2D isotropic simple harmonics: operator method

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Here we discuss the eigenstates of 2D isotropic simple harmonics using the creation and annihilation operators. The eigenstates are the simultaneous ones of the Hamiltonian and angular momentum.

1. Hamiltonian and angular momentum in terms of creation operator and annihilation operator and angular momentum

We consider the Hamiltonian for the 2D motion of a particle of mass μ in a isotropic harmonic oscillator potential

$$\begin{aligned}\hat{H} &= \\ &= \frac{1}{2\mu}(\hat{p}_x{}^2 + \hat{p}_y{}^2) + \frac{1}{2}\mu\omega_0{}^2(\hat{x}^2 + \hat{y}^2) \\ &= (\frac{1}{2\mu}\hat{p}_x{}^2 + \frac{1}{2}\mu\omega_0{}^2\hat{x}^2) + (\frac{1}{2\mu}\hat{p}_y{}^2 + \frac{1}{2}\mu\omega_0{}^2\hat{y}^2) \\ &= \hat{H}_x + \hat{H}_y\end{aligned}$$

We introduce the creation and annihilation operators such that

$$\hat{x} = \frac{1}{\sqrt{2}\beta}\left(\hat{a}_1 + \hat{a}_1{}^+\right) = \sqrt{\frac{\hbar}{2\mu\omega_0}}\left(\hat{a}_1 + \hat{a}_1{}^+\right),$$

$$\hat{y} = \frac{1}{\sqrt{2}\beta}\left(\hat{a}_2 + \hat{a}_2{}^+\right) = \sqrt{\frac{\hbar}{2\mu\omega_0}}\left(\hat{a}_2 + \hat{a}_2{}^+\right),$$

$$\hat{p}_x = \frac{1}{\sqrt{2}\beta}\frac{\mu\omega_0}{i}\left(\hat{a}_1 - \hat{a}_1{}^+\right) = \frac{1}{i}\sqrt{\frac{\mu\hbar\omega_0}{2}}\left(\hat{a}_1 - \hat{a}_1{}^+\right),$$

$$\hat{p}_y = \frac{1}{\sqrt{2}\beta}\frac{\mu\omega_0}{i}\left(\hat{a}_2 - \hat{a}_2{}^+\right) = \frac{1}{i}\sqrt{\frac{\mu\hbar\omega_0}{2}}\left(\hat{a}_2 - \hat{a}_2{}^+\right),$$

where

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \qquad \hat{a}^+|n\rangle = \sqrt{n+1}|n+1\rangle,$$

$$\hat{a}|0\rangle = 0, \qquad \hat{a}|1\rangle = |0\rangle, \qquad \hat{a}|2\rangle = \sqrt{2}|1\rangle, \qquad \hat{a}|3\rangle = \sqrt{3}|2\rangle,$$

$$\hat{a}^+|0\rangle = |1\rangle, \qquad \hat{a}^+|1\rangle = \sqrt{2}|2\rangle, \ \hat{a}^+|2\rangle = \sqrt{3}|3\rangle, \qquad \hat{a}^+|3\rangle = 2|4\rangle.$$

Then the Hamiltonian van be written as

$$\begin{aligned}
\hat{H} &= \frac{1}{2\mu}(\hat{p}_x^2 + \hat{p}_y^2) + \frac{1}{2}\mu\omega_0^2(\hat{x}^2 + \hat{y}^2) \\
&= -\frac{\hbar\omega_0}{4}[\left(\hat{a}_1 - \hat{a}_1^{\,+}\right)\left(\hat{a}_1 - \hat{a}_1^{\,+}\right) + \left(\hat{a}_2 - \hat{a}_2^{\,+}\right)\left(\hat{a}_2 - \hat{a}_2^{\,+}\right)] \\
&\quad + \frac{\hbar\omega_0}{4}[\left(\hat{a}_1 + \hat{a}_1^{\,+}\right)\left(\hat{a}_1 + \hat{a}_1^{\,+}\right) + \left(\hat{a}_2 + \hat{a}_2^{\,+}\right)\left(\hat{a}_2 + \hat{a}_2^{\,+}\right)] \\
&= \hbar\omega_0(\hat{a}_1^{\,+}\hat{a}_1 + \hat{a}_2^{\,+}\hat{a}_2 + \hat{1}) \\
&= \hbar\omega_0(\hat{N}_1 + \hat{N}_2 + \hat{1})
\end{aligned}$$

where

$$[\hat{a}_1, \hat{a}_1^{\,+}] = \hat{1}, \qquad [\hat{a}_2, \hat{a}_2^{\,+}] = \hat{1}.$$

The energy eigenstate is defined by

$$\begin{aligned}
\hat{H}|n_1, n_2\rangle &= \hbar\omega_0(\hat{N}_1 + \hat{N}_2 + \hat{1})|n_1, n_2\rangle \\
&= \hbar\omega_0(n+1)|n_1, n_2\rangle
\end{aligned}$$

where

$$E(n) = \hbar\omega_0(n+1),$$

with integer n (=0, 1, 2, …) such that

$$n_1 + n_2 = n.$$

The angular momentum is also written as

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$
$$= \frac{\hbar}{2i}(\hat{a}_1 + \hat{a}_1^+)(\hat{a}_2 - \hat{a}_2^+) - \frac{\hbar}{2i}(\hat{a}_2 + \hat{a}_2^+)(\hat{a}_1 - \hat{a}_1^+)$$
$$= i\hbar(\hat{a}_1\hat{a}_2^+ - \hat{a}_1^+\hat{a}_2)$$

From a symmetry argument, we find that $[\hat{H}, \hat{L}_z] = 0$. Then we have a simultaneous eigenket of $\hat{H}$ and $\hat{L}_z$ such that

$$\hat{H}|\psi\rangle = E|\psi\rangle \quad \text{and} \quad \hat{L}_z|\psi\rangle = m\hbar|\psi\rangle .$$

2. The eigenstates of $E = 2\hbar\omega$ with the combination of states $|10\rangle, |01\rangle$)

When $E = 2\hbar\omega_0$, there are two states $|n_1 = 1, n_2 = 0\rangle$ and $|n_1 = 0, n_2 = 1\rangle$. These two states are degenerate states with the same energy. The combination of these two states leads to the eigenstate of $\hat{H}$.

$$\hat{L}_z|10\rangle = i\hbar(\hat{a}_1\hat{a}_2^+ - \hat{a}_1^+\hat{a}_2)|10\rangle = i\hbar\hat{a}_1\hat{a}_2^+|10\rangle = i\hbar|01\rangle$$

$$\hat{L}_z|01\rangle = i\hbar(\hat{a}_1\hat{a}_2^+ - \hat{a}_1^+\hat{a}_2)|01\rangle = (-i\hbar)\hat{a}_1^+\hat{a}_2)|01\rangle = -i\hbar|10\rangle$$

$$\hat{L}_z = \hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

under the basis of $|10\rangle$ and $|01\rangle$. The eigenstate and eigenvalue for $\hat{L}_z$ is evaluated with the use of Mathematica.

For $m = 1$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}[|10\rangle + i|01\rangle\} .$$

For $m = -1$

$$|\psi_{-1}\rangle = \frac{1}{\sqrt{2}}[|10\rangle - i|01\rangle\} .$$

((**Note**))

We assume that

$$|\psi\rangle = c_1|1,0\rangle + c_2|0,1\rangle = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Since $[\hat{H}, \hat{L}_z] = 0$, $|\psi\rangle$ is also the eigenstate of $\hat{L}_z$;

$$\hat{L}_z|\psi\rangle = \lambda|\psi\rangle.$$

Since

$$\hat{L}_z|1,0\rangle = i\hbar|0,1\rangle, \qquad \hat{L}_z|0,1\rangle = -i\hbar|1,0\rangle,$$

we have the matrix representation of $\hat{L}_z$ under the basis of $|1,0\rangle$ and $|0,1\rangle$,

$$\hat{L}_z = \hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \hbar\hat{\sigma}_y.$$

Therefore the eigenvalue and eigemstate of $\hat{L}_z$ is

$$|+\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}}(|1,0\rangle + i|0,1\rangle), \quad |-\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}}(|1,0\rangle - i|0,1\rangle).$$

((**Mathematica**))

```
Clear["Global`*"]; A = ⅈ ℏ ( 0  -1
                             1   0 );
eq1 = Eigensystem[A]
```

$\{\{-\hbar, \hbar\}, \{\{i, 1\}, \{-i, 1\}\}\}$

```
ψ1 = ⅈ Normalize[eq1[[2, 2]]]
```

$\left\{\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right\}$

```
ψ1 = -ⅈ Normalize[eq1[[2, 1]]]
```

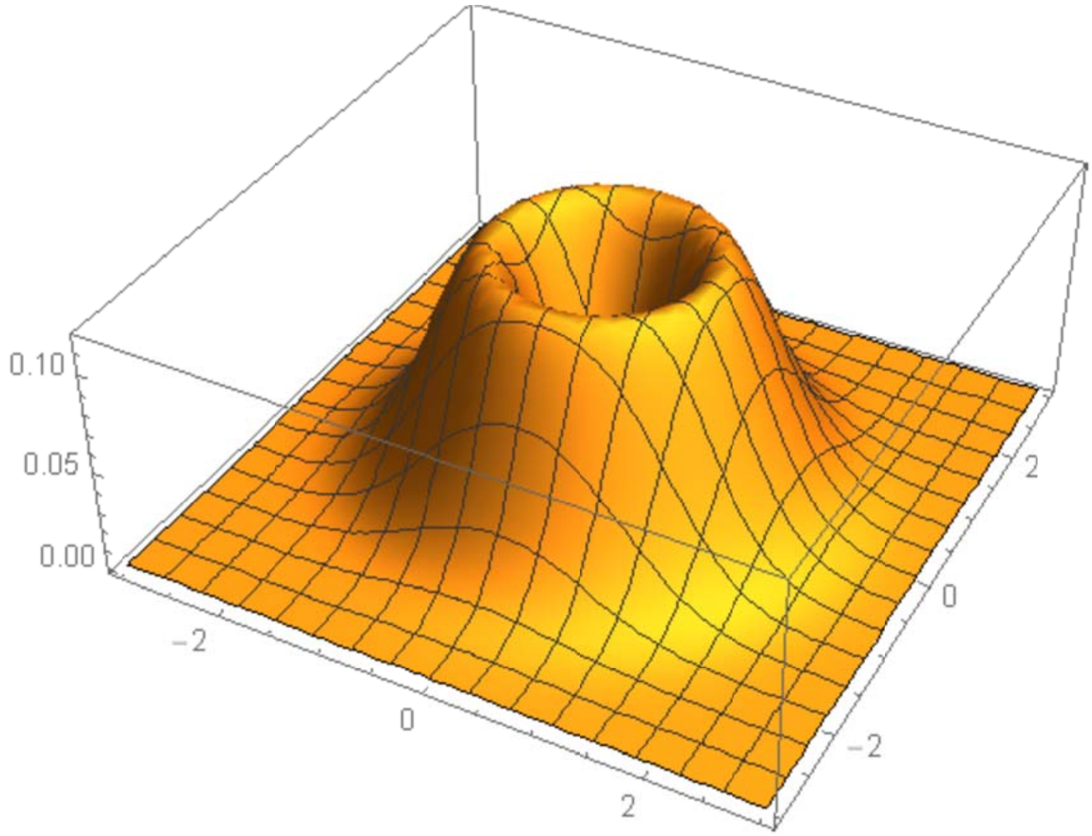
$\left\{\frac{1}{\sqrt{2}}, -\frac{i}{\sqrt{2}}\right\}$

((**Plot3D**))
We make a plot3D of the probability density:

$$\left|\langle x, y|\psi_1\rangle\right|^2 \qquad (E = 2\hbar\omega\text{, and } m = 1)$$

as a function of *x* and *y*, where

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}[|10\rangle + i|01\rangle\}$$



where

$$\langle x, y | n_1, n_2 \rangle = 2^{-n} \pi^{-1/2} (n_1!)^{-1/2} (n_2!)^{-1/2} \exp(-\frac{x^2 + y^2}{2}) H[n_1, x] H[n_2, y]$$

3. The eigenstates of $E = 3\hbar\omega$ with the combination of states $|20\rangle$, $|11\rangle$, and $|02\rangle$)

When $E = 3\hbar\omega_0$, there are three states $|n_1 = 2, n_2 = 0\rangle$, $|n_1 = 1, n_2 = 1\rangle$, and $|n_1 = 0, n_2 = 2\rangle$. These three states are degenerate states with the same energy. The combination of these three states leads to the eigenstate of $\hat{H}$.

$$\hat{L}_z |20\rangle = i\hbar(\hat{a}_1 \hat{a}_2^+ - \hat{a}_1{}^+ \hat{a}_2)|20\rangle = i\hbar \hat{a}_1 \hat{a}_2^+ |20\rangle = i\hbar\sqrt{2}|11\rangle,$$

$$\hat{L}_z |11\rangle = i\hbar(\hat{a}_1 \hat{a}_2^+ - \hat{a}_1{}^+ \hat{a}_2)|11\rangle = i\hbar\sqrt{2}(|02\rangle - |20\rangle),$$

$$\hat{L}_z |02\rangle = i\hbar(\hat{a}_1 \hat{a}_2^+ - \hat{a}_1{}^+ \hat{a}_2)|02\rangle = -i\hbar \hat{a}_1{}^+ \hat{a}_2 |02\rangle = -\hbar\sqrt{2}|11\rangle,$$

$$\hat{L}_z = i\hbar \begin{pmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}.$$

under the basis of $|20\rangle$, $|11\rangle$, and $|02\rangle$. We use the Mathematica to get the eigenstates and eigenvalues.

For $m = 2$

$$|\psi_2\rangle = \frac{1}{2}|20\rangle + \frac{i}{\sqrt{2}}|11\rangle - \frac{1}{2}|02\rangle$$

For $m = 0$

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}|20\rangle + \frac{1}{\sqrt{2}}|02\rangle$$

For $m = -2$

$$|\psi_{-2}\rangle = \frac{1}{2}|20\rangle - \frac{i}{\sqrt{2}}|11\rangle - \frac{1}{2}|02\rangle$$

((**Mathematica**))

```
B = ⅈ ℏ √2 (0 -1 0; 1 0 -1; 0 1 0); eq2 = Eigensystem[B]
```

$$B = i\hbar\sqrt{2}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\left\{\{-2\hbar,\ 2\hbar,\ 0\},\ \left\{\left\{-1,\ i\sqrt{2},\ 1\right\},\ \left\{-1,\ -i\sqrt{2},\ 1\right\},\ \{1,\ 0,\ 1\}\right\}\right\}$$

```
ψ1 = - Normalize[eq2[[2, 2]]]
```

$$\left\{\frac{1}{2},\ \frac{i}{\sqrt{2}},\ -\frac{1}{2}\right\}$$

```
ψ2 = Normalize[eq2[[2, 3]]]
```

$$\left\{\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right\}$$

```
ψ3 = - Normalize[eq2[[2, 1]]]
```

$$\left\{\frac{1}{2},\ -\frac{i}{\sqrt{2}},\ -\frac{1}{2}\right\}$$

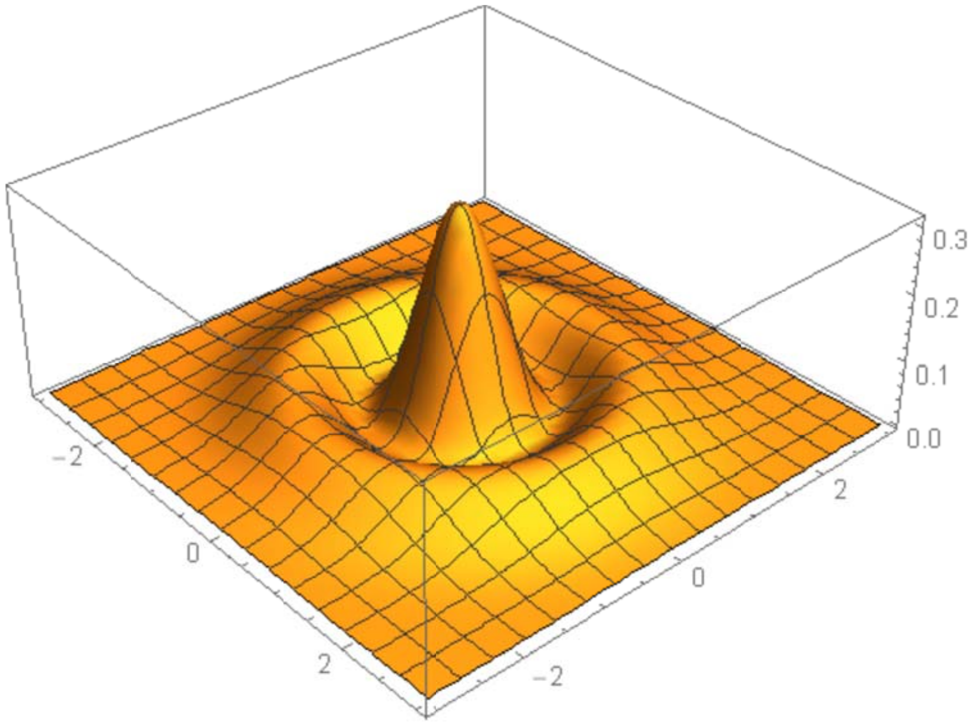
((**Plot3D**))

We make a Plot3D of the probability density:

$$\left|\langle x, y|\psi_0\rangle\right|^2 \qquad (E = 3\hbar\omega\text{, and } m = 0)$$

in the *x* and *y* plane where

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}|20\rangle + \frac{1}{\sqrt{2}}|02\rangle$$



4. Eigenstates of $E = 4\hbar\omega$ with the combination of states $|30\rangle$, $|21\rangle$, $|12\rangle$, and $|03\rangle$)

When $E = 4\hbar\omega_0$, there are four states $|n_1 = 3, n_2 = 0\rangle$, $|n_1 = 2, n_2 = 1\rangle$, $|n_1 = 1, n_2 = 2\rangle$, and, $|n_1 = 0, n_2 = 3\rangle$. These four states are degenerate states with the same energy. The combination of these four states leads to the eigenstate of $\hat{H}$.

$$\hat{L}_z|30\rangle = i\hbar(\hat{a}_1\hat{a}_2^+ - \hat{a}_1{}^+\hat{a}_2)|30\rangle = i\hbar\hat{a}_1\hat{a}_2^+|30\rangle = i\hbar\sqrt{3}|21\rangle,$$

$$\hat{L}_z|21\rangle = i\hbar(\hat{a}_1\hat{a}_2^+ - \hat{a}_1{}^+\hat{a}_2)|21\rangle = i\hbar(2|12\rangle - \sqrt{3}|3,0\rangle),$$

$$\hat{L}_z|12\rangle = i\hbar(\hat{a}_1\hat{a}_2^+ - \hat{a}_1{}^+\hat{a}_2)|12\rangle = i\hbar(\sqrt{3}|03\rangle - 2|21\rangle),$$

$$\hat{L}_z|03\rangle = i\hbar(\hat{a}_1\hat{a}_2^+ - \hat{a}_1{}^+\hat{a}_2)|03\rangle = -i\hbar\hat{a}_1{}^+\hat{a}_2|30\rangle = -i\hbar\sqrt{3}|12\rangle,$$

$$\hat{L}_z = i\hbar\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}.$$

under the basis of $|30\rangle$, $|21\rangle$, $|12\rangle$, and $|03\rangle$. We use the Mathematica to get the eigenstates and eigenvalues.

For $m = 3$

$$|\psi_3\rangle = \frac{1}{2\sqrt{2}}[|30\rangle + i\sqrt{3}|21\rangle - \sqrt{3}|12\rangle - i|03\rangle].$$

For $m = 1$

$$|\psi_1\rangle = \frac{1}{2\sqrt{2}}[\sqrt{3}|30\rangle + i|21\rangle + |12\rangle + i\sqrt{3}|03\rangle].$$

For $m = -1$

$$|\psi_{-1}\rangle = \frac{1}{2\sqrt{2}}[\sqrt{3}|30\rangle - i|21\rangle + |12\rangle - i\sqrt{3}|03\rangle].$$

For $m = -3$

$$|\psi_{-3}\rangle = \frac{1}{2\sqrt{2}}[|30\rangle - i\sqrt{3}|21\rangle - \sqrt{3}|12\rangle + i|03\rangle].$$

((**Mathematica**))

$$C1 = i\hbar \begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix};$$

eq3 = Eigensystem[C1]

$$\left\{\{-3\hbar, 3\hbar, -\hbar, \hbar\}, \left\{\{-i, -\sqrt{3}, i\sqrt{3}, 1\}, \{i, -\sqrt{3}, -i\sqrt{3}, 1\}, \left\{i, \frac{1}{\sqrt{3}}, \frac{i}{\sqrt{3}}, 1\right\}, \left\{-i, \frac{1}{\sqrt{3}}, -\frac{i}{\sqrt{3}}, 1\right\}\right\}\right\}$$

ψ1 = -i Normalize[eq3[[2, 2]]] // Simplify

$$\left\{\frac{1}{2\sqrt{2}}, \frac{1}{2} i \sqrt{\frac{3}{2}}, -\frac{\sqrt{\frac{3}{2}}}{2}, -\frac{i}{2\sqrt{2}}\right\}$$

ψ2 = i Normalize[eq3[[2, 4]]] // Simplify

$$\left\{\frac{\sqrt{\frac{3}{2}}}{2}, \frac{i}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2} i \sqrt{\frac{3}{2}}\right\}$$

ψ3 = -i Normalize[eq3[[2, 3]]] // Simplify

$$\left\{\frac{\sqrt{\frac{3}{2}}}{2}, -\frac{i}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, -\frac{1}{2} i \sqrt{\frac{3}{2}}\right\}$$

ψ4 = i Normalize[eq3[[2, 1]]] // Simplify

$$\left\{\frac{1}{2\sqrt{2}}, -\frac{1}{2} i \sqrt{\frac{3}{2}}, -\frac{\sqrt{\frac{3}{2}}}{2}, \frac{i}{2\sqrt{2}}\right\}$$

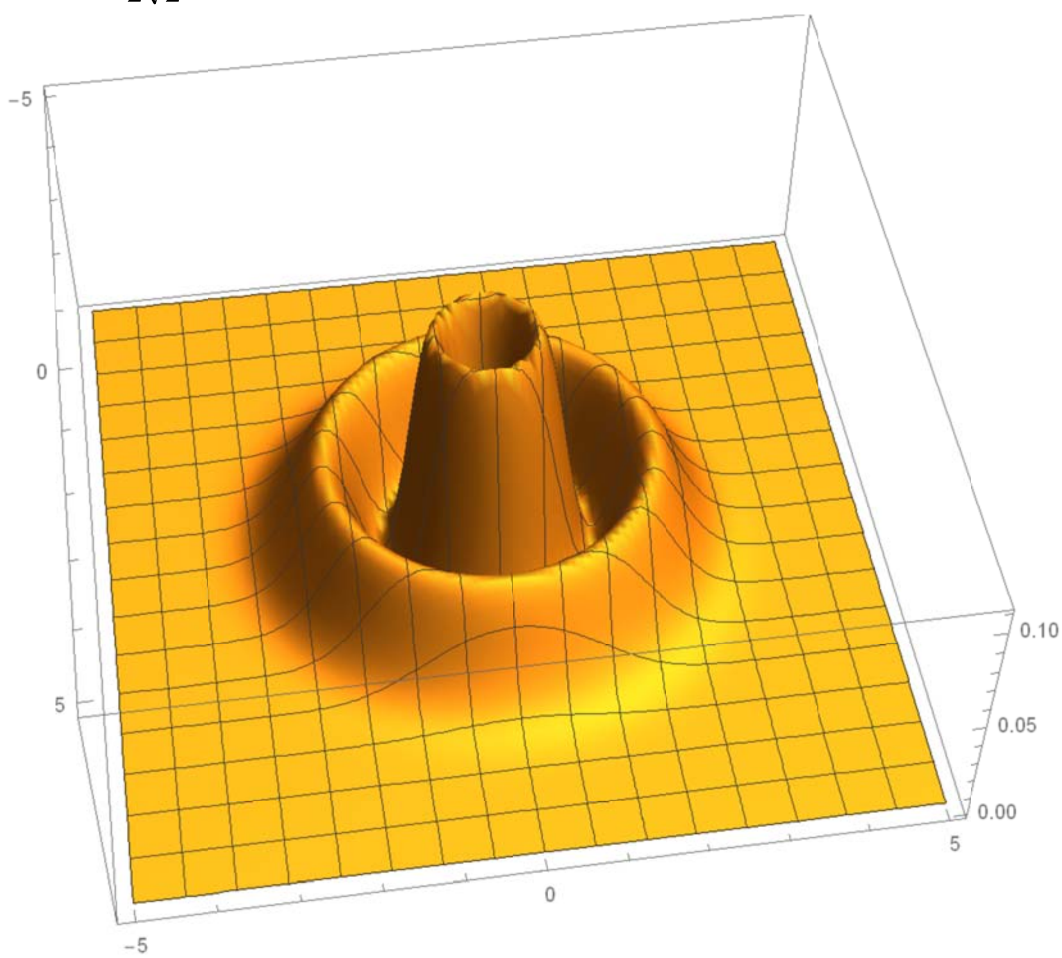
((**Plot3D**))

We make a Plot3D of the probability density,

$$|\langle x, y|\psi_1\rangle|^2 \qquad (E = 4\hbar\omega\text{, and } m = 1)$$

where

$$|\psi_1\rangle = \frac{1}{2\sqrt{2}}[\sqrt{3}|30\rangle + i|21\rangle + |12\rangle + i\sqrt{3}|03\rangle]$$



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