Confining circle and Circular well

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The confining circle is the 2D analog of the spherical box and is also the zero-height, 2D version of the cylindrical box considered

1. Cylindrical co-ordinate system

The position of a point in the space having Cartesian coordinates x, y, and z may be expressed in terms of cylindrical co-ordinates

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z.$$

The position vector $\boldsymbol{r}$ is written as

$$\boldsymbol{r} = \rho\cos\phi\boldsymbol{e}_x + \rho\sin\phi\boldsymbol{e}_y + z\boldsymbol{e}_z$$

$$d\boldsymbol{r} = \sum_{j=1}^{3}\boldsymbol{e}_j h_j dq_j = \boldsymbol{e}_\rho d\rho + \boldsymbol{e}_\phi \rho d\phi + \boldsymbol{e}_z dz$$

where

$$h_1 = h_\rho = 1$$
$$h_2 = h_\phi = \rho$$
$$h_3 = h_z = 1$$

The unit vectors are written as

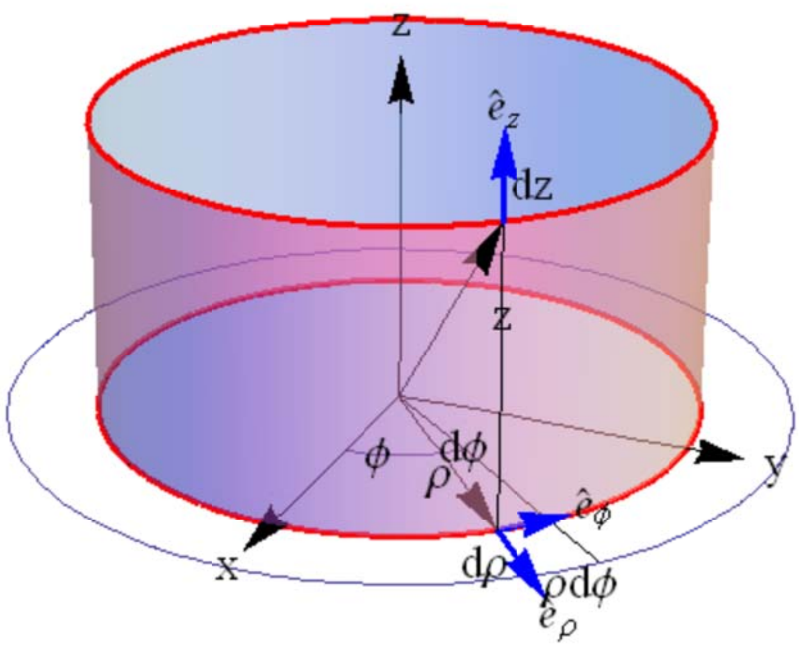
$$\boldsymbol{e}_\rho = \frac{1}{h_\rho}\frac{\partial \boldsymbol{r}}{\partial \rho} = \frac{\partial \boldsymbol{r}}{\partial \rho} = \cos\phi\boldsymbol{e}_x + \sin\phi\boldsymbol{e}_y$$
$$\boldsymbol{e}_\phi = \frac{1}{h_\phi}\frac{\partial \boldsymbol{r}}{\partial \phi} = \frac{1}{\rho}\frac{\partial \boldsymbol{r}}{\partial \phi} = -\sin\phi\boldsymbol{e}_x + \cos\phi\boldsymbol{e}_y$$
$$\boldsymbol{e}_z = \frac{1}{h_z}\frac{\partial \boldsymbol{r}}{\partial z} = \frac{\partial \boldsymbol{r}}{\partial z} = \boldsymbol{e}_z$$

The above expression can be described using a matrix A as

$$\begin{pmatrix} \boldsymbol{e}_\rho \\ \boldsymbol{e}_\phi \\ \boldsymbol{e}_z \end{pmatrix} = \boldsymbol{A} \begin{pmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{pmatrix}.$$

or by using the inverse matrix A^{-1} as

$$\begin{pmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{pmatrix} = \boldsymbol{A}^{-1} \begin{pmatrix} \boldsymbol{e}_\rho \\ \boldsymbol{e}_\phi \\ \boldsymbol{e}_z \end{pmatrix} = \boldsymbol{A}^T \begin{pmatrix} \boldsymbol{e}_\rho \\ \boldsymbol{e}_\phi \\ \boldsymbol{e}_z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_\rho \\ \boldsymbol{e}_\phi \\ \boldsymbol{e}_z \end{pmatrix}.$$



The differential operations involving ∇ are as follows.

$$\nabla\psi = \mathbf{e}_\rho \frac{\partial\psi}{\partial\rho} + \mathbf{e}_\phi \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_z \frac{\partial\psi}{\partial z},$$

$$\nabla\cdot\boldsymbol{V} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho V_\rho) + \frac{1}{\rho}\frac{\partial}{\partial\phi}V_\phi + \frac{\partial}{\partial z}V_z,$$

$$\nabla \times \boldsymbol{V} = \frac{1}{\rho}\begin{vmatrix} \boldsymbol{e}_\rho & \rho\boldsymbol{e}_\phi & \boldsymbol{e}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ V_\rho & \rho V_\phi & V_z \end{vmatrix},$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial\psi}{\partial\rho}) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2},$$

where $\boldsymbol{V}$ is a vector and ψ is a scalar.

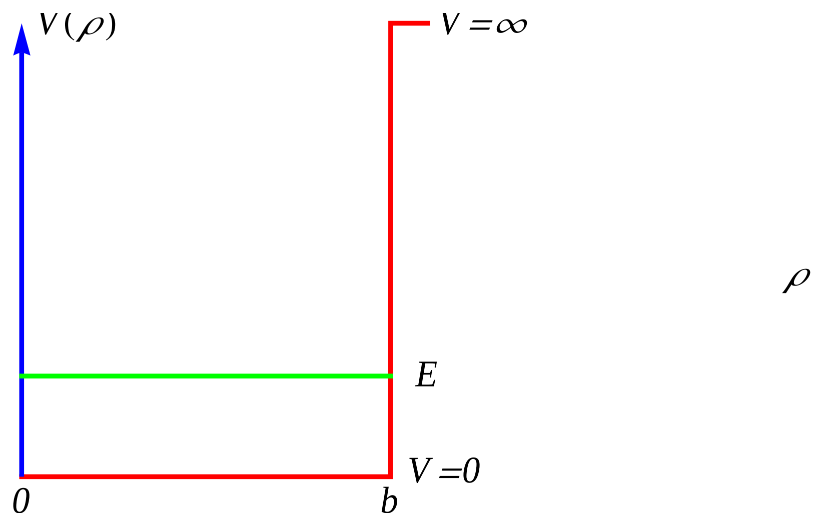
2. Schrödinger equation in the Confining circle (the 2D plane)

We consider the Schrödinger equation

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi + V\psi = E\psi,$$

where the wavefunction depends only on ρ and ϕ,

$$\psi = \psi(\rho, \phi)$$



The potential energy is defined by

$$V = 0 \quad \text{for } \rho < b, \qquad V = \infty \quad \text{for } \rho > b,$$

Using the cylindrical co-ordinate, the differential equation for ψ can be written as

$$-\frac{\hbar^2}{2\mu}[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial\psi}{\partial\rho}) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2}] = E\psi.$$

The angular momentum $\hat{L}_z$ is defined by

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}.$$

We note that under the rotation around the *z* axis,

$$\langle\psi'|\hat{H}|\psi'\rangle = \langle\psi|\hat{H}|\psi\rangle,$$

where

$$|\psi'\rangle = \hat{R}|\psi\rangle = (1 - \frac{i}{\hbar}\hat{J}_z\delta\phi)|\psi\rangle.$$

Then we have

$$[\hat{H}, \hat{J}_z] = 0.$$

So the wavefunction is the simultaneous eigenket of both H and L_z (orbital angular momentum)

$$\psi(\rho,\phi) = R(\rho)\Phi(\phi), \qquad \text{(separation variable)}$$

with

$$\frac{\hbar}{i}\frac{\partial}{\partial\phi}\Phi(\phi) = m\hbar\Phi(\phi), \qquad \Phi(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi}$$

where m is an integer; m = 0, ±1, ±2,… The radial wavefunction satisfies the differential equation given by

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial R}{\partial\rho}) - \frac{m^2}{\rho^2}R + \frac{2\mu E}{\hbar^2}R = 0,$$

or

$$(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{m^2}{\rho^2} + k^2)R(\rho) = 0, \qquad \text{for } 0 < \rho < b$$

with

$$E = \frac{\hbar^2 k^2}{2\mu} = \frac{\hbar^2}{2\mu b^2}(kb)^2.$$

where the boundary condition is given by $R(b) = 0$.

((**Note**)) We put $x = k\rho$. Then we get

$$(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + 1 - \frac{m^2}{x^2})R(x) = 0$$

The solution of this differential equation is $J_m(x)$, and $N_m(x)$.

Then the solution of the differential equation is given by

$$R(\rho) = C_1 J_m(k\rho) + C_2 N_m(k\rho),$$

where $J_m(x)$ is the Bessel function and $N_m(x)$ is the Neumann function. Note that $N_m(x)$ becomes infinite at $x = 0$. Thus we remove the Neumann function from the solution. Then we have

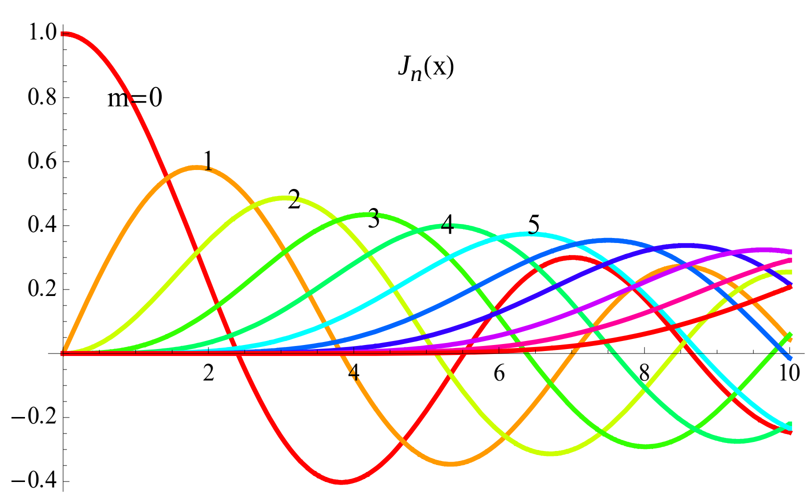
$$R(\rho) = C_1 J_m(k\rho).$$

This function satisfies the boundary condition such that

$$J_m(k_{mk} b) = 0,$$

where

$$E_{mk} = \frac{\hbar^2}{2\mu b^2}(k_{mk} b)^2,$$



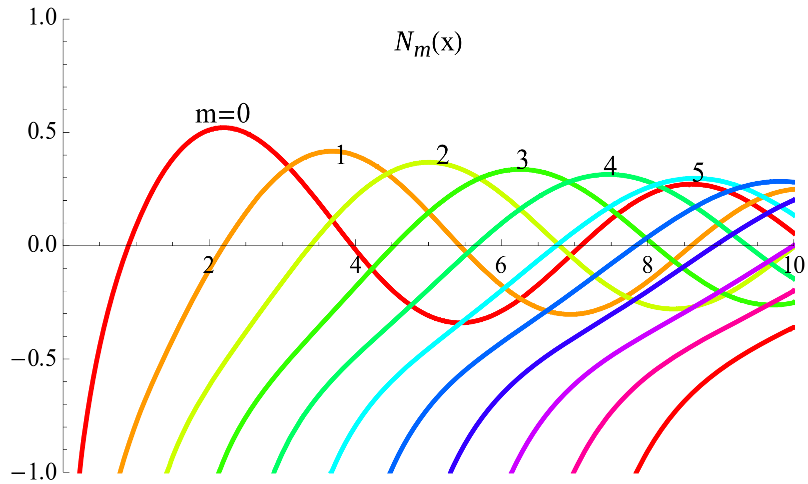


Table values of $Z_{mk} = k_{mk}b$ that yield $J_m(k_{mk}b) = 0$

$$E_{mk} = \frac{\hbar^2}{2\mu b^2} Z_{mk}{}^2$$

k	m=0	m=1	m=2	m=3
1.	2.40483	3.83171	5.13562	6.38016
2.	5.52008	7.01559	8.41724	9.76102
3.	8.65373	10.1735	11.6198	13.0152
4.	11.7915	13.3237	14.796	16.2235
5.	14.9309	16.4706	17.9598	19.4094
6.	18.0711	19.6159	21.117	22.5827
7.	21.2116	22.7601	24.2701	25.7482
8.	24.3525	25.9037	27.4206	28.9084
9.	27.4935	29.0468	30.5692	32.0649
10.	30.6346	32.1897	33.7165	35.2187

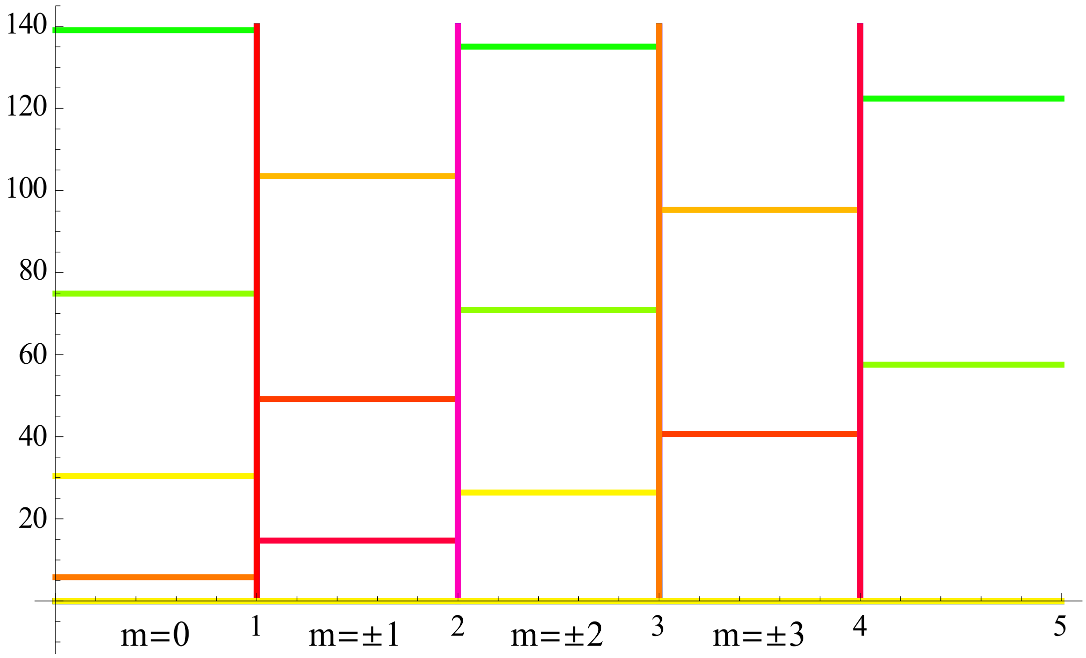
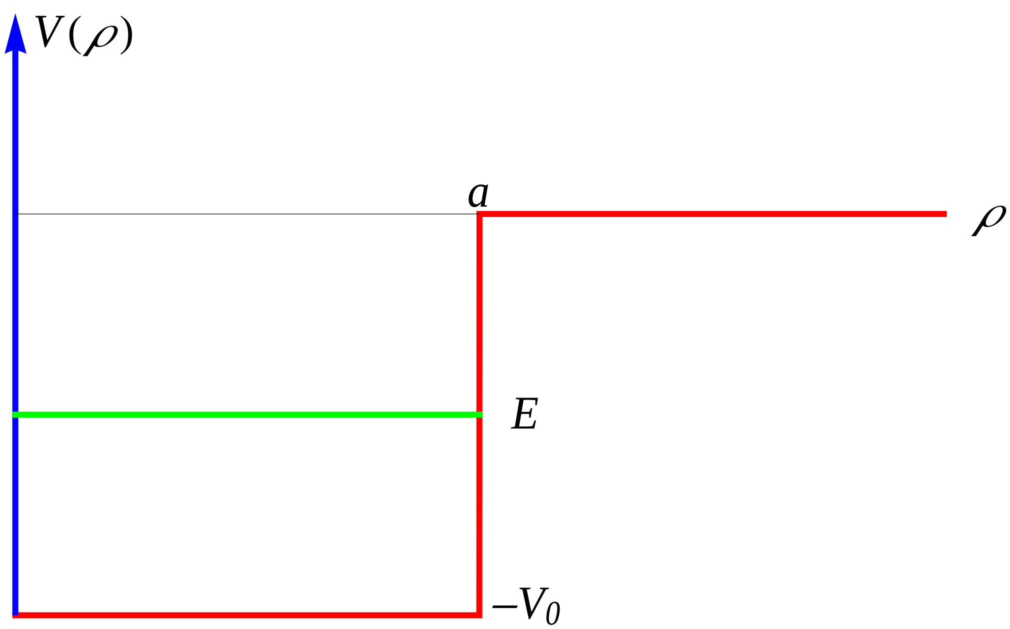


Fig. Plot of the normalized energy $\frac{2\mu b^2 E_{mn}}{\hbar^2}$ for each m. $m = \pm 0, \pm 1, \pm 2, \pm 3, \pm 4,\ldots$.

3. Bound states in the 2D square well



((**Schiff**))

It is shown that a 1D square well potential has a bound state for any positive V_0a^2, and that a 3D square well potential has a bound state only for $V_0a^2 > \frac{\pi^2\hbar^2}{8\mu}$. What is the analogous situation for a 2D square well potential? What, if any, is the physical significance of these results?

We start with the Schrodinger equation

$$-\frac{\hbar^2}{2\mu}[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial\psi}{\partial\rho})+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2}]+V(\rho)\psi = E\psi ,$$

where

$$V(\rho) = -V_0 \qquad \text{for } \rho < a\text{, and 0 for } \rho > a .$$

(a) $\rho < a$

We have the differential equation

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial\psi}{\partial\rho})+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} = -\frac{2\mu}{\hbar^2}(E+V_0)\psi = -k^2\psi ,$$

where

$$E + V_0 = \frac{\hbar^2k^2}{2\mu} .$$

The wavefunction is the simultaneous eigenket of both H and L_z.

$$\psi(\rho,\phi) = R(\rho)\Phi(\phi) , \qquad \text{(separation variable)}$$

with

$$\frac{\hbar}{i}\frac{\partial}{\partial\phi}\Phi(\phi) = m\hbar\Phi(\phi) , \qquad \Phi(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi} ,$$

where m is integer; m = 0, ±1, ±2,… The radial wavefunction satisfies the differential equation given by

$$(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\frac{m^2}{\rho^2}+k^2)R(\rho) = 0 . \qquad \text{for } 0 < \rho < a$$

The solution of this differential equation is obtained as

$$R(\rho) = C_m J_m(k\rho). \tag{1}$$

(b) $\rho > a$

We have the differential equation

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial\psi}{\partial\rho}) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} = -\frac{2\mu}{\hbar^2}E\psi = \kappa^2\psi \, ,$$

or

$$(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{m^2}{\rho^2} - \kappa^2)R(\rho) = 0 \, ,$$

where

$$\kappa^2 = -\frac{2\mu}{\hbar^2}E \, .$$

((**Note**)) We put $x = k\rho$. Then we get

$$(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - 1 - \frac{m^2}{x^2})R(x) = 0$$

The solution of this differential equation is $I_m(x)$, and $K_m(x)$, which are the modified Bessel function of the first and second kind. BesselI[n,x]. BesselK[n,x].

The solution of this differential equation is obtained as the modified Bessel function

$$R(\rho) = A_m I_m(\kappa\rho) + B_m K_m(\kappa\rho) \, .$$

Note that only $K_m(\kappa\rho)$ becomes zero for large $\kappa\rho$. So our solution for $\rho > a$ is given by

$$R(\rho) = B_m K_m(\kappa\rho).$$

Using the boundary condition at $\rho = a$, we determine the energy eigenvalues. We note that the wave function and its derivative should be continuous at $\rho = a$.

$$A_m J_m(ka) = B_m K_m(\kappa\rho) \, ,$$

$$A_m k J_m'(ka) = B_m \kappa K_m'(\kappa\rho),$$

or

$$ka\frac{1}{J_m(x)}\frac{\partial J_m(x)}{\partial x}|_{x=ka} = \kappa a\frac{1}{K_m(x)}\frac{\partial K_m(x)}{\partial x}|_{x=\kappa a} \qquad (1)$$

where

$$\xi = ka, \qquad \eta = \kappa a.$$

From the conditions of

$$E = -\frac{\hbar^2}{2\mu}\kappa^2, \text{ and } \qquad E + V_0 = \frac{\hbar^2}{2\mu}k^2,$$

we have

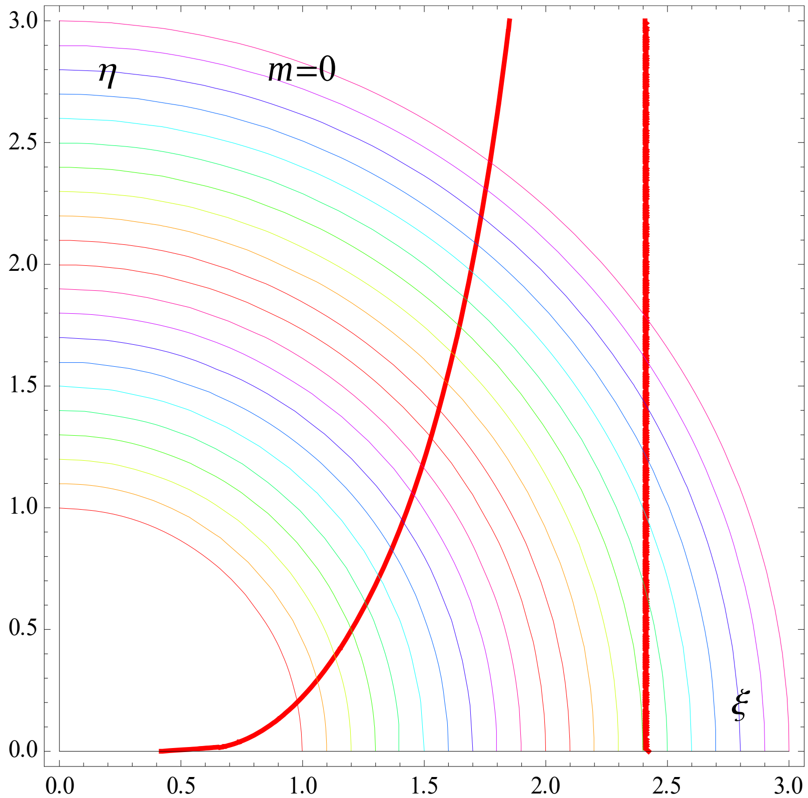
$$(ka)^2 + (\kappa a)^2 = \frac{2\mu V_0}{\hbar^2}a^2 = r_0^{\ 2},$$

or

$$\xi^2 + \eta^2 = \frac{2\mu V_0}{\hbar^2}a^2 = r_0^{\ 2}. \qquad (2)$$

We solve the problem using the graphs. These graphs can be drawn in the (ξ, η) plane by using the Mathematica (ContourPlot), where the radius r_0 is changed as a parameter.

4. 2D square well with *m* = 0



The bound state of the 2D square well can be found

$$\frac{2\mu V_0}{\hbar^2}a^2 = r_0^2 > (0.6)^2 = 0.36, \qquad \text{(2D case)}$$

for $m = 0$. This is in contrast with the case of the bound state for the 3D square well with $m = 0$. The bound state occurs when

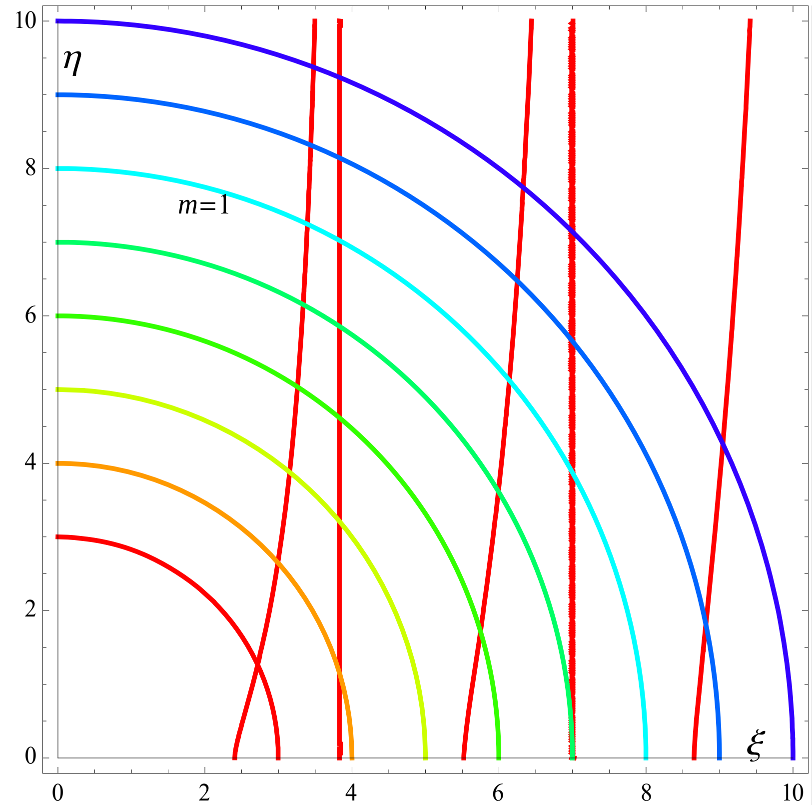
$$\frac{2\mu V_0 a^2}{\hbar^2} > \left(\frac{\pi}{2}\right)^2 = 2.4674. \qquad \text{(3D case)}$$

As shown in the APPENDIX, the bound state of the 1D square well can occur for any positive value of V_0;

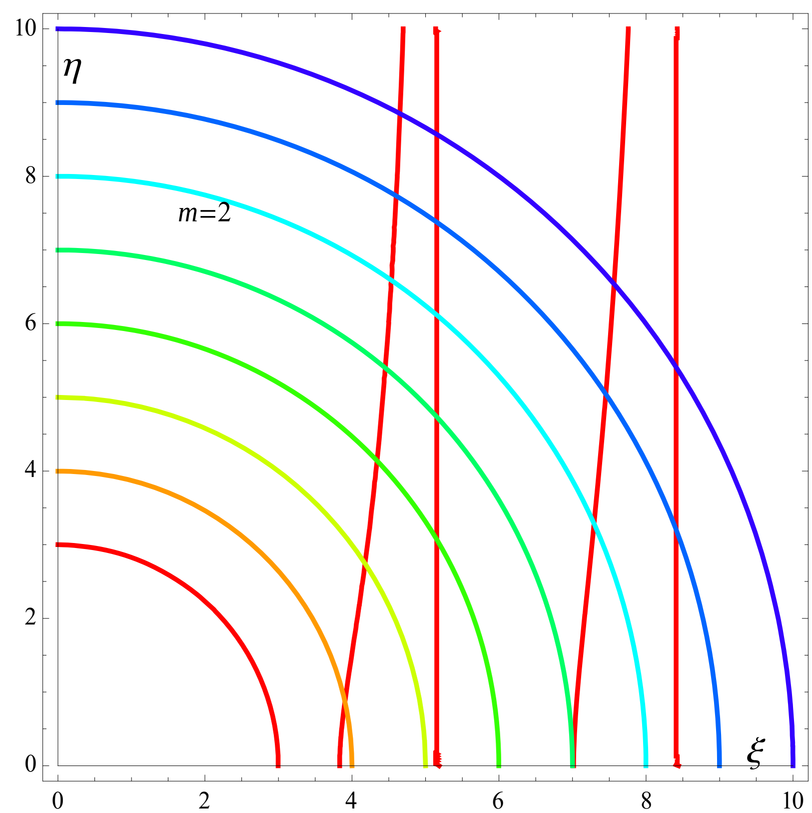
$$\frac{2\mu V_0 a^2}{\hbar^2} > 0\,. \qquad \text{(1D case)}$$

5. 2D square well with $m = 1$ and 2

(a) $m = 1$



(b) $m = 2$



APPENDIX-I
The bound state of 1D symmetric square well

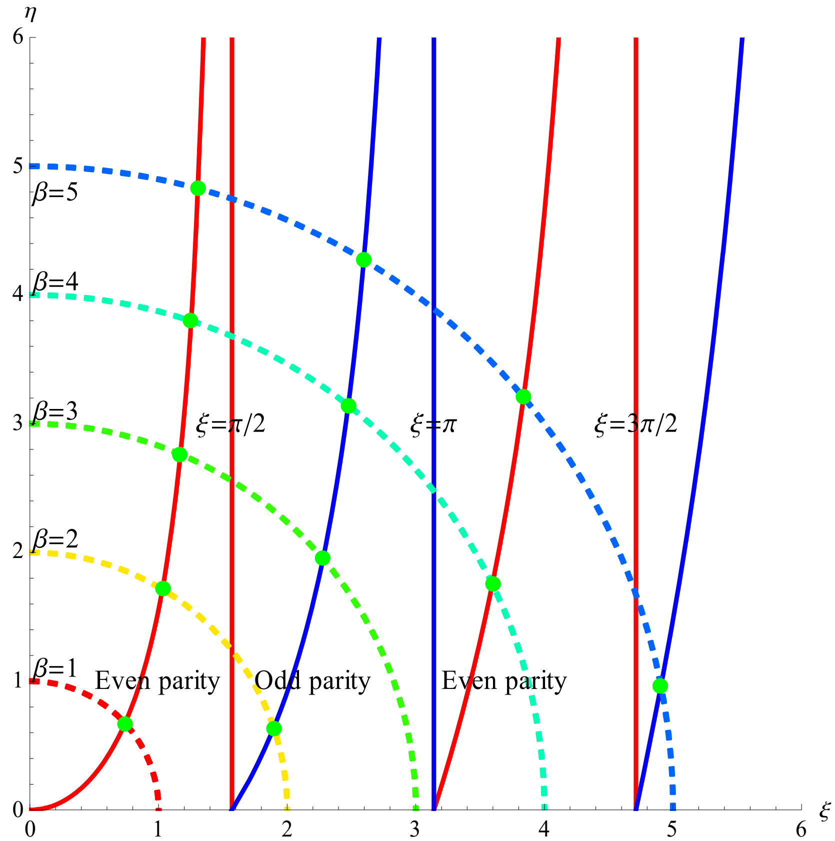


Fig. Graphical solution. One solution with even parity for $0<\beta<\pi/2$. One solution with even parity and one solution with odd parity for $\pi/2<\beta<\pi$. Two solutions with even parity and one solution with odd parity for $\pi<\beta<3\pi/2$. Two solutions with even parity and two solutions with odd parity for $3\pi/2<\beta<2\pi$. $\eta=\xi\tan\xi$ for the even parity (red lines). $\eta=-\xi\cot\xi$ for the odd parity (blue lines). The circles are denoted by $\xi^2+\eta^2=\beta^2$. The parameter β is changed as β = 1, 2, 3, 4, and 5. $\beta=\sqrt{\dfrac{2mV_0a^2}{\hbar^2}}$. $\varepsilon=\dfrac{|E|}{V_0}=\dfrac{\eta^2}{\beta^2}=1-\dfrac{\xi^2}{\beta^2}$. $\xi = ka$ and $\eta = \kappa a$.

APPENDIX-II Vector analysis in the cylindrical co-ordinates

$$\nabla\psi(\rho,\phi,z)=\boldsymbol{e}_\rho\frac{\partial\psi}{\partial\rho}+\boldsymbol{e}_\phi\frac{1}{\rho}\frac{\partial\psi}{\partial\phi}+\boldsymbol{e}_z\frac{\partial\psi}{\partial z},$$

$$\nabla\cdot\boldsymbol{V}=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho V_\rho)+\frac{1}{\rho}\frac{\partial}{\partial\phi}(V_\phi)+\frac{\partial}{\partial z}(V_z)$$

$$\nabla^2\psi(\rho,\phi,\mathrm{z}) = \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial \mathrm{z}^2}$$
$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial\psi}{\partial\rho}) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial \mathrm{z}^2}$$

$$\nabla\times\boldsymbol{V} = \frac{1}{\rho}\begin{vmatrix} \boldsymbol{e}_\rho & \rho\boldsymbol{e}_\phi & \boldsymbol{e}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial \mathrm{z}} \\ V_\rho & \rho V_\phi & V_z \end{vmatrix}$$

$$\begin{aligned}
\boldsymbol{L}\psi &= \frac{\hbar}{i}\boldsymbol{r}\times\nabla\psi(\rho,\phi,\mathrm{z}) \\
&= \frac{\hbar}{i}(\boldsymbol{e}_\rho\rho + \boldsymbol{e}_z \mathrm{z})\times(\boldsymbol{e}_\rho\frac{\partial\psi}{\partial\rho} + \boldsymbol{e}_\phi\frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \boldsymbol{e}_z\frac{\partial\psi}{\partial \mathrm{z}}) \\
&= \frac{\hbar}{i}\rho(\boldsymbol{e}_\rho\times\boldsymbol{e}_\rho\frac{\partial\psi}{\partial\rho} + \boldsymbol{e}_\rho\times\boldsymbol{e}_\phi\frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \boldsymbol{e}_\rho\times\boldsymbol{e}_z\frac{\partial\psi}{\partial \mathrm{z}}) + \frac{\hbar}{i}\mathrm{z}(\boldsymbol{e}_z\times\boldsymbol{e}_\rho\frac{\partial\psi}{\partial\rho} + \boldsymbol{e}_z\times\boldsymbol{e}_\phi\frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \boldsymbol{e}_z\times\boldsymbol{e}_z\frac{\partial\psi}{\partial \mathrm{z}}) \\
&= \frac{\hbar}{i}(\boldsymbol{e}_z\frac{\partial\psi}{\partial\phi} - \boldsymbol{e}_\phi\rho\frac{\partial\psi}{\partial \mathrm{z}}) + \frac{\hbar}{i}(\mathrm{z}\boldsymbol{e}_\phi\frac{\partial\psi}{\partial\rho} - \boldsymbol{e}_\rho\frac{\mathrm{z}}{\rho}\frac{\partial\psi}{\partial\phi}) \\
&= i\hbar[\boldsymbol{e}_\rho\frac{\mathrm{z}}{\rho}\frac{\partial\psi}{\partial\phi} + \boldsymbol{e}_\phi(\rho\frac{\partial}{\partial \mathrm{z}} - \mathrm{z}\frac{\partial}{\partial\rho})\psi - \boldsymbol{e}_z\frac{\partial\psi}{\partial\phi}]
\end{aligned}$$

$$\boldsymbol{L}\psi(\rho,\phi,\mathrm{z}) = i\hbar[\boldsymbol{e}_\rho\frac{\mathrm{z}}{\rho}\frac{\partial\psi}{\partial\phi} + \boldsymbol{e}_\phi(\rho\frac{\partial}{\partial \mathrm{z}} - \mathrm{z}\frac{\partial}{\partial\rho})\psi - \boldsymbol{e}_z\frac{\partial\psi}{\partial\phi}]$$

$$L_z\psi(\rho,\phi,\mathrm{z}) = \boldsymbol{e}_z\cdot\boldsymbol{L}\psi(\rho,\phi,\mathrm{z}) = -i\hbar\frac{\partial\psi}{\partial\phi}$$

$$\boldsymbol{L}^2\psi(\rho,\mathrm{z},\phi) = \hbar^2\{2\mathrm{z}\frac{\partial\psi}{\partial \mathrm{z}} + \frac{1}{\rho^2}[-\rho^4\frac{\partial^2\psi}{\partial \mathrm{z}^2} - (\mathrm{z}^2+\rho^2)\frac{\partial^2\psi}{\partial\phi^2}$$
$$+\rho[(-\mathrm{z}^2+\rho^2)\frac{\partial\psi}{\partial\rho} + \mathrm{z}\rho(2\rho\frac{\partial^2\psi}{\partial\rho\partial \mathrm{z}} - \mathrm{z}\frac{\partial^2\psi}{\partial\rho^2})\}$$

((**Mathematica**))　　**Cylindrical co-ordinates**

Vector analysis
Angular momentum in the cylindrical coordinates
Here we use the angular momentum operatior in the unit of ℏ=1

```
Clear["Global`"]; r1 = Sqrt[z^2 + ρ^2];
ux = {Cos[ϕ], -Sin[ϕ], 0};
uy = { Sin[ϕ], Cos[ϕ], 0}; uz = {0, 0, 1};
r = {ρ, 0, z}; ur = 1/r1 {ρ, 0, z};
Gra :=
 Grad[#, {ρ, ϕ, z}, "Cylindrical"] &;
Lap :=
 Laplacian[#, {ρ, ϕ, z},
   "Cylindrical"] &;
Curla :=
 Curl[#, {ρ, ϕ, z}, "Cylindrical"] &;
Diva :=
 Div[#, {ρ, ϕ, z}, "Cylindrical"] &;
```

Vector analysis in the cylindrical coordinate

```
eq1 = Lap[ψ[ρ, ϕ, z]] // Simplify
```

$$\psi^{(0,0,2)}[\rho, \phi, z] + \frac{\psi^{(0,2,0)}[\rho, \phi, z]}{\rho^2} + \frac{\psi^{(1,0,0)}[\rho, \phi, z]}{\rho} + \psi^{(2,0,0)}[\rho, \phi, z]$$

```
eq2 = Gra[ψ[ρ, ϕ, z]] // Simplify
```

$$\left\{\psi^{(1,0,0)}[\rho, \phi, z], \frac{\psi^{(0,1,0)}[\rho, \phi, z]}{\rho}, \psi^{(0,0,1)}[\rho, \phi, z]\right\}$$

```
B = {Bρ[ρ, ϕ, z], Bϕ[ρ, ϕ, z],
    Bz[ρ, ϕ, z]};
      eq3 = Curla[B] // Simplify
```

$$\left\{-B\phi^{(0,0,1)}[\rho, \phi, z] + \frac{Bz^{(0,1,0)}[\rho, \phi, z]}{\rho}, B\rho^{(0,0,1)}[\rho, \phi, z] - Bz^{(1,0,0)}[\rho, \phi, z], \frac{1}{\rho}\left(B\phi[\rho, \phi, z] - B\rho^{(0,1,0)}[\rho, \phi, z] + \rho\, B\phi^{(1,0,0)}[\rho, \phi, z]\right)\right\}$$

```
      eq4 = Diva[B] // Simplify
```

$$Bz^{(0,0,1)}[\rho, \phi, z] + \frac{B\rho[\rho, \phi, z] + B\phi^{(0,1,0)}[\rho, \phi, z]}{\rho} + B\rho^{(1,0,0)}[\rho, \phi, z]$$

Angular momentum in the cylindrical coordinate

```
L := (-ⅈ Cross[r, Gra[#]]) &;
Lx := (ux.L[#] &) // Simplify;
Ly := (uy.L[#] &) // Simplify;
Lz := (uz.L[#] &) // Simplify;
```

```
L[ψ[ρ, ϕ, z]] // Simplify
```

$$\left\{\frac{i\, z\, \psi^{(0,1,0)}[\rho, \phi, z]}{\rho},\; i\left(\rho\, \psi^{(0,0,1)}[\rho, \phi, z] - z\, \psi^{(1,0,0)}[\rho, \phi, z]\right),\; -i\, \psi^{(0,1,0)}[\rho, \phi, z]\right\}$$

```
Lx[ψ[ρ, ϕ, z]] // FullSimplify
```

$$i\left(-\rho\, \mathrm{Sin}[\phi]\, \psi^{(0,0,1)}[\rho, \phi, z] + \frac{z\, \mathrm{Cos}[\phi]\, \psi^{(0,1,0)}[\rho, \phi, z]}{\rho} + z\, \mathrm{Sin}[\phi]\, \psi^{(1,0,0)}[\rho, \phi, z]\right)$$

Ly[ψ[ρ, ϕ, z]] // FullSimplify

$$i\left(\rho\,\mathrm{Cos}[\phi]\,\psi^{(0,0,1)}[\rho,\phi,z]+\frac{z\,\mathrm{Sin}[\phi]\,\psi^{(0,1,0)}[\rho,\phi,z]}{\rho}-z\,\mathrm{Cos}[\phi]\,\psi^{(1,0,0)}[\rho,\phi,z]\right)$$

Lx[ψ[ρ, ϕ, z]] + i Ly[ψ[ρ, ϕ, z]] // FullSimplify

$$\frac{1}{\rho}e^{i\phi}\left(-\rho^2\,\psi^{(0,0,1)}[\rho,\phi,z]+i\,z\,\psi^{(0,1,0)}[\rho,\phi,z]+z\,\rho\,\psi^{(1,0,0)}[\rho,\phi,z]\right)$$

Lx[ψ[ρ, ϕ, z]] − i Ly[ψ[ρ, ϕ, z]] // FullSimplify

$$\frac{1}{\rho}e^{-i\phi}\left(\rho^2\,\psi^{(0,0,1)}[\rho,\phi,z]+i\,z\,\psi^{(0,1,0)}[\rho,\phi,z]-z\,\rho\,\psi^{(1,0,0)}[\rho,\phi,z]\right)$$

Lz[ψ[ρ, ϕ, z]] // Simplify

$$-i\,\psi^{(0,1,0)}[\rho,\phi,z]$$

The commutation of the angular momentum in the cylindrical coordinate

eq5 = Lx[Ly[ψ[ρ, ϕ, z]]] − Ly[Lx[ψ[ρ, ϕ, z]]] − i Lz[ψ[ρ, ϕ, z]] // Simplify

0

L^2 in the cylindrical coordinate

```
eq6 =
 Lx[Lx[ψ[ρ, ϕ, z]]] +
   Ly[Ly[ψ[ρ, ϕ, z]]] +
   Lz[ Lz[ψ[ρ, ϕ, z]]] // FullSimplify
```

$$2\, z\, \psi^{(0,0,1)}[\rho, \phi, z] + \frac{1}{\rho^2}\Big(-\rho^4\, \psi^{(0,0,2)}[\rho, \phi, z] - \left(z^2+\rho^2\right) \psi^{(0,2,0)}[\rho, \phi, z] + \rho\,\Big(\left(-z^2+\rho^2\right)\psi^{(1,0,0)}[\rho, \phi, z] + z\,\rho\,\big(2\,\rho\,\psi^{(1,0,1)}[\rho, \phi, z] - z\,\psi^{(2,0,0)}[\rho, \phi, z]\big)\Big)\Big)$$