Addition of angular momentum

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Here we discuss the addition of angular momentum, such as the orbital angular momentum $\boldsymbol{L}$ and the spin angular momentum $\boldsymbol{S}$. The addition of the angular momentum is encountered in all area of the modern physics, especially in quantum mechanics. The total angular momentum $\hat{\boldsymbol{J}}$ is expressed by

$$\hat{\boldsymbol{J}} = \hat{\boldsymbol{J}}_1 + \hat{\boldsymbol{J}}_2$$

in terms of the angular momenta $\hat{\boldsymbol{J}}_1$ and $\hat{\boldsymbol{J}}_2$.We show that the simultaneous eigenket $|j,m\rangle$ of $\hat{\boldsymbol{J}}$ and $\hat{J}_z$ can be expressed in terms of the linear combination of $|j_1,m_1\rangle \otimes |j_2,m_2\rangle$ with the so-called Clebsch-Gordan co-efficient, where $|j_1,m_1\rangle$ is the simultaneous eigenket of $\hat{\boldsymbol{J}}_1$ and $\hat{J}_{1z}$, and the simultaneous eigenket of $\hat{\boldsymbol{J}}_2$ and $\hat{J}_{2z}$. The Clebsch-Gordan co-efficient $\langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m\rangle$ is discussed.

1. Clebsch-Gordan (CG) Coefficient

Let $\hat{\boldsymbol{J}}_1$ and $\hat{\boldsymbol{J}}_2$ be the two angular momenta, and let

$$[\hat{\boldsymbol{J}}_1, \hat{\boldsymbol{J}}_2] = 0 ,$$

or

$$[\hat{J}_{1i}, \hat{J}_{2j}] = 0 . \qquad (i, j = x, y, z)$$

Define the total angular momentum

$$\hat{\boldsymbol{J}} = \hat{\boldsymbol{J}}_1 + \hat{\boldsymbol{J}}_2 ,$$

where

$$\hat{\boldsymbol{J}}_1^{\,2} | j_1, m_1\rangle = \hbar^2 j_1 (j_1 + 1) | j_1, m_1\rangle ,$$

$$\hat{J}_{1z}| j_1, m_1\rangle = \hbar m_1 | j_1, m_1\rangle,$$

$$\hat{J}_{1\pm}| j_1, m_1\rangle = \hbar\sqrt{j_1(j_1+1) - m_1(m_1 \pm 1)}| j_1, m_1 \pm 1\rangle,$$

Similarly,

$$\hat{\boldsymbol{J}}_2{}^2| j_2, m_2\rangle = \hbar^2 j_2(j_2+1)| j_2, m_2\rangle,$$

$$\hat{J}_{2z}| j_2, m_2\rangle = \hbar m_2 | j_2, m_2\rangle,$$

$$\hat{J}_{2\pm}| j_2, m_2\rangle = \hbar\sqrt{j_2(j_2+1) - m_2(m_2 \pm 1)}| j_2, m_2 \pm 1\rangle.$$

We now consider a new ket defined by

$$| j_1, j_2; m_1, m_2\rangle = | j_1, m_1\rangle \otimes | j_2, m_2\rangle.$$

The total angular momentum: commutation relation

$$\hat{\boldsymbol{J}} = \hat{\boldsymbol{J}}_1 + \hat{\boldsymbol{J}}_2$$

$$[\hat{J}_z, \hat{\boldsymbol{J}}_1{}^2] = [\hat{J}_z, \hat{\boldsymbol{J}}_2{}^2] = 0$$

$$[\hat{\boldsymbol{J}}^2, \hat{\boldsymbol{J}}_1{}^2] = [\hat{\boldsymbol{J}}^2, \hat{\boldsymbol{J}}_2{}^2] = 0$$

((Note))

$$[\hat{J}_z, \hat{\boldsymbol{J}}_1{}^2] = [\hat{J}_{1z} + \hat{J}_{2z}, \hat{\boldsymbol{J}}_1{}^2] = [\hat{J}_{1z}, \hat{\boldsymbol{J}}_1{}^2] = 0$$

$$[\hat{J}_z{}^2, \hat{\boldsymbol{J}}_1{}^2] = -[\hat{\boldsymbol{J}}_1{}^2, \hat{J}_z{}^2] = -([\hat{\boldsymbol{J}}_1{}^2, \hat{J}_z]\hat{J}_z + \hat{J}_z[\hat{\boldsymbol{J}}_1{}^2, \hat{J}_z]) = 0$$

$$[\hat{\boldsymbol{J}}^2, \hat{\boldsymbol{J}}_1{}^2] = [\hat{J}_x{}^2 + \hat{J}_y{}^2 + \hat{J}_z{}^2, \hat{\boldsymbol{J}}_1{}^2] = 0$$

Furthermore, $\hat{J}_{1z}$ and $\hat{J}_{2z}$ commute with $\hat{J}_z$.

$$[\hat{J}_{1z}, \hat{J}_z] = [\hat{J}_{2z}, \hat{J}_z] = 0,$$

but not with $\hat{\boldsymbol{J}}^2$.

$$\hat{\boldsymbol{J}}^2 = \hat{\boldsymbol{J}}_1{}^2 + \hat{\boldsymbol{J}}_2{}^2 + 2\hat{\boldsymbol{J}}_1 \cdot \hat{\boldsymbol{J}}_2 = \hat{\boldsymbol{J}}_1{}^2 + \hat{\boldsymbol{J}}_2{}^2 + 2\hat{J}_{1z}\hat{J}_{2z} + \hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+}$$

$$[\hat{J}_{1z}, \hat{\boldsymbol{J}}^2] = [\hat{J}_{1z}, \hat{\boldsymbol{J}}_1{}^2 + \hat{\boldsymbol{J}}_2{}^2 + 2\hat{J}_{1z}\hat{J}_{2z} + \hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+}]$$
$$= [\hat{J}_{1z}, \hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+}]$$
$$= [\hat{J}_{1z}, \hat{J}_{1+}]\hat{J}_{2-} + [\hat{J}_{1z}, \hat{J}_{1-}]\hat{J}_{2+}$$
$$= \hbar(\hat{J}_{1+}\hat{J}_{2-} - \hat{J}_{1-}\hat{J}_{2+})$$

$$[\hat{J}_{2z}, \hat{\boldsymbol{J}}^2] = [\hat{J}_{2z}, \hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+}]$$
$$= \hat{J}_{1+}[\hat{J}_{2z}, \hat{J}_{2-}] + \hat{J}_{1-}[\hat{J}_{2z}, \hat{J}_{2+}]$$
$$= -\hbar(\hat{J}_{1+}\hat{J}_{2-} - \hat{J}_{1-}\hat{J}_{2+})$$

Thus we have

$$[\hat{J}_z, \hat{\boldsymbol{J}}^2] = 0$$

Since

$$[\hat{\boldsymbol{J}}^2, \hat{J}_z] = [\hat{\boldsymbol{J}}^2, \hat{\boldsymbol{J}}_1{}^2] = [\hat{\boldsymbol{J}}^2, \hat{\boldsymbol{J}}_2{}^2] = [\hat{\boldsymbol{J}}_1{}^2, \hat{\boldsymbol{J}}_2{}^2] = [\hat{\boldsymbol{J}}_1{}^2, \hat{J}_z] = [\hat{\boldsymbol{J}}_2{}^2, \hat{J}_z] = 0,$$

We have simultaneous eigenkets of $\hat{\boldsymbol{J}}^2, \hat{\boldsymbol{J}}_1{}^2, \hat{\boldsymbol{J}}_2{}^2, \hat{J}_z$. We use the eigenket,

$$|j_1, j_2; j, m\rangle$$

to denote the basis.

$$\hat{\boldsymbol{J}}_1{}^2 |j_1, j_2; j, m\rangle = \hbar^2 j_1(j_1+1)|j_1, j_2; j, m\rangle$$

$$\hat{\boldsymbol{J}}_2{}^2 |j_1, j_2; j, m\rangle = \hbar^2 j_2(j_2+1)|j_1, j_2; j, m\rangle$$

$$\hat{\boldsymbol{J}}^2 |j_1, j_2; j, m\rangle = \hbar^2 j(j+1)|j_1, j_2; j, m\rangle$$

$$\hat{J}_z |j_1, j_2; j, m\rangle = \hbar m|j_1, j_2; j, m\rangle$$

Here we have the relation

$$|j_1, j_2; j, m\rangle = \sum_{m_1}\sum_{m_2} |j_1, j_2; m_1, m_2\rangle\langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m\rangle$$

where we use the closure relation.

$$\sum_{m_1}\sum_{m_2} | j_1, j_2; m_1, m_2\rangle\langle j_1, j_2; m_1, m_2 | = 1 \ :$$

$\langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m\rangle$: Clebsch Gordan co-efficient

For simplicity we use

$$| j_1, j_2; m_1, m_2\rangle = | j, m\rangle,$$

$$| j_1, j_2; m_1, m_2\rangle = | j_1, m_1\rangle \otimes | j_2, m_2\rangle = | j_1, m_1\rangle | j_2, m_2\rangle$$

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2. Properties of CG co-efficient

(1) The co-efficient vanishes unless $m = m_1 + m_2$.

((**Proof**))

Note that

$$(\hat{J}_z - \hat{J}_{1z} - \hat{J}_{2z}) | j_1, j_2; j, m\rangle = 0$$

$$\langle j_1, j_2; m_1, m_2 | \hat{J}_z - \hat{J}_{1z} - \hat{J}_{2z} | j_1, j_2; j, m\rangle = 0$$

or

$$(m - m_1 - m_2)\langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m\rangle = 0$$

For $m - m_1 - m_2 \neq 0$,

$$\langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m\rangle = 0$$

(2) The CG co-efficient vanishes unless

$$|j_1 - j_2| \le j \le j_1 + j_2$$

triangle inequality, which tells us the possible range of values of j when we add two angular momenta j_1 and j_2 whose values are fixed.

((**Proof**))

For a proof of the triangle rule, let us consider the possible values of m. Since $m = m_1 + m_2$, its maximum value is $j_1 + j_2$. The value is realized for the single state $| j_1, j_2; j_1, j_2\rangle$. The total J of this state is $j = j_1 + j_2$. The next highest value of m, $j_1 + j_2 - 1$ is realized for the linear combinations of two states in the uncoupled representations; $| j_1, j_2; j_1 - 1, j_2\rangle$

and $|j_1, j_2; j_1, j_2 - 1\rangle$. One of the linear combinations belongs to $j = j_1 + j_2$ and the second one to $j = j_1 + j_2 - 1$.

For $m = j_1 + j_2 - 2$, there are three linear combinations of the three uncoupled states; $|j_1, j_2; j_1 - 2, j_2\rangle$, $|j_1, j_2; j_1 - 1, j_2 - 1\rangle$, and $|j_1, j_2; j_1, j_2 - 2\rangle$, corresponding to three of j, $j = j_1 + j_2$, $j = j_1 + j_2 - 1$, and $j = j_1 + j_2 - 2$.

If we continue this process, we can see that each time we lower m by 1, a new value of j appears. The argument continues until we reach a stage where we can no longer go one more step down in one of them to make new states. We assume that $j_1 > j_2$.

$j = j_1 + j_2$; $m = j_1 + j_2, j_1 + j_2 - 1, ..., -(j_1 + j_2)$

$j = j_1 + j_2 - 1$, $m = j_1 + j_2 - 1, j_1 + j_2 - 2, ... -(j_1 + j_2 - 1)$,

....

$j = j_1 - j_2$, $m = j_1 - j_2, j_1 - j_2 - 1, ..., -(j_1 - j_2)$.

Here we show that if the triangle rule is valid, then the dimension of the space spanned by $\{|j_1, j_2; j, m\rangle\}$ is the same as that of the space spanned by $\{|j_1, j_2; m_1, m_2\rangle\}$.

For the (m_1, m_2) way of counting, we obtain $N = (2j_1 + 1)(2j_2 + 1)$.

As for the $\{j, m)$ way of counting, note that for each j, there are $(2j+1)$ states. According to the inequality ($|j_1 - j_2| \le j \le j_1 + j_2$), j runs (we assume $j_1 > j_2$) from $j_1 - j_2$ to $j_1 + j_2$.

$$
\begin{aligned}
N &= \sum_{j=j_1-j_2}^{j_1+j_2} (2j+1) = \sum_{j=0}^{j_1+j_2} (2j+1) - \sum_{j=0}^{j_1-j_2} (2j+1) \\
&= (j_1 + j_2 + 1)^2 - (j_1 - j_2)^2 \\
&= (2j_1 + 1)(2j_2 + 1)
\end{aligned}
$$

((Note))

$$D_{j_1} \times D_{j_2} = D_{j_1+j_2} + D_{j_1+j_2-1} + ... + D_{|j_1-j_2|}$$

where

$D_{j_1+j_2}$; $j = j_1 + j_2$; $m = j_1 + j_2, j_1 + j_2 - 1, ..., -(j_1 + j_2)$,

$D_{j_1+j_2-1}$; $j = j_1 + j_2 - 1$; $m = j_1 + j_2 - 1, j_1 + j_2 - 2, ..., -(j_1 + j_2 - 1)$,

....

$D_{|j_1-j_2|}$; $j = |j_1 - j_2|$; $m = |j_1 - j_2|, |j_1 - j_2| - 1, ..., -|j_1 - j_2|$,

The CG coefficients form an unitary matrix. Furthermore, the matrix elements are taken to be real by convention. A real unitary matrix is orthogonal. We have the orthogonal condition.

$$\langle j_1, j_2, j, m | j_1, j_2, m_1, m_2 \rangle = \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle^* = \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle$$

Closure relation

$$\begin{aligned}\langle j_1, j_2, m_1, m_1 | j_1, j_2, m_1', m_2' \rangle &= \sum_{j,m} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \langle j_1, j_2; j, m | j_1, j_2; m_1', m_2' \rangle \\ &= \sum_{j,m} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m \rangle \\ &= \delta_{m_1, m_1'} \delta_{m_2, m_2'}\end{aligned}$$

or

$$\sum_{j,m} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m \rangle = \delta_{m_1, m_1'} \delta_{m_2, m_2'}$$

Similarly,

$$\sum_{m_1, m_2} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \langle j_1, j_2; m_1, m_2 | j_1, j_2; j', m' \rangle = \delta_{j, j'} \delta_{m, m'}$$

As a special case of this, we may set $j = j'$, $m' = m = m_1 + m_2$.

$$\sum_{\substack{m_1, m_2 \\ m = m_1 + m_2}} \left| \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \right|^2 = 1$$

which is just the normalization condition of $|j_1, j_2; j, m\rangle$.

3 Recursion relation-1 for the CG co-efficients

((First step))

$$\hat{J}_{\pm}\left| j_1, j_2; j, m \right\rangle = (\hat{J}_{1\pm} + \hat{J}_{2\pm}) \sum_{\substack{m_1', m_2' \\ m_1'+m_2'=m}} \left| j_1, j_2; m_1', m_2' \right\rangle \left\langle j_1, j_2; m_1', m_2' \right| j_1, j_2; j, m \rangle$$

$$\sqrt{(j \mp m)(j \pm m + 1)}\left| j_1, j_2; j, m \pm 1 \right\rangle$$

$$= \sum_{\substack{m_1', m_2' \\ m_1'+m_2'=m}} [\sqrt{(j_1 \mp m_1')(j_1 \pm m_1'+1)}\left| j_1, j_2; m_1'\pm 1, m_2' \right\rangle \left\langle j_1, j_2; m_1', m_2' \right| j_1, j_2; j, m \rangle$$

$$+ \sqrt{(j_2 \mp m_2')(j_2 \pm m_2'+1)}\left| j_1, j_2; m_1', m_2'\pm 1 \right\rangle \left\langle j_1, j_2; m_1', m_2' \right| j_1, j_2; j, m \rangle]$$

((Second step))

Next step is to multiply by $\left\langle j_1, j_2; m_1, m_2 \right|$ on the left and to use orthonormality.

$$\sqrt{(j \mp m)(j \pm m + 1)}\left\langle j_1, j_2; m_1, m_2 \right| j_1, j_2; j, m \pm 1 \rangle$$

$$= \sum_{\substack{m_1', m_2' \\ m_1'+m_2'=m}} [\sqrt{(j_1 \mp m_1')(j_1 \pm m_1'+1)}\left\langle j_1, j_2; m_1, m_2 \right| j_1, j_2; m_1'\pm 1, m_2' \rangle \left\langle j_1, j_2; m_1', m_2' \right| j_1, j_2; j, m \rangle$$

$$+ \sqrt{(j_2 \mp m_2')(j_2 \pm m_2'+1)}\left\langle j_1, j_2; m_1, m_2 \right| j_1, j_2; m_1', m_2'\pm 1 \rangle \left\langle j_1, j_2; m_1', m_2' \right| j_1, j_2; j, m \rangle]$$

or

$$\sqrt{(j \mp m)(j \pm m + 1)}\left\langle j_1, j_2; m_1, m_2 \right| j_1, j_2; j, m \pm 1 \rangle$$

$$= \sum_{\substack{m_1', m_2' \\ m_1'+m_2'=m}} [\sqrt{(j_1 \mp m_1')(j_1 \pm m_1'+1)}\delta_{m_1, m_1'\pm 1}\delta_{m_2, m_2'}\left\langle j_1, j_2; m_1', m_2' \right| j_1, j_2; j, m \rangle$$

$$+ \sqrt{(j_2 \mp m_2')(j_2 \pm m_2'+1)}\delta_{m_1, m_1'}\delta_{m_2, m_2'\pm 1}\left\langle j_1, j_2; m_1', m_2' \right| j_1, j_2; j, m \rangle]$$

or

$$\sqrt{(j\mp m)(j\pm m+1)}\langle j_1,j_2;m_1,m_2|j_1,j_2;j,m\pm 1\rangle$$

$$=\sqrt{[j_1\mp(m_1\mp 1))(j_1\pm(m_1\mp 1)+1]}\langle j_1,j_2;m_1\mp 1,m_2|j_1,j_2;j,m\rangle$$

$$+\sqrt{[j_2\mp(m_2\mp 1)(j_2\pm(m_2\mp 1)+1]}\langle j_1,j_2;m_1,m_2\mp 1|j_1,j_2;j,m\rangle$$

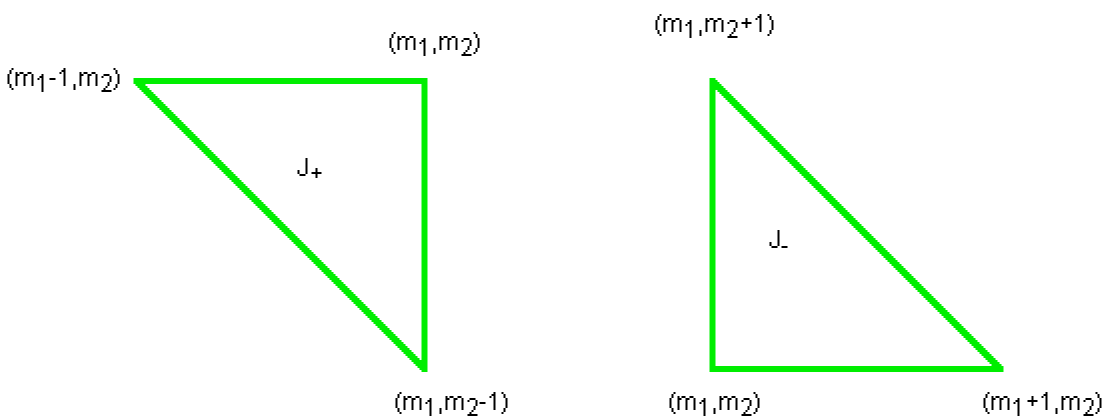
((Note)) We use a more convenient expression

$$\sqrt{(j\mp m+1)(j\pm m)}\langle j_1,j_2;m_1,m_2|j_1,j_2;j,m\rangle$$

$$=\sqrt{(j_1\mp m_1+1)(j_1\pm m_1)}\langle j_1,j_2;m_1\mp 1,m_2|j_1,j_2;j,m\mp 1\rangle$$

$$+\sqrt{(j_2\mp m_2+1)(j_2\pm m_2)}\langle j_1,j_2;m_1,m_2\mp 1|j_1,j_2;j,m\mp 1\rangle$$

This recursion relations are related to the CG coefficients at (m_1, m_2), ($m_1\mp 1$, m_2), and (m_1, $m_2\mp 1$), where m_1+m_2 = m.



4 Recursion relation-2 for the CG co-efficients

$$\hat{\boldsymbol{J}}^2|j_1,j_2;j,m\rangle=\hat{\boldsymbol{J}}^2\sum_{\substack{m_1',m_2'\\ m_1'+m_2'=m}}|j_1,j_2;m_1',m_2'\rangle\langle j_1,j_2;m_1',m_2'|j_1,j_2;j,m\rangle$$

$$\hat{\boldsymbol{J}}^2=\hat{\boldsymbol{J}}_1^{\,2}+\hat{\boldsymbol{J}}_2^{\,2}+2\hat{J}_{1z}\hat{J}_{2z}+\hat{J}_{1+}\hat{J}_{2-}+\hat{J}_{1-}\hat{J}_{2+}$$

$$\hat{\boldsymbol{J}}^2|j_1,j_2;j,m\rangle=\hbar^2 j(j+1)|j_1,j_2;j,m\rangle$$

$$\sum_{\substack{m_1',m_2' \\ m_1'+m_2'=m}} \hat{\mathbf{J}}^2 \left| j_1, j_2; m_1', m_2' \right\rangle \left\langle j_1, j_2; m_1', m_2' \right| j_1, j_2; j, m \rangle$$

$$= \sum_{\substack{m_1',m_2' \\ m_1'+m_2'=m}} (\hat{\mathbf{J}}_1{}^2 + \hat{\mathbf{J}}_2{}^2 + 2\hat{J}_{1z}\hat{J}_{2z} + \hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+})$$

$$\times \left| j_1, j_2; m_1', m_2' \right\rangle \left\langle j_1, j_2; m_1', m_2' \right| j_1, j_2; j, m \rangle$$

Here

$$\sum_{\substack{m_1',m_2' \\ m_1'+m_2'=m}} (\hat{\mathbf{J}}_1{}^2 + \hat{\mathbf{J}}_2{}^2 + 2\hat{J}_{1z}\hat{J}_{2z}) \left| j_1, j_2; m_1', m_2' \right\rangle \left\langle j_1, j_2; m_1', m_2' \right| j_1, j_2; j, m \rangle$$

$$= \hbar^2 \sum_{\substack{m_1',m_2' \\ m_1'+m_2'=m}} \{ j_1(j_1+1) + j_2(j_2+1) + 2m_1'm_2' \}$$

$$\times \left| j_1, j_2; m_1', m_2' \right\rangle \left\langle j_1, j_2; m_1', m_2' \right| j_1, j_2; j, m \rangle$$

$$\sum_{\substack{m_1',m_2' \\ m_1'+m_2'=m}} (\hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+}) \left| j_1, j_2; m_1', m_2' \right\rangle \left\langle j_1, j_2; m_1', m_2' \right| j_1, j_2; j, m \rangle$$

$$= \hbar^2 \sum_{\substack{m_1',m_2' \\ m_1'+m_2'=m}} \{ \sqrt{(j_1 - m_1')(j_1 + m_1'+1)} \sqrt{(j_2 + m_2')(j_2 - m_2'+1)}$$

$$\times \left| j_1, j_2; m_1'+1, m_2'-1 \right\rangle \left\langle j_1, j_2; m_1', m_2' \right| j_1, j_2; j, m \rangle$$

$$+ \hbar^2 \sum_{\substack{m_1',m_2' \\ m_1'+m_2'=m}} \{ \sqrt{(j_1 + m_1')(j_1 - m_1'+1)} \sqrt{(j_2 - m_2')(j_2 + m_2'+1)}$$

$$\times \left| j_1, j_2; m_1'-1, m_2'+1 \right\rangle \left\langle j_1, j_2; m_1', m_2' \right| j_1, j_2; j, m \rangle$$

Note that

$$\left\langle j_1, j_2; m_1, m_2 \right| j_1, j_2; m_1', m_2' \rangle = \delta_{m_1, m_1'} \delta_{m_2, m_2'}$$

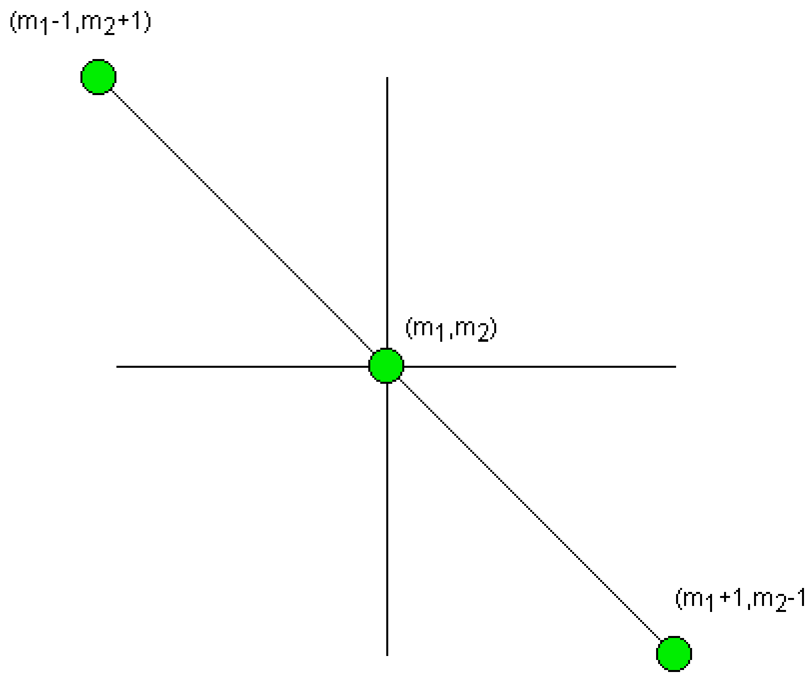
We multiply the above equation by bra $\left\langle j_1, j_2; m_1, m_2 \right|$

$$\{ j(j+1) - j_1(j_1+1) - j_2(j_2+1) - 2m_1 m_2) \} \left\langle j_1, j_2; m_1, m_2 \right| j_1, j_2; j, m \rangle$$

$$= \sqrt{(j_1 - m_1 + 1)(j_1 + m_1)} \sqrt{(j_2 + m_2 + 1)(j_2 - m_2)} \left\langle j_1, j_2; m_1 - 1, m_2 + 1 \right| j_1, j_2; j, m \rangle$$

$$+ \sqrt{(j_1 + m_1 + 1)(j_1 - m_1)} \sqrt{(j_2 - m_2 + 1)(j_2 + m_2)} \left\langle j_1, j_2; m_1 + 1, m_2 - 1 \right| j_1, j_2; j, m \rangle$$

This recursion relations are related to the CG coefficients at (m_1, m_2), (m_1-1, m_2+1), and (m_1+1, m_2-1), where $m_1+m_2 = m$.



$$D_{j_1} \times D_{j_2} = D_{j_1+j_2} + D_{j_1+j_2-1} + ... + D_{|j_1-j_2|}$$

For fixed j, $m = j, j\text{-}1, ..., \text{-}j$.

In the (m_1, m_2) plane, we plot the boundary of the allowed region determined by

$$|m_1| \le j_1, \qquad |m_2| \le j_2$$

$$m_1 + m_2 = m$$

The J_+ recursion relation tells us that the co-efficient at (m_1, m_2) is related to those at (m_1-1, m_2) and (m_1, m_2-1). Likewise, the J_- recursion relation (lower sign) relates the three co-efficients whose m_1, m_2 values are given in Fig.

(Recursion relations + Normalization conditions)→uniquely determine the CG co-efficients.

$$\left|l_1, l_2; j, m\right\rangle = \left|j, m\right\rangle = \sum_{\substack{m_1, m_2 \\ m_1+m_2=m}} C_{m_1,m_2} \left|l_1, m_1\right\rangle \left|l_2, m_2\right\rangle$$

5 CG values: $j_1 = 1/2, j_2 = 1/2$

$j_1 = 1/2, j_2 = 1/2$ ($|m_1| \le 1/2$, $|m_2| \le 1/2$)

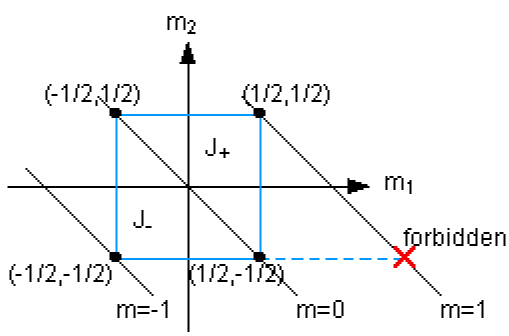
$D_{1/2} \times D_{1/2} = D_1 + D_0 \rightarrow j = 1$ and $j = 0$

(i) $j = 1$

$m = 1, 0, -1$

where

$m = m_1 + m_2$



$$\left|\frac{1}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle \qquad (m = 1)$$

$$\frac{\left|\frac{1}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle + \left|\frac{1}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle}{\sqrt{2}} \qquad (m = 0)$$

$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle \qquad (m = -1)$$

(ii) $j = 0$ ($m = 0$)

$$\frac{\left|\frac{1}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle-\left|\frac{1}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle}{\sqrt{2}} \qquad (m=0)$$

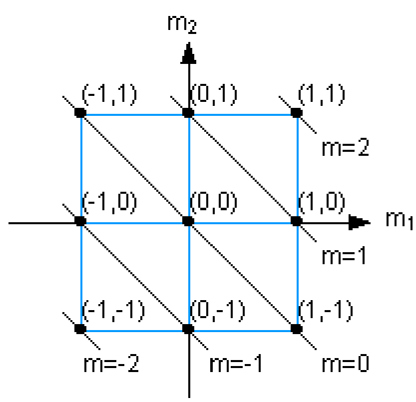
6 CG values: $j_1 = 1, j_2 = 1$

$j_1 = 1, j_2 = 1$ ($|m_1| \leq 1, |m_2| \leq 1$)

$$D_1 \times D_1 = D_2 + D_1 + D_0$$

(i) $j = 2$ ($|m| \leq 2$)

$m = m_1 + m_2$



$$|1,1\rangle|1,1\rangle \qquad (m=2)$$

$$\frac{|1,1\rangle|1,0\rangle+|1,0\rangle|1,1\rangle}{\sqrt{2}} \qquad (m=1)$$

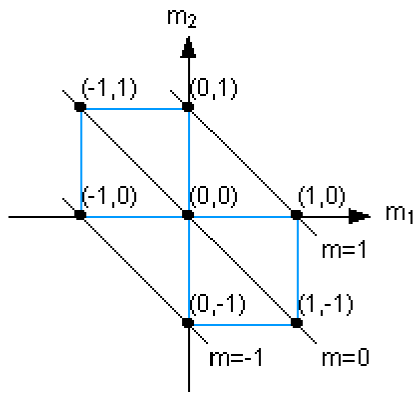
$$\frac{|1,1\rangle|1,-1\rangle+2|1,0\rangle|1,0\rangle+|1,-1\rangle|1,1\rangle}{\sqrt{6}} \qquad (m=0)$$

$$\frac{|1,0\rangle|1,-1\rangle+|1,-1\rangle|1,0\rangle}{\sqrt{2}} \qquad (m=-1)$$

$$|1,-1\rangle|1,-1\rangle \qquad (m=-2)$$

Symmetric for the particle exchange

(ii) $j = 1$ ($|m| \leq 1$)



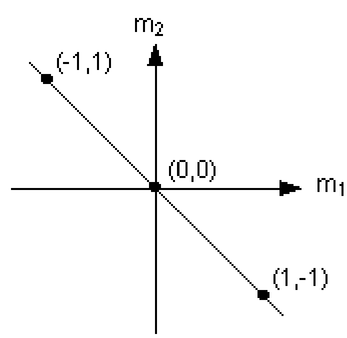
$$\frac{|1,1\rangle|1,0\rangle - |1,0\rangle|1,1\rangle}{\sqrt{2}} \qquad (m = 1)$$

$$\frac{|1,1\rangle|1,-1\rangle - |1,-1\rangle|1,1\rangle}{\sqrt{2}} \qquad (m = 0)$$

$$\frac{|1,0\rangle|1,-1\rangle - |1,-1\rangle|1,0\rangle}{\sqrt{2}} \qquad (m = -1)$$

Anti-symmetric for the particle exchange

(iii) $j = 0$ ($m = 0$)



$$\frac{|1,1\rangle|1,-1\rangle - |1,0\rangle|1,0\rangle + |1,-1\rangle|1,1\rangle}{\sqrt{3}}$$

Symmetric for the particle exchange

---

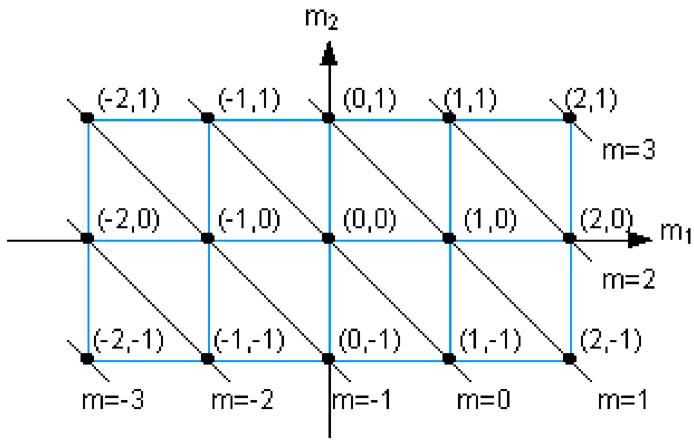
7 CG values: $j_1 = 2, j_2 = 1$

$j_1 = 2, j_2 = 1$ ($|m_1| \le 2, |m_2| \le 1$)

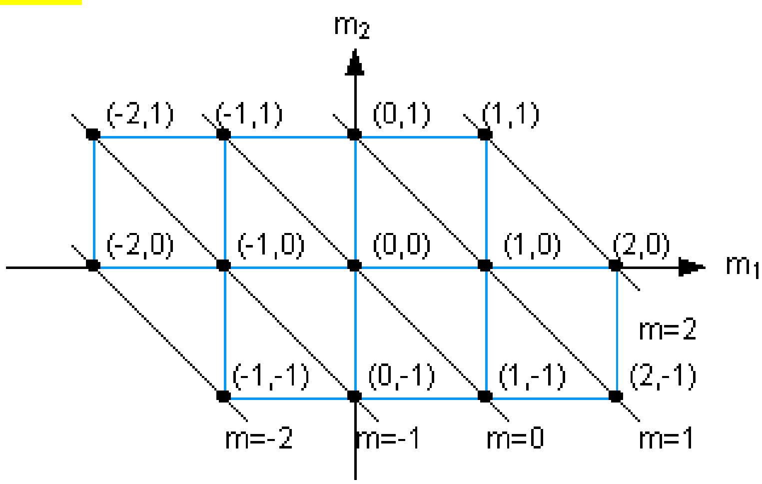
$D_2 \times D_1 = D_3 + D_2 + D_1$

(i) $j = 3$ ($|m| \le 3$)

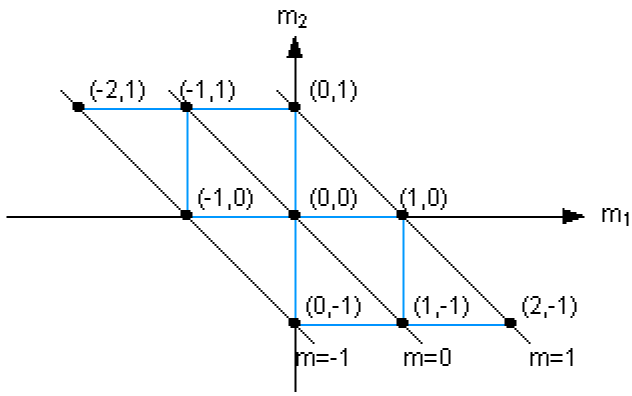
$m = m_1 + m_2$



(ii) $j = 2$ ($|m| \le 2$)



(iii) $j = 1$ (($|m| \le 1$)



8. Addition of orbital and spin angular momentum

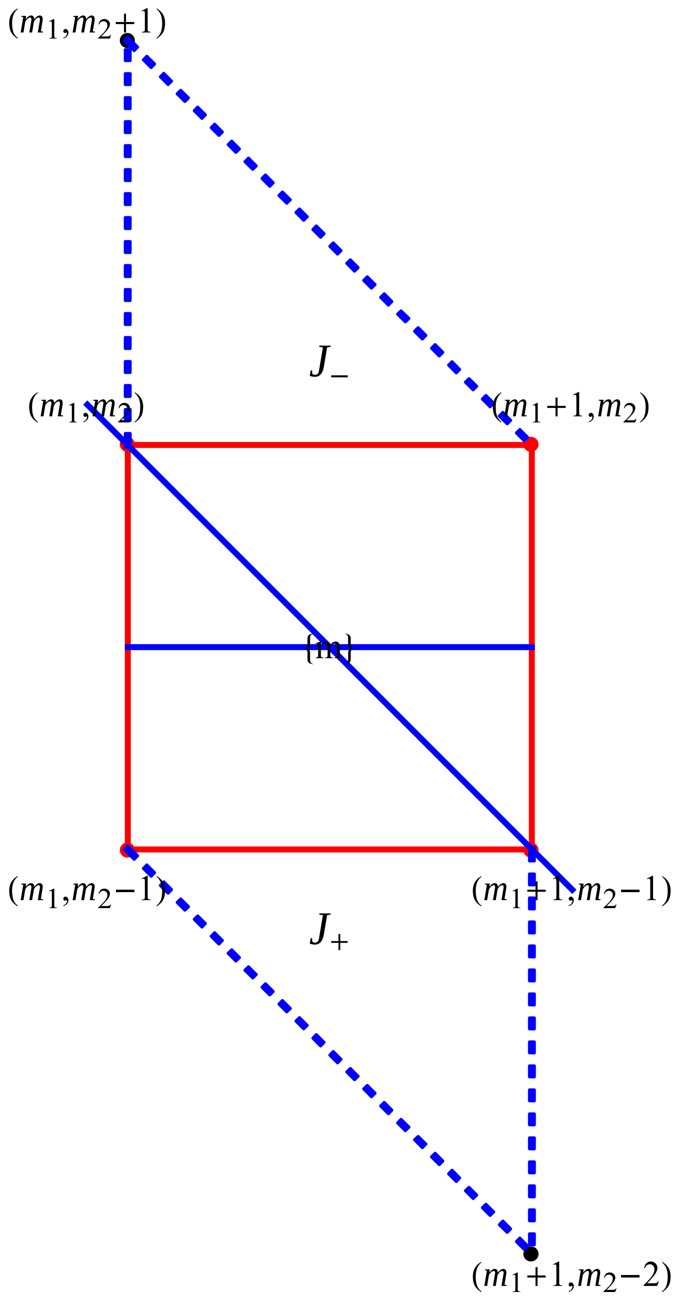
We consider one electron with the orbital angular momentum ***L*** (typically $l = 1$, *p* electron)and the spin angular momentum ***S***. The total angular momentum *J* is an addition of ***L*** and ***S***.

$$\hat{J} = \hat{L} + \hat{S}$$

$$D_l \times D_{1/2} = D_{l+1/2} + D_{l-1/2}$$

9 *J*₋ recursion

We use the J_- recursion for the diagram below.



$$\sqrt{(j+m+1)(j-m)}\langle j_1, j_2; m_1, m_2 \mid j_1, j_2; j, m\rangle$$

$$= \sqrt{(j_1+m_1+1)(j_1-m_1)}\langle j_1, j_2; m_1+1, m_2 \mid j_1, j_2; j, m+1\rangle$$

$$+ \sqrt{(j_2+m_2+1)(j_2-m_2)}\langle j_1, j_2; m_1, m_2+1 \mid j_1, j_2; j, m+1\rangle$$

We note that

$$\langle j_1, j_2; m_1, m_2+1 | j_1, j_2; j, m+1\rangle = 0$$

Then we have

$$\sqrt{(j+m+1)(j-m)}\langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m\rangle$$

$$= \sqrt{(j_1+m_1+1)(j_1-m_1)}\langle j_1, j_2; m_1+1, m_2 | j_1, j_2; j, m+1\rangle$$

We assume that

$j_1 = l$, j_2=1/2, j= l+1/2

m_1=m - 1/2, m_2 = 1/2

Then we have

$$\sqrt{(l+\frac{1}{2}+m+1)(l+\frac{1}{2}-m)}\left\langle j_1 = l, j_2 = \frac{1}{2}; m-\frac{1}{2}, \frac{1}{2}\right| \left. j_1 = l, j_2 = \frac{1}{2}; j = l+\frac{1}{2}, m\right\rangle$$

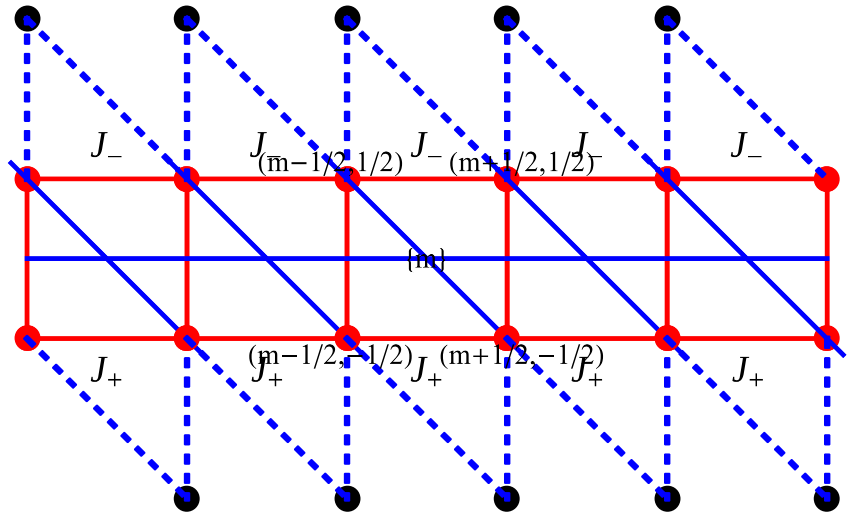
$$= \sqrt{(l+m+\frac{1}{2})(l-m+\frac{1}{2})}\left\langle j_1 = l, j_2 = \frac{1}{2}; m+\frac{1}{2}, \frac{1}{2}\right| \left. j_1 = l, j_2 = \frac{1}{2}; j = l+\frac{1}{2}, m+1\right\rangle$$

or

$$\left\langle j_1 = l, j_2 = \frac{1}{2}; m-\frac{1}{2}, \frac{1}{2}\right| \left. j_1 = l, j_2 = \frac{1}{2}; j = l+\frac{1}{2}, m\right\rangle$$

$$= \sqrt{\frac{l+m+\frac{1}{2}}{l+m+\frac{3}{2}}}\left\langle j_1 = l, j_2 = \frac{1}{2}; m+\frac{1}{2}, \frac{1}{2}\right| \left. j_1 = l, j_2 = \frac{1}{2}; j = l+\frac{1}{2}, m+1\right\rangle$$

This procedure can be continued until $m = l + 1/2$.



Then we have

$$\left\langle j_1 = l, j_2 = \frac{1}{2}; m - \frac{1}{2}, \frac{1}{2} \right| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m \Big\rangle$$

$$= \sqrt{\frac{l+m+\frac{1}{2}}{l+m+\frac{3}{2}}}\sqrt{\frac{l+m+\frac{3}{2}}{l+m+\frac{5}{2}}}\left\langle j_1 = l, j_2 = \frac{1}{2}; m + \frac{3}{2}, \frac{1}{2} \right| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m + 2 \Big\rangle$$

$$= \sqrt{\frac{l+m+\frac{1}{2}}{l+m+\frac{3}{2}}}\sqrt{\frac{l+m+\frac{3}{2}}{l+m+\frac{5}{2}}}\sqrt{\frac{l+m+\frac{5}{2}}{l+m+\frac{7}{2}}}\left\langle j_1 = l, j_2 = \frac{1}{2}; m + \frac{5}{2}, \frac{1}{2} \right| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2} j, m + 3 \Big\rangle$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}}\left\langle j_1 = l, j_2 = \frac{1}{2}; l, \frac{1}{2} \right| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, l + \frac{1}{2} \Big\rangle$$

We choose

$$\left\langle j_1 = l, j_2 = \frac{1}{2}; l, \frac{1}{2} \right| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, l + \frac{1}{2} \Big\rangle = 1$$

Then we have

$$\left\langle j_1 = l, j_2 = \frac{1}{2}; m - \frac{1}{2}, \frac{1}{2} \right| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m \bigg\rangle = \sqrt{\frac{l + m + \frac{1}{2}}{2l + 1}}$$

10 J_+ recursion

We use the J_+ recursion for the diagram.

$$\sqrt{(j-m+1)(j+m)}\langle j_1, j_2; m_1, m_2 \,|\, j_1, j_2; j, m\rangle$$

$$= \sqrt{(j_1 - m_1 + 1)(j_1 + m_1)}\langle j_1, j_2; m_1 - 1, m_2 \,|\, j_1, j_2; j, m-1\rangle$$

$$+ \sqrt{(j_2 - m_2 + 1)(j_2 + m_2)}\langle j_1, j_2; m_1, m_2 - 1 \,|\, j_1, j_2; j, m-1\rangle$$

We assume that

$j_1 = l$, $j_2 = 1/2$, $j = l + 1/2$

$m_1 = m + 1/2$, $m_2 = -1/2$

Then we have

$$\sqrt{(l - m + \frac{2}{3})(l + m + \frac{1}{2})}\left\langle j_1 = l, j_2 = \frac{1}{2}; m + \frac{1}{2}, -\frac{1}{2} \right| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m \bigg\rangle$$

$$= \sqrt{(l - m + \frac{1}{2})(l + m + \frac{1}{2})}\left\langle j_1 = l, j_2 = \frac{1}{2}; m - \frac{1}{2}, -\frac{1}{2} \right| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m - 1 \bigg\rangle$$

or

$$\left\langle j_1 = l, j_2 = \frac{1}{2}; m + \frac{1}{2}, -\frac{1}{2} \right| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m \bigg\rangle$$

$$= \sqrt{\frac{l - m + \frac{1}{2}}{l - m + \frac{3}{2}}}\left\langle j_1 = l, j_2 = \frac{1}{2}; m - \frac{1}{2}, -\frac{1}{2} \right| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m - 1 \bigg\rangle$$

This procedure can be continued until $m = -l - 1/2$.

$$\left\langle j_1=l, j_2=\frac{1}{2}; m+\frac{1}{2}, -\frac{1}{2} \middle| j_1=l, j_2=\frac{1}{2}; j=l+\frac{1}{2}, m \right\rangle$$

$$= \sqrt{\frac{l-m+\frac{1}{2}}{l-m+\frac{3}{2}}} \left\langle j_1=l, j_2=\frac{1}{2}; m-\frac{1}{2}, -\frac{1}{2} \middle| j_1=l, j_2=\frac{1}{2}; j=l+\frac{1}{2}, m-1 \right\rangle$$

$$= \sqrt{\frac{l-m+\frac{1}{2}}{l-m+\frac{3}{2}}} \sqrt{\frac{l-m+\frac{3}{2}}{l-m+\frac{5}{2}}} \left\langle j_1=l, j_2=\frac{1}{2}; m-\frac{3}{2}, -\frac{1}{2} \middle| j_1=l, j_2=\frac{1}{2}; j=l+\frac{1}{2}, m-2 \right\rangle$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \sqrt{\frac{l-m+\frac{1}{2}}{2l+1}} \left\langle j_1=l, j_2=\frac{1}{2}; -l, -\frac{1}{2} \middle| j_1=l, j_2=\frac{1}{2}; j=l+\frac{1}{2}, -l-\frac{1}{2} \right\rangle$$

We choose

$$\left\langle j_1=l, j_2=\frac{1}{2}; -l, -\frac{1}{2} \middle| j_1=l, j_2=\frac{1}{2}; j=l+\frac{1}{2}, -l-\frac{1}{2} \right\rangle = 1$$

Then we have

$$\left\langle j_1=l, j_2=\frac{1}{2}; m+\frac{1}{2}, -\frac{1}{2} \middle| j_1=l, j_2=\frac{1}{2}; j=l+\frac{1}{2}, m \right\rangle = \sqrt{\frac{l-m+\frac{1}{2}}{2l+1}}$$

11 Derivation of Clebsch-Gordan coefficient)

We assume that

$$|\, j=l+1/2, m\rangle = \alpha |m_l = m-1/2, m_s = 1/2\rangle + \beta |m_l = m+1/2, m_s = -1/2\rangle$$

$$|\, j=l-1/2, m\rangle = \gamma |m_l = m-1/2, m_s = 1/2\rangle + \delta |m_l = m+1/2, m_s = -1/2\rangle$$

where α, β, γ, and δ are real. The normalization condition:

$$\alpha^2 + \beta^2 = 1$$

$$\gamma^2 + \delta^2 = 1$$

The condition of the orthogonality:

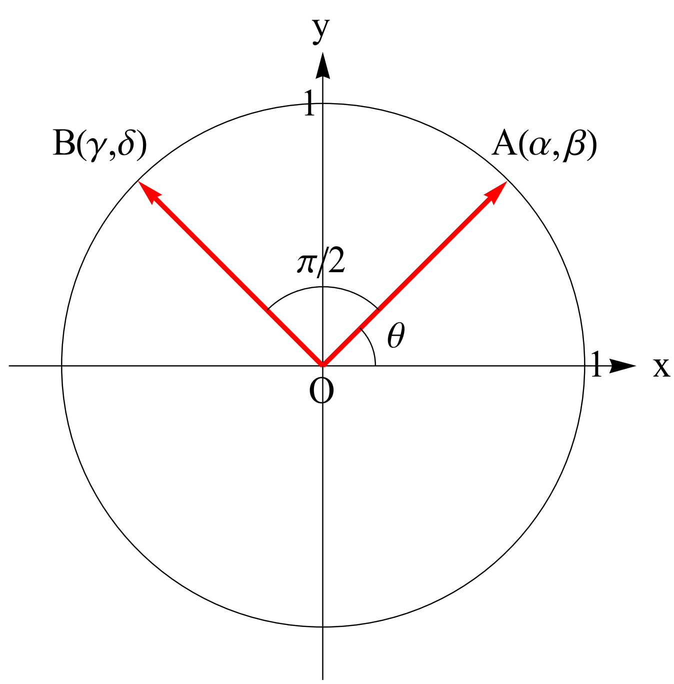
$$\alpha\gamma + \beta\delta = 0$$

There are four unknown parameters and three equations. So we need one more equation. In Appendix, we show that

$$\frac{\alpha}{\beta} = \sqrt{\frac{l+m+\frac{1}{2}}{l-m+\frac{1}{2}}}\,.$$

We want to determine the values of β, γ, and δ. To this end, we assume that

$$\alpha = \cos\theta,$$
$$\beta = \sin\theta$$
$$\gamma = \cos(\theta + \pi/2) = -\sin\theta,$$
$$\delta = \sin(\theta + \pi/2) = \cos\theta$$



Since

$$\alpha^2 + \beta^2 = 1$$

we have

$$\beta = \sqrt{1 - \frac{l+m+\frac{1}{2}}{2l+1}} = \sqrt{\frac{l-m+\frac{1}{2}}{2l+1}}$$

Then we get

$$\alpha = \delta = \sqrt{\frac{l+m+1/2}{2l+1}}, \qquad \beta = -\gamma = \sqrt{\frac{l-m+1/2}{2l+1}}$$

The final result is

$$\left| j = l+1/2, m \right\rangle = \sqrt{\frac{l+m+1/2}{2l+1}} \left| m_l = m-1/2, m_s = 1/2 \right\rangle + \sqrt{\frac{l-m+1/2}{2l+1}} \left| m_l = m+1/2, m_s = -1/2 \right\rangle$$

$$\left| j = l-1/2, m \right\rangle = -\sqrt{\frac{l-m+1/2}{2l+1}} \left| m_l = m-1/2, m_s = 1/2 \right\rangle + \sqrt{\frac{l+m+1/2}{2l+1}} \left| m_l = m+1/2, m_s = -1/2 \right\rangle$$

or

$$\left| j = l+1/2, m \right\rangle = \begin{pmatrix} \sqrt{\frac{l+m+1/2}{2l+1}} \\ \sqrt{\frac{l-m+1/2}{2l+1}} \end{pmatrix},$$

$$\left| j = l-1/2, m \right\rangle == \begin{pmatrix} -\sqrt{\frac{l-m+1/2}{2l+1}} \\ \sqrt{\frac{l+m+1/2}{2l+1}} \end{pmatrix}$$

We note that

$$\left| m_l = m-1/2, m_s = 1/2 \right\rangle = \alpha \left| j = l+1/2, m \right\rangle - \beta \left| j = l-1/2, m \right\rangle$$

$$\left| m_l = m+1/2, m_s = -1/2 \right\rangle = \beta \left| j = l+1/2, m \right\rangle + \alpha \left| j = l-1/2, m \right\rangle$$

12 Mathematica: The use of Standard Conventions of Edmonds

It is much easier to calculate the Clebsch-Gordan co-efficient by using Mathematica.

(i) ClebschGordan[{*l*,*m*-1/2},{1/2,1/2},{*l*+1/2,*m*}]

$$\alpha = -\frac{(-1)^{2(m+l)}\sqrt{1+2m+2l}}{\sqrt{2+4l}}$$

Since l is an integer and m is an half-integer, we have $(-1)^{2(m+l)} = -1$. Then α is simplified as

$$\alpha = \frac{\sqrt{1+2m+2l}}{\sqrt{2+4l}}$$

(ii) ClebschGordan[{*l*,*m*+1/2},{1/2,-1/2},{*l*+1/2,*m*}]

$$\beta = -\frac{(-1)^{2(m+l)}\sqrt{1-2m+2l}}{\sqrt{2+4l}}$$

or

$$\beta = \frac{\sqrt{1-2m+2l}}{\sqrt{2+4l}}$$

(iii) ClebschGordan[{*l*,*m*-1/2},{1/2, 1/2},{*l*-1/2,*m*}]

$$\gamma = -\frac{\sqrt{1-2m+2l}}{\sqrt{2+4l}}.$$

(iv) ClebschGordan[{*l*,*m*+1/2},{1/2, -1/2},{*l*-1/2,*m*}]

$$\delta = \frac{\sqrt{1+2m+2l}}{\sqrt{2+4l}}.$$

13. Clebsch-Gordan coefficient with *l* and *s* = 1

$D_l \times D_1 = D_{l+1} + D_l + D_{l-1}$

(i) ClebschGordan[{*l*,*m*-1},{1, 1},{*l*+1,*m*}]

$$\frac{(-1)^{2(m+l)}\sqrt{m+l}\sqrt{1+m+l}}{\sqrt{2}\sqrt{1+l}\sqrt{1+2l}}.$$

(ii) ClebschGordan[{l,m},{1, 0},{l+1,m}]

$$\frac{(-1)^{2(m+l)}\sqrt{1-m+l}\sqrt{1+m+l}}{\sqrt{1+l}\sqrt{1+2l}}.$$

(iii) ClebschGordan[{l,m+1},{1, -1},{l+1,m}]

$$\frac{(-1)^{2(m+l)}\sqrt{-m+l}\sqrt{1-m+l}}{\sqrt{2}\sqrt{1+l}\sqrt{1+2l}}.$$

(iv) ClebschGordan[{l,m-1},{1, 1},{l,m}]

$$-\frac{\sqrt{m+l}\sqrt{1-m+l}}{\sqrt{2}\sqrt{l}\sqrt{1+2l}}.$$

(v) ClebschGordan[{l,m},{1, 0},{l,m}]

$$\frac{m}{\sqrt{l}\sqrt{1+2l}}$$

(vi) ClebschGordan[{l,m+1},{1, -1},{l,m}]

$$\frac{(-1)^{2(m+l)}\sqrt{-m+l}\sqrt{1+m+l}}{\sqrt{2}\sqrt{l}\sqrt{1+2l}}$$

(vii) ClebschGordan[{l,m-1},{1, 1},{l-1,m}]

$$\frac{\sqrt{-m+l}\sqrt{1-m+l}}{\sqrt{2}\sqrt{l}\sqrt{1+2l}}$$

(viii) ClebschGordan[{l,m},{1, 0},{l-1,m}]

$$-\frac{\sqrt{-m+l}\sqrt{m+l}}{\sqrt{l}\sqrt{1+2l}}$$

(ix) ClebschGordan[{l,m+1},{1, -1},{l-1,m}]

$$\frac{\sqrt{m+l}\sqrt{1+m+l}}{\sqrt{2}\sqrt{l}\sqrt{1+2l}}$$

4. Typical example: Na D lines

(a) For the electron with 3 s state (l = 0, s = 1/2)

$D_0 \times D_{1/2} = D_{1/2}$

Thus we have j = 1/2. The state is described by $^2S_{1/2}$.

$$|j=1/2, m=1/2\rangle = |m_l=0, m_s=1/2\rangle$$

$$|j=1/2, m=-1/2\rangle = |m_l=0, m_s=-1/2\rangle$$

(b) For the electron with 3 p state (l = 1, s = 1/2)

$D_1 \times D_{1/2} = D_{3/2} + D_{1/2}$

Thus we have j = 3/2 and j = 1/2. The state is described by $^2P_{3/2}$ and $^2P_{1/2}$

(i) j = 3/2

$$|j=3/2, m=-3/2\rangle = |m_l=-1, m_s=-1/2\rangle$$

$$|j=3/2, m=-1/2\rangle = \sqrt{\frac{2}{3}}|m_l=0, m_s=-1/2\rangle + \frac{1}{\sqrt{3}}|m_l=-1, m_s=1/2\rangle$$

$$|j=3/2, m=1/2\rangle = \frac{1}{\sqrt{3}}|m_l=1, m_s=-1/2\rangle + \sqrt{\frac{2}{3}}|m_l=0, m_s=1/2\rangle$$

$$|j=3/2, m=3/2\rangle = |m_l=1, m_s=1/2\rangle$$

(ii) j = 1/2

$$|j=1/2, m=-1/2\rangle = \frac{1}{\sqrt{3}}|m_l=0, m_s=-1/2\rangle - \sqrt{\frac{2}{3}}|m_l=-1, m_s=1/2\rangle$$

$$|j=1/2, m=1/2\rangle = \sqrt{\frac{2}{3}}|m_l=1, m_s=-1/2\rangle - \frac{1}{\sqrt{3}}|m_l=0, m_s=1/2\rangle$$

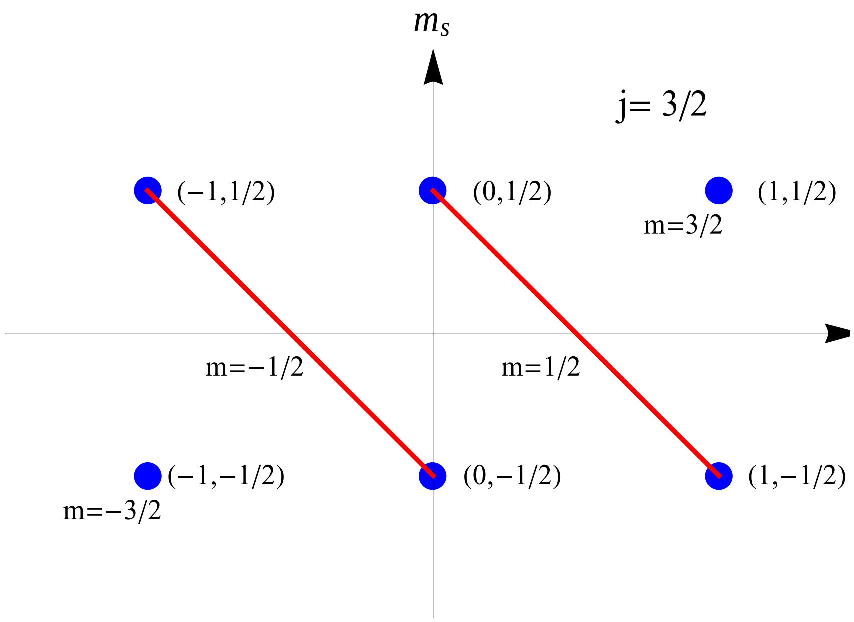


Fig. Clebsh-Gordan diagram for $|j, m\rangle$ with j = 3/2. m = 3/2, 1/2, -1/2, and -3/2.

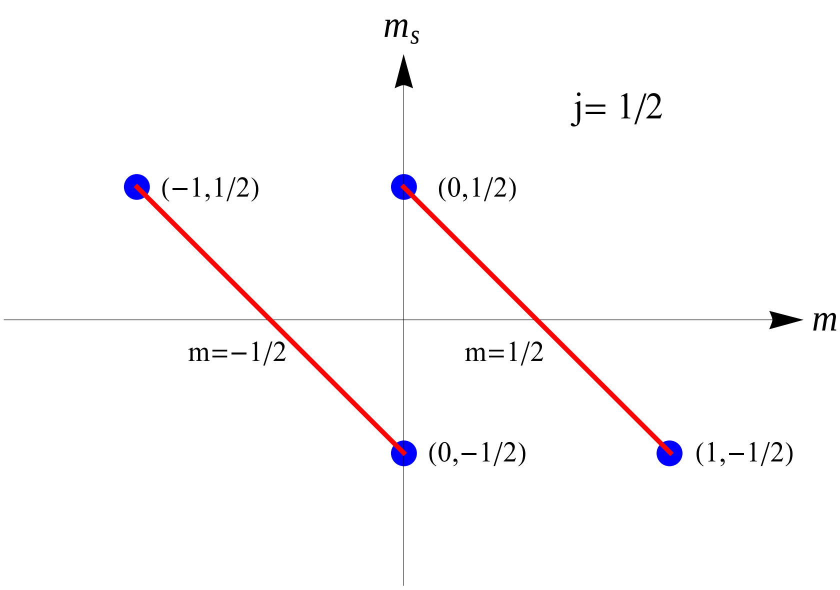


Fig. Clebsh-Gordan diagram for $|j, m\rangle$ with j = 1/2. m = 1/2 and -1/2.

5. Mathematica

ClebschGordan

`ClebschGordan[{j1, m1}, {j2, m2}, {j, m}]`

gives the Clebsch-Gordan coefficient for the decomposition of $|j, m\rangle$ in terms of $|j_1, m_1\rangle |j_2, m_2\rangle$.

▼ **Details**

- The Clebsch-Gordan coefficients vanish except when $m = m_1 + m_2$ and the j_i satisfy a triangle inequality.
- The parameters of `ClebschGordan` can be integers, half-integers, or symbolic expressions.
- *Mathematica* uses the standard conventions of Edmonds for the phase of the Clebsch-Gordan coefficients.

We can use the Mathematica to obtain the Clebsch-Gordan co-efficient. We show some typical programs using Mathematica.

5.1 $j_1 = 1/2$ and $j_2 = 1/2$

Clebsch-Gordan coefficient

```
Clear["Global`*"];

CCGG[{j1_, m1_}, {j2_, m2_}, {j_, m_}] :=
 Module[{s1},
  s1 = If[Abs[m1] ≤ j1 && Abs[m2] ≤ j2 && Abs[m] ≤ j,
    ClebschGordan[{j1, m1}, {j2, m2}, {j, m}], 0]]

CG[{j_, m_}, j1_, j2_] :=
 Sum[CCGG[{j1, m1}, {j2, m - m1}, {j, m}] a[j1 , m1]
   b[j2, m - m1], {m1, -j1, j1}]
```

j1 = 1/2 and j2 = 1/2

j = 1

```
j1 = 1 / 2; j2 = 1 / 2;

CG[{1, 1}, j1, j2]
```

$a\left[\frac{1}{2},\frac{1}{2}\right]b\left[\frac{1}{2},\frac{1}{2}\right]$

```
CG[{1, 0}, j1, j2]
```

$\frac{a\left[\frac{1}{2},\frac{1}{2}\right]b\left[\frac{1}{2},-\frac{1}{2}\right]}{\sqrt{2}}+\frac{a\left[\frac{1}{2},-\frac{1}{2}\right]b\left[\frac{1}{2},\frac{1}{2}\right]}{\sqrt{2}}$

```
CG[{1, -1}, j1, j2]
```

$a\left[\frac{1}{2},-\frac{1}{2}\right]b\left[\frac{1}{2},-\frac{1}{2}\right]$

j =0

```
j1 = 1 / 2; j2 = 1 / 2;

CG[{0, 0}, j1, j2]
```

$\frac{a\left[\frac{1}{2},\frac{1}{2}\right]b\left[\frac{1}{2},-\frac{1}{2}\right]}{\sqrt{2}}-\frac{a\left[\frac{1}{2},-\frac{1}{2}\right]b\left[\frac{1}{2},\frac{1}{2}\right]}{\sqrt{2}}$

5.2 $j_1 = 1$ and $j_2 = 1/2$

```
Clear["Global`*"];

CCGG[{j1_, m1_}, {j2_, m2_}, {j_, m_}] :=
 Module[{s1},
  s1 = If[Abs[m1] ≤ j1 && Abs[m2] ≤ j2 && Abs[m] ≤ j,
    ClebschGordan[{j1, m1}, {j2, m2}, {j, m}], 0]]

CG[{j_, m_}, j1_, j2_] :=
 Sum[CCGG[{j1, m1}, {j2, m - m1}, {j, m}] a[j1 , m1]
   b[j2, m - m1], {m1, -j1, j1}]

j1 = 1; j2 = 1 / 2;
```

j = 3/2;

```
CG[{3 / 2, 3 / 2}, j1, j2]
```

$a[1, 1]\, b\left[\frac{1}{2}, \frac{1}{2}\right]$

```
CG[{3 / 2, 1 / 2}, j1, j2]
```

$\frac{a[1, 1]\, b\left[\frac{1}{2}, -\frac{1}{2}\right]}{\sqrt{3}} + \sqrt{\frac{2}{3}}\, a[1, 0]\, b\left[\frac{1}{2}, \frac{1}{2}\right]$

```
CG[{3 / 2, -1 / 2}, j1, j2]
```

$\sqrt{\frac{2}{3}}\, a[1, 0]\, b\left[\frac{1}{2}, -\frac{1}{2}\right] + \frac{a[1, -1]\, b\left[\frac{1}{2}, \frac{1}{2}\right]}{\sqrt{3}}$

```
CG[{3 / 2, -3 / 2}, j1, j2]
```

$a[1, -1]\, b\left[\frac{1}{2}, -\frac{1}{2}\right]$

j = 1/2;

```
CG[{1 / 2, 1 / 2}, j1, j2]
```

$\sqrt{\frac{2}{3}}\, a[1, 1]\, b\left[\frac{1}{2}, -\frac{1}{2}\right] - \frac{a[1, 0]\, b\left[\frac{1}{2}, \frac{1}{2}\right]}{\sqrt{3}}$

```
CG[{1 / 2, -1 / 2}, j1, j2]
```

$\frac{a[1, 0]\, b\left[\frac{1}{2}, -\frac{1}{2}\right]}{\sqrt{3}} - \sqrt{\frac{2}{3}}\, a[1, -1]\, b\left[\frac{1}{2}, \frac{1}{2}\right]$

5.3 $j_1 = 1$ and $j_2 = 1$

```
Clear["Global`*"];

CCGG[{j1_, m1_}, {j2_, m2_}, {j_, m_}] :=
 Module[{s1},
  s1 = If[Abs[m1] ⩽ j1 && Abs[m2] ⩽ j2 && Abs[m] ⩽ j,
    ClebschGordan[{j1, m1}, {j2, m2}, {j, m}], 0]]

CG[{j_, m_}, j1_, j2_] :=
 Sum[CCGG[{j1, m1}, {j2, m - m1}, {j, m}] a[j1 , m1]
   b[j2, m - m1], {m1, -j1, j1}]

j1 = 1; j2 = 1;
```

j =2;

```
CG[{2, 2}, j1, j2]
```

a[1, 1] b[1, 1]

```
CG[{2, 1}, j1, j2]
```

$$\frac{a[1, 1]\, b[1, 0]}{\sqrt{2}} + \frac{a[1, 0]\, b[1, 1]}{\sqrt{2}}$$

CG[{2, 0}, j1, j2]

$$\frac{a[1,1]\,b[1,-1]}{\sqrt{6}} + \sqrt{\frac{2}{3}}\,a[1,0]\,b[1,0] + \frac{a[1,-1]\,b[1,1]}{\sqrt{6}}$$

CG[{2, -1}, j1, j2]

$$\frac{a[1,0]\,b[1,-1]}{\sqrt{2}} + \frac{a[1,-1]\,b[1,0]}{\sqrt{2}}$$

CG[{2, -2}, j1, j2]

a[1, -1] b[1, -1]

j = 1;

CG[{1, 1}, j1, j2]

$$\frac{a[1,1]\,b[1,0]}{\sqrt{2}} - \frac{a[1,0]\,b[1,1]}{\sqrt{2}}$$

CG[{1, 0}, j1, j2]

$$\frac{a[1,1]\,b[1,-1]}{\sqrt{2}} - \frac{a[1,-1]\,b[1,1]}{\sqrt{2}}$$

CG[{1, -1}, j1, j2]

$$\frac{a[1,0]\,b[1,-1]}{\sqrt{2}} - \frac{a[1,-1]\,b[1,0]}{\sqrt{2}}$$

j =0;

CG[{0, 0}, j1, j2]

$$\frac{a[1,1]\,b[1,-1]}{\sqrt{3}} - \frac{a[1,0]\,b[1,0]}{\sqrt{3}} + \frac{a[1,-1]\,b[1,1]}{\sqrt{3}}$$

APPENDIX-I
Mathematica

▪

j1=3/2 and j2=1/2

```
Clear["Global`*"];

CCGG[{j1_, m1_}, {j2_, m2_}, {j_, m_}] :=
 Module[{s1},
  s1 = If[Abs[m1] ≤ j1 && Abs[m2] ≤ j2 && Abs[m] ≤ j,
    ClebschGordan[{j1, m1}, {j2, m2}, {j, m}], 0]]

ψ[j_, m_, j1_, j2_] := Sum[CCGG[{j1, m1}, {j2, m - m1}, {j, m}] a[j1 , m1] b[j2, m - m1],
  {m1, -j1, j1}]

ψ[2, 2, 3 / 2, 1 / 2]
```

$a\left[\frac{3}{2}, \frac{3}{2}\right] b\left[\frac{1}{2}, \frac{1}{2}\right]$

```
ψ[2, 1, 3 / 2, 1 / 2]
```

$\frac{1}{2} a\left[\frac{3}{2}, \frac{3}{2}\right] b\left[\frac{1}{2}, -\frac{1}{2}\right] + \frac{1}{2} \sqrt{3} a\left[\frac{3}{2}, \frac{1}{2}\right] b\left[\frac{1}{2}, \frac{1}{2}\right]$

ψ[2, 0, 3 / 2, 1 / 2]

$$\frac{a\left[\frac{3}{2}, \frac{1}{2}\right] b\left[\frac{1}{2}, -\frac{1}{2}\right]}{\sqrt{2}} + \frac{a\left[\frac{3}{2}, -\frac{1}{2}\right] b\left[\frac{1}{2}, \frac{1}{2}\right]}{\sqrt{2}}$$

ψ[2, -1, 3 / 2, 1 / 2]

$$\frac{1}{2}\sqrt{3}\, a\left[\frac{3}{2}, -\frac{1}{2}\right] b\left[\frac{1}{2}, -\frac{1}{2}\right] + \frac{1}{2} a\left[\frac{3}{2}, -\frac{3}{2}\right] b\left[\frac{1}{2}, \frac{1}{2}\right]$$

ψ[2, -2, 3 / 2, 1 / 2]

$$a\left[\frac{3}{2}, -\frac{3}{2}\right] b\left[\frac{1}{2}, -\frac{1}{2}\right]$$

ψ[2, 1, 3 / 2, 1 / 2]

$$\frac{1}{2} a\left[\frac{3}{2}, \frac{3}{2}\right] b\left[\frac{1}{2}, -\frac{1}{2}\right] + \frac{1}{2}\sqrt{3}\, a\left[\frac{3}{2}, \frac{1}{2}\right] b\left[\frac{1}{2}, \frac{1}{2}\right]$$

ψ[2, 0, 3 / 2, 1 / 2]

$$\frac{a\left[\frac{3}{2}, \frac{1}{2}\right] b\left[\frac{1}{2}, -\frac{1}{2}\right]}{\sqrt{2}} + \frac{a\left[\frac{3}{2}, -\frac{1}{2}\right] b\left[\frac{1}{2}, \frac{1}{2}\right]}{\sqrt{2}}$$

ψ[2, -1, 3 / 2, 1 / 2]

$$\frac{1}{2}\sqrt{3}\, a\left[\frac{3}{2}, -\frac{1}{2}\right] b\left[\frac{1}{2}, -\frac{1}{2}\right] + \frac{1}{2} a\left[\frac{3}{2}, -\frac{3}{2}\right] b\left[\frac{1}{2}, \frac{1}{2}\right]$$

APPENDIX II
Claculation of the ratio α/β

$$|j = l + 1/2, m\rangle = \alpha|m_l = m - 1/2, m_s = 1/2\rangle + \beta|m_l = m + 1/2, m_s = -1/2\rangle$$

$$|j = l - 1/2, m\rangle = \gamma|m_l = m - 1/2, m_s = 1/2\rangle + \delta|m_l = m + 1/2, m_s = -1/2\rangle$$

$$\hat{J}_-|j, m\rangle = \hbar\sqrt{(j + m)(j - m + 1)}|j, m - 1\rangle$$

$$\hat{J}_+|j, m\rangle = \hbar\sqrt{(j - m)(j + m + 1)}|j, m + 1\rangle$$

$$\hat{S}_+|-\rangle = \hbar|+\rangle$$

$$\hat{S}_-|+\rangle = \hbar|-\rangle$$

$$\hat{J}_+|j,m\rangle = \hbar\sqrt{(j-m)(j+m+1)}|j,m+1\rangle$$

$$\hat{\boldsymbol{J}}^2|j,m\rangle = \hbar^2 j(j+1)(|j,m\rangle$$

We demand that

$$\hat{\boldsymbol{J}}^2\left|j=l+\frac{1}{2},m\right\rangle = \hbar^2 j(j+1)\left|j=l+\frac{1}{2},m\right\rangle = \hbar^2(l+\frac{1}{2})(l+\frac{3}{2})\left|j=l+\frac{1}{2},m\right\rangle$$

where

$$\left|j=l+\frac{1}{2},m\right\rangle = \alpha\left|m_l=m-\frac{1}{2},m_s=\frac{1}{2}\right\rangle + \beta\left|m_l=m+\frac{1}{2},m_s=-\frac{1}{2}\right\rangle.$$

Here we note that

$$\hat{\boldsymbol{J}} = \hat{\boldsymbol{L}} + \hat{\boldsymbol{S}}$$

$$\hat{\boldsymbol{J}}^2 = \hat{\boldsymbol{L}}^2 + \hat{\boldsymbol{S}}^2 + 2\hat{\boldsymbol{L}}\cdot\hat{\boldsymbol{S}} = \hat{\boldsymbol{L}}^2 + \hat{\boldsymbol{S}}^2 + 2\hat{L}_z\cdot\hat{S}_z + (\hat{L}_+\hat{S}_- + \hat{L}_-\hat{S}_+)$$

Then we get

$$\begin{aligned}\hat{\boldsymbol{J}}^2\alpha\left|m_l=m-\frac{1}{2},m_s=\frac{1}{2}\right\rangle &= \hbar^2\alpha[l(l+1)+\frac{3}{4}+(m-\frac{1}{2})+\hat{L}_+\hat{S}_-]\left|m_l=m-\frac{1}{2},m_s=\frac{1}{2}\right\rangle \\ &= \hbar^2\alpha[l(l+1)+\frac{3}{4}+(m-\frac{1}{2})\left|m_l=m-\frac{1}{2},m_s=\frac{1}{2}\right\rangle \\ &\quad + \hbar^2\alpha\sqrt{(l+m+\frac{1}{2})(l-m+\frac{1}{2})}\left|m_l=m+\frac{1}{2},m_s=-\frac{1}{2}\right\rangle\end{aligned}$$

$$\begin{aligned}\hat{\boldsymbol{J}}^2\beta\left|m_l=m+\frac{1}{2},m_s=-\frac{1}{2}\right\rangle &= \hbar^2\beta[l(l+1)+\frac{3}{4}-(m+\frac{1}{2})+\hat{L}_-\hat{S}_+]\left|m_l=m+\frac{1}{2},m_s=-\frac{1}{2}\right\rangle \\ &= \hbar^2\beta[l(l+1)+\frac{3}{4}-(m+\frac{1}{2})\left|m_l=m+\frac{1}{2},m_s=-\frac{1}{2}\right\rangle \\ &\quad + \hbar^2\beta\sqrt{(l+m+\frac{1}{2})(l-m+\frac{1}{2})}\left|m_l=m-\frac{1}{2},m_s=\frac{1}{2}\right\rangle\end{aligned}$$

or

$$\hbar^2(l+\frac{1}{2})(l+\frac{3}{2})[\alpha\left|m_l=m-\frac{1}{2},m_s=\frac{1}{2}\right\rangle+\beta\left|m_l=m+\frac{1}{2},m_s=-\frac{1}{2}\right\rangle]$$

$$=\hbar^2\alpha[l(l+1)+\frac{3}{4}+(m-\frac{1}{2})]\left|m_l=m-\frac{1}{2},m_s=\frac{1}{2}\right\rangle$$

$$+\hbar^2\alpha\sqrt{(l+m+\frac{1}{2})(l-m+\frac{1}{2})}\left|m_l=m+\frac{1}{2},m_s=-\frac{1}{2}\right\rangle$$

$$+\hbar^2\beta[l(l+1)+\frac{3}{4}-(m+\frac{1}{2})]\left|m_l=m+\frac{1}{2},m_s=-\frac{1}{2}\right\rangle$$

$$+\hbar^2\beta\sqrt{(l+m+\frac{1}{2})(l-m+\frac{1}{2})}\left|m_l=m-\frac{1}{2},m_s=\frac{1}{2}\right\rangle$$

or

$$[\alpha\sqrt{(l+m+\frac{1}{2})(l-m+\frac{1}{2})}-\beta(l+m+\frac{1}{2})]\left|m_l=m-1/2,m_s=\frac{1}{2}\right\rangle$$
$$+[\beta\sqrt{(l+m+\frac{1}{2})(l-m+\frac{1}{2})}-\alpha(l-m+\frac{1}{2})]\left|m_l=m+1/2,m_s=-\frac{1}{2}\right\rangle$$
$$=0$$

Then we have

$$\frac{\alpha}{\beta}=\sqrt{\frac{l+m+\frac{1}{2}}{l-m+\frac{1}{2}}}$$