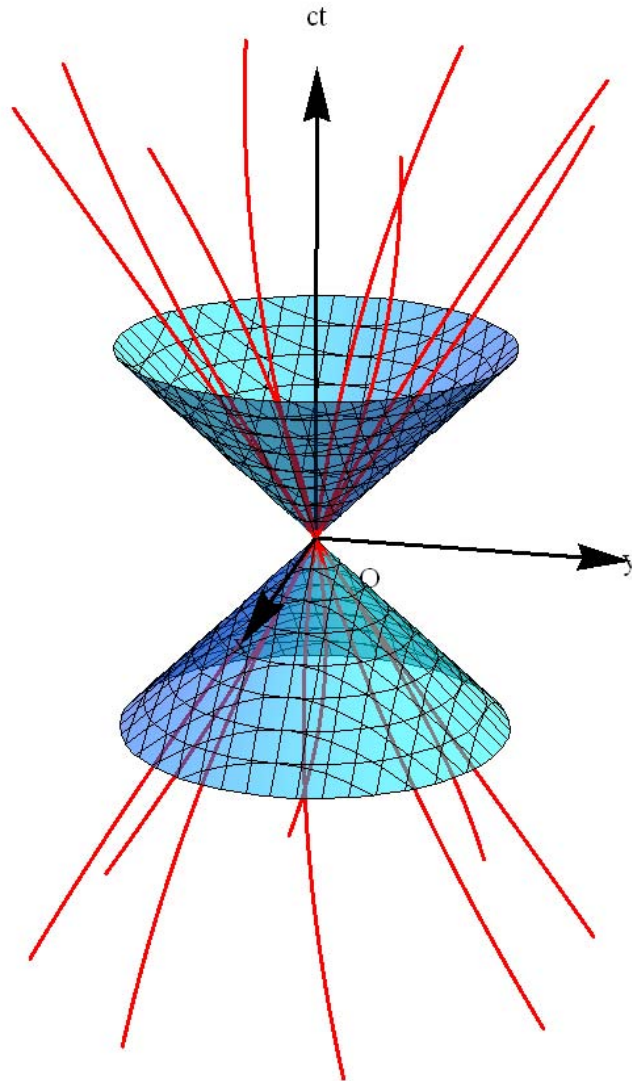


Fig.18 World lines passing through the origin from the past-region to future region. The world lines should lie inside the light (null) cone.

- 7. Scaling factor k (I)**
- (a) Derivation of the scaling factor k**

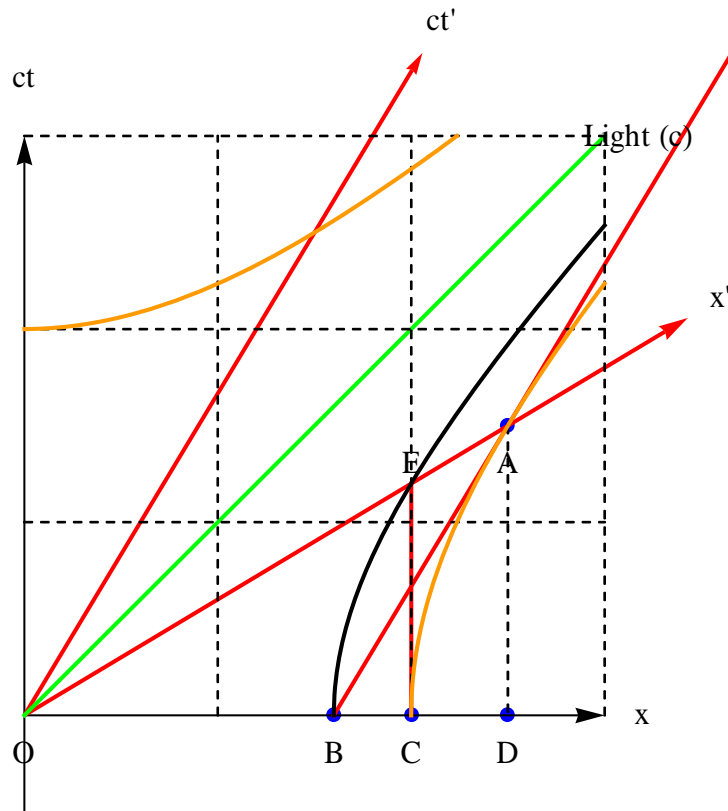


Fig.19 $(\overline{OB})_S = \frac{1}{\gamma}(\overline{OC})_S$. $(\overline{OD})_{S'} = \gamma(\overline{OC})_S$

Let us take a look at the hyperbola,

$$x^2 - y^2 = \sigma^2 = x'^2 - y'^2$$

This curve passes at the points C ($x = \sigma, y = 0$) in the S frame and A ($x' = \sigma, y' = 0$).

$$(\overline{OC})_S = \sigma, \quad (\overline{OA})_{S'} = \sigma.$$

This means that the unit length of the S' frame is different from that of the S frame.

What is the scaling factor between the unit lengths of the S and S' frames? We now find the coordinate of the point A in the (x, y) plane, by solving two equations given by

$$x^2 - y^2 = \sigma^2, \quad y = \tan \theta x.$$

We get

$$x_A = \frac{\sigma \cos \theta}{\sqrt{\cos(2\theta)}}, \quad y_A = \frac{\sigma \sin \theta}{\sqrt{\cos(2\theta)}}$$

Then the distance $(\overline{OA})_S$ is given by

$$(\overline{OA})_S = \sigma \sqrt{\sec(2\theta)}.$$

Then we have the scaling factor

$$k = \frac{(\overline{OA})_S}{(\overline{OA})_{S'}} = \sqrt{\sec(2\theta)} = \frac{1}{\sqrt{\cos^2 \theta - \sin^2 \theta}} = \sqrt{\frac{1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}}$$

where

$$\cos \theta = \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}}, \quad \text{and} \quad \sin \theta = \frac{\frac{v}{c}}{\sqrt{1 + \frac{v^2}{c^2}}}$$

(b) Ratio $(\overline{OB})_S / (\overline{OC})_S$

We consider the tangential line of $x^2 - y^2 = \sigma^2$ at the point A.

$$2x - 2yy' = 0, \quad \text{or} \quad y' = \frac{x}{y}$$

The tangential line at the point A is given by

$$y - y_A = \cot \theta (x - x_A)$$

This line intersects at the point B in the x axis,

$$x_B = x_A - y_A \tan \theta = \frac{\sigma(\cos \theta - \sin \theta \tan \theta)}{\sqrt{\cos(2\theta)}} = \sigma \sqrt{\sec(2\theta)} (\cos \theta - \sin \theta \tan \theta).$$

Then the distance $(\overline{OB})_s$ is given by

$$(\overline{OB})_s = \sigma \sqrt{\sec(2\theta)} (\cos \theta - \sin \theta \tan \theta)$$

The ratio is given by

$$\frac{(\overline{OB})_s}{(\overline{OC})_s} = \sqrt{\sec(2\theta)} (\cos \theta - \sin \theta \tan \theta) = \sqrt{\frac{1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}} \frac{1 - \frac{v^2}{c^2}}{\sqrt{1 + \frac{v^2}{c^2}}} = \frac{1}{\gamma} = \sqrt{1 - \frac{v^2}{c^2}} < 1$$

(Lorentz contraction)

(c) Ratio $(\overline{OD})_s / (\overline{OC})_s$

We now consider the point E which is located just above the point C

$$(\overrightarrow{OE})_s = (\sigma, \sigma \tan \theta) = (x_E, y_E)$$

This point is on the hyperbola given by

$$x_E^2 - y_E^2 = \sigma^2 (1 - \tan^2 \theta) = \sigma^2 \left(1 - \frac{\sin^2 \theta}{\cos^2 \theta}\right) = \sigma^2 \frac{\cos(2\theta)}{\cos^2 \theta} = x_B^2$$

which means that this hyperbola intersects at the point B in the x axis. We also consider the point D which is located just below the point A at the x axis,

$$(\overline{OD})_s = \frac{\sigma \cos \theta}{\sqrt{\cos(2\theta)}}$$

Then we get

$$\frac{(\overline{OD})_s}{(\overline{OC})_s} = \sqrt{\sec(2\theta)} \cos \theta = \sqrt{\frac{1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}} \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma < 1$$

8. Scaling factor k (II)

(a) Derivation of the scaling factor k

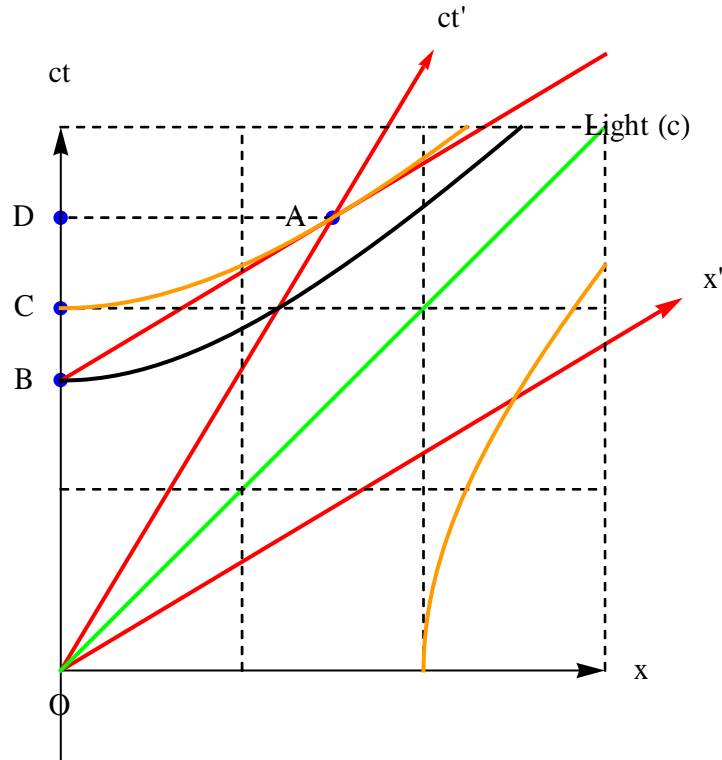


Fig.20 $(\overline{OB})_S = \frac{1}{\gamma}(\overline{OC})_S$. $(\overline{OD})_S = \gamma(\overline{OC})_S$.

We consider the equation of the hyperbola

$$x^2 - y^2 = -\sigma^2 = x'^2 - y'^2$$

where $y = ct$ and $y' = ct'$.

$$(\overline{OC})_S = \sigma$$

The hyperbola passes through the points $C (x = 0, y = \sigma)$ in the S frame and $A (x' = 0, y' = \sigma)$ in the S' frame. This means that the unit length of the S' frame is different from that of the S frame.

We now find the coordinate of the point A, by solving two equations,

$$x^2 - y^2 = -\sigma^2, \quad y = \cot \theta x,$$

leading to

$$x_A = \frac{\sigma \sin \theta}{\sqrt{\cos(2\theta)}} = \sigma \sin \theta \sqrt{\sec(2\theta)}, \quad y_A = \frac{\sigma \cos \theta}{\sqrt{\cos(2\theta)}} = \sigma \cos \theta \sqrt{\sec(2\theta)}$$

Then the distance \overline{OA} is given by

$$(\overline{OA})_S = \sigma \sqrt{\sec(2\theta)}$$

The coordinate of the point D is

$$y_D = (\overline{OD})_S = (\overline{OA})_S \cos \theta = \sigma \cos \theta \sqrt{\sec(2\theta)}$$

Then we obtain the scaling factor between the lengths of the S frame and S' frame,

$$k = \frac{(\overline{OA})_S}{(\overline{OA})_{S'}} = \sqrt{\sec(2\theta)} = \sqrt{\frac{1}{\cos^2 \theta - \sin^2 \theta}} = \sqrt{\frac{1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}}$$

where

$$(\overline{OA})_{S'} = \sigma$$

(b) Ratio $(\overline{OD})_S = \gamma(\overline{OC})_S$ and $(\overline{OB})_S = \frac{1}{\gamma}(\overline{OC})_S$

How about the coordinate of the point B? We consider the tangential line of $x^2 - y^2 = \sigma^2$ at the point A.

$$2x - 2yy' = 0, \quad \text{or} \quad y' = \frac{x}{y}$$

The tangential line at the point A is given by

$$y - y_A = \tan \theta (x - x_A)$$

This line intersects at the point B in the x axis,

$$y_B = y_A - x_A \cot \theta = \frac{\sigma(\cos \theta - \sin \theta \tan \theta)}{\sqrt{\cos(2\theta)}} = \sigma \frac{\sqrt{\cos(2\theta)}}{\cos \theta}.$$

Using the length calculated in the S frame we have

$$\frac{(\overline{OD})_S}{(\overline{OC})_S} = \cos \theta \sqrt{\sec(2\theta)} = \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} \sqrt{\frac{1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma > 1$$

(Time dilation)

and

$$\frac{(\overline{OB})_S}{(\overline{OC})_S} = \sqrt{\sec(2\theta)}(\cos \theta - \sin \theta \tan \theta) = \frac{1 - \frac{v^2}{c^2}}{\sqrt{1 + \frac{v^2}{c^2}}} \sqrt{\frac{1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}} = \sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{\gamma} < 1$$

where

$$(\overline{OD})_S = y_A = \sigma \cos \theta \sqrt{\sec(2\theta)}, \quad (\overline{OB})_S = \sigma \frac{\sqrt{\cos(2\theta)}}{\cos \theta}.$$

9. Lorentz contraction

Let us turn to the visualization of Lorentz contraction using the Minkowski diagrams.

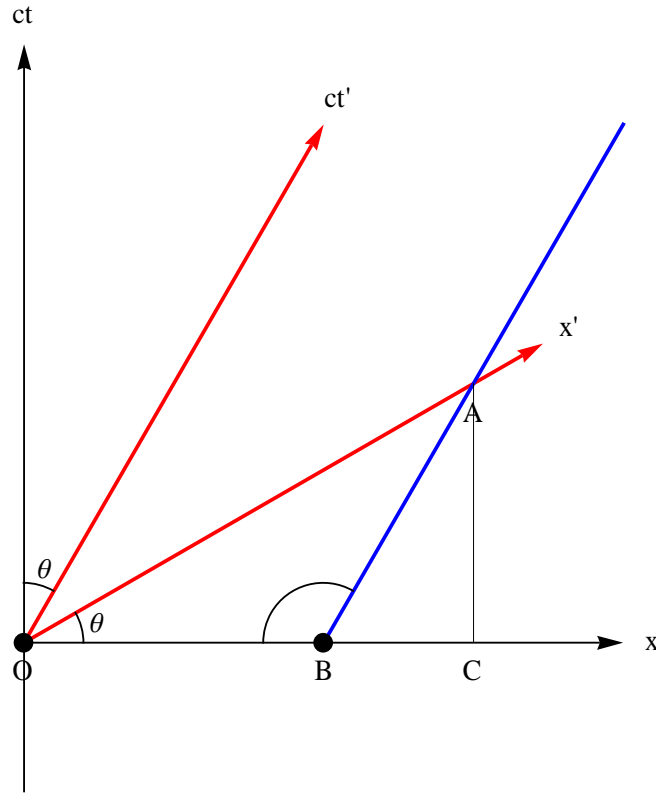


Fig.21 Lorentz contraction.

From the above figure, we get

$$(\overline{OC})_{S'} = (\overline{OA})_{S'} \cos \theta$$

$$(\overline{AC})_{S'} = (\overline{OA})_{S'} \sin \theta$$

Since $\angle BAC = \theta$, we get

$$(\overline{BC})_{S'} = (\overline{AC})_{S'} \tan \theta = (\overline{OA})_{S'} \sin \theta \tan \theta$$

$$\begin{aligned} (\overline{OB})_{S'} &= (\overline{OC})_{S'} - (\overline{BC})_{S'} \\ &= (\overline{OA})_{S'} (\cos \theta - \sin \theta \tan \theta) \\ &= (\overline{OA})_{S'} \frac{\cos(2\theta)}{\cos \theta} \end{aligned}$$

Then we have

$$\frac{(\overline{OB})_{S'}}{(\overline{OA})_{S'}} = \frac{\cos(2\theta)}{\cos\theta} = \frac{1 - \frac{v^2}{c^2}}{\sqrt{1 + \frac{v^2}{c^2}}}$$

We note that

$$(\overline{OB})_S = k(\overline{OB})_{S'} \quad (\text{scaling factor})$$

where k is the scaling factor

$$k = \sqrt{\frac{1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}}$$

Then we have

$$(\overline{OB})_S = k(\overline{OB})_{S'} = \sqrt{\frac{1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}} \frac{1 - \frac{v^2}{c^2}}{\sqrt{1 + \frac{v^2}{c^2}}} (\overline{OA})_{S'} = \sqrt{1 - \frac{v^2}{c^2}} (\overline{OA})_{S'}$$

10. Time dilation

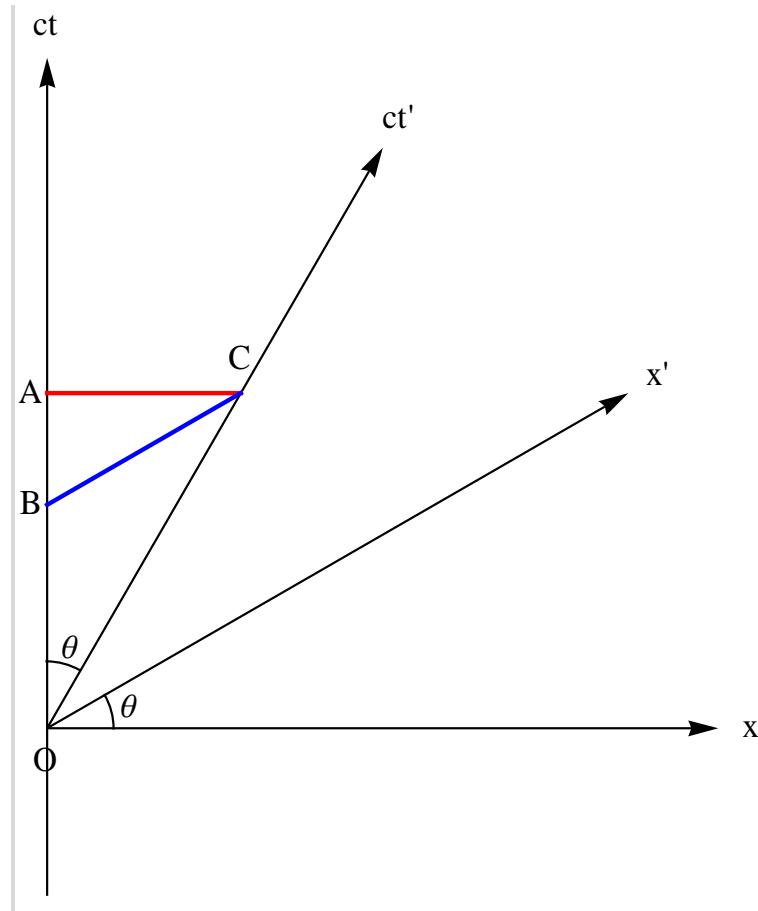


Fig.22 Time dilation

We consider the geometry of the Minkowski diagram for the time dilation,

$$(\overline{AC})_{S'} = (\overline{OC})_{S'} \sin \theta,$$

$$(\overline{OA})_{S'} = (\overline{OC})_{S'} \cos \theta,$$

and

$$(\overline{AB})_{S'} = (\overline{AC})_{S'} \tan \theta = (\overline{OC})_{S'} \sin \theta \tan \theta$$

or

$$\begin{aligned}
(\overline{OB})_{S'} &= (\overline{OA})_{S'} - (\overline{AB})_{S'} \\
&= (\overline{OC})_{S'} \cos \theta - (\overline{OC})_{S'} \sin \theta \tan \theta \\
&= (\overline{OC})_{S'} \left(\cos \theta - \frac{\sin^2 \theta}{\cos \theta} \right) = (\overline{OC})_{S'} \frac{\cos(2\theta)}{\cos \theta}
\end{aligned}$$

where

$$\tan \theta = \frac{v}{c}.$$

Then we get the relation

$$\frac{(\overline{OA})_{S'}}{(\overline{OB})_{S'}} = \frac{(\overline{OC})_{S'} \cos \theta}{(\overline{OC})_{S'} \frac{\cos(2\theta)}{\cos \theta}} = \frac{\cos^2 \theta}{\cos(2\theta)} = \frac{\cos^2 \theta}{\cos^2 \theta - \sin^2 \theta} = \frac{1}{1 - \tan^2 \theta} = \frac{1}{1 - \frac{v^2}{c^2}} = \gamma^2 > 1$$

Here we note that the scaling factor k is defined as.

$$(\overline{OA})_S = k(\overline{OA})_{S'}$$

where

$$k = \sqrt{\frac{1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}}$$

Then we have

$$(\overline{OA})_S = k(\overline{OC})_{S'} \cos \theta = \sqrt{\frac{1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}} (\overline{OC})_{S'} \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} = \frac{(\overline{OC})_{S'}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where

$$(\overline{OA})_S = ct, \quad (\overline{OC})_{S'} = ct'$$

11. Lorentz transformation and scaling factor k

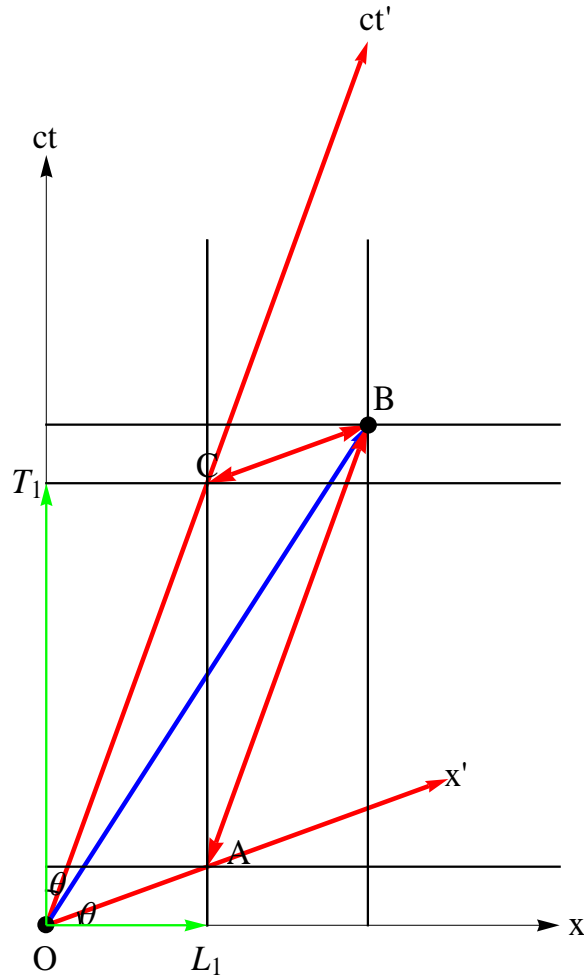


Fig.23 Lorentz transformation. A ($x', ct' = 0$). B($x', ct'; x, ct$). C($x' = 0, ct'$).

We consider the Minkowski space-time diagram (described above)

$$\overline{OA}_{S'} = x', \quad \overline{AB}_{S'} = ct'$$

For $\overline{OB}_{S'} = (x, ct)_{S'}$,

$$(x)_{S'} = x' \cos \theta + ct' \sin \theta = \frac{x' + ct'}{\sqrt{1 + \frac{v^2}{c^2}}} = k(x)_S = \sqrt{\frac{1 - \frac{v^2}{c^2}}{1 + \frac{v^2}{c^2}}} (x)_S$$

$$(ct)_{S'} = x' \sin \theta + ct' \cos \theta = \frac{\frac{v}{c} x' + ct'}{\sqrt{1 + \frac{v^2}{c^2}}} = k(ct)_S = (ct)_S \sqrt{\frac{1 - \frac{v^2}{c^2}}{1 + \frac{v^2}{c^2}}}$$

where we use the scaling factor k . Then we get the Lorentz transformation as

$$(x)_S = x = \frac{x' + ct'}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (ct)_S = ct = \frac{ct' + \frac{v}{c} x'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

12. Relativity of simultaneity

In physics, the relativity of simultaneity is the concept that simultaneity—whether two events occur at the same time—is not absolute, but depends on the observer's reference frame. According to the special theory of relativity, it is impossible to say in an absolute sense whether two events occur at the same time if those events are separated in space.

- (1) Events that occur at two separate places at the same time (as seen in the S frame) do not happen at the same time as viewed in the S' frame.

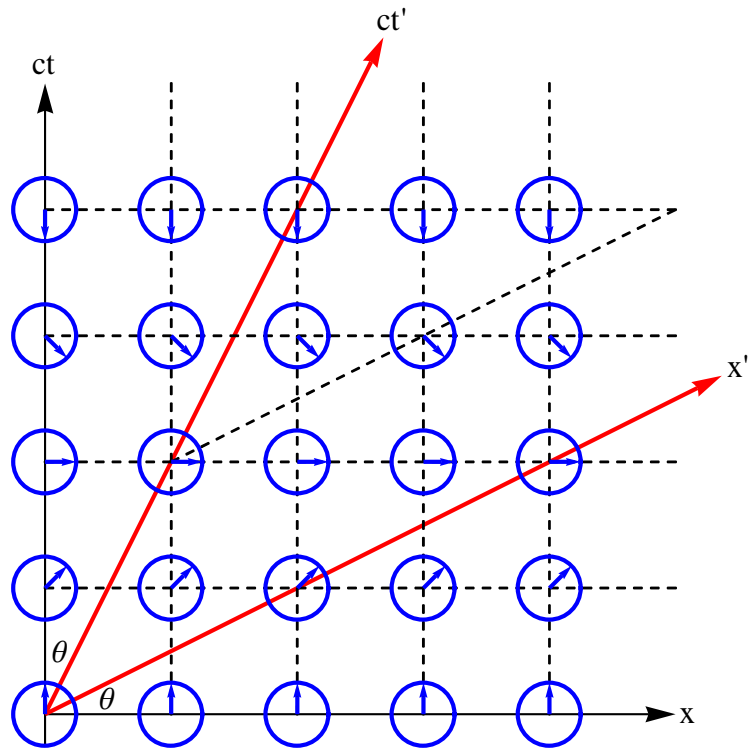


Fig.24 Relativity of simultaneity-I

$$\Delta t' = \frac{\Delta t - \frac{v}{c^2} \Delta x}{\sqrt{1 - \frac{v^2}{c^2}}}$$

- (2) Events that occur at two separate places at the same time in the S' frame do not happen at the same time, as viewed in the S frame.

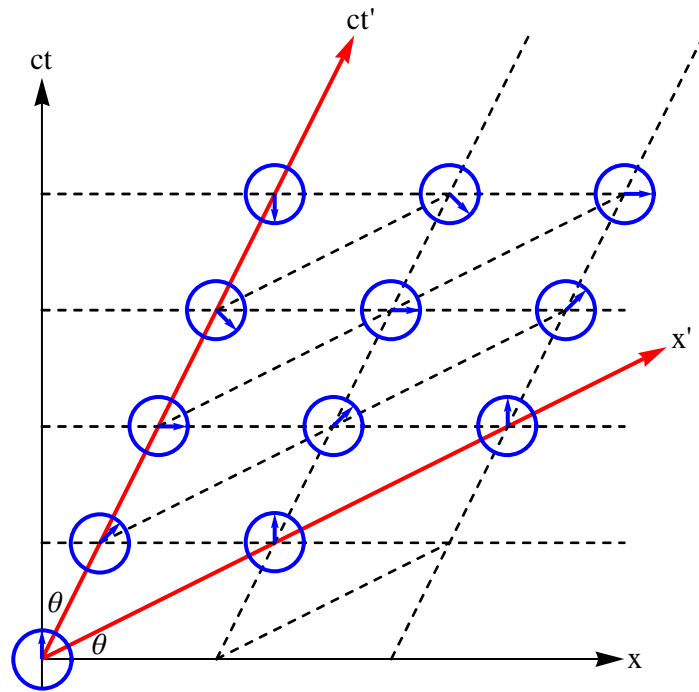


Fig.25 Relativity of simultaneity-II

$$\Delta t = \frac{\Delta t' + \frac{v}{c^2} \Delta x'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

13. Train and light

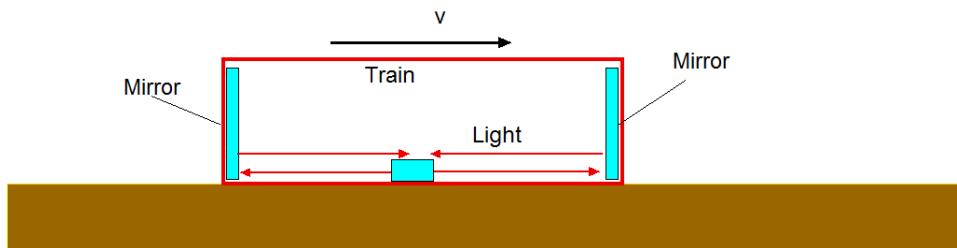


Fig.26

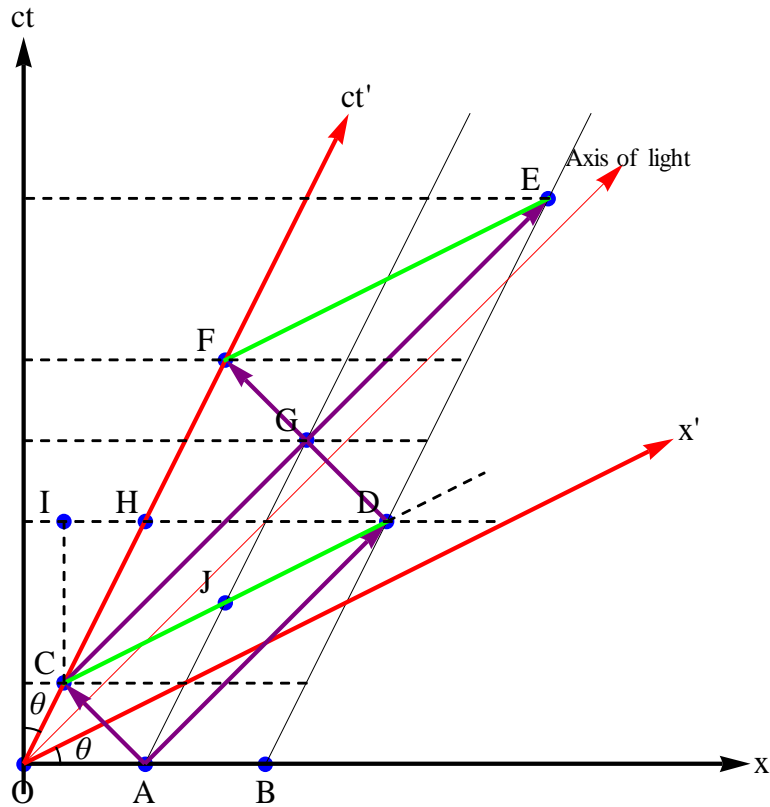


Fig.27

At $t = 0$ ($t' = 0$), a light is emitted from the midpoint (the point A) of the train which moves along the positive x axis at the constant velocity v . There are two mirrors at both the ends of train. The a light pulse is emitted in the backward direction and is reflected back by a mirror at one of the end (the event C). Simultaneously another light pulse is emitted in the forward direction and is reflected back by a mirror at the other end (the event D). We note that

$$ct_D > ct_C$$

in the S frame, while

$$ct'_D = ct'_C.$$

in the S' frame, After the reflection of the light pulses by the mirrors, two light pulses meet again at the point G. We note that the line CD is parallel to the x' axis. In other words the length CD (L_0) is the length of train in the S' frame. The point J is the midpoint of the points C and D.

We consider the triangle CHD. The length HD in the S' frame can be calculated as

$$(\overline{HD})_{S'} = (\overline{CD})_{S'} (\cos \theta - \sin \theta \tan \theta) = (\overline{CD})_{S'} \frac{1 - \frac{v^2}{c^2}}{\sqrt{1 + \frac{v^2}{c^2}}}$$

Using the scaling factor k , we have

$$(\overline{HD})_S = k(\overline{HD})_{S'} = (\overline{CD})_{S'} \frac{1 - \frac{v^2}{c^2}}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{\sqrt{1 + \frac{v^2}{c^2}}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\gamma} (\overline{CD})_{S'}$$

or

$$L = \frac{1}{\gamma} L' \quad (\text{Lorentz contraction}).$$

We also get a time dilation from the triangle CHI.

$$(\overline{CI})_S = (\overline{CH})_{S'} \cos \theta = (\overline{CH})_{S'} \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}}$$

Using the scaling factor k , we have

$$(\overline{CI})_S = k(\overline{CI})_{S'} = (\overline{CH})_{S'} \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{\sqrt{1 + \frac{v^2}{c^2}}}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma (\overline{CH})_{S'}$$

or

$$ct = \gamma ct' \quad (\text{time dilation}).$$

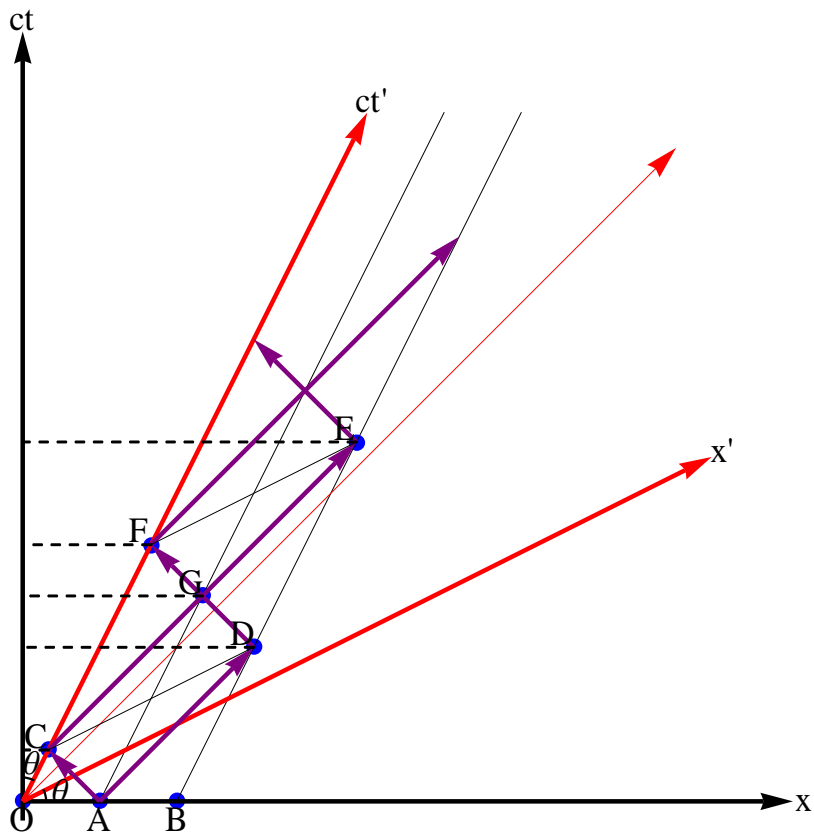


Fig.28 Repetition process.

14. Scaling factor κ for the Doppler effect

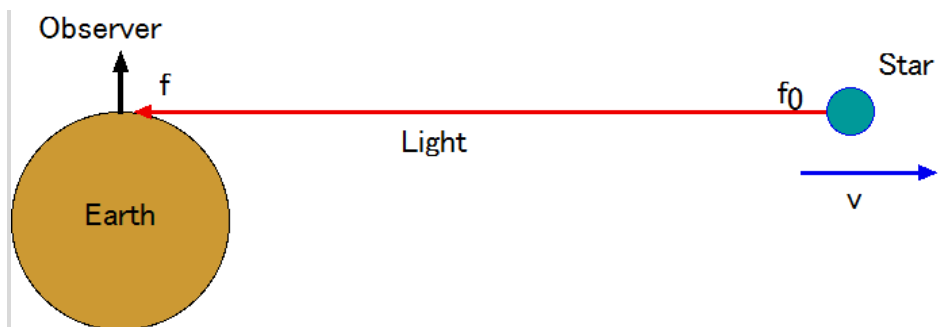


Fig.29

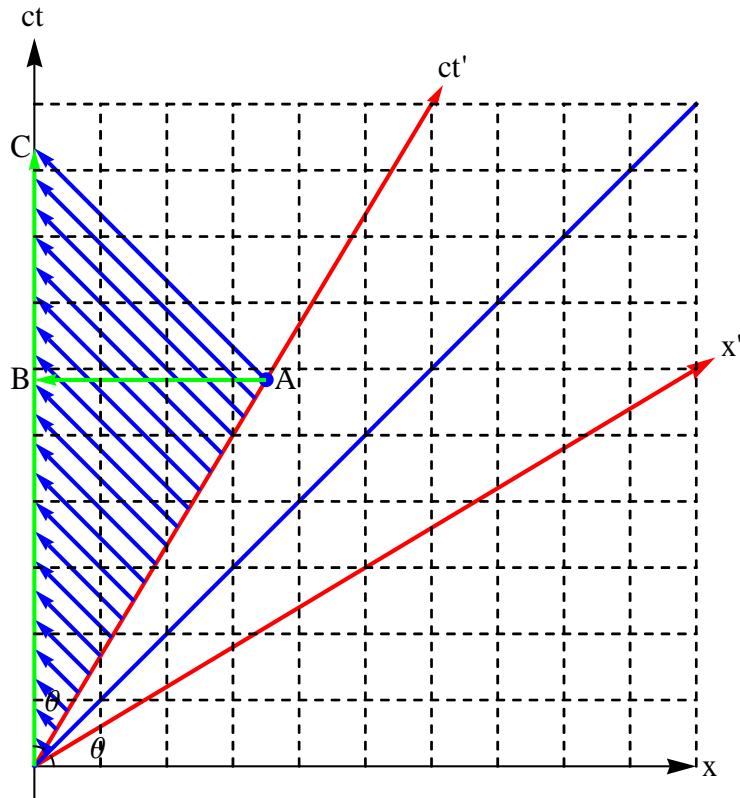


Fig.30 Doppler effect

We consider a moving light source that flashes with frequency f_0 . The observer at rest in the S frame receives the light signal with a different frequency f (this is called the Doppler effect of light). First we calculate the distances in the S' frame;

$$\begin{aligned}
 (\overline{OC})_{S'} &= (\overline{OB} + \overline{BC})_{S'} = (\overline{OA})_{S'} \cos \theta + (\overline{AB})_{S'} \tan \frac{\pi}{4} \\
 &= ct' \cos \theta + ct' \sin \theta = ct' \frac{1 + \frac{v}{c}}{\sqrt{1 + \frac{v^2}{c^2}}}
 \end{aligned}$$

Using the scaling factor

$$(\overline{OC})_S = \sqrt{\frac{1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}} (\overline{OC})_{S'}$$

we have

$$(\overline{OC})_S = ct = \sqrt{\frac{1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}} (\overline{OC})_{S'} = \sqrt{\frac{1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}} ct' \frac{1 + \frac{v}{c}}{\sqrt{1 + \frac{v^2}{c^2}}} = \sqrt{\frac{1 + \frac{v}{c}}{1 + \frac{v}{c}}} ct'$$

The frequency of the light observed at the S frame is related to that of the light emitted in the S' frame is

$$(\overline{OC})_S = ct = \sqrt{\frac{1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}} (\overline{OC})_{S'} = \sqrt{\frac{1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}} ct' \frac{1 + \frac{v}{c}}{\sqrt{1 + \frac{v^2}{c^2}}} = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} ct'$$

((Formula of the Doppler effect))

$$\frac{f}{f_0} = \frac{ct'}{ct} = \frac{1}{\kappa} = \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} < 1$$

$$t = \kappa t'$$

with

$$\kappa = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} > 1$$

15. Derivation of Lorentz transformation using the scaling factor κ

We consider the case when the laser on the Earth (at rest) emits a light pulse. The observer moves away from the light source on Earth (the S frame) at the velocity v , and receives the light signal.

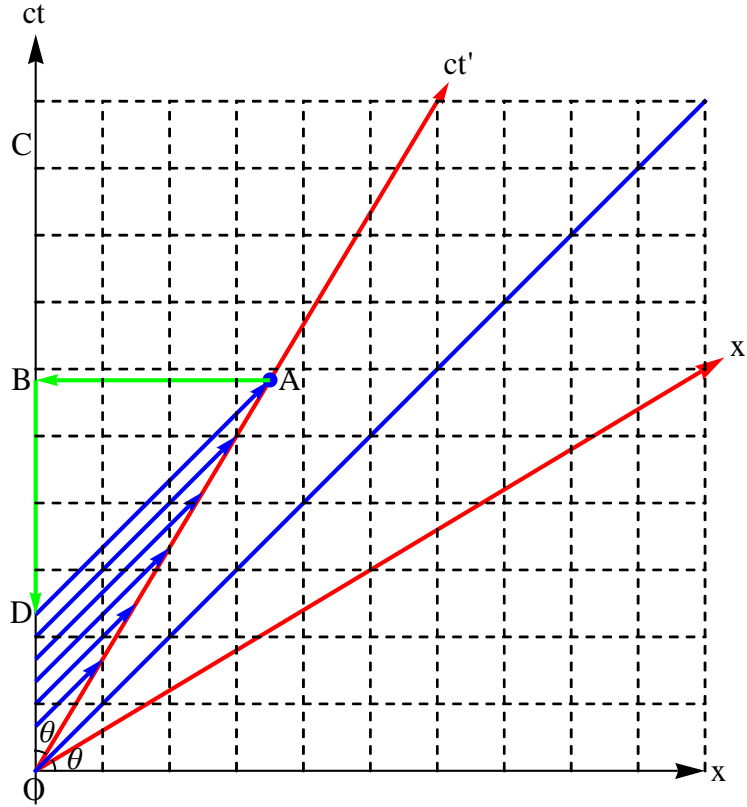


Fig.31

$$(\overline{OA})_{S'} = ct', \quad (\overline{BA})_{S'} = (\overline{DB})_{S'} = ct' \sin \theta$$

$$(\overline{OB})_{S'} = ct' \cos \theta$$

Then we have

$$(\overline{OD})_{S'} = (\overline{OB})_{S'} - (\overline{DB})_{S'} = ct'(\cos \theta - \sin \theta) = \frac{ct'}{\sqrt{1 + \frac{v^2}{c^2}}} \left(1 - \frac{v}{c}\right)$$

Using the scaling factor k , we get

$$(\overline{OD})_S = k(\overline{OD})_{S'} = \frac{ct'}{\sqrt{1 + \frac{v^2}{c^2}}} \left(1 - \frac{v}{c}\right) \sqrt{\frac{1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}} = ct' \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} = \frac{1}{\kappa} ct' < ct'$$

16. Derivation of the Lorentz transformation from κ

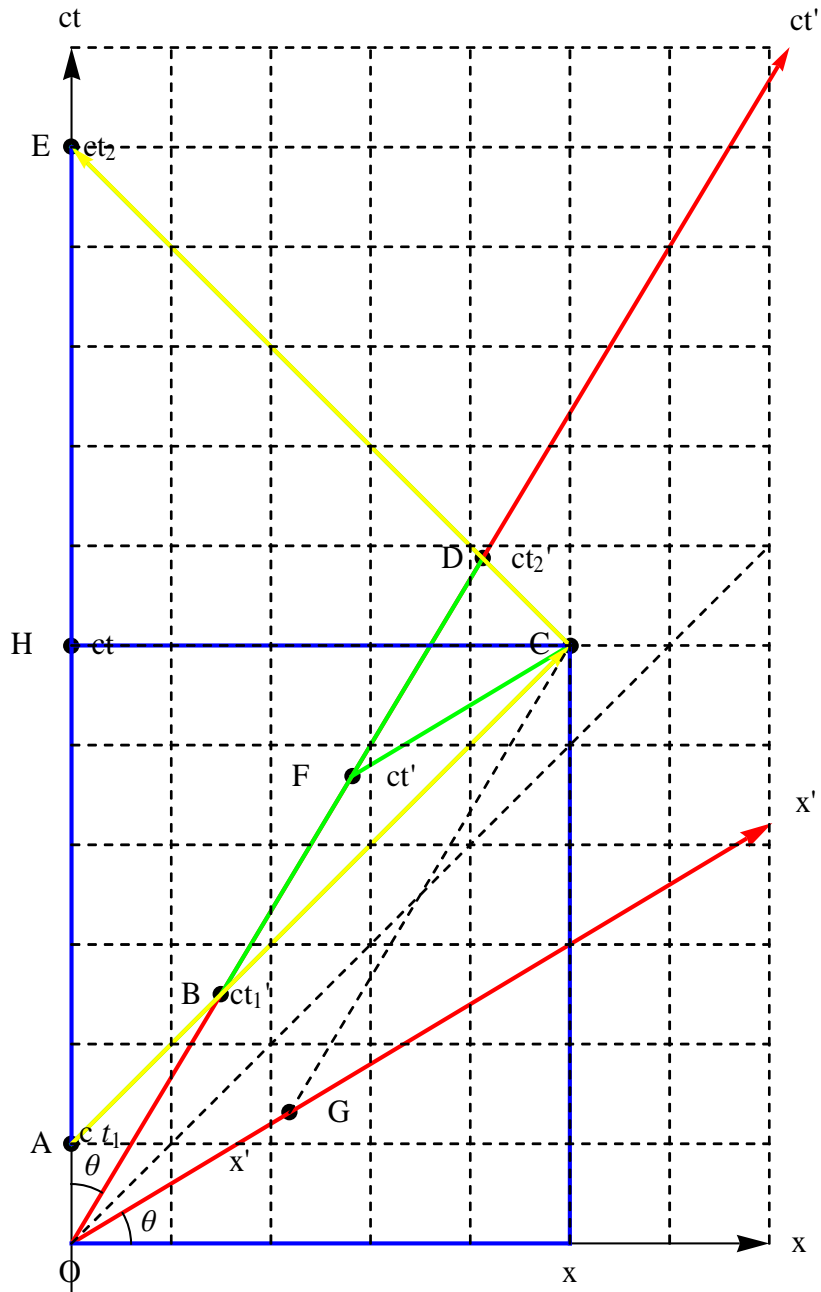


Fig.32

We consider two spaceships (A and B). The spaceship B moves at the velocity v away from the spaceship S (the S frame), forming the S' frame. Suppose that the spaceship A (in the S frame) emits the light pulse at the time t_A . The spaceship B (in the S' frame) encounters the light

pulse at the point B (in the S' frame) at $t' = t_B'$. The light pulse is reflected by a mirror at the point C, and returns back to the spaceship A at $t = t_A$ (in the S frame). Note that the spaceship B again encounters the light pulse at the point D (in the S' frame).

From the simple geometry, it is found that

$$(\overline{BF})_{S'} = (\overline{FD})_{S'} = (\overline{FC})_{S'}$$

and

$$(\overline{CH})_{S'} = (\overline{AH})_{S'} = (\overline{HE})_{S'}$$

By using the Doppler factor κ ,

$$\kappa = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}$$

we get

$$(\overline{OB})_{S'} = ct_1' = \kappa(\overline{OA})_S = \kappa ct_1, \text{ and } (\overline{OD})_{S'} = ct_2' = \frac{1}{\kappa}(\overline{OE})_S = \frac{1}{\kappa} ct_2.$$

or

$$t_1' = \kappa t_1, \quad t_2' = \frac{1}{\kappa} t_2$$

In the S-frame;

$$t = \frac{t_1 + t_2}{2}, \quad x = \frac{c(t_2 - t_1)}{2}$$

or

$$t_1 = t - \frac{x}{c}, \quad t_2 = t + \frac{x}{c}$$

In the S' frame

$$t' = \frac{t_1' + t_2'}{2}, \quad x' = \frac{c(t_2' - t_1')}{2}$$

or

$$t_1' = t' - \frac{x'}{c}, \quad t_2' = t' + \frac{x'}{c}$$

Then we have two equations,

$$t' - \frac{x'}{c} = \kappa \left(t - \frac{x}{c} \right), \quad t' + \frac{x'}{c} = \frac{1}{\kappa} \left(t + \frac{x}{c} \right)$$

Solving these equations for x' and t' , we get the Lorentz transformation

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad ct' = \frac{ct - \frac{v}{c}x}{\sqrt{1 - \frac{v^2}{c^2}}}$$

17. Physical meaning of the time dilation

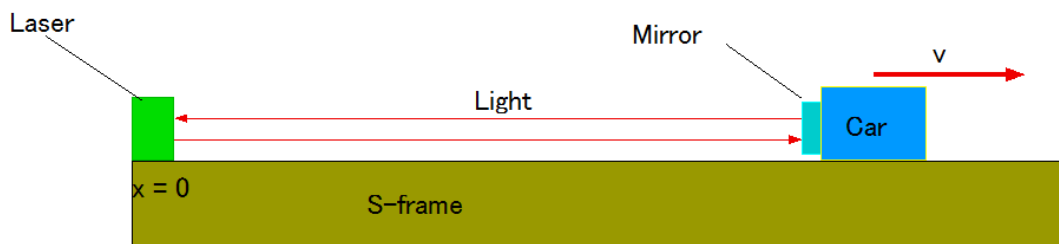


Fig.33

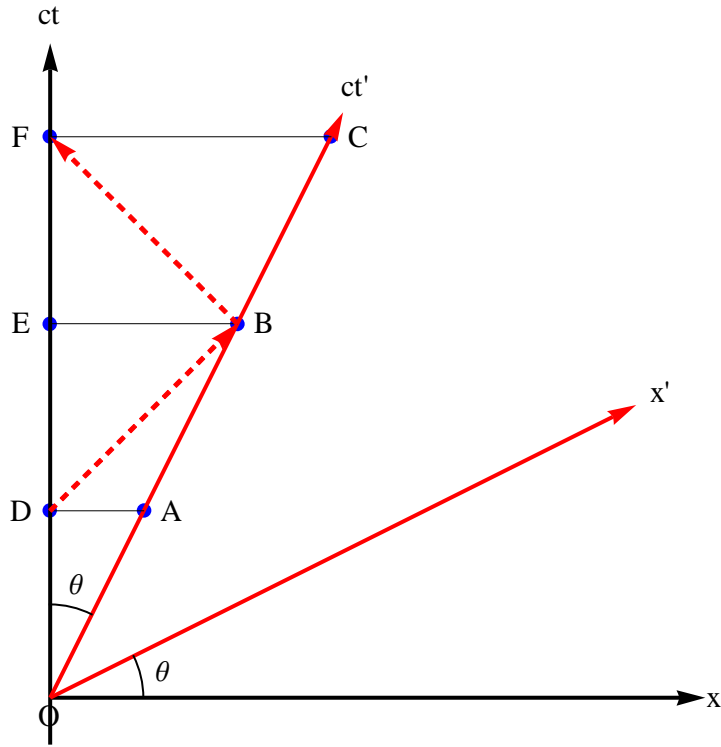


Fig.34 $\tan\theta = v/c = 1/2$. $(\overline{DE})_{S'} = (\overline{EB})_{S'} = (\overline{EF})_{S'}$. In the S' frame.

Here the inertial frame S', moving with constant speed $v < c$ to the right, is represented as “seen” in the frame S. The initial conditions are $(x, ct) = (0, 0) = (x', ct')$. A light source at the spatial origin ($x = 0$) in the S frame emits a light pulse towards the S' frame ($t = t_D$). There exists a mirror that reflects it back. Your clock (in the S frame) says that the pulse is emitted at the event D ($0, ct_D$) and is back at the event F ($0, ct_F$). According to the postulate of constant speed of light, one can get that

$$ct_E = \frac{ct_F + ct_D}{2}$$

((Discussion))

The event E ($0, ct_E$) and the event B are simultaneous in the S frame.

$$(\overline{OA})_{S'} = ct_1'$$

$$(\overline{OD})_{S'} = ct_1' \cos\theta = \frac{ct_1'}{\sqrt{1 + \frac{v^2}{c^2}}}$$

$$(\overline{DE})_{S'} = (\overline{EB})_{S'} = (\overline{EF})_{S'} = \alpha$$

Then we have

$$\left(\overline{OE}\right)_{S'} = ct_1' \cos \theta + \alpha, \quad \left(\overline{OF}\right)_{S'} = ct_1' \cos \theta + 2\alpha$$

$$\left(\overline{EB}\right)_{S'} = \alpha$$

Since the event B is on the $x' = 0$ axis, we get the relation

$$\tan \theta = \frac{v}{c} = \frac{\alpha}{ct_1' \cos \theta + \alpha}$$

or

$$\alpha = \frac{ct_1' \sin \theta}{1 - \tan \theta}$$

Using the value of α , we have

$$\begin{aligned} \left(\overline{OE}\right)_{S'} &= ct_1' \left(\cos \theta + \frac{\sin \theta}{1 - \tan \theta} \right) = ct_1' \left(\frac{\cos \theta}{1 - \tan \theta} \right) \\ &= \frac{ct_1'}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{1}{1 - \frac{v}{c}} \end{aligned}$$

$$\left(\overline{EB}\right)_{S'} = ct_1' \left(\frac{\sin \theta \cos \theta}{1 - \tan \theta} \right)$$

$$\begin{aligned} \left(\overline{OF}\right)_{S'} &= ct_1' \left(\cos \theta + \frac{2 \sin \theta}{1 - \tan \theta} \right) = ct_1' \left(\frac{\sin \theta + \cos \theta}{1 - \tan \theta} \right) \\ &= ct_1' \frac{1 + \frac{v}{c}}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{1}{1 - \frac{v}{c}} \end{aligned}$$

$$\left(\overline{FC}\right)_{S'} = ct_1' \left(\frac{\sin \theta + \cos \theta}{1 - \tan \theta} \right) \tan \theta$$

$$(\overline{OB})_{S'} = (\overline{OE})_{S'} \frac{1}{\cos \theta} = ct_1' \left(\frac{\cos \theta}{1 - \tan \theta} \right) \frac{1}{\cos \theta} = \frac{ct_1'}{1 - \tan \theta}$$

Using the scaling factor k , we have

$$\begin{aligned} ct_F = (\overline{OF})_{S'} &= kct_1' \frac{1 + \frac{v}{c}}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{1}{(1 - \frac{v}{c})} = ct_1' \frac{1 + \frac{v}{c}}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{1}{(1 - \frac{v}{c})} \sqrt{\frac{1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}} \\ &= \gamma ct_1' \left(\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}} \right) \end{aligned}$$

$$\begin{aligned} ct_E = (\overline{OE})_{S'} &= kct_1' \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{1}{1 - \frac{v}{c}} = ct_1' \frac{1}{\sqrt{1 + \frac{v^2}{c^2}}} \frac{1}{(1 - \frac{v}{c})} \sqrt{\frac{1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}} \\ &= \frac{ct_1'}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{1}{1 - \frac{v}{c}} = \frac{\gamma ct_1'}{1 - \frac{v}{c}} \end{aligned}$$

$$ct_D = (\overline{OD})_{S'} = kct_1' \cos \theta = \frac{ct_1'}{\sqrt{1 + \frac{v^2}{c^2}}} \sqrt{\frac{1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}} = \frac{ct_1'}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma ct_1'$$

Then we get the relation

$$ct_D + ct_F = \gamma ct_1' \left(1 + \frac{1 + \frac{v}{c}}{1 - \frac{v}{c}} \right) = \gamma ct_1' \frac{2}{1 - \frac{v}{c}} = 2ct_E,$$

or

$$t_E = \frac{1}{2}(t_D + t_F)$$

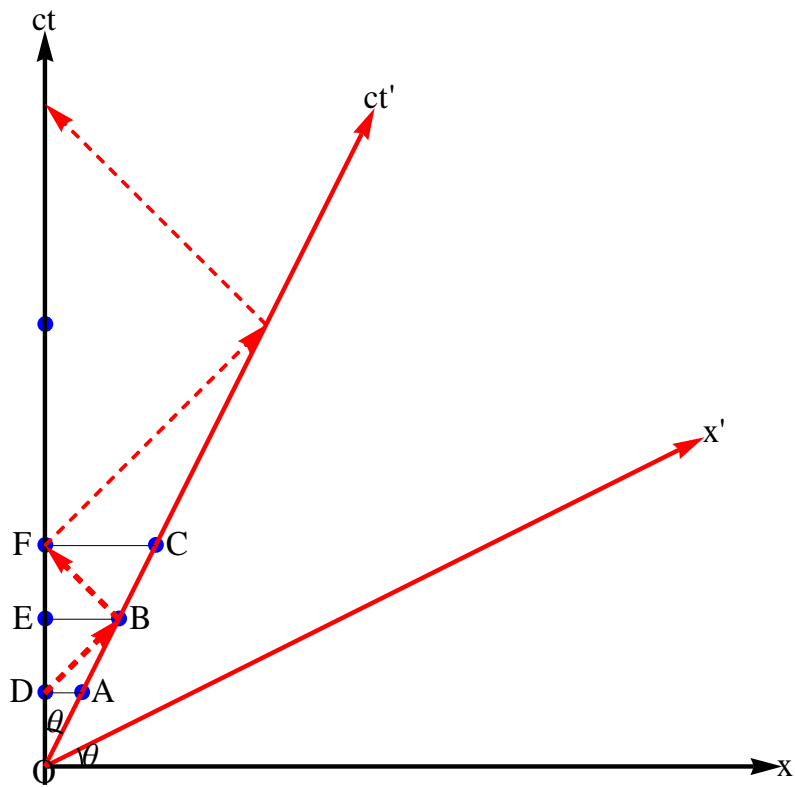


Fig.35 Moving of the Fabry-Perot interferometer at the velocity v along the x axis. The light emits from the origin of the S frame (on the ground).

18. The relativity of simultaneity

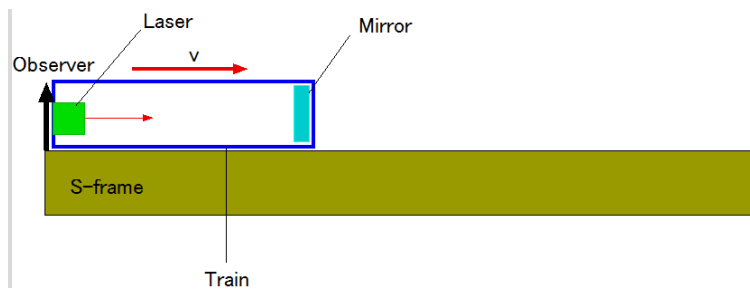


Fig.36

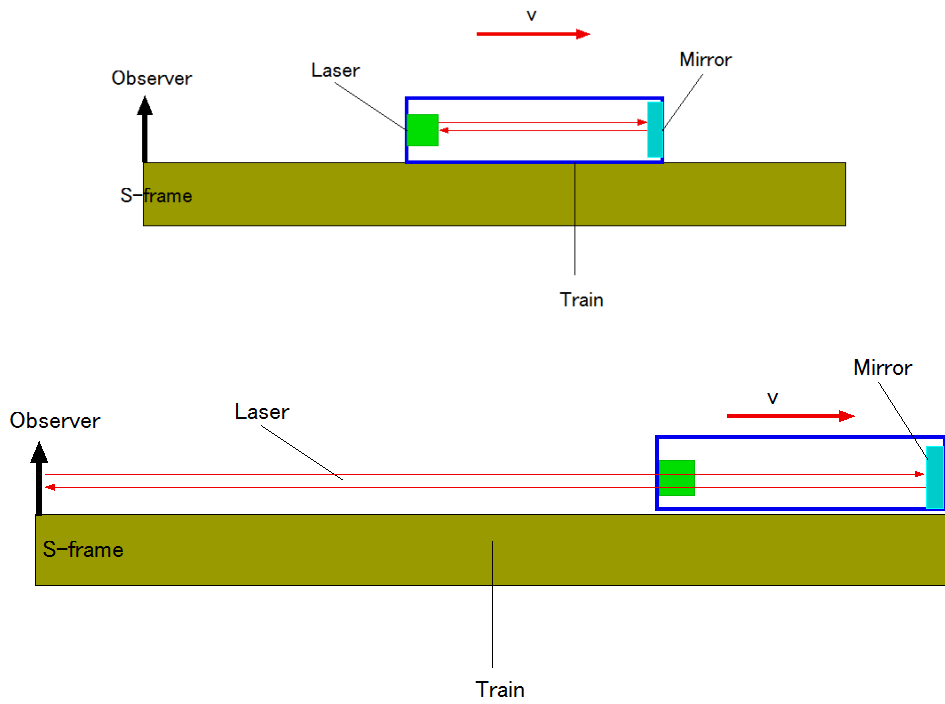


Fig.37

Suppose that the train starts to move at the origin. It runs at the velocity v close to the speed of light c . There are an emitter of light at the end of the train and a mirror at the other end of train. Suppose that one sends a light signal to the mirror. The light is reflected by the mirror and is returned back to the position of emitter.

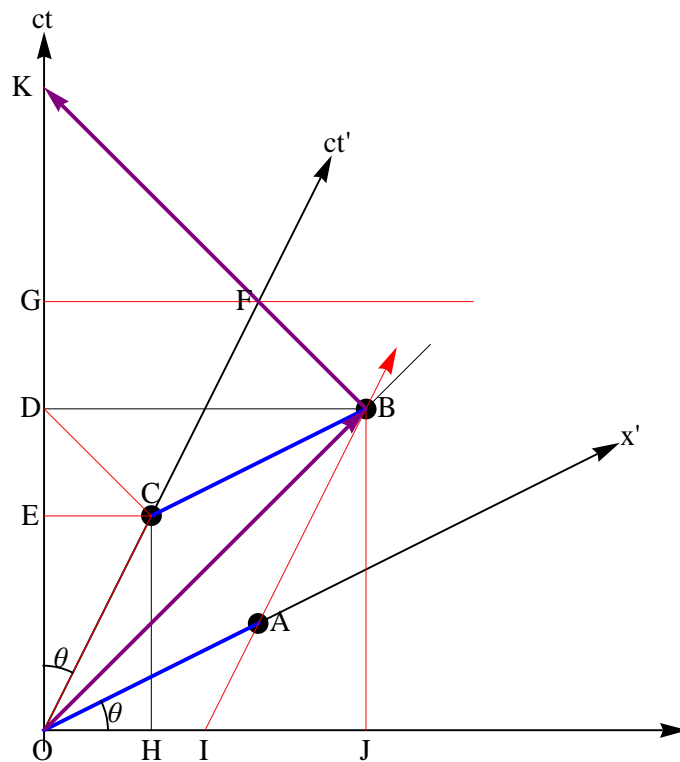


Fig.38 A light ray emitted at the common origin of both frames and reaches the right end of the train at the event B. Let a light source be places at the left of the train and a mirror at the right end.

The event B: (x', ct') in the S' frame. The event B is on the world of the light:

$$x' = ct'.$$

Note that

$$\overline{CE} = \overline{DE}, \quad \overline{BD} = \overline{DK}$$

since $\angle KBD = \angle DCE = \frac{\pi}{4}$.

The point A;

$$\left(\overrightarrow{OA}\right)_{S'} = (x', ct' = 0)$$

The point C;

$$\left(\overrightarrow{OC}\right)_{S'} = (x' = 0, ct')$$

The point B;

$$\begin{aligned}\left(\overrightarrow{OB}\right)_{S'} &= (ct' \sin \theta + x' \cos \theta) \hat{i} + (ct' \cos \theta + x' \sin \theta) \hat{j} \\ &= ct'(\sin \theta + \cos \theta)(\hat{i} + \hat{j})\end{aligned}$$

The point C:

$$\left(\overrightarrow{OC}\right)_{S'} = ct'(\sin \hat{\theta} + \cos \hat{\theta})$$

The event A:

$$\left(\overrightarrow{OA}\right)_{S'} = x'(\cos \hat{\theta} + \sin \hat{\theta}) = ct'(\cos \hat{\theta} + \sin \hat{\theta})$$

The event I;

$$\left(\overrightarrow{OI}\right)_{S'} = x'(\cos \theta - \sin \theta \tan \theta) \hat{i} = ct'(\cos \theta - \sin \theta \tan \theta) \hat{i}$$

The event J:

$$\left(\overrightarrow{OJ}\right)_{S'} = (x' \cos \theta + ct' \sin \theta) \hat{i} = ct'(\cos \theta + \sin \theta) \hat{i}$$

The event E:

$$\left(\overrightarrow{OE}\right)_{S'} = ct' \cos \hat{\theta}$$

The event D:

$$\left(\overrightarrow{OD}\right)_{S'} = ct'(\sin \theta + \cos \theta) \hat{j}$$

The event K:

$$\begin{aligned}\left(\overrightarrow{OK}\right)_{S'} &= [ct'(\cos \theta + \sin \theta) + (ct' \cos \theta + x' \sin \theta)] \hat{j} \\ &= 2ct'(\cos \theta + \sin \theta) \hat{j}\end{aligned}$$

The event F:

$$\left(\overrightarrow{OF}\right)_{S'} = (x'+ct')[\sin\hat{\theta} + \cos\theta]\hat{j} = 2ct'[\sin\hat{\theta} + \cos\theta]\hat{j}$$

The event G:

$$\left(\overrightarrow{OG}\right)_{S'} = (x'+ct')\cos\hat{\theta} = 2ct'\cos\hat{\theta}$$

$$\begin{aligned}\left(\overrightarrow{AC}\right)_{S'} &= ct'(\sin\hat{\theta} + \cos\hat{\theta}) - ct'(\cos\hat{\theta} + \sin\hat{\theta}) \\ &= ct'(\sin\theta - \cos\theta)(\hat{i} - \hat{j})\end{aligned}$$

$$\begin{aligned}\left(\overrightarrow{AD}\right)_{S'} &= ct'(\sin\theta + \cos\theta)\hat{j} - ct'(\cos\hat{\theta} + \sin\hat{\theta}) \\ &= ct'\cos\theta(\hat{i} - \hat{j})\end{aligned}$$

Then we have

$$\left(\overrightarrow{OG}\right)_S = k\left(\overrightarrow{OG}\right)_{S'} = 2kct'\cos\theta = \frac{2ct'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\left(\overrightarrow{OE}\right)_S = k\left(\overrightarrow{OE}\right)_{S'} = kct'\cos\theta = \frac{ct'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\left(\overrightarrow{OD}\right)_S = k\left(\overrightarrow{OD}\right)_{S'} = kct'(\sin\theta + \cos\theta) = \frac{ct'(1+\frac{v}{c})}{\sqrt{1-\frac{v^2}{c^2}}}$$

where k is the scaling factor.

In conclusion,

- (a) $2t'$ is the time taken in the S' frame. A clock on the spaceship 1 measures how long the round trip takes.

(b)
$$\frac{1}{c}(\overline{OG})_S = \frac{2t'}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma(2t')$$
 is the time measure by an observer on the S frame. A clock

on the observer on the S frame measures how long the round trip takes. (time dilation).

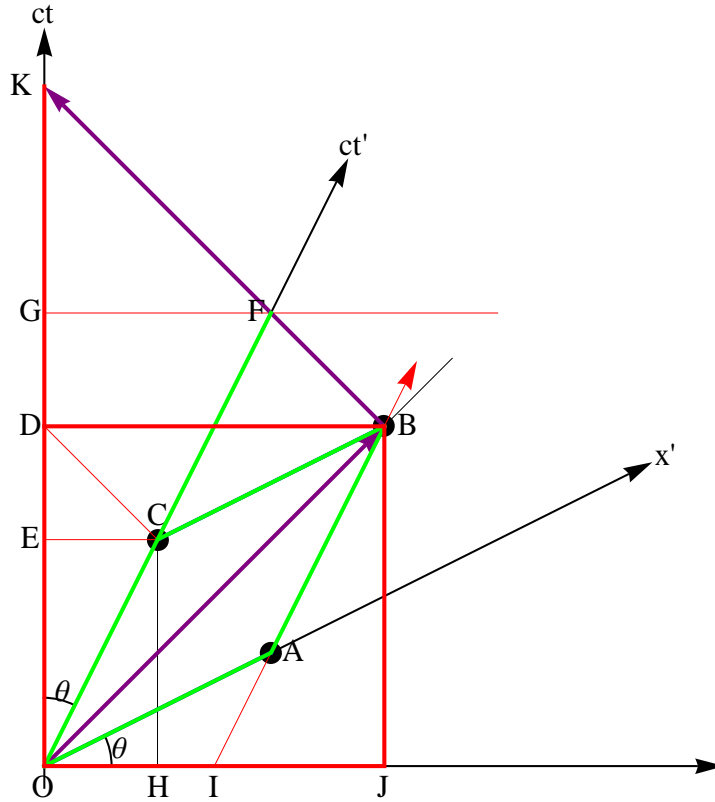


Fig.39 $(\overline{OA})_{S'} = (\overline{AB})_{S'} = (\overline{OC})_{S'} = (\overline{CB})_{S'} = (\overline{CF})_{S'}$ (green lines).
 $(\overline{OJ})_{S'} = (\overline{JB})_{S'} = (\overline{OD})_{S'} = (\overline{DB})_{S'} = (\overline{DK})_{S'}$ (red lines).

((Discussion))

$$(\overline{OI})_S = L, \quad (\overline{OA})_{S'} = L_0$$

$$(\overline{OD})_S = ct_D, \quad (\overline{OJ})_{S'} = ct_D$$

$$(\overline{OE})_S = ct_E, \quad (\overline{OC})_{S'} = ct_C'$$

Then we get

$$ct_D = L + ct_D \tan \theta = L + vt_D$$

or

$$L = \left(1 - \frac{v}{c}\right) ct_D$$

Using the Doppler effect, we get

$$ct_D = ct_C' \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}$$

The we have

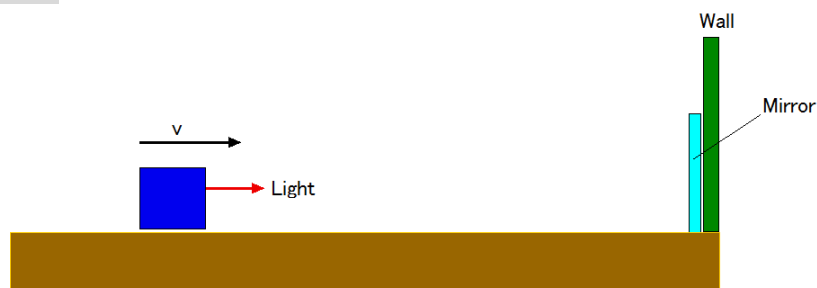
$$L = \left(1 - \frac{v}{c}\right) ct_D = \left(1 - \frac{v}{c}\right) ct_C' \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} = \sqrt{1 - \frac{v^2}{c^2}} ct_C'$$

Since $L_0 = ct_C'$, the length L is expressed by

$$L = \sqrt{1 - \frac{v^2}{c^2}} L_0$$

(Lorentz contraction)

19. Application



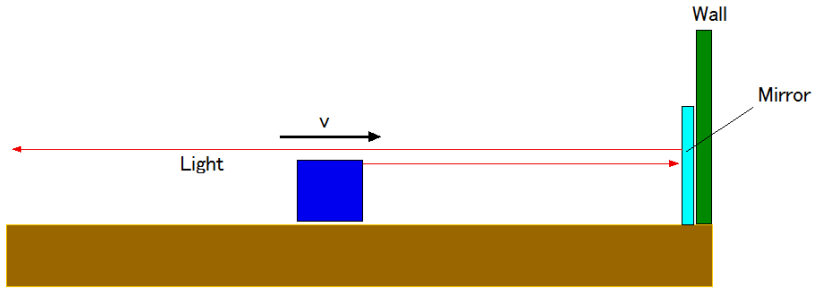


Fig.41

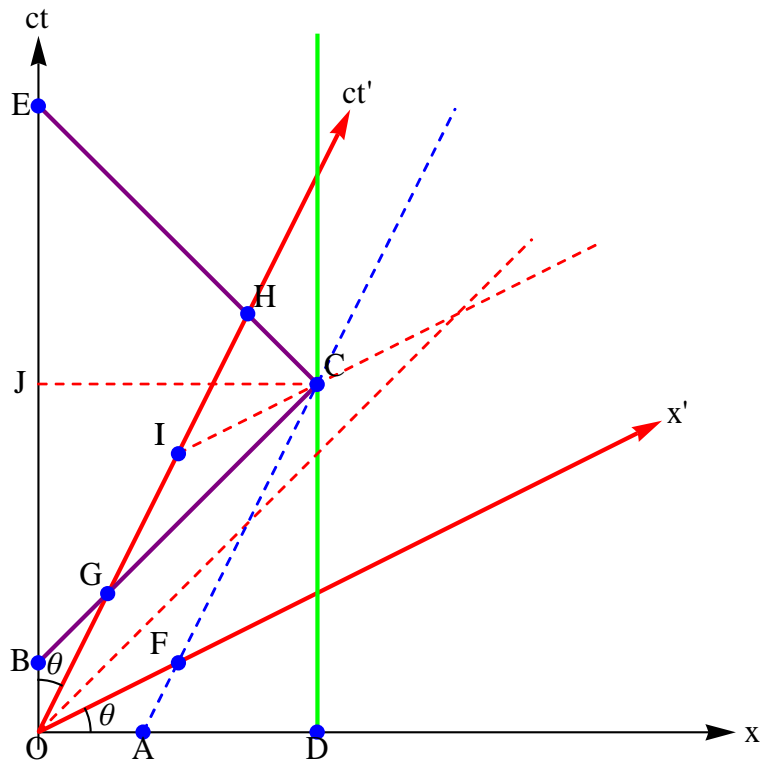


Fig.42

20. Station master at the station and conductors in the train

A station master with a watch stands at the station. Two conductors with watches are in the train. Suppose that the train passes through the station at the speed of v (close to the speed of light). How can we measure the length of the train by using watches?

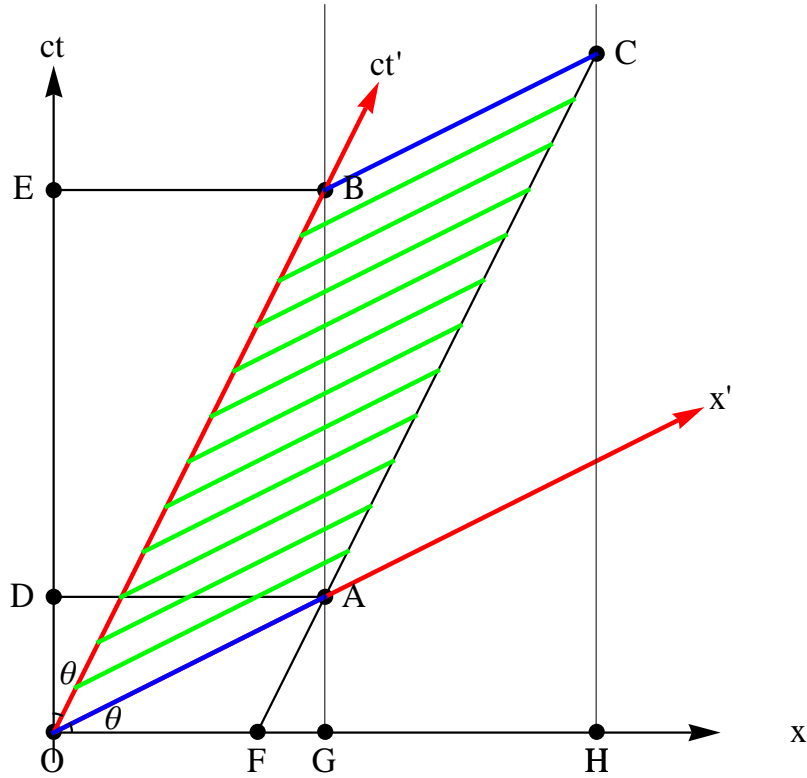


Fig.43

The station master (in the S frame) is located at the point G on the x axis. The length \overline{OA} is the length of the train measured in the S' frame. The station master starts to measure the time when the head of the train passes through the point G where the station master stands ($t = t_D$). He finishes measuring the time when the tail of the train passes through the point G ($t = t_E$). On the other hand, one conductor is at the head of the train and the other conductor is at the tail of the train. The clock of one conductor is simultaneous to that of the other conductor. The two conductors measure the time when the train passes through the station. The conductors start to measure the time when the conductor at the head of the train passes through the station master ($t'_A = 0$). The conductor at the tail of the train measures the time elapsed when he passes through the station master.

$$(\overline{OA})_{S'} = L_0,$$

$$(\overline{OB})_{S'} = ct'$$

$$(\overline{OE})_{S'} = ct' \cos \theta$$

$$(\overline{OG})_{S'} = (\overline{DA})_{S'} = L_0 \cos \theta$$

$$(\overline{OD})_{S'} = (\overline{GA})_{S'} = L_0 \sin \theta$$

$$(\overline{DE})_{S'} = (\overline{OE})_{S'} - (\overline{OD})_{S'} = ct' \cos \theta - L_0 \sin \theta.$$

Then we have

$$t = (\overline{DE})_S = k(\overline{DE})_{S'} = k(ct' \cos \theta - L_0 \sin \theta) = \frac{ct' - \frac{v}{c} L_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

This is equivalent to the Lorentz transformation. When we know t and t' , we can evaluate the length of L_0 .

21. Space time interval

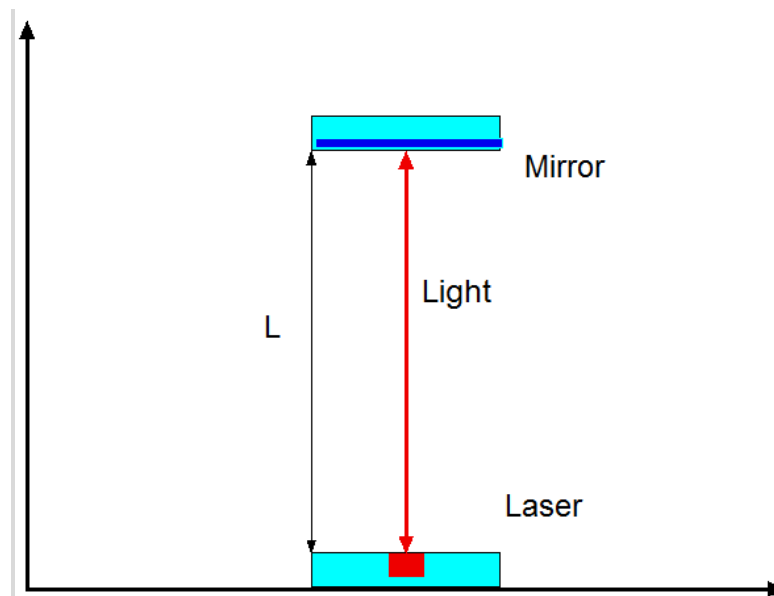


Fig.44 The S' frame ($x' - y'$).

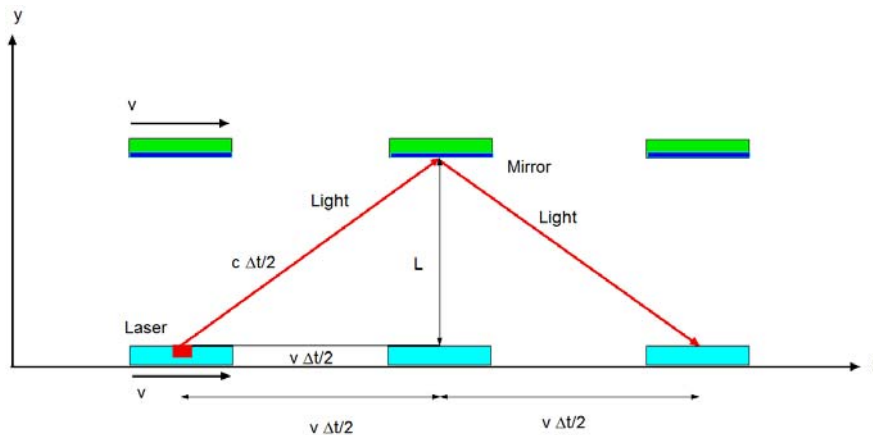


Fig.45 The S frame (x - y).

Two spaceships are travelling together through the galaxy at close to the speed of light. Mounted on one ship is a laser that can fire the pulse of light, and the other mirror. The pilot of the first spaceship fires the pulse to the mirror, and watches as it is reflected back. A clock on board measures how long the round trip takes. But now suppose that he does this as the ships passing an observer nearby asteroid. According to relativity theory this observer sees the pulse moving through space at exactly the same speed that the pilot does, namely the speed of light. But he also sees the pulse travelling a longer distance because from his prospective he must add the forward motion of the ships to the motion of the pulse between them. So he measures a longer time interval for this round trip than the pilot does, because he's watching the pulse go farther without going any faster. This effect is called time dilation. If two observers are moving with respect to one another each perceives that the other's time is flowing more slowly.

(<http://www.youtube.com/watch?v=KHjpBjgIMVk>)

We assume that two spaceships moves along the x axis at the same velocity v relative to the ground (the S frame). The distance between two spaceships remain unchanged in the S' frame. At $t = 0$, the spaceship 1 emits the light signal to the spaceship 2. The spaceship 2 has a mirror. The light signal is reflected by the mirror and is sent back to the spaceship 1.

In the S' frame,

$$\Delta t' = \frac{2L}{c}.$$

In the S frame,

$$\frac{c\Delta t}{2} = \sqrt{L^2 + \left(\frac{v\Delta t}{2}\right)^2}$$

or

$$\Delta t = \frac{\frac{2L}{c}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{\Delta t'}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (\text{time dilation})$$

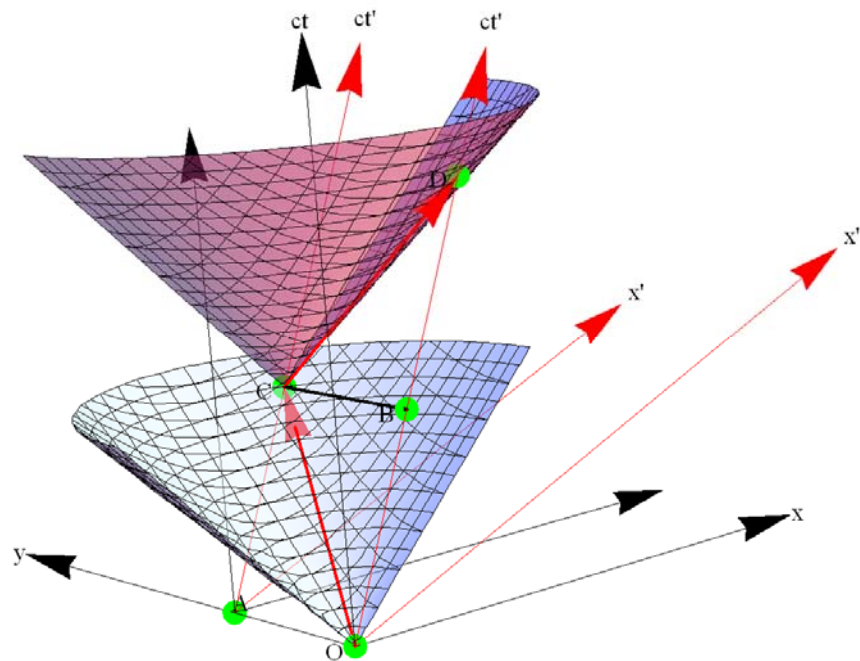


Fig.46 Space-time diagram for the two airplanes moving along the same direction.

22. World distance

The world line is a line connecting two space-time coordinates in the Minkowski space-time diagram. We consider two space-time coordinates P (x_1, y_1, z_1, ct_1) and Q (x_2, y_2, z_2, ct_2) . The world distance is defined by

$$s_{12}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_2 - t_1)^2$$

This world distance is invariant under the Lorentz transformation;

$$s_{12}^2 = (x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2 - c^2(t_2' - t_1')^2 = s_{12}^2$$

where the space-time coordinates $(x_1', y_1', z_1', ct_1')$ and $(x_2', y_2', z_2', ct_2')$ are related to (x_1, y_1, z_1, ct_1) and (x_2, y_2, z_2, ct_2) through a Lorentz transformation, respectively. We use the $(+++)$ signature pseudometric form.

When the light starts from the point P (x_1, y_1, z_1, ct_1) at time t_1 and reaches the Q (x_2, y_2, z_2, ct_2) at time t_2 , we have

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_2 - t_1)^2 = s_{12}^2 = 0.$$

In other words, the world distance between two space-time coordinates connecting by the light is zero.

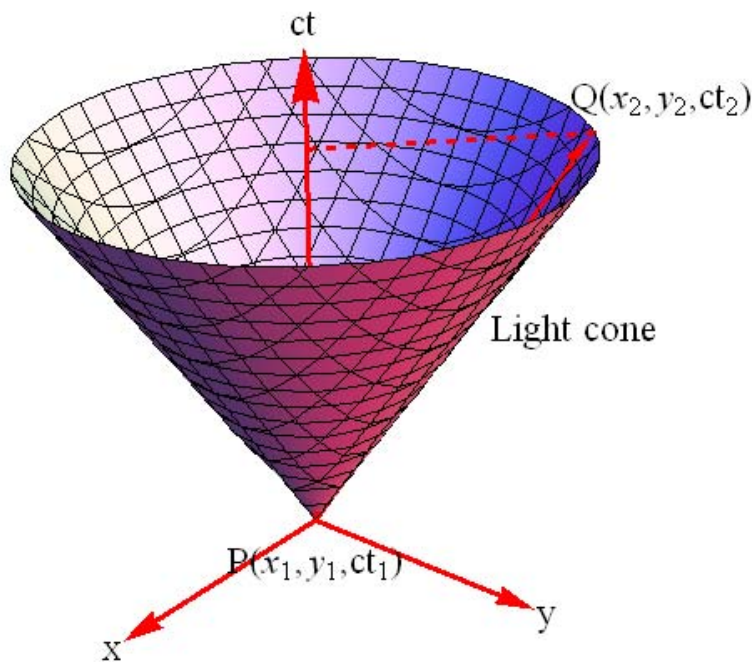


Fig.47 Light cone (null cone) of the 2D space-time diagram for the element P. Time-like region ($ds^2 < 0$) and null cone ($ds^2 = 0$).

The world distance s_{12}^2 is time-like when $s_{12}^2 > 0$, space-like when $s_{12}^2 < 0$, and light-like when $s_{12}^2 = 0$. For an infinitesimal world distance, we have

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2$$

23. Proper time

We consider the clock which stays at the same place in the S' frame. The space-time coordinates are expressed by $(x_1', y_1', z_1', ct_1')$ and $(x_2', y_2', z_2', ct_2')$ with $x_1' = x_2'$, $y_1' = y_2'$, and $z_1' = z_2'$, in the S' frame. The world distance between them is given by

$$s_{12}^2 = s_{12}^2 = -c^2(t_2' - t_1')^2$$

This is invariant under the Lorentz transformation. The time $(t_2' - t_1')$ is intrinsic to the clock itself and is called a proper time.

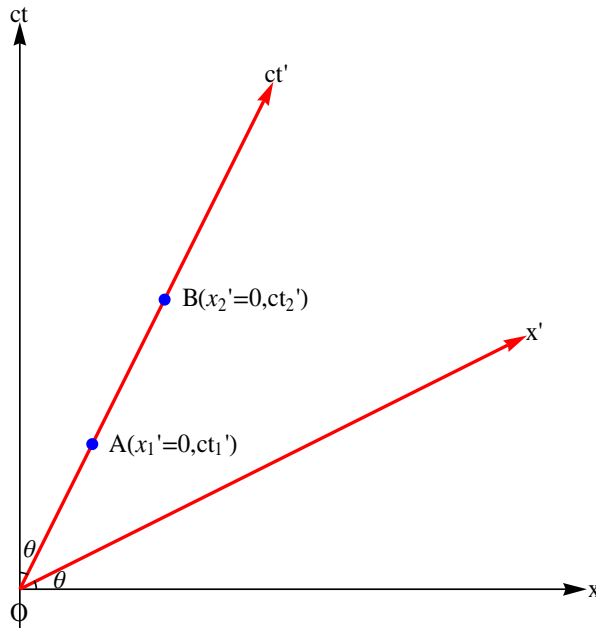


Fig.48 Elements A and B lie on the world line (constant velocity v). $\tan \theta = \frac{v}{c}$. The proper time τ_{AB} is defined by $\tau_{12} = c(t_2' - t_1')$.

Suppose that the coordinate of the event A is described by

$$(x_1, ct_1) \text{ in the S frame and } (x_1'=0, ct_1') \text{ in the S' frame,}$$

and the coordinate of the event B is described by

$$(x_2, ct_2) \text{ in the S frame and } (x_2'=0, ct_2') \text{ in the S' frame,}$$

Then we have

$$(x_2 - x_1)^2 - c^2(t_2 - t_1)^2 = (x_2' - x_1')^2 - c^2(t_2' - t_1')^2 = -c^2(t_2' - t_1')^2$$

or

$$(x_2 - x_1)^2 = c^2(t_2 - t_1)^2 - c^2(t_2' - t_1')^2 \geq 0,$$

or

$$c^2(t_2 - t_1)^2 \geq c^2(t_2' - t_1')^2.$$

The time difference $(t_2 - t_1)$ is called the *coordinate time*. We consider the special case when $v = c$.

$$s_{12}^2 = s_{12}'^2 = -c^2(t_2' - t_1')^2 = 0.$$

In other words, no time passes in the S' frame moving with the velocity of light ($v = c$).

In general case

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2 = -c^2 d\tau^2 \quad [\text{the } (+++)-\text{signature pseudo-metric}]$$

or

$$d\tau = \sqrt{1 - \frac{v^2}{c^2}} dt,$$

where

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}.$$

Then the total proper time can be calculated as

$$\tau = \int d\tau = \int_{t_1}^{t_2} \sqrt{1 - \frac{v^2}{c^2}} dt.$$

24. Triangle inequality

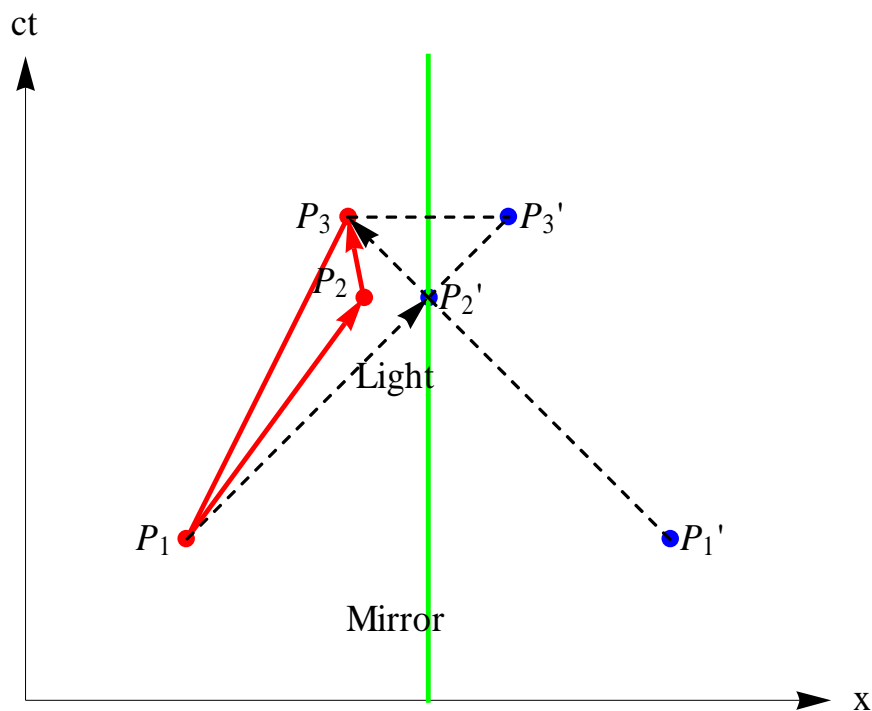


Fig.49 The world lines P_1P_2 , P_2P_3 , P_3P_1 are all future-time like. $\tau_{13} \geq \tau_{12} + \tau_{23}$ for the proper times. The world lines P_1P_2' and P_3P_2' are light-like; $\tau_{12'} = 0$ and $\tau_{2'3} = 0$. The mirror is set up so that the world lines P_1P_2' and P_3P_2' are light-like.

Suppose that there are three points P_1 , P_2 , and P_3 which are all time-like. When $t_1 < t_2 < t_3$, the proper times τ_{12} , τ_{23} , and τ_{31} satisfies the inequality

$$\tau_{13} \geq \tau_{12} + \tau_{23}.$$

In the above Fig. we put a mirror in the x axis. We choose a point P_2' such that the light emitted from the point P_1 is reflected by mirror and is reached the point P_3 . Since P_1P_2' and $P_2'P_3$ are the light world lines, we get

$$\tau_{12}' = \tau_{23}' = 0$$

which leads leading to

$$\tau_{13} \geq \tau_{12}' + \tau_{23}' = 0$$

The inequality indicates that the sum of the proper time along the straight line connecting between P_1 and P_3 is longer than that along any curve connecting between the points P_1 and P_3 .

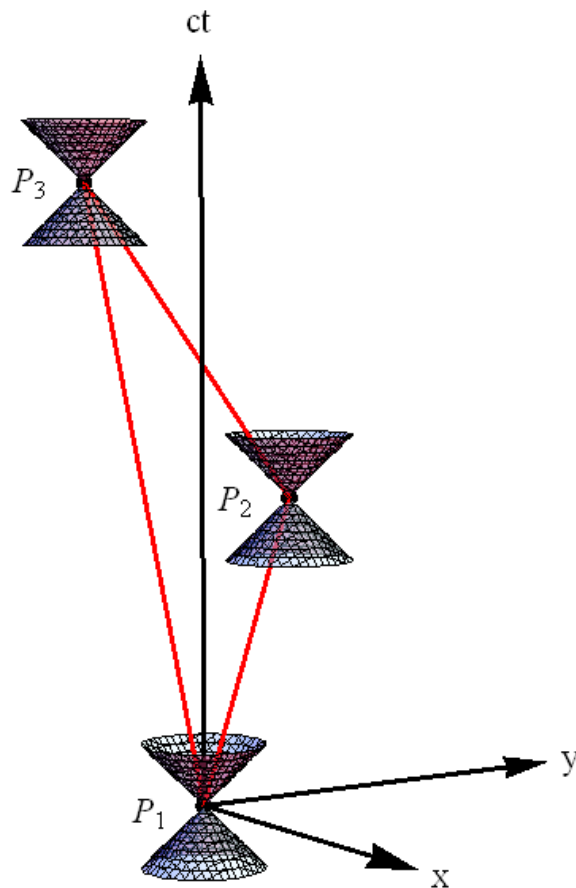


Fig.50 In Lorentz geometry, with P_1P_2 , P_2P_3 , P_3P_1 all future-time like, the inequality $\tau_{13} \geq \tau_{12} + \tau_{23}$ holds for the proper times. The interpretation of this is the so-called clock paradox.

((Roger Penrose))

The space traveler's world line is the broken path $P_1P_2P_3$, whereas the inhabitant of Earth have the world line P_1P_3 . We see that, according to the inequality, the space travel's clock indeed registers a shorter total elapsed time than those on Earth. Some people worry that the acceleration of the rocket ship is not properly accounted for in this description, and indeed I have idealized things so that the astronaut appears to be subjected to an impulsive (i.e. infinite) acceleration at the event P_3 (which ought to be fatal!). However, this issue is easily dealt with by simply smoothing over the corners of the triangle. The time difference is no greatly affected, as is obvious in the corresponding situation for the Euclidean smoothed-off triangle depicted in the above figure. It used to be frequently argued that it would be necessary to pass to Einstein's general relativity in order to handle acceleration, but this is completely wrong. The answer for the clock times is obtained using the formula $\int ds$ (with $ds > 0$) in both theories. The astronaut is allowed to accelerate in special relativity, just in general relativity. The distinction simply lies in what actual metric is being used in order to evaluate the quantity ds ; i.e., it depends on the actual g_{ij} . We are working in special relativity provided that this metric is the flat metric of Minkowski geometry. Physically, this means that the gravitational fields can be neglected. When we need to take the gravitational fields into account, we must introduce the curved metric of Einstein's general relativity.

25. Twin paradox

Tom and Jim are two identical twins. Jim goes on a space trip, moving through a galaxy at great velocity v , to return home after a long journey. Tom stays at home. At a certain time Jim comes back. Because Jim has been moving, his clock has been running more slowly and therefore for his less time has elapsed since he left. He will find his brother (Tom) much more aged than himself.

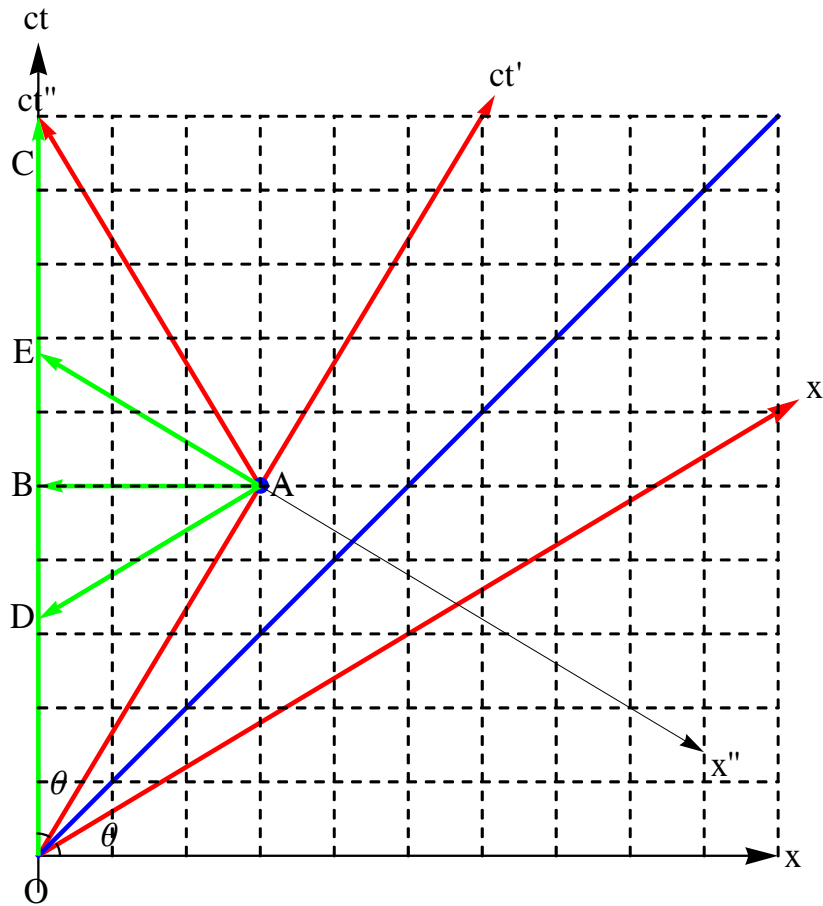


Fig.51

Jim gets on the rocket moving at the velocity v . In the above figure, the x axis (with the same ct') is parallel to the AD line (x' axis). At the event A , Jim jumps from his rocket to the other rocket which moves back to the Earth at the velocity of $-v$. The x'' axis (the same ct'') is parallel to the line AE . The rocket moves along the AC line and returns back to the Earth.

$$(\overline{OC})_s = \frac{2ct'}{\sqrt{1 - \frac{v^2}{c^2}}} > 2ct'$$

Note that

$$(\overline{OC})_s - (\overline{DE})_s = \frac{2ct'}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{2ct' \frac{v^2}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} = 2ct' \sqrt{1 - \frac{v^2}{c^2}} < 2ct'$$

$$T_0 = \int_0^T dt \sqrt{1 - \left[\frac{v(t)}{c}\right]^2},$$

while the twin that stayed home has aged by

$$T = \int_0^T dt > T_0$$

26. Pole and barn

In this figure, there is the black rest frame. The light green area is the barn, being at rest (S frame). The two black arrows (G and H) pointing up correspond to the world lines of the front and back doors of the barn. The pole (green line) is moving with a constant velocity (v) in the positive x axis. It is at rest with respect to the red frame (S' frame). The two red arrows pointing diagonally upward on the right represent the world lines of the endpoints of the pole.

In the S frame, the length is measured along horizontal equal time line, and we see that the pole fits exactly in the barn (B). Both endpoints are inside the barn. For the S' frame, the story is very different. At B, the front end of the pole reaches the black door, while the other endpoint of the pole has not yet entered the barn. The moving observer concludes correctly that the pole does not fit in the barn. The clue is thus that length measurements by definition involves the notion of simultaneity. Both observers spoke the truth, or at least, they spoke their own truth.

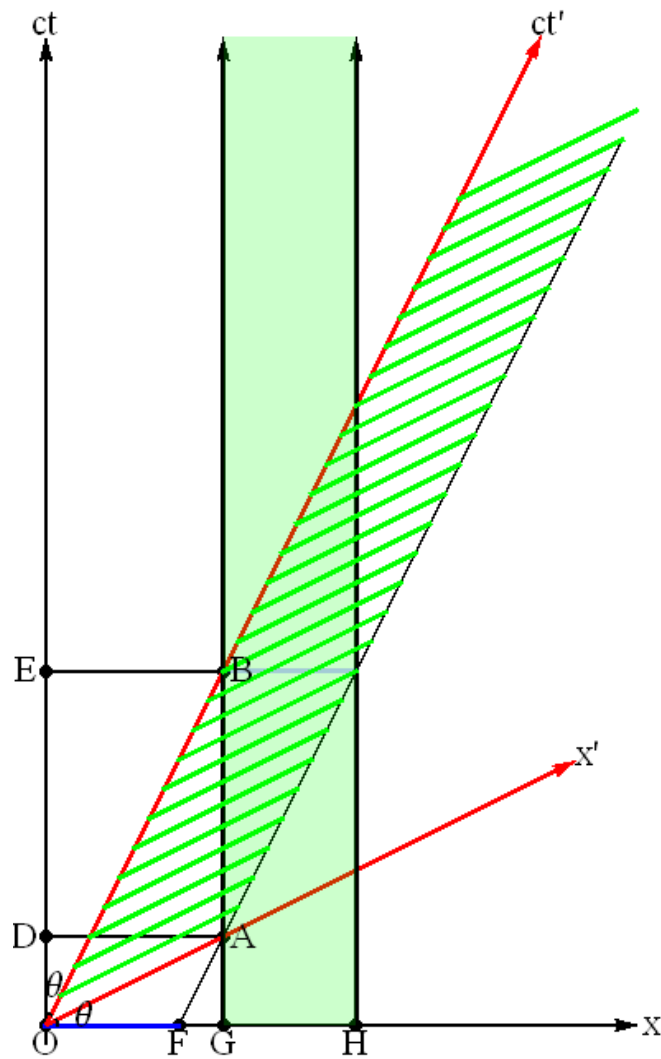


Fig.53 Barn and pole

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