Addition theorem for spherical harmonics
Masatsugu Sei Suzuki, Department of Physics
SUNY at Binghamton
(Date: January 25, 2016)

Let

$$\left|\boldsymbol{n}_1\right\rangle = \hat{R}(\theta_1,\phi_1)\left|\boldsymbol{e}_z\right\rangle = \left|\Re(\theta_1,\phi_1)\boldsymbol{e}_z\right\rangle$$

with the geometrical rotation defined by

$$\Re(\theta_1,\phi_1) = \Re_z(\phi_1)\Re_y(\theta_1) = \begin{pmatrix} \cos\theta_1\cos\phi_1 & -\sin\phi_1 & \sin\theta_1\cos\phi_1 \\ \cos\theta_1\sin\phi_1 & \cos\phi_1 & \sin\theta_1\sin\phi_1 \\ -\sin\theta_1 & 0 & \cos\theta_1 \end{pmatrix}$$

$$\Re_z(\phi_1) = \begin{pmatrix} \cos\phi_1 & -\sin\phi_1 & 0 \\ \sin\phi_1 & \cos\phi_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Re_y(\theta_1) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

The unit vector $\boldsymbol{n}_1$ is defined by

$$\begin{aligned} \boldsymbol{n}_1 &= \Re(\theta_1,\phi_1)\boldsymbol{e}_z \\ &= \begin{pmatrix} \cos\theta_1\cos\phi_1 & -\sin\phi_1 & \sin\theta_1\cos\phi_1 \\ \cos\theta_1\sin\phi_1 & \cos\phi_1 & \sin\theta_1\sin\phi_1 \\ -\sin\theta_1 & 0 & \cos\theta_1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \sin\theta_1\cos\phi_1 \\ \sin\theta_1\sin\phi_1 \\ \cos\theta_1 \end{pmatrix} \end{aligned}$$

Another ket $\left|\boldsymbol{n}_2\right\rangle$ is defined by

$$\left|\boldsymbol{n}_2\right\rangle = \hat{R}(\theta_2,\phi_2)\left|\boldsymbol{e}_z\right\rangle = \left|\Re(\theta_2,\phi_2)\boldsymbol{e}_z\right\rangle$$

where the geometrical rotation matrix is given by

$$\Re(\theta_2,\phi_2)=\Re_z(\phi_2)\Re_y(\theta_2)=\begin{pmatrix}\cos\theta_2\cos\phi_2 & -\sin\phi_2 & \sin\theta_2\cos\phi_2\\ \cos\theta_2\sin\phi_2 & \cos\phi_2 & \sin\theta_2\sin\phi_2\\ -\sin\theta_2 & 0 & \cos\theta_2\end{pmatrix}$$

Note that

$$\begin{aligned}\boldsymbol{n}_2 &= \Re(\theta_2,\phi_2)\boldsymbol{e}_z\\ &=\begin{pmatrix}\cos\theta_2\cos\phi_2 & -\sin\phi_2 & \sin\theta_2\cos\phi_2\\ \cos\theta_2\sin\phi_2 & \cos\phi_2 & \sin\theta_2\sin\phi_2\\ -\sin\theta_2 & 0 & \cos\theta_2\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\\ &=\begin{pmatrix}\sin\theta_2\cos\phi_2\\ \sin\theta_2\sin\phi_2\\ \cos\theta_2\end{pmatrix}\end{aligned}$$

Then we have the inner product as

$$\boldsymbol{n}_1\cdot\boldsymbol{n}_2=\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)$$

We assume a new ket defined by

$$\left|\boldsymbol{n}'\right\rangle=\hat{R}^{-1}(\theta_1,\phi_1)\left|\boldsymbol{n}_2\right\rangle=\left|\Re^{-1}(\theta_1,\phi_1)\boldsymbol{n}_2\right\rangle,$$

and

$$\left\langle\boldsymbol{n}'\right|=\left\langle\boldsymbol{n}_2\right|\hat{R}(\theta_1,\phi_1)$$

Note that the matrix $\Re^{-1}(\theta_1,\phi_1)$ is given by

$$\Re^{-1}(\theta_1,\phi_1)=\begin{pmatrix}\cos\theta_1\cos\phi_1 & \cos\theta_1\sin\phi_1 & -\sin\theta_1\\ -\sin\phi_1 & \cos\phi_1 & 0\\ \sin\theta_1\cos\phi_1 & \sin\theta_1\sin\phi_1 & \cos\theta_1\end{pmatrix}$$

and

$$
\begin{aligned}
\boldsymbol{n}' &= \Re^{-1}(\theta_1,\phi_1)\boldsymbol{n}_2 \\
&= \begin{pmatrix} -\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2\cos(\phi_1-\phi_2) \\ -\sin\theta_2\sin(\phi_1-\phi_2) \\ \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2) \end{pmatrix} \\
&= \Re(\Theta,\Phi)\boldsymbol{e}_z \\
&= \begin{pmatrix} \sin\Theta\cos\Phi \\ \sin\Theta\sin\Phi \\ \cos\Theta \end{pmatrix}
\end{aligned}
$$

Since

$$\boldsymbol{n}'\cdot\boldsymbol{e}_z = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)$$

it is found that

$$\boldsymbol{n}_1\cdot\boldsymbol{n}_2 = \boldsymbol{n}'\cdot\boldsymbol{e}_z = \cos\Theta$$

In other words, Θ is the angle between $\boldsymbol{n}'$ and $\boldsymbol{e}_z$ and it is also the angle between $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$.

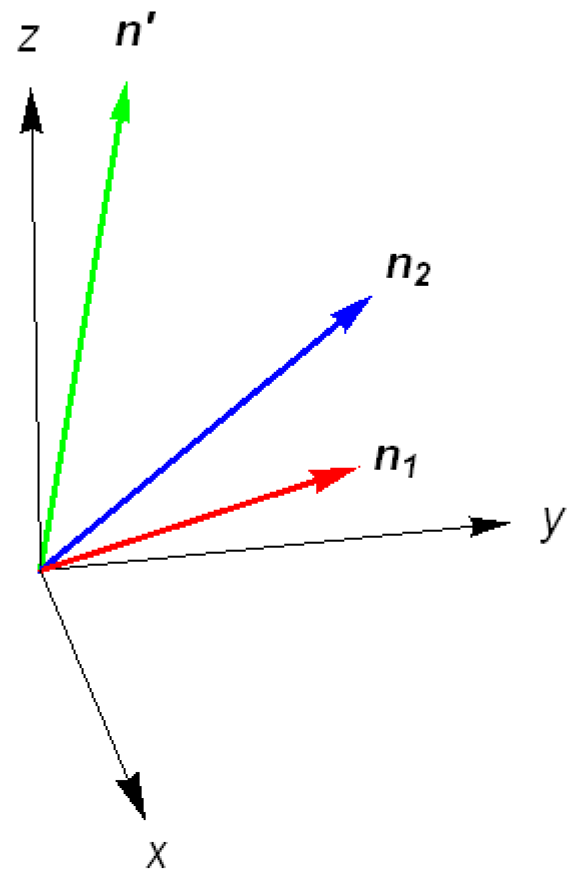

Fig. Three vectors $\boldsymbol{n}_1 = \Re(\theta_1, \phi_1)\boldsymbol{e}_z$. $\boldsymbol{n}_2 = \Re(\theta_2, \phi_2)\boldsymbol{e}_z$. $\boldsymbol{n}' = \Re^{-1}(\theta_1, \phi_1)\boldsymbol{n}_2 = \Re(\Theta, \Phi)\boldsymbol{e}_z$. The Mathematica is used for this figure. $\theta_1 = 60°$. $\theta_2 = 45°$. $\phi_1 = 30°$. $\phi_2 = 50°$. Without Mathematica, it is hard for me to get the line for the unit vector ***n***'.

Using the closure relation, we get

$$\begin{aligned} Y_l^m(\Theta, \Phi) &= \langle \boldsymbol{n}' | lm \rangle \\ &= \langle \boldsymbol{n}_2 | \hat{R}(\theta_1, \phi_1) | l, m \rangle \\ &= \sum_{m'} \langle \boldsymbol{n}_2 | l, m' \rangle \langle l, m' | \hat{R}(\theta_1, \phi_1) | l, m \rangle \\ &= \sum_{m'} Y_l^{m'}(\theta_2, \phi_2) \langle l, m' | \hat{R}(\theta_1, \phi_1) | l, m \rangle \end{aligned} \tag{1}$$

where

$$D_{m'm}^{(l)}(\theta_1, \phi_1) = \langle l, m' | \hat{R}(\theta_1, \phi_1) | l, m \rangle$$

Equation (1) relates spherical harmonics in three different directions. The most useful case is m = 0;

$$D^{(l)}_{m'm=0}(\theta_1,\phi_1) = e^{-im'\phi_1} d^{(l)}_{m'm=0}(\theta_1)$$

where

$$d^{(l)}_{m'm=0}(\theta_1) = e^{im'\phi_1}\sqrt{\frac{4\pi}{2l+1}}[Y_l^{m'}(\theta_1,\phi_1)]^*$$

and

$$d^{(l)}_{m'=0,m=0}(\theta_1) = \sqrt{\frac{4\pi}{2l+1}}[Y_l^{m'=0'}(\theta_1,\phi_1)]^* = P_l(\cos\theta_1)$$

Then we have

$$\begin{aligned} Y_l^{m=0}(\Theta,\Phi) &= \sum_{m'} Y_l^{m'}(\theta_2,\phi_2) D^{(l)}_{m'm=0}(\theta_1,\phi_1) \\ &= \sum_{m'} Y_l^{m'}(\theta_2,\phi_2) e^{-im'\phi_1} d^{(l)}_{m'm=0}(\theta_1) \\ &= \sqrt{\frac{4\pi}{2l+1}}\sum_{m'} e^{-im'\phi_1} e^{im'\phi_1}[Y_l^{m'}(\theta_1,\phi_1)]^* Y_l^{m'}(\theta_2,\phi_2) \\ &= \sqrt{\frac{4\pi}{2l+1}}\sum_{m}[Y_l^{m}(\theta_1,\phi_1)]^* Y_l^{m}(\theta_2,\phi_2) \end{aligned}$$

which leads to the addition theorem for the spherical harmonics

$$P_l(\cos\Theta) = \frac{4\pi}{2l+1}\sum_{m}[Y_l^{m}(\theta_1,\phi_1)]^* Y_l^{m}(\theta_2,\phi_2)$$

(a)

$$|\boldsymbol{n}'\rangle = \hat{R}|\boldsymbol{n}\rangle, \qquad |\boldsymbol{n}\rangle = \hat{R}^{+}|\boldsymbol{n}'\rangle, \qquad \langle\boldsymbol{n}| = \langle\boldsymbol{n}'|\hat{R}$$

Using the closure relation, we get

$$\hat{R}|l,m\rangle = \sum_{m'} |l,m'\rangle\langle l,m'|\hat{R}|l,m\rangle$$

and

$$\langle \boldsymbol{n}'|\hat{R}|l,m\rangle = \sum_{m'} \langle \boldsymbol{n}'|l,m'\rangle\langle l,m'|\hat{R}|l,m\rangle$$

Noting that $\langle \boldsymbol{n}| = \langle \boldsymbol{n}'|\hat{R}$, we get

$$\langle \boldsymbol{n}|l,m\rangle = \sum_{m'} \langle \boldsymbol{n}'|l,m'\rangle\langle l,m'|\hat{R}|l,m\rangle$$

or

$$Y_l^m(\boldsymbol{n}) = \sum_{m'} Y_l^{m'}(\boldsymbol{n}')D_{m',m}^{(l)}(\hat{R})$$

(b)

$$|\boldsymbol{n}\rangle = \hat{R}|\boldsymbol{n}'\rangle ,, \quad \langle \boldsymbol{n}| = \langle \boldsymbol{n}'|\hat{R}^{+}$$

Using the closure relation, we get

$$\hat{R}^{+}|l,m\rangle = \sum_{m'} |l,m'\rangle\langle l,m'|\hat{R}^{+}|l,m\rangle$$

and

$$\langle \boldsymbol{n}'|\hat{R}^{+}|l,m\rangle = \sum_{m'} \langle \boldsymbol{n}'|l,m'\rangle\langle l,m'|\hat{R}^{+}|l,m\rangle$$

Noting that $\langle \boldsymbol{n}| = \langle \boldsymbol{n}'|\hat{R}^{+}$, we get

$$\langle \boldsymbol{n}|l,m\rangle = \sum_{m'} \langle \boldsymbol{n}'|l,m'\rangle\langle l,m'|\hat{R}|l,m\rangle$$

or

$$Y_l^m(\boldsymbol{n}) = \sum_{m'} Y_l^{m'}(\boldsymbol{n}')D_{m',m}^{(l)}(\hat{R})$$