Addition of angular momentum with the use of Kronecker product

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(Date: August 20, 2014)

For the addition of angular momentum for many spin systems, conventionally, we use the Clebsch-Gordan coefficients. Hereas an alternative method (using Mathematica), we present a different method with the use of the KroneckerProduct. Using this method we will discuss the addition of angular-momentum of n spins (spin 1/2) with n = 2, 3, 4, and 5. This method consists of two steps. The z component of the total angular momentum $(\boldsymbol{M}_{\text{tot}})_z$ and the magnitude of the total angular momentum $(\boldsymbol{M}_{\text{tot}})^2$ are expressed in terms of the KroneckerProduct of for n spins (n = 2, 3, 4, and 5). These are expressed by matrices. The eigenvalues and the eigenkets of these matrices can be determined using the Mathematica with the use of Eigensystem. Our results will be compared with those reported previously in the published books including Schiff's book, Tomonaga's book, and Mizushima's book, and so on.

It should be noted that the KroneckerProduct is non-commutative mathematically;, $\hat{A}\otimes\hat{B}\neq\hat{B}\otimes\hat{A}$. In the present case, both $(\boldsymbol{M}_{\text{tot}})_z$ and $(\boldsymbol{M}_{\text{tot}})^2$ always have the operators with the symmetric form of $\hat{A}\otimes\hat{A}$ and $(\hat{A}\otimes\hat{B}+\hat{B}\otimes\hat{A})$ or a special form of $\hat{A}\otimes\hat{I}$ or $+\hat{I}\otimes\hat{A}$, which are commutable under the exchange of the position sites. Thus the non-commutative nature for the KroneckerProduct is irrelevant in solving the eigenvalue problems of $(\boldsymbol{M}_{\text{tot}})_z$ and $(\boldsymbol{M}_{\text{tot}})^2$.

Note that we use the word, KroneckerProduct since we use the Mathematica. It is conventionally called as Kronecker product or direct product, or tensor product. The KroneckerProduct is widely used in the field of density operator, quantum entanglement, and quantum computer, and quantum teleportation.

1. Clebsch-Gordan coefficients for the addition of the angular momenta

We consider the addition of the angular momenta ($\hat{\boldsymbol{J}}_1$ and $\hat{\boldsymbol{J}}_2$) in quantum mechanics. These two angular momenta coupled together form a total angular momentum $\hat{\boldsymbol{J}}$ given by

$$\hat{\boldsymbol{J}}=\hat{\boldsymbol{J}}_1+\hat{\boldsymbol{J}}_2$$

where $\hat{\boldsymbol{J}}_1$ and $\hat{\boldsymbol{J}}_2$ are characterized by the quantum numbers j1, m1, and j2, m2,

$$\hat{\boldsymbol{J}}_1^{\,2}\left|j_1,m_1\right\rangle=\hbar^2 j_1(j_1+1)\left|j_1,m_1\right\rangle, \qquad \hat{J}_{1z}\left|j_1,m_1\right\rangle=\hbar m_1\left|j_1,m_1\right\rangle$$

and

$$\hat{\boldsymbol{J}}_2^{\,2}\left|j_2,m_2\right\rangle=\hbar^2 j_2(j_2+1)\left|j_2,m_2\right\rangle, \qquad \hat{J}_{2z}\left|j_2,m_2\right\rangle=\hbar m_2\left|j_2,m_2\right\rangle$$

Note that the uncoupled-basis $\left| j_1, m_1 \right\rangle$ is the simultaneous eigenket of $\hat{\boldsymbol{J}}_1^{\,2}$ and $\hat{J}_{1z}$ with the eigenvlaues j_1 amd m_1 (= j_1, j_1 - 1, …, - j_1), and that the uncoupled-basis $\left| j_2, m_2 \right\rangle$ is the simultaneous eigenket of $\hat{\boldsymbol{J}}_2^{\,2}$ and $\hat{J}_{2z}$ with the eigenvlaues j_2 amd m_2 (= j_2, j_2 - 1, …, - j_2), where j_1 and j_2 are either positive integers (including 0) or positive half integers. The coupled-basis eigenket of the total angular momentum is given by $\left| j, m \right\rangle$ and is expressed as

$$\left| j, m \right\rangle = \sum_{m1,m2} \left\langle j_1, j_2; m_1, m_2 \right| j_1, j_2; j, m \rangle \left| j_1, m_1 \right\rangle \otimes \left| j_2, m_2 \right\rangle$$

using the Clebsch-Gordan coefficient $\left\langle j_1, j_2; m_1, m_2 \right| j_1, j_2; j, m \rangle$, where , $m = m_1 + m_2$ and

m = j, j-1, j-2, …, -j+1, -j.

Note that the Kronecker product $\left| j_1, m_1 \right\rangle \otimes \left| j_2, m_2 \right\rangle$ is the two-particle state which is combined together with the individual ket vector $\left| j_1, m_1 \right\rangle$ and $\left| j_2, m_2 \right\rangle$. The symbol $\otimes$ is also called as a direct product or tensor product. We use the name of Kronecker product. The value of j is related to j_1 and j_2 as

$$j = j_1 + j_2,\ j_1 + j_2 - 1,\ j_1 + j_2 - 2, \ldots\ldots., |j_1 - j_2|, \text{ or}$$

$$|j_1 - j_2| \le j \le j_1 + j_2$$

which is known as the triangle inequality, reminiscent of ordinary vector addition. For each value of j, m takes the (2j+1) values; m = j, j-1, j - 2,… -j. In the notation of group theory, this can be symbolically written as

$$D_{j_1} \times D_{j_2} = D_{j_1 + j_2} + D_{j_1 + j_2 - 1} + \ldots.. + D_{|j_1 - j_2|}$$

Note that we use D_j instead of conventional form $D^{(j)}$. The dimensionality of the space spanned by $\left| j_1, m_1 \right\rangle \otimes \left| j_2, m_2 \right\rangle$ is the same as that of the space spanned by $\left| j, m \right\rangle$;

$$N = (2j_1 + 1)(2j_2 + 1)$$

2. KroneckerProduct for the addition of angular momentum

The concept of the Kronecker product have been extensively used for the description of quantum mechanics. The advance of computational physics through mathemnatics makes it easy for one to study the quantum mechanics. The Kronecker product is extensively used in the important topics of quantum mechanics, such as addition of angular momentum, Clebsch-Gordan coefficient, quantum entanglements, quantum information, even quantum teleportation. We make use of the Mathematica programs to solve the eigenvalue problems.

Here we discuss the addition of the angular momentum without the use of Clebsh-Gordan coefficient. We show how to construct the wave functions of many particles (with the angular spin momentum **S**) with the number of particles (n = 2, 3, 4, 5, …) which are simultaneous eigenkets of the square of the total angular momentum vector and the z component of the total angular momentum. Theese operators can be described by the Kronecker products.

$$\boldsymbol{M}^2 = (\boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{S}_3 + ... + \boldsymbol{S}_n)^2 ,$$

$$M_z = S_1 + S_2 + S_3 + ... + S_n ,$$

Next we use the mathematica (KroneckerProduct) to get the matrices of $\hat{\boldsymbol{M}}_{tot}{}^2$ and $\hat{M}_z$. The size of matrices increases with increasing the number (N) of spins. In the case of spin 1/2, the size of matrix is 2^N. We solve the eigenvalue problem for $\hat{M}_{tot}{}^2$ and $\hat{M}_z$ by using the Mathematica (Eigensystem program). Because of the commutable nature of these operators, we will get the the simultaneous eigenkets of of $\hat{\boldsymbol{M}}_{tot}{}^2$ and $\hat{M}_z$. To this end, we need to get the appropriate expressions of $\hat{\boldsymbol{M}}_{tot}{}^2$ and $\hat{M}_z$ in terms of the KroneckerProduct.

1. Two spins with spin 1/2
2. Three spins with spin 1/2
3. Four spins with spin 1/2
4. Five spins with spin 1/2
5. Two particles with spin 1
6. Three particles with spin 1
7. Two particles with spin 1 and spin 1/2
8. Two particles with spin 3/2 and 1/2

where S = 1/2. The spin operators (with spin 1/2) is defined by

$$\hat{S}_x = \frac{\hbar}{2}\hat{\sigma}_x , \qquad \hat{S}_y = \frac{\hbar}{2}\hat{\sigma}_y , \qquad \hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z , \qquad \hat{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where the Pauli matrices are given by

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

3. Permutation matrix

We define the matrix defined by

$$\begin{aligned} A &= \sum_{i=1,j=1}^{2} E_{ij} \otimes E_{ji} \\ &= E_{11} \otimes E_{11} + E_{12} \otimes E_{21} + E_{21} \otimes E_{12} + E_{22} \otimes E_{22} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

where

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

This matrix is known as a permutation matrix. This is the same as the Dirac spin operator as shown below.

4. Dirac spin exchange operator

The Dirac spin exchange operator is defined by

$$\begin{aligned} \hat{P} &= \frac{1}{2}(\hat{I}_2 \otimes \hat{I}_2 + \hat{\sigma}_x \otimes \hat{\sigma}_x + \hat{\sigma}_y \otimes \hat{\sigma}_y + \hat{\sigma}_z \otimes \hat{\sigma}_z) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

This matrix is exactly the same as the permutation matrix. We note that

$$\hat{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = E_{11} + E_{22},$$

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = E_{12} + E_{21}$$

$$\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -iE_{12} + iE_{21}$$

$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = E_{11} - E_{22}$$

$$\begin{aligned} \hat{I}_2 \otimes \hat{I}_2 &= (E_{11} + E_{22}) \otimes (E_{11} + E_{22}) \\ &= E_{11} \otimes E_{11} + E_{11} \otimes E_{22} + E_{22} \otimes E_{11} + E_{22} \otimes E_{22} \end{aligned}$$

$$\begin{aligned} \hat{\sigma}_x \otimes \hat{\sigma}_x &= (E_{12} + E_{21}) \otimes (E_{12} + E_{21}) \\ &= E_{12} \otimes E_{12} + E_{12} \otimes E_{21} + E_{21} \otimes E_{12} + E_{21} \otimes E_{21} \end{aligned}$$

$$\begin{aligned} \hat{\sigma}_y \otimes \hat{\sigma}_y &= -(-E_{12} + E_{21}) \otimes (-E_{12} + E_{21}) \\ &= -E_{12} \otimes E_{12} + E_{12} \otimes E_{21} + E_{21} \otimes E_{12} - E_{21} \otimes E_{21} \end{aligned}$$

$$\begin{aligned} \hat{\sigma}_z \otimes \hat{\sigma}_z &= (E_{11} - E_{22}) \otimes (E_{11} - E_{22}) \\ &= E_{11} \otimes E_{11} - E_{11} \otimes E_{22} - E_{22} \otimes E_{11} + E_{22} \otimes E_{22} \end{aligned}$$

Then we get

$$\hat{P} = E_{11} \otimes E_{11} + E_{12} \otimes E_{21} + E_{21} \otimes E_{12} + E_{22} \otimes E_{22}$$

5. Dirac spin exchange operator

The Dirac spin exchange operator is defined by

$$\hat{P} = \frac{1}{2}[\hat{I}_2 \otimes \hat{I}_2 + \hat{\sigma}_x \otimes \hat{\sigma}_x + \hat{\sigma}_y \otimes \hat{\sigma}_y + \hat{\sigma}_z \otimes \hat{\sigma}_z]$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The eigenket for spin 1/2 is defined as

$$\psi_1 = \alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \psi_2 = \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

There are four state vectors defined by

$$\xi_1 = \alpha \otimes \alpha = \alpha\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \xi_2 = \alpha \otimes \beta = \alpha\beta = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\xi_3 = \beta \otimes \alpha = \beta\alpha = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \xi_4 = \beta \otimes \beta = \beta\beta = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

We note that

$$\hat{P}\xi_1 = \xi_1, \qquad \hat{P}\xi_2 = \xi_3, \qquad \hat{P}\xi_3 = \xi_2, \qquad \hat{P}\xi_4 = \xi_4$$

This matrix is exactly the same as the permutation matrix. We already know that ξ_1 and ξ_2 are the eigenket of $\hat{P}$ with the eigenvalues 1 and -1. Nevertheless, we solve the eigenvalue problem for the matrix $\hat{P}$ using the Mathematica with KroneckerProduct and Eigensystem.

Eigenvalue (+1);

$$|\Phi_1\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad |\Phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \qquad |\Phi_3\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \qquad \text{(symmetric)}$$

Eigenvalue (-1);

$$|\Phi_4\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} \qquad \text{(antisymmetric)}$$

The unitary matrix is given by

$$\hat{U} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\\ 0 & 0 & 1 & 0\end{pmatrix}, \qquad \hat{U}^{+}\hat{P}\hat{U} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

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6. Spin state of two particles with spin 1/2

(a) Method using the Clebsch-Gordan coefficient

Conventionbally, we can find the eigenkets of the two paticles with spin 1/2 by using the Clebsch-Gordan co-efficients. To this end, we use the Mathematica. The results are as follows.

$$|j=1, m=1\rangle = \alpha\alpha$$

$$|j=1, m=0\rangle = \frac{\alpha\beta + \beta\alpha}{\sqrt{2}}$$

$$|j=1, m=-1\rangle = \beta\beta$$

$$|j=0, m=0\rangle = \frac{\alpha\beta - \beta\alpha}{\sqrt{2}}$$

((**Mathematica**))

```
Clear["Global`*"];
CCGG[{j1_, m1_}, {j2_, m2_}, {j_, m_}] :=
 Module[{s1},
  s1 = If[Abs[m1] ≤ j1 && Abs[m2] ≤ j2 &&
     Abs[m] ≤ j, ClebschGordan[{j1, m1},
     {j2, m2}, {j, m}], 0]];
CG[j_, m_, j1_, j2_] :=
 Sum[CCGG[{j1, m1}, {j2, m - m1},
    {j, m}] a[j1 , m1] b[j2, m - m1],
  {m1, -j1, j1}];
rule1 = {a[1/2, 1/2] → α1, a[1/2, -1/2] → β1 ,
  b[1/2, 1/2] → α2, b[1/2, -1/2] → β2 };
```

j1 = 1/2; j2 = 1/2; j = 1

```
j1 = 1 / 2; j2 = 1 / 2; j = 1;
Table[{j, m, CG[j, m, j1, j2]}, {m, -j, j, 1}] /.
   rule1 // Simplify // TableForm
```

1	-1	$\beta 1\, \beta 2$
1	0	$\frac{\alpha 2\, \beta 1+\alpha 1\, \beta 2}{\sqrt{2}}$
1	1	$\alpha 1\, \alpha 2$

j1 = 1/2; j2 = 1/2; j = 0

```
j1 = 1 / 2; j2 = 1 / 2; j = 0;
Table[{j, m, CG[j, m, j1, j2]}, {m, -j, j, 1}] /.
   rule1 // Simplify // TableForm
```

0	0	$\frac{-\alpha 2\, \beta 1+\alpha 1\, \beta 2}{\sqrt{2}}$

(b) The use of Kronecker product

For two particles with spin 1/2, there are four states. In group theory, we have

$$D_{1/2} \times D_{1/2} = D_1 + D_0 ,$$

which leads to $j = 1$ ($m = 1, 0, -1$), and $j = 0$ ($m = 0$). Suppose that α and β are the eigenkets of one particle with spin 1/2,

$$\psi_1 = \alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \psi_2 = \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The eigenkets of two particles with spin 1/2 can be expressed in terms of linear combination of the combined two spin spates ξ_1, ξ_2, ξ_3, and ξ_4, defined by

$$\xi_1 = \alpha \otimes \alpha = \alpha\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \xi_2 = \alpha \otimes \beta = \alpha\beta = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\xi_3 = \beta \otimes \alpha = \beta\alpha = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \xi_4 = \beta \otimes \beta = \beta\beta = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The magnitude of the total angular momentum and the *z*-component of the total angular momentum, can be expressed by using the Kronecker product,

$$\begin{aligned} \boldsymbol{M}^2 &= 2(S_x \otimes S_x + S_y \otimes S_y + S_z \otimes S_z) \\ &\quad + 2S(S+1) I_2 \otimes I_2 \\ &= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} M_z &= S_z \otimes I_2 + I_2 \otimes S_z \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

We solve the eigenvalue problem by using Mathematica. The simultaneous eigenkets for both $\boldsymbol{M}^2$ and M_z can be obtained as

$$\left|\Phi_1\right\rangle = \left| j=1, m=1\right\rangle = U\xi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \alpha\alpha$$

$$\left|\Phi_2\right\rangle = \left| j=1, m=0\right\rangle = U\xi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(\alpha\beta + \beta\alpha)$$

$$\left|\Phi_3\right\rangle = \left| j=1, m=-1\right\rangle = U\xi_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \beta\beta$$

$$\left|\Phi_4\right\rangle = \left| j=0, m=0\right\rangle = U\xi_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(\alpha\beta - \beta\alpha)$$

The unitary operator U is given by

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Then the matrices $\boldsymbol{S}_{tot}{}^2$ and $S_{total}{}^z$ can be diagonalized with the use of the unitary operator as

$$U^+\boldsymbol{S}_{tot}{}^2 U = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad U^+ S_{total}{}^z U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

These matrices can be written as block-diagonal form. It is formed of (3x3) matrix denoted by the irreducible representation D_1 (for j = 1) and (1x1) matrix denoted by the irreducible representation D_0 (for j = 0). In fact, the Clebsh-Gordan coefficients define a unitary operator that allows one to decompose into a direct sum of irreducible representation of the angular momentum.

7. Mathematica for the two particles with spin 1/2

We show a Mathematica program which is used to obtain the simultaneous eigenkets of $\boldsymbol{M}^2$ and M_z for the two particles with spin 1/2. The detail is as follows..

((**Mathematica**))

```
Clear["Global`*"]; ℏ = 1; S = 1 / 2;
exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]}; α = (1
                                                              0);
β = (0 ; Sx = ℏ/2 PauliMatrix[1]; Sy = ℏ/2 PauliMatrix[2]; Sz = ℏ/2 PauliMatrix[3];
     1)
I2 = IdentityMatrix[2];

ξ1 = KroneckerProduct[α, α]; ξ2 = KroneckerProduct[α, β];
ξ3 = KroneckerProduct[β, α];
ξ4 = KroneckerProduct[β, β];
```

Squres of the magnitude of the total spin

```
MT = 2 (KroneckerProduct[Sx, Sx] + KroneckerProduct[Sy, Sy]
      + KroneckerProduct[Sz, Sz]) + 2 S (S + 1) KroneckerProduct[I2, I2];

MT // MatrixForm
```

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

The z component of the total spin

```
Mz = (KroneckerProduct[Sz, I2] + KroneckerProduct[I2, Sz]);

Mz // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Eigenvalue problem

```
eq1 = Eigensystem[MT] // Simplify

{{2, 2, 2, 0}, {{0, 0, 0, 1}, {0, 1, 1, 0}, {1, 0, 0, 0}, {0, -1, 1, 0}}}

χ1 = eq1[[2, 3]]; χ2 = eq1[[2, 2]]; χ3 = eq1[[2, 1]]; χ4 = eq1[[2, 4]];
eq2 = Orthogonalize[{χ1, χ2, χ3, χ4}]
```

$$\left\{\{1, 0, 0, 0\}, \left\{0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right\}, \{0, 0, 0, 1\}, \left\{0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right\}\right\}$$

```
Φ1 = eq2[[1]]; Φ2 = eq2[[2]]; Φ3 = eq2[[3]]; Φ4 = -eq2[[4]];
```

Confiming that the eigenkets of MT are also the eigenkets of Mz (simultaneous eigenkets)

```
Mz.Φ1 - Φ1
```

{0, 0, 0, 0}

```
Mz.Φ2
```

{0, 0, 0, 0}

```
Mz.Φ3 + Φ3
```

{0, 0, 0, 0}

```
Mz.Φ4
```

{0, 0, 0, 0}

```
Φ1 // MatrixForm
```

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

```
Φ2 // MatrixForm
```

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

```
Φ3 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

```
Φ4 // MatrixForm
```

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

```
ξ1 // MatrixForm
```

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

```
ξ2 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

```
ξ3 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

```
ξ4 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

```
K1 = Φ1; K2 = Φ2; K3 = Φ3; K4 = Φ4; f1 = p1 ξ1 + p2 ξ2 + p3 ξ3 + p4 ξ4;
g1 = p1 η1 + p2 η2 + p3 η3 + p4 η4;
rule1 = {η1 → αα, η2 → αβ, η3 → βα, η4 → ββ};
```

```
seq1 = Solve[K1 == f1, {p1, p2, p3, p4}] // Flatten;
g1 /. seq1 /. rule1
```

$\alpha\alpha$

```
seq2 = Solve[K2 == f1, {p1, p2, p3, p4}] // Flatten;
g1 /. seq2 /. rule1 // Simplify
```

$$\frac{\alpha\beta + \beta\alpha}{\sqrt{2}}$$

```
seq3 = Solve[K3 == f1, {p1, p2, p3, p4}] // Flatten;
g1 /. seq3 /. rule1 // Simplify
```

$\beta\beta$

```
seq4 = Solve[K4 == f1, {p1, p2, p3, p4}] // Flatten;
g1 /. seq4 /. rule1 // Simplify
```

$$\frac{\alpha\beta - \beta\alpha}{\sqrt{2}}$$

Unitary operator U
UH is the hermite conjugate of U

```
UT1 = {K1, K2, K3, K4}; U = Transpose[UT1]; UH = UT1*;
U // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

```
U.ξ1 // MatrixForm
```

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

```
U.ξ2 // MatrixForm
```

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

```
U.ξ3 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

```
U.ξ4 // MatrixForm
```

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

```
UH.MT.U // MatrixForm
```

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
UH.Mz.U // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

---

8. Spin state of three particles with spin 1/2

(i) The use of KroneckerProduct

$$D_{1/2} \times D_{1/2} \times D_{1/2} = (D_1 + D_0) \times D_{1/2}$$
$$= D_{3/2} + 2D_{1/2}$$

There are 8 states which are linear combination of ξ_1, ξ_2, ξ_3, ξ_4, ξ_5, ξ_6, ξ_7, ξ_8, where

$$\xi_1 = \alpha \otimes \alpha \otimes \alpha = \alpha\alpha\alpha\ , \quad \xi_2 = \alpha\alpha\beta\ , \quad \xi_3 = \alpha\beta\alpha \quad \xi_4 = \alpha\beta\beta\ ,$$
$$\xi_5 = \beta\alpha\alpha\ , \quad \xi_6 = \beta\alpha\beta\ , \quad \xi_7 = \beta\beta\alpha \quad \xi_8 = \beta\beta\beta\ ,$$

$$
\begin{aligned}
\hat{\boldsymbol{M}}^2 &= 2(S_x \otimes S_x \otimes I_2 + S_y \otimes S_y \otimes I_2 + S_z \otimes S_z \otimes I_2 \\
&+ S_x \otimes I_2 \otimes S_x + S_y \otimes I_2 \otimes S_y + S_z \otimes I_2 \otimes S_z \\
&+ I_2 \otimes S_x \otimes S_x + I_2 \otimes S_y \otimes S_y + I_2 \otimes S_z \otimes S_z) \\
&+ 3S(S+1) I_2 \otimes I_2 \otimes I_2 \\
&= \begin{pmatrix}
15/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 7/4 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 7/4 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 7/4 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 7/4 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 7/4 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 7/4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 15/4
\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\hat{M}_z &= S_z \otimes I_2 \otimes I_2 + I_2 \otimes S_z \otimes I_2 + I_2 \otimes I_2 \otimes S_z \\
&= \begin{pmatrix}
3/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -3/2
\end{pmatrix}
\end{aligned}
$$

where we use the Mathematica (KroneckerProduct and Eigensystem) to solve the eigenvalue problem.

$$\left| j = 3/2, m = 3/2 \right\rangle = U\xi_1 = \alpha\alpha\alpha$$

$$\left| j = 3/2, m = 1/2 \right\rangle = U\xi_2 = \frac{1}{\sqrt{3}}(\alpha\alpha\beta + \alpha\beta\alpha + \beta\alpha\alpha +)$$

$$\left| j = 3/2, m = -1/2 \right\rangle = U\xi_3 = \frac{1}{\sqrt{3}}(\alpha\beta\beta + \beta\alpha\beta + \beta\beta\alpha)$$

$$\left| j = 3/2, m = -3/2 \right\rangle = U\xi_4 = \beta\beta\beta$$

$$|j=1/2, m=1/2\rangle = U\xi_5 = \frac{1}{\sqrt{2}}(\alpha\alpha\beta - \beta\alpha\alpha)$$

$$|j=1/2, m=-1/2\rangle = U\xi_6 = \frac{1}{\sqrt{2}}(\alpha\beta\beta - \beta\beta\alpha)$$

$$|j=1/2, m=1/2\rangle = U\xi_7 = \frac{1}{\sqrt{6}}(\alpha\alpha\beta - 2\alpha\beta\alpha + \beta\alpha\alpha)$$

$$|j=1/2, m=-1/2\rangle = U\xi_8 = \frac{1}{\sqrt{6}}(\alpha\beta\beta - 2\beta\alpha\beta + \beta\beta\alpha)$$

The unitary operator is given by

$\hat{U} =$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\hat{U}^{+}\hat{\boldsymbol{M}}^{2}\hat{U} =$

$$\begin{pmatrix} \frac{15}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{15}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{15}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{15}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} \end{pmatrix}$$

$\hat{U}^{+}\hat{M}_{z}\hat{U} =$

$$\begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

These matrices can be written as block-diagonal form. It is formed of (4x4) matrix denoted by the irreducible representation $D_{3/2}$ (for j = 3/2) and two (2x2) matrices denoted by the irreducible representation $D_{1/2}$ (for j = 1/2).

(ii) The use of Clebsch-Gordan co-efficient

```
Clear["Global`*"]; CCGG[{j1_, m1_}, {j2_, m2_}, {j_, m_}] :=
 Module[{s1},
  s1 = If[Abs[m1] ≤ j1 && Abs[m2] ≤ j2 && Abs[m] ≤ j,
    ClebschGordan[{j1, m1}, {j2, m2}, {j, m}], 0]];
CG2[j1_, j2_, j_, a1_, a2_] :=
 Table[Sum[CCGG[{j1, k1}, {j2, k2}, {j, k1 + k2}] a1[j1, k1] a2[j2, k2]
    KroneckerDelta[k1 + k2, m], {k1, -j1, j1}, {k2, -j2, j2}], {m, -j, j}];
rule1 = {b1[1/2, 1/2] → α1, b1[1/2, -1/2] → β1 , b2[1/2, 1/2] → α2, b2[1/2, -1/2] → β2 ,
   b3[1/2, 1/2] → α3, b3[1/2, -1/2] → β3 };
```

j1=1, j2=1/2 j = 3/2

```
j1 = 1; j2 = 1 / 2; j = 3 / 2;
Table[
   {m, Sum[b1[j2, k2] CCGG[{j1, k1}, {j2, k2}, {j, k1 + k2}]
      CG2[1 / 2, 1 / 2, j1, b2, b3][[k1 + j1 + 1]] KroneckerDelta[k1 + k2, m],
     {k1, -j1, j1}, {k2, -j2, j2}]}, {m, j, -j, -1}] /. rule1 // Simplify //
 TableForm
```

$\frac{3}{2}$	$\alpha1\ \alpha2\ \alpha3$
$\frac{1}{2}$	$\frac{\alpha2\ \alpha3\ \beta1+\alpha1\ \alpha3\ \beta2+\alpha1\ \alpha2\ \beta3}{\sqrt{3}}$
$-\frac{1}{2}$	$\frac{\alpha3\ \beta1\ \beta2+\alpha2\ \beta1\ \beta3+\alpha1\ \beta2\ \beta3}{\sqrt{3}}$
$-\frac{3}{2}$	$\beta1\ \beta2\ \beta3$

j1=1, j2=1/2 j = 1/2

```
j1 = 1; j2 = 1 / 2; j = 1 / 2;
Table[
   {m, Sum[b1[j2, k2] CCGG[{j1, k1}, {j2, k2}, {j, k1 + k2}]
      CG2[1 / 2, 1 / 2, j1, b2, b3][[k1 + j1 + 1]] KroneckerDelta[k1 + k2, m],
     {k1, -j1, j1}, {k2, -j2, j2}]}, {m, j, -j, -1}] /. rule1 // Simplify //
 TableForm
```

$\frac{1}{2}$	$\frac{2\ \alpha2\ \alpha3\ \beta1-\alpha1\ \alpha3\ \beta2-\alpha1\ \alpha2\ \beta3}{\sqrt{6}}$
$-\frac{1}{2}$	$\frac{\alpha3\ \beta1\ \beta2+\alpha2\ \beta1\ \beta3-2\ \alpha1\ \beta2\ \beta3}{\sqrt{6}}$

j1=0, j2=1/2 j = 1/2

```
j1 = 0; j2 = 1 / 2; j = 1 / 2;
Table[
    {m, Sum[b1[j2, k2] CCGG[{j1, k1}, {j2, k2}, {j, k1 + k2}]
       CG2[1 / 2, 1 / 2, j1, b2, b3][[k1 + j1 + 1]] KroneckerDelta[k1 + k2, m],
      {k1, -j1, j1}, {k2, -j2, j2}]}, {m, j, -j, -1}] /. rule1 // Simplify //
 TableForm
```

$\frac{1}{2}$	$\frac{\alpha1\ (-\alpha3\ \beta2+\alpha2\ \beta3)}{\sqrt{2}}$
$-\frac{1}{2}$	$\frac{\beta1\ (-\alpha3\ \beta2+\alpha2\ \beta3)}{\sqrt{2}}$

We show that the eigenkets obtained from the method of the Clebsch-Gordan co-efficient are the same as those obtained from the method of the KroneckerProduct.

We have the eigenkets from the Clebsh-Gordan, in the order of $1\rightarrow 2\rightarrow 3$ (1, 2, and 3 denote the names of particles).

$$\left|j=3/2, m=3/2\right\rangle = \alpha\alpha\alpha$$

$$\left|j=3/2, m=1/2\right\rangle = \frac{1}{\sqrt{3}}(\beta\alpha\alpha+\alpha\beta\alpha+\alpha\alpha\beta)$$

$$\left|j=3/2, m=-1/2\right\rangle = \frac{1}{\sqrt{3}}(\alpha\beta\beta+\beta\alpha\beta+\beta\beta\alpha)$$

$$\left|j=3/2, m=-3/2\right\rangle = \beta\beta\beta$$

$$\left|j=1/2, m=1/2\right\rangle = \frac{1}{\sqrt{2}}(\alpha\alpha\beta-\alpha\beta\alpha)$$

$$\left|j=1/2, m=-1/2\right\rangle = \frac{1}{\sqrt{2}}(\beta\alpha\beta-\beta\beta\alpha)$$

$$\left|j=1/2, m=1/2\right\rangle = \frac{1}{\sqrt{6}}(\alpha\alpha\beta-2\beta\alpha\alpha+\alpha\beta\alpha)$$

$$\left|j=1/2, m=-1/2\right\rangle = \frac{1}{\sqrt{6}}(\beta\alpha\beta-2\alpha\beta\beta+\beta\beta\alpha)$$

Here we redefine the notation such that

$$\alpha_1\beta_2\alpha_3 \to \beta_2\alpha_1\alpha_3 = \beta\alpha\alpha \,, \qquad \text{(Interchange between 1 and 2)}$$

in the order of $2 \to 1 \to 3$, instead of the order $1 \to 2 \to 3$. Then we have

$$\left| j = 3/2, m = 3/2 \right\rangle = \alpha\alpha\alpha$$

$$\left| j = 3/2, m = 1/2 \right\rangle = \frac{1}{\sqrt{3}}(\beta\alpha\alpha + \alpha\beta\alpha + \alpha\alpha\beta)$$

$$\left| j = 3/2, m = -1/2 \right\rangle = \frac{1}{\sqrt{3}}(\alpha\beta\beta + \beta\alpha\beta + \beta\beta\alpha)$$

$$\left| j = 3/2, m = -3/2 \right\rangle = \beta\beta\beta$$

$$\left| j = 1/2, m = 1/2 \right\rangle = \frac{1}{\sqrt{2}}(\alpha\alpha\beta - \beta\alpha\alpha)$$

$$\left| j = 1/2, m = -1/2 \right\rangle = \frac{1}{\sqrt{2}}(\alpha\beta\beta - \beta\beta\alpha)$$

$$\left| j = 1/2, m = 1/2 \right\rangle = \frac{1}{\sqrt{6}}(\alpha\alpha\beta - 2\alpha\beta\alpha + \beta\alpha\alpha)$$

$$\left| j = 1/2, m = -1/2 \right\rangle = \frac{1}{\sqrt{6}}(\alpha\beta\beta - 2\beta\alpha\beta + \beta\beta\alpha)$$

which are the same as those derived from the use of KroneckerProduct.

9. Comparison of our results (3 spins) of the eigenkets with those from other sources

(a) Schiff L.I.

In Schiff's book, the eigenkets are given as follows. When the notations of eigenkets are changed from the order of $1 \to 2 \to 3$ to the order of $2 \to 1 \to 3$ (interchange between 1 and 2), the eigenkets shown in Schiff's book are the same as those obtained by the KroneckerProduct, except for the sign for some eigenkets

$$\left| j = 3/2, m = 3/2 \right\rangle = \alpha\alpha\alpha \to \alpha\alpha\alpha \qquad \text{(Interchange between 1and 2)}$$

$$\left| j=3/2, m=1/2 \right\rangle = \frac{1}{\sqrt{3}}(\alpha\alpha\beta + \alpha\beta\alpha + \beta\alpha\alpha)$$

$$\rightarrow \frac{1}{\sqrt{3}}(\alpha\alpha\beta + \beta\alpha\alpha + \alpha\beta\alpha)$$ (interchange between 1 and 2)

$$\left| j=3/2, m=-1/2 \right\rangle = \frac{1}{\sqrt{3}}(\beta\alpha\beta + \beta\beta\alpha + \alpha\beta\beta)$$

$$\rightarrow \frac{1}{\sqrt{3}}(\alpha\beta\beta + \beta\beta\alpha + \beta\alpha\beta)$$ (interchange between 1 and 2)

$$\left| j=3/2, m=-3/2 \right\rangle = \beta\beta\beta \rightarrow \beta\beta\beta$$ (Interchange between 1and 2)

$$\left| j=1/2, m=1/2 \right\rangle = \frac{1}{\sqrt{2}}(\alpha\alpha\beta - \alpha\beta\alpha)$$

$$\rightarrow \frac{1}{\sqrt{2}}(\alpha\alpha\beta - \beta\alpha\alpha)$$ (interchange between 1 and 2)

$$\left| j=1/2, m=-1/2 \right\rangle = \frac{1}{\sqrt{2}}(\beta\alpha\beta - \beta\beta\alpha)$$

$$\rightarrow \frac{1}{\sqrt{2}}(\alpha\beta\beta - \beta\beta\alpha)$$ (interchange between 1 and 2)

$$\left| j=1/2, m=1/2 \right\rangle = \frac{1}{\sqrt{6}}(2\beta\alpha\alpha - \alpha\alpha\beta - \alpha\beta\alpha)$$

$$\rightarrow -\frac{1}{\sqrt{6}}(\alpha\alpha\beta - 2\alpha\beta\alpha + \beta\alpha\alpha)$$ (interchange between 1 and 2)

$$\left| j=1/2, m=-1/2 \right\rangle = \frac{1}{\sqrt{6}}(\beta\alpha\beta + \beta\beta\alpha - 2\alpha\beta\beta)$$

$$\rightarrow \frac{1}{\sqrt{6}}(\alpha\beta\beta + \beta\beta\alpha - 2\beta\alpha\beta)$$ (interchange between 1 and 2)

(b) Tomonaga S., Harrison J.F.

In Tomonaga's book, the eigenkets are given as follows. When the notations of eigenkets are changed from the order of $1 \rightarrow 2 \rightarrow 3$ to the order of $1 \rightarrow 3 \rightarrow 2$, the eigenkets shown in Tomaoaga's book are the same as those obtained by the KroneckerProduct, except for the sign for some eigenkets

$$|j=3/2, m=3/2\rangle = \alpha\alpha\alpha \to \alpha\alpha\alpha$$ (Interchange between 2 and 3)

$$|j=3/2, m=1/2\rangle = \frac{1}{\sqrt{3}}(\alpha\alpha\beta + \alpha\beta\alpha + \beta\alpha\alpha)$$

$$\to \frac{1}{\sqrt{3}}(\alpha\beta\alpha + \alpha\alpha\beta + \beta\alpha\alpha)$$ (Interchange between 2 and 3)

$$|j=3/2, m=-1/2\rangle = \frac{1}{\sqrt{3}}(\beta\alpha\beta + \beta\beta\alpha + \alpha\beta\beta)$$

$$\to \frac{1}{\sqrt{3}}(\beta\beta\alpha + \beta\alpha\beta + \alpha\beta\beta)$$ (Interchange between 2 and 3)

$$|j=3/2, m=-3/2\rangle = \beta\beta\beta \to \beta\beta\beta$$ (Interchange between 2 and 3)

$$|j=1/2, m=1/2\rangle = \frac{1}{\sqrt{2}}(\alpha\beta\alpha - \beta\alpha\alpha)$$

$$\to \frac{1}{\sqrt{2}}(\alpha\alpha\beta - \beta\alpha\alpha)$$ (Interchange between 2 and 3)

$$|j=1/2, m=-1/2\rangle = \frac{1}{\sqrt{2}}(\beta\alpha\beta - \alpha\beta\beta)$$

$$\to \frac{1}{\sqrt{2}}(\beta\beta\alpha - \alpha\beta\beta)$$ (Interchange between 2 and 3)

$$|j=1/2, m=1/2\rangle = \frac{1}{\sqrt{6}}(\alpha\beta\alpha + \beta\alpha\alpha - 2\alpha\alpha\beta)$$

$$\to \frac{1}{\sqrt{6}}(\alpha\alpha\beta + \beta\alpha\alpha - 2\alpha\beta\alpha)$$ (Interchange between 2 and 3)

$$|j=1/2, m=-1/2\rangle = \frac{1}{\sqrt{6}}(\beta\alpha\beta + \alpha\beta\beta - 2\beta\beta\alpha)$$

$$\to \frac{1}{\sqrt{6}}(\beta\beta\alpha + \alpha\beta\beta - 2\beta\alpha\beta)$$ (Interchange between 2 and 3)

(c) Mizushima M.

In Schiff's book, the eigenkets are the same as those given by Schiff.

$$\left| j = 3/2, m = 3/2 \right\rangle = \alpha\alpha\alpha$$

$$\left| j = 1/2, m = 1/2 \right\rangle = \frac{1}{\sqrt{2}}(\alpha\alpha\beta - \alpha\beta\alpha)$$

$$\left| j = 1/2, m = 1/2 \right\rangle = \frac{1}{\sqrt{6}}(\alpha\alpha\beta + \alpha\beta\alpha - 2\beta\alpha\alpha)$$

M. Mizushima, Quantum Mechanics of Atomic Spectra and Atomic Structure (W.A. Benjamin, 1970)

10. Spin states of four particles with *S* = 1/2

$$\begin{aligned} D_{1/2} \times D_{1/2} \times D_{1/2} \times D_{1/2} &= (D_{3/2} + 2D_{1/2}) \times D_{1/2} \\ &= D_2 + D_1 + 2(D_1 + D_0) \\ &= D_2 + 3D_1 + 2D_0 \end{aligned}$$

There are 16 (= 2^4) states.

$$\xi_1 = \alpha \otimes \alpha \otimes \alpha \otimes \alpha = \alpha\alpha\alpha\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\xi_2 = \alpha\alpha\alpha\beta$

$\xi_3 = \alpha\alpha\beta\alpha$

$\xi_4 = \alpha\alpha\beta\beta$

$\xi_5 = \alpha\beta\alpha\alpha$

$\xi_6 = \alpha\beta\alpha\beta$

$\xi_7 = \alpha\beta\beta\alpha$

$\xi_8 = \alpha\beta\beta\beta$

$\xi_9 = \beta\alpha\alpha\alpha$

$\xi_{10} = \beta\alpha\alpha\beta$

$\xi_{11} = \beta\alpha\beta\alpha$

$\xi_{12} = \beta\alpha\beta\beta$

$\xi_{13} = \beta\beta\alpha\alpha$

$\xi_{14} = \beta\beta\alpha\beta$

$\xi_{15} = \beta\beta\beta\alpha$

$\xi_{16} = \beta\beta\beta\beta$

$$
\begin{aligned}
\hat{\boldsymbol{M}}^2 = {} & 2(S_x \otimes S_x \otimes I_2 \otimes I_2 + S_y \otimes S_y \otimes I_2 \otimes I_2 + S_z \otimes S_z \otimes I_2 \otimes I_2 \\
& + S_x \otimes I_2 \otimes S_x \otimes I_2 + S_y \otimes I_2 \otimes S_y \otimes I_2 + S_z \otimes I_2 \otimes S_z \otimes I_2 \\
& + S_x \otimes I_2 \otimes I_2 \otimes S_x + S_y \otimes I_2 \otimes I_2 \otimes S_y + S_z \otimes I_2 \otimes I_2 \otimes S_z \\
& + I_2 \otimes S_x \otimes S_x \otimes I_2 + I_2 \otimes S_y \otimes S_y \otimes I_2 + I_2 \otimes S_z \otimes S_z \otimes I_2) \\
& + I_2 \otimes S_x \otimes I_2 \otimes S_x + I_2 \otimes S_y \otimes I_2 \otimes S_y + I_2 \otimes S_z \otimes I_2 \otimes S_z) \\
& + I_2 \otimes I_2 \otimes S_x \otimes S_x + I_2 \otimes I_2 \otimes S_y \otimes S_y + I_2 \otimes I_2 \otimes S_z \otimes S_z) \\
& + 4S(S+1) I_2 \otimes I_2 \otimes I_2 \otimes I_2 \\
& =
\end{aligned}
$$

$$
\begin{pmatrix}
6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 3 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 3 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 3 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6
\end{pmatrix}
$$

$$
\hat{M}_z = S_z \otimes I_2 \otimes I_2 \otimes I_2 + I_2 \otimes S_z \otimes I_2 \otimes I_2 + I_2 \otimes I_2 \otimes S_z \otimes I_2 + I_2 \otimes I_2 \otimes I \otimes S_z
$$

$$\begin{pmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2
\end{pmatrix}$$

$\hat{U} =$

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{2\sqrt{3}} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2\sqrt{3}} \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & \frac{1}{2\sqrt{3}} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{3}} \\
0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{2} & \frac{1}{2\sqrt{3}} \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{2\sqrt{3}} & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{2\sqrt{3}} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{2} & \frac{1}{2\sqrt{3}} \\
0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{3}} \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2\sqrt{3}} \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 & \frac{1}{2\sqrt{3}} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{2\sqrt{3}} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$\hat{U}^{+}\hat{\boldsymbol{M}}^{2}\hat{U} =$

$$\begin{pmatrix}
6&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&6&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&6&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&6&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&6&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

$\hat{U}^{+}\hat{M}_z\hat{U} =$

$$\begin{pmatrix}
2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

These matrices can be written as block-diagonal form. It is formed of one (5x5) matrix denoted by the irreducible representation D_2 (for j = 2), three (3x3) matrices denoted by the irreducible representation D_1 (for j = 1), and two (1x1) matrices denoted by the irreducible representation D_0 (for j = 0).

$| j = 2, m = 2\rangle = U\xi_1 = \alpha\alpha\alpha\alpha$

$$\left| j=2, m=1 \right\rangle = U\xi_2 = \frac{1}{2}(\alpha\alpha\alpha\beta + \alpha\alpha\beta\alpha + \alpha\beta\alpha\alpha + \beta\alpha\alpha\alpha)$$

$$\left| j=2, m=0 \right\rangle = U\xi_3 = \frac{1}{\sqrt{6}}(\alpha\alpha\beta\beta + \alpha\beta\alpha\beta + \alpha\beta\beta\alpha + \beta\alpha\alpha\beta + \beta\alpha\beta\alpha + \beta\beta\alpha\alpha)$$

$$\left| j=2, m=-1 \right\rangle = U\xi_4 = \frac{1}{2}(\alpha\beta\beta\beta + \beta\alpha\beta\beta + \beta\beta\alpha\beta + \beta\beta\beta\alpha)$$

$$\left| j=2, m=-2 \right\rangle = U\xi_5 = \beta\beta\beta\beta$$

$$\left| j=1, m=1 \right\rangle = U\xi_6 = \frac{1}{\sqrt{2}}(\alpha\alpha\alpha\beta - \beta\alpha\alpha\alpha)$$

$$\left| j=1, m=0 \right\rangle = U\xi_7 = \frac{1}{\sqrt{2}}(\alpha\alpha\beta\beta - \beta\beta\alpha\alpha)$$

$$\left| j=1, m=-1 \right\rangle = U\xi_8 = \frac{1}{\sqrt{2}}(\alpha\beta\beta\beta - \beta\beta\beta\alpha)$$

$$\left| j=1, m=1 \right\rangle = U\xi_9 = \frac{1}{\sqrt{6}}(\alpha\alpha\alpha\beta - 2\alpha\beta\alpha\alpha + \beta\alpha\alpha\alpha)$$

$$\left| j=1, m=0 \right\rangle = U\xi_{10} = \frac{1}{\sqrt{2}}(\alpha\beta\alpha\beta - \beta\alpha\beta\alpha)$$

$$\left| j=1, m=-1 \right\rangle = U\xi_{11} = \frac{1}{\sqrt{6}}(\alpha\beta\beta\beta - 2\beta\beta\alpha\beta + \beta\beta\beta\alpha)$$

$$\left| j=1, m=1 \right\rangle = U\xi_{12} = \frac{1}{2\sqrt{3}}(\alpha\alpha\alpha\beta - 3\alpha\alpha\beta\alpha + \alpha\beta\alpha\alpha + \beta\alpha\alpha\alpha)$$

$$\left| j=1, m=0 \right\rangle = U\xi_{13} = \frac{1}{\sqrt{2}}(\alpha\beta\beta\alpha - \beta\alpha\alpha\beta)$$

$$\left| j=1, m=-1 \right\rangle = U\xi_{14} = \frac{1}{2\sqrt{3}}(\alpha\beta\beta\beta - 3\beta\alpha\beta\beta + \beta\beta\alpha\beta + \beta\beta\beta\alpha)$$

$$\left| j=0, m=0 \right\rangle = U\xi_{15} = \frac{1}{2}(\alpha\alpha\beta\beta - \alpha\beta\beta\alpha - \beta\alpha\alpha\beta + \beta\beta\alpha\alpha)$$

$$\left| j=0, m=0 \right\rangle = U\xi_{16} = \frac{1}{2\sqrt{3}}(\alpha\alpha\beta\beta - 2\alpha\beta\alpha\beta + \alpha\beta\beta\alpha + \beta\alpha\alpha\beta - 2\beta\alpha\beta\alpha + \beta\beta\alpha\alpha)$$

11. **Comparison** our results with those obtained by Harrisson and Mizushima

(a) J.F. Harrison

$\alpha\alpha\alpha\alpha$ (any interchange is OK)

$$\frac{1}{\sqrt{2}}(\alpha\beta\alpha\alpha - \beta\alpha\alpha\alpha)$$

$$\rightarrow \frac{1}{\sqrt{2}}(\alpha\alpha\alpha\beta - \beta\alpha\alpha\alpha)$$ (Interchange of 2 and 4)

$$\left| j=1, m=1 \right\rangle = U\xi_6 = \frac{1}{\sqrt{2}}(\alpha\alpha\alpha\beta - \beta\alpha\alpha\alpha)$$

$$\frac{1}{\sqrt{6}}(\beta\alpha\alpha\alpha + \alpha\beta\alpha\alpha - 2\alpha\alpha\beta\alpha)$$

$$\rightarrow \frac{1}{\sqrt{6}}(\beta\alpha\alpha\alpha + \alpha\alpha\alpha\beta - 2\alpha\beta\alpha\alpha)$$ (Interchange of 2 and 4)

$$\left| j=1, m=1 \right\rangle = U\xi_9 = \frac{1}{\sqrt{6}}(\alpha\alpha\alpha\beta - 2\alpha\beta\alpha\alpha + \beta\alpha\alpha\alpha)$$

$$\frac{1}{2}(\alpha\beta - \beta\alpha)(\alpha\beta - \beta\alpha) = \frac{1}{2}(\alpha\beta\alpha\beta - \alpha\beta\beta\alpha - \beta\alpha\alpha\beta + \beta\alpha\beta\alpha)$$

$$\rightarrow \frac{1}{2}(\alpha\alpha\beta\beta - \alpha\beta\beta\alpha - \beta\alpha\alpha\beta + \beta\beta\alpha\alpha)$$ (Interchange of 2 and 3)

$$\left| j=0, m=0 \right\rangle = U\xi_{15} = \frac{1}{2}(\alpha\alpha\beta\beta - \alpha\beta\beta\alpha - \beta\alpha\alpha\beta + \beta\beta\alpha\alpha)$$

$$\frac{1}{\sqrt{12}}[2\alpha\alpha\beta\beta + 2\beta\beta\alpha\alpha - \alpha\beta\alpha\beta - \beta\alpha\beta\alpha - \alpha\beta\beta\alpha - \beta\alpha\alpha\beta)$$

$$\to \frac{1}{\sqrt{12}}[2\alpha\beta\alpha\beta + 2\beta\alpha\beta\alpha - \alpha\alpha\beta\beta - \beta\beta\alpha\alpha - \alpha\beta\beta\alpha - \beta\alpha\alpha\beta) \qquad \text{(Interchange of 2 and 3)}$$

$$\left| j = 0, m = 0 \right\rangle = U\xi_{16} = \frac{1}{2\sqrt{3}}(\alpha\alpha\beta\beta - 2\alpha\beta\alpha\beta + \alpha\beta\beta\alpha + \beta\alpha\alpha\beta - 2\beta\alpha\beta\alpha + \beta\beta\alpha\alpha)$$

(b) Mizushima

In Mizushima's book, the eigenkets are given as follows. When the notations of eigenkets are changed from the order of $1 \to 2 \to 3 \to 4$ to the order of $1 \to 3 \to 2 \to 4$ (the interchange between 2 and 3), the resulting eigenkets coincide with our eigenkets.

$$\left| j = 0, m = 0 \right\rangle = \frac{1}{2\sqrt{3}}(\alpha\beta\alpha\beta - 2\alpha\alpha\beta\beta + \alpha\beta\beta\alpha + \beta\alpha\alpha\beta - 2\beta\beta\alpha\alpha + \beta\alpha\beta\alpha)$$

$$\to \frac{1}{2\sqrt{3}}(\alpha\alpha\beta\beta - 2\alpha\beta\alpha\beta + \alpha\beta\beta\alpha + \beta\alpha\alpha\beta - 2\beta\alpha\beta\alpha + \beta\beta\alpha\alpha)$$

(Interchange between 2 and 3).

which is the same as our result from KroneckerProduct method;

$$\left| j = 0, m = 0 \right\rangle = U\xi_{16} = \frac{1}{2\sqrt{3}}(\alpha\alpha\beta\beta - 2\alpha\beta\alpha\beta + \alpha\beta\beta\alpha + \beta\alpha\alpha\beta - 2\beta\alpha\beta\alpha + \beta\beta\alpha\alpha)$$

Ref. M. Mizushima, Quantum Mechanics of Atomic Spectra and Atomic Structure (W.A. Benjamin, 1970). p.395

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12. Spin states of five particles with spin 1/2

$$\begin{aligned} D_{1/2} \times D_{1/2} \times D_{1/2} \times D_{1/2} \times D_{1/2} &= (D_2 + 3D_1 + 2D_0) \times D_{1/2} \\ &= D_{5/2} + D_{3/2} + 3(D_{3/2} + D_{1/2}) + 2D_{1/2} \\ &= D_{5/2} + 4D_{3/2} + 5D_{1/2} \end{aligned}$$

$$\xi_1 = \alpha \otimes \alpha \otimes \alpha \otimes \alpha \otimes \alpha = \alpha\alpha\alpha\alpha\alpha$$

$$\xi_2 = \alpha\alpha\alpha\alpha\beta$$

$$\xi_3 = \alpha\alpha\alpha\beta\alpha$$

$\xi_4 = \alpha\alpha\alpha\beta\beta$
$\xi_5 = \alpha\alpha\beta\alpha\alpha$
$\xi_6 = \alpha\alpha\beta\alpha\beta$
$\xi_7 = \alpha\alpha\beta\beta\alpha$
$\xi_8 = \alpha\alpha\beta\beta\beta$
$\xi_9 = \alpha\beta\alpha\alpha\alpha$
$\xi_{10} = \alpha\beta\alpha\alpha\beta$
$\xi_{11} = \alpha\beta\alpha\alpha\beta$
$\xi_{12} = \alpha\beta\alpha\beta\alpha$
$\xi_{13} = \alpha\beta\beta\alpha\alpha$
$\xi_{14} = \alpha\beta\beta\alpha\beta$
$\xi_{15} = \alpha\beta\beta\beta\alpha$
$\xi_{16} = \alpha\beta\beta\beta\beta$
$\xi_{17} = \beta\alpha\alpha\alpha\alpha$
$\xi_{18} = \beta\alpha\alpha\alpha\beta$
$\xi_{19} = \beta\alpha\alpha\beta\alpha$
$\xi_{20} = \beta\alpha\alpha\beta\beta$
$\xi_{21} = \beta\alpha\beta\alpha\alpha$
$\xi_{22} = \beta\alpha\beta\alpha\beta$
$\xi_{23} = \beta\alpha\beta\beta\alpha$
$\xi_{24} = \beta\alpha\beta\beta\beta$
$\xi_{25} = \beta\beta\alpha\alpha\alpha$
$\xi_{26} = \beta\beta\alpha\alpha\beta$
$\xi_{27} = \beta\beta\alpha\beta\alpha$
$\xi_{28} = \beta\beta\alpha\beta\beta$
$\xi_{29} = \beta\beta\beta\alpha\alpha$
$\xi_{30} = \beta\beta\beta\alpha\beta$
$\xi_{31} = \beta\beta\beta\beta\alpha$
$\xi_{32} = \beta\beta\beta\beta\beta$

$$\left| j = 5/2, m = 5/2 \right\rangle = U\xi_1$$
$$= \alpha\alpha\alpha\alpha\alpha$$

$$\left| j=5/2, m=3/2 \right\rangle = U\xi_2$$
$$= \frac{1}{\sqrt{5}}(\alpha\alpha\alpha\alpha\beta + \alpha\alpha\alpha\beta\alpha + \alpha\alpha\beta\alpha\alpha + \alpha\beta\alpha\alpha\alpha + \beta\alpha\alpha\alpha\alpha)$$

$$\left| j=5/2, m=1/2 \right\rangle = U\xi_3$$
$$= \frac{1}{\sqrt{10}}(\alpha\alpha\alpha\beta\beta + \alpha\alpha\beta\alpha\beta + \alpha\alpha\beta\beta\alpha + \alpha\beta\alpha\alpha\beta + \alpha\beta\alpha\beta\alpha$$
$$+ \alpha\beta\beta\alpha\alpha + \beta\alpha\alpha\alpha\beta + \beta\alpha\alpha\beta\alpha + \beta\alpha\beta\alpha\alpha + \beta\beta\alpha\alpha\alpha)$$

$$\left| j=5/2, m=-1/2 \right\rangle = U\xi_4$$
$$= \frac{1}{\sqrt{10}}(\alpha\alpha\beta\beta\beta + \alpha\beta\alpha\beta\beta + \alpha\beta\beta\alpha\beta + \alpha\beta\beta\beta\alpha + \beta\alpha\alpha\beta\beta$$
$$+ \beta\alpha\beta\alpha\beta + \beta\alpha\beta\beta\alpha + \beta\beta\alpha\alpha\beta + \beta\beta\alpha\beta\alpha + \beta\beta\beta\alpha\alpha)$$

$$\left| j=5/2, m=-3/2 \right\rangle = U\xi_5$$
$$= \frac{1}{\sqrt{5}}(\alpha\beta\beta\beta\beta + \beta\alpha\beta\beta\beta + \beta\beta\alpha\beta\beta + \beta\beta\beta\alpha\beta + \beta\beta\beta\beta\alpha)$$

$$\left| j=5/2, m=-5/2 \right\rangle = U\xi_6 = \beta\beta\beta\beta\beta$$

$$\left| j=3/2, m=3/2 \right\rangle = U\xi_7$$
$$= \frac{1}{\sqrt{2}}[\alpha\alpha\alpha\alpha\beta - \beta\alpha\alpha\alpha\alpha]$$

$$\left| j=3/2, m=3/2 \right\rangle = U\xi_8$$
$$= \frac{1}{\sqrt{6}}(\alpha\alpha\alpha\alpha\beta - 2\alpha\beta\alpha\alpha\alpha + \beta\alpha\alpha\alpha\alpha)$$

$$\left| j=3/2, m=3/2 \right\rangle = U\xi_9$$
$$= \frac{1}{2\sqrt{3}}(\alpha\alpha\alpha\alpha\beta - 3\alpha\alpha\beta\alpha\alpha + \alpha\beta\alpha\alpha\alpha + \beta\alpha\alpha\alpha\alpha)$$

$$\left| j=3/2, m=3/2 \right\rangle = U\xi_{10}$$

$$= \frac{1}{2\sqrt{5}}(\alpha\alpha\alpha\alpha\beta - 4\alpha\alpha\alpha\beta\alpha + \alpha\alpha\beta\alpha\alpha + \alpha\beta\alpha\alpha\alpha + \beta\alpha\alpha\alpha\alpha)$$

$$\left| j=3/2, m=1/2 \right\rangle$$

$$U\xi_{11} = \frac{1}{2\sqrt{6}}(2\alpha\alpha\alpha\beta\beta + 2\alpha\alpha\beta\alpha\beta + 2\alpha\alpha\beta\beta\alpha - \alpha\beta\alpha\alpha\beta - \alpha\beta\alpha\beta\alpha$$
$$- \alpha\beta\beta\alpha\alpha - 3\beta\beta\alpha\alpha\alpha)$$

$$U\xi_{12} = \frac{1}{2\sqrt{42}}[6\alpha\alpha\alpha\beta\beta - 2\alpha\alpha\beta\alpha\beta - 2\alpha\alpha\beta\beta\alpha + 5\alpha\beta\alpha\alpha\beta + 5\alpha\beta\alpha\beta\alpha$$
$$- 3\alpha\beta\beta\alpha\alpha - 8\beta\alpha\beta\alpha\alpha - \beta\beta\alpha\alpha\alpha]$$

$$U\xi_{13} = \frac{1}{3\sqrt{14}}[\alpha\alpha\alpha\beta\beta - 5\alpha\alpha\beta\alpha\beta + 2\alpha\alpha\beta\beta\alpha - 5\alpha\beta\alpha\alpha\beta + 2\alpha\beta\alpha\beta\alpha$$
$$- 4\alpha\beta\beta\alpha\alpha + 7\beta\alpha\alpha\beta\alpha + \beta\alpha\beta\alpha\alpha + \beta\beta\alpha\alpha\alpha]$$

$$U\xi_{14} = \frac{1}{3\sqrt{10}}[\alpha\alpha\alpha\beta\beta + \alpha\alpha\beta\alpha\beta - 4\alpha\alpha\beta\beta\alpha + \alpha\beta\alpha\alpha\beta - 4\alpha\beta\alpha\beta\alpha$$
$$- 4\alpha\beta\beta\alpha\alpha + 6\beta\alpha\alpha\alpha\beta + \beta\alpha\alpha\beta\alpha + \beta\alpha\beta\alpha\alpha + \beta\beta\alpha\alpha\alpha]$$

$$\left| j=3/2, m=-1/2 \right\rangle$$

$$U\xi_{15} = \frac{1}{2\sqrt{3}}(\alpha\alpha\beta\beta\beta + \alpha\beta\alpha\beta\beta - \alpha\beta\beta\alpha\beta - \alpha\beta\beta\beta\alpha$$
$$+ 2\beta\alpha\alpha\beta\beta - 2\beta\beta\beta\alpha\alpha)$$

$$U\xi_{16} = \frac{1}{2\sqrt{3}}(\alpha\alpha\beta\beta\beta - \alpha\beta\alpha\beta\beta + \alpha\beta\beta\alpha\beta - \alpha\beta\beta\beta\alpha$$
$$+ 2\beta\alpha\beta\alpha\beta - 2\beta\beta\alpha\beta\alpha)$$

$$U\xi_{17} = \frac{1}{3\sqrt{2}}(2\alpha\alpha\beta\beta\beta + \alpha\beta\alpha\beta\beta + \alpha\beta\beta\alpha\beta + 2\alpha\beta\beta\beta\alpha - \beta\alpha\alpha\beta\beta - \beta\alpha\beta\alpha\beta - 2\beta\beta\alpha\alpha\beta$$
$$- \beta\beta\alpha\beta\alpha - \beta\beta\beta\alpha\alpha)$$

$$U\xi_{18} = \frac{1}{3\sqrt{10}}(\alpha\alpha\beta\beta\beta - 4\alpha\beta\alpha\beta\beta - 4\alpha\beta\beta\alpha\beta + \alpha\beta\beta\beta\alpha + \beta\alpha\alpha\beta\beta + \beta\alpha\beta\alpha\beta + 6\beta\alpha\beta\beta\alpha - 4\beta\beta\alpha\alpha\beta + \beta\beta\alpha\beta\alpha + \beta\beta\beta\alpha\alpha)$$

$\left| j = 3/2, m = -3/2 \right\rangle$

$$U\xi_{19} = \frac{1}{\sqrt{2}}(\alpha\beta\beta\beta\beta - \beta\beta\beta\beta\alpha)$$

$$U\xi_{20} = \frac{1}{\sqrt{6}}(\alpha\beta\beta\beta\beta - 2\beta\beta\beta\alpha\beta + \beta\beta\beta\beta\alpha)$$

$$U\xi_{21} = \frac{1}{2\sqrt{3}}(\alpha\beta\beta\beta\beta - 3\beta\beta\alpha\beta\beta + \beta\beta\beta\alpha\beta + \beta\beta\beta\beta\alpha)$$

$$U\xi_{22} = \frac{1}{2\sqrt{5}}(\alpha\beta\beta\beta\beta - 4\beta\alpha\beta\beta\beta + \beta\beta\alpha\beta\beta + \beta\beta\beta\alpha\beta + \beta\beta\beta\beta\alpha)$$

$\left| j = 1/2, m = 1/2 \right\rangle$

$$U\xi_{23} = \frac{1}{\sqrt{6}}(\alpha\alpha\alpha\beta\beta + \alpha\alpha\beta\alpha\beta - \alpha\alpha\beta\beta\alpha - \alpha\beta\alpha\alpha\beta - \beta\alpha\alpha\alpha\beta + \beta\beta\alpha\alpha\alpha)$$

$$U\xi_{24} = \frac{1}{\sqrt{10}}(\alpha\alpha\alpha\beta\beta - \alpha\alpha\beta\alpha\beta - \alpha\alpha\beta\beta\alpha + \alpha\beta\alpha\alpha\beta - \beta\alpha\alpha\alpha\beta + 2\beta\alpha\beta\alpha\alpha - \beta\beta\alpha\alpha\alpha)$$

$$U\xi_{25} = \frac{1}{2\sqrt{15}}(3\alpha\alpha\alpha\beta\beta - 3\alpha\alpha\beta\alpha\beta + 2\alpha\alpha\beta\beta\alpha - 2\alpha\beta\alpha\alpha\beta + 2\beta\alpha\alpha\alpha\beta - 5\beta\alpha\alpha\beta\alpha + \beta\alpha\beta\alpha\alpha + 2\beta\beta\alpha\alpha\alpha)$$

$$U\xi_{26} = \frac{1}{6}(\alpha\alpha\alpha\beta\beta - \alpha\alpha\beta\alpha\beta - 2\alpha\alpha\beta\beta\alpha - 2\alpha\beta\alpha\alpha\beta + 4\alpha\beta\beta\alpha\alpha + 2\beta\alpha\alpha\alpha\beta + \beta\alpha\alpha\beta\alpha - \beta\alpha\beta\alpha\alpha - 2\beta\beta\alpha\alpha\alpha)$$

$$U\xi_{27} = \frac{1}{3\sqrt{2}}(\alpha\alpha\alpha\beta\beta - \alpha\alpha\beta\alpha\beta + \alpha\alpha\beta\beta\alpha + \alpha\beta\alpha\alpha\beta - 3\alpha\beta\alpha\beta\alpha + \alpha\beta\beta\alpha\alpha - \beta\alpha\alpha\alpha\beta + \beta\alpha\alpha\beta\alpha - \beta\alpha\beta\alpha\alpha + \beta\beta\alpha\alpha\alpha)$$

$\left| j = 1/2, m = -1/2 \right\rangle$

$$U\xi_{28} = \frac{1}{\sqrt{6}}(\alpha\alpha\beta\beta\beta + \alpha\beta\alpha\beta\beta - \alpha\beta\beta\alpha\beta - \alpha\beta\beta\beta\alpha - \beta\alpha\alpha\beta\beta + \beta\beta\beta\alpha\alpha)$$

$$U\xi_{29} = \frac{1}{\sqrt{10}}(\alpha\alpha\beta\beta\beta - \alpha\beta\alpha\beta\beta + \alpha\beta\beta\alpha\beta - \alpha\beta\beta\beta\alpha - \beta\alpha\alpha\beta\beta + 2\beta\beta\alpha\beta\alpha - \beta\beta\beta\alpha\alpha)$$

$$U\xi_{30} = \frac{1}{2\sqrt{15}}(2\alpha\alpha\beta\beta\beta - 2\alpha\beta\alpha\beta\beta - 3\alpha\beta\beta\alpha\beta + 3\alpha\beta\beta\beta\alpha - 2\beta\alpha\alpha\beta\beta + 5\beta\beta\alpha\alpha\beta - \beta\beta\alpha\beta\alpha - 2\beta\beta\beta\alpha\alpha)$$

$$U\xi_{31} = \frac{1}{6}(2\alpha\alpha\beta\beta\beta - 2\alpha\beta\alpha\beta\beta - \alpha\beta\beta\alpha\beta + \alpha\beta\beta\beta\alpha + 2\beta\alpha\alpha\beta\beta - 4\beta\alpha\beta\beta\alpha - \beta\beta\alpha\alpha\beta + \beta\beta\alpha\beta\alpha + 2\beta\beta\beta\alpha\alpha)$$

$$U\xi_{32} = \frac{1}{3\sqrt{2}}(\alpha\alpha\beta\beta\beta - \alpha\beta\alpha\beta\beta + \alpha\beta\beta\alpha\beta - \alpha\beta\beta\beta\alpha + \beta\alpha\alpha\beta\beta - 3\beta\alpha\beta\alpha\beta + \beta\alpha\beta\beta\alpha + \beta\beta\alpha\alpha\beta - \beta\beta\alpha\beta\alpha + \beta\beta\beta\alpha\alpha)$$

13. Comparison

(a) J.F. Harrison

The results (denoted by yellow) reported by Harrison are given as follows.

$j = 5/2$

$$\alpha\alpha\alpha\alpha\alpha$$

$j = 3/2$

$$\frac{1}{\sqrt{2}}(\alpha\beta\alpha\alpha\alpha - \beta\alpha\alpha\alpha\alpha)$$

$$\rightarrow \frac{1}{\sqrt{2}}(\alpha\alpha\alpha\alpha\beta - \beta\alpha\alpha\alpha\alpha)$$ (Interchange between 2 and 5)

which is the same as our result

$$\left| j = 3/2, m = 3/2 \right\rangle = U\xi_7 = \frac{1}{\sqrt{2}}[\alpha\alpha\alpha\alpha\beta - \beta\alpha\alpha\alpha\alpha]$$

$$\frac{1}{\sqrt{6}}(\beta\alpha\alpha\alpha\alpha + \alpha\beta\alpha\alpha\alpha - 2\alpha\alpha\beta\alpha\alpha)$$

$$\rightarrow \frac{1}{\sqrt{6}}(\beta\alpha\alpha\alpha\alpha + \alpha\alpha\alpha\alpha\beta - 2\alpha\beta\alpha\alpha\alpha)$$ (Interchange between 2 and 3)

which is the same as our result

$$\left| j = 3/2, m = 3/2 \right\rangle = U\xi_8 = \frac{1}{\sqrt{6}}(\alpha\alpha\alpha\alpha\beta - 2\alpha\beta\alpha\alpha\alpha + \beta\alpha\alpha\alpha\alpha)$$

$$\frac{1}{2\sqrt{3}}(\beta\alpha\alpha\alpha\alpha + \alpha\beta\alpha\alpha\alpha + \alpha\alpha\beta\alpha\alpha - 3\alpha\alpha\alpha\beta\alpha)$$

$$\rightarrow \frac{1}{2\sqrt{3}}(\beta\alpha\alpha\alpha\alpha + \alpha\beta\alpha\alpha\alpha + \alpha\alpha\alpha\beta\alpha - 3\alpha\alpha\beta\alpha\alpha)$$ (Interchange between 3 and 4)

$$\rightarrow \frac{1}{2\sqrt{3}}(\beta\alpha\alpha\alpha\alpha + \alpha\beta\alpha\alpha\alpha + \alpha\alpha\alpha\alpha\beta - 3\alpha\alpha\beta\alpha\alpha)$$ (Interchange between 4 and 5)

which is the same as our result

$$\left| j = 3/2, m = 3/2 \right\rangle = U\xi_9 = \frac{1}{2\sqrt{3}}(\alpha\alpha\alpha\alpha\beta - 3\alpha\alpha\beta\alpha\alpha + \alpha\beta\alpha\alpha\alpha + \beta\alpha\alpha\alpha\alpha)$$

$$\frac{1}{\sqrt{20}}(\beta\alpha\alpha\alpha\alpha + \alpha\beta\alpha\alpha\alpha + \alpha\alpha\beta\alpha\alpha + \alpha\alpha\alpha\beta\alpha - 4\alpha\alpha\alpha\alpha\beta)$$

$$\rightarrow \frac{1}{\sqrt{20}}(\beta\alpha\alpha\alpha\alpha + \alpha\beta\alpha\alpha\alpha + \alpha\alpha\beta\alpha\alpha + \alpha\alpha\alpha\alpha\beta - 4\alpha\alpha\alpha\beta\alpha)$$ (Interchange between 4 and 5)

which is the same as our result

$$\left| j=3/2, m=3/2 \right\rangle = U\xi_{10}$$
$$= \frac{1}{2\sqrt{5}}(\alpha\alpha\alpha\alpha\beta - 4\alpha\alpha\alpha\beta\alpha + \alpha\alpha\beta\alpha\alpha + \alpha\beta\alpha\alpha\alpha + \beta\alpha\alpha\alpha\alpha)$$

j = 1/2

$$\frac{1}{2}(\alpha\beta\alpha\beta\alpha - \alpha\beta\beta\alpha\alpha - \beta\alpha\alpha\beta\alpha + \beta\alpha\beta\alpha\alpha)$$

$$\frac{1}{2\sqrt{3}}(2\alpha\alpha\beta\beta\alpha + 2\beta\beta\alpha\alpha\alpha - \alpha\beta\alpha\beta\alpha - \beta\alpha\beta\alpha\alpha - \alpha\beta\beta\alpha\alpha - \beta\alpha\alpha\beta\alpha)$$

$$\frac{1}{2\sqrt{3}}(\alpha\beta\beta\alpha\alpha + \alpha\beta\alpha\beta\alpha - 2\alpha\beta\alpha\alpha\beta - \beta\alpha\beta\alpha\alpha - \beta\alpha\alpha\beta\alpha + 2\beta\alpha\alpha\alpha\beta))$$

$$\frac{1}{\sqrt{6}}(\alpha\alpha\alpha\beta\beta + \alpha\alpha\beta\alpha\beta + \beta\beta\alpha\alpha\alpha - \alpha\beta\alpha\alpha\beta - \beta\alpha\alpha\alpha\beta - \alpha\alpha\beta\beta\alpha)$$

which is the same as

$$U\xi_{23} = \frac{1}{\sqrt{6}}(\alpha\alpha\alpha\beta\beta + \alpha\alpha\beta\alpha\beta - \alpha\alpha\beta\beta\alpha - \alpha\beta\alpha\alpha\beta - \beta\alpha\alpha\alpha\beta + \beta\beta\alpha\alpha\alpha)$$

$$\frac{1}{2\sqrt{3}}(2\alpha\alpha\beta\alpha\beta - 2\alpha\alpha\alpha\beta\beta + \alpha\beta\alpha\beta\alpha + \beta\alpha\alpha\beta\alpha - \alpha\beta\beta\alpha\alpha - \beta\alpha\beta\alpha\alpha)$$

(b) Mizushima M.

We find the table of the eigenkets for five S= 1/2 spins in the Mizushima's book. Here we compare our results with those reported by Mizushima. The agreement of these result with our results is not always good.

j = 5/2

$\alpha\alpha\alpha\alpha\alpha$

$j = 3/2$

$$\frac{1}{2}(\alpha\alpha\alpha\alpha\beta - \alpha\alpha\alpha\beta\alpha + \alpha\alpha\beta\alpha\alpha - \alpha\beta\alpha\alpha\alpha)$$

$$\frac{1}{2}(\alpha\alpha\alpha\alpha\beta - \alpha\alpha\alpha\beta\alpha - \alpha\alpha\beta\alpha\alpha + \alpha\beta\alpha\alpha\alpha)$$

$$\frac{1}{2}(\alpha\alpha\alpha\alpha\beta + \alpha\alpha\alpha\beta\alpha - \alpha\alpha\beta\alpha\alpha - \alpha\beta\alpha\alpha\alpha)$$

$$\frac{1}{\sqrt{20}}(\alpha\alpha\alpha\alpha\beta + \alpha\alpha\alpha\beta\alpha + \alpha\alpha\beta\alpha\alpha + \alpha\beta\alpha\alpha\alpha - 4\beta\alpha\alpha\alpha\alpha)$$

$j = 1/2$

$$\frac{1}{\sqrt{12}}(\alpha\beta\beta\alpha\alpha + \alpha\alpha\beta\beta\alpha + \alpha\alpha\alpha\beta\beta + \alpha\beta\alpha\alpha\beta - 2\alpha\beta\alpha\beta\alpha - 2\alpha\alpha\beta\alpha\beta)$$

$$\frac{1}{2}(\alpha\beta\beta\alpha\alpha - \alpha\alpha\beta\beta\alpha + \alpha\alpha\alpha\beta\beta - \alpha\beta\alpha\alpha\beta)$$

$$\frac{1}{\sqrt{6}}(\beta\alpha\alpha\alpha\beta - \beta\beta\alpha\alpha\alpha - \beta\alpha\beta\alpha\alpha + \beta\alpha\alpha\beta\alpha + \alpha\beta\beta\alpha\alpha - \alpha\alpha\alpha\beta\beta)$$

$$\frac{1}{\sqrt{6}}(\beta\alpha\alpha\alpha\beta + \beta\beta\alpha\alpha\alpha - \beta\alpha\beta\alpha\alpha - \beta\alpha\alpha\beta\alpha + \alpha\alpha\beta\beta\alpha - \alpha\beta\alpha\alpha\beta)$$

$$\frac{1}{\sqrt{6}}(\beta\alpha\alpha\alpha\beta - \beta\beta\alpha\alpha\alpha + \beta\alpha\beta\alpha\alpha - \beta\alpha\alpha\beta\alpha + \alpha\beta\alpha\beta\alpha - \alpha\alpha\beta\alpha\beta)$$

Ref. M. Mizushima, Quantum Mechanics of Atomic Spectra and Atomic Structure (W.A. Benjamin, 1970) p.395

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7. Two particles wth the angular momentum $L = \hbar$.

One particle state for the angular momentum $\hbar$,

$$|l=1,m=1\rangle=\alpha=\begin{pmatrix}1\\0\\0\end{pmatrix},\qquad |l=1,m=0\rangle=\beta=\begin{pmatrix}0\\1\\0\end{pmatrix},\qquad |l=1,m=-1\rangle=\gamma=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

The wave function of two particles with the angular momentum $\hbar$.

$$\xi_1=\alpha\otimes\alpha=\alpha\alpha$$
$$\xi_2=\alpha\otimes\beta=\alpha\beta$$
$$\xi_3=\alpha\otimes\gamma=\alpha\gamma$$
$$\xi_4=\beta\otimes\alpha=\beta\alpha$$
$$\xi_5=\beta\otimes\beta=\beta\beta$$
$$\xi_6=\beta\otimes\gamma=\beta\gamma$$
$$\xi_7=\gamma\otimes\alpha=\beta\alpha$$
$$\xi_8=\gamma\otimes\beta=\beta\beta$$
$$\xi_9=\gamma\otimes\gamma=\beta\gamma$$

We now consider the state of two particles with the angular momentum $\hbar$.

$$D_1\times D_1=D_2+D_1+D_0$$

$$L_x=\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix},\quad L_y=\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix},\qquad L_y=\hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$$

(a) The use of Clebsch-Gordan coefficient

```
Clear["Global`*"]; CCGG[{j1_, m1_}, {j2_, m2_}, {j_, m_}] :=
 Module[{s1},
  s1 = If[Abs[m1] ≤ j1 && Abs[m2] ≤ j2 && Abs[m] ≤ j,
    ClebschGordan[{j1, m1}, {j2, m2}, {j, m}], 0]];
CG[j_, m_, j1_, j2_] := Sum[CCGG[{j1, m1}, {j2, m - m1},
    {j, m}] a[j1 , m1] b[j2, m - m1], {m1, -j1, j1}];
rule1 = {a[1, 1] → α1, a[1, 0] → β1 , a[1, -1] → γ1 , b[1, 1] → α2, b[1, 0] → β2,
  b[1, -1] → γ2};
```

j1 = 1 and j2 = 1; j = 2

```
j1 = 1; j2 = 1; j = 2;
Table[{j, m, CG[j, m, j1, j2]}, {m, j, -j, -1}] /. rule1 // Simplify //
 TableForm
```

2	2	$\alpha1\,\alpha2$
2	1	$\frac{\alpha2\,\beta1+\alpha1\,\beta2}{\sqrt{2}}$
2	0	$\frac{2\,\beta1\,\beta2+\alpha2\,\gamma1+\alpha1\,\gamma2}{\sqrt{6}}$
2	-1	$\frac{\beta2\,\gamma1+\beta1\,\gamma2}{\sqrt{2}}$
2	-2	$\gamma1\,\gamma2$

j1 = 1 and j2 = 1; j = 1

```
j1 = 1; j2 = 1; j = 1;
Table[{j, m, CG[j, m, j1, j2]}, {m, j, -j, -1}] /. rule1 // Simplify //
 TableForm
```

1	1	$\frac{-\alpha2\,\beta1+\alpha1\,\beta2}{\sqrt{2}}$
1	0	$\frac{-\alpha2\,\gamma1+\alpha1\,\gamma2}{\sqrt{2}}$
1	-1	$\frac{-\beta2\,\gamma1+\beta1\,\gamma2}{\sqrt{2}}$

j1 = 1 and j2 = 1; j = 0

```
j1 = 1; j2 = 1; j = 0; Table[{j, m, CG[j, m, j1, j2]}, {m, j, -j, 1}] /. rule1 //
 TableForm
```

0	0	$-\frac{\beta1\,\beta2}{\sqrt{3}}+\frac{\alpha2\,\gamma1}{\sqrt{3}}+\frac{\alpha1\,\gamma2}{\sqrt{3}}$

(b) The use of Kronecker product

$$\hat{\boldsymbol{M}}^2 = 2(L_x \otimes L_x + L_y \otimes L_y + L_z \otimes L_z)$$
$$+ 2L(L+1) I_3 \otimes I_3$$

$$\begin{pmatrix} 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 4 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

$$\hat{M}_z = L_z \oplus L_z = L_z \otimes I_3 + I_3 \otimes L_z$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

with $L = 1$ and I_3 is the identity matrix (3x3),where $\hbar = 1$.

The unitary operator U is

U=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{3}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$U^{+}\boldsymbol{M}^{2}U =$

$$\begin{pmatrix} 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$U^{+}M_{z}U =$

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

These matrices can be written as block-diagonal form. It is formed of one (5x5) matrix denoted by the irreducible representation D_2 (for $j = 2$), one (3x3) matrix denoted by the irreducible representation D_1 (for $j = 1$), and one (1x1) matrices denoted by the irreducible representation D_0 (for $j = 0$).

$$\left| j=2, m=2 \right\rangle = U\xi_1 = \alpha\alpha$$

$$\left| j=2, m=1 \right\rangle = U\xi_2 = \frac{1}{\sqrt{2}}(\alpha\beta + \beta\alpha)$$

$$\left| j=2, m=0 \right\rangle = U\xi_3 = \frac{1}{\sqrt{6}}(\alpha\gamma + 2\beta\beta + \gamma\alpha)$$

$$\left| j=2, m=-1 \right\rangle = U\xi_4 = \frac{1}{\sqrt{2}}(\beta\gamma + \gamma\beta)$$

$$\left| j=2, m=-2 \right\rangle = U\xi_5 = \gamma\gamma$$

$$\left| j=1, m=1 \right\rangle = U\xi_6 = \frac{1}{\sqrt{2}}(\alpha\beta - \beta\alpha)$$

$$\left| j=1, m=0 \right\rangle = U\xi_7 = \frac{1}{\sqrt{2}}(\alpha\gamma - \gamma\alpha)$$

$$\left| j=1, m=-1 \right\rangle = U\xi_8 = \frac{1}{\sqrt{2}}(\beta\gamma - \gamma\beta)$$

$$\left| j=0, m=0 \right\rangle = U\xi_9 = \frac{1}{\sqrt{3}}(\alpha\gamma - \beta\beta + \gamma\alpha)$$

---

2. Three particles wth the angular momentum $L = \hbar$.

The wave function of three particles with the angular momentum $\hbar$.

$$\xi_1 = \alpha \otimes \alpha \otimes \alpha = \alpha\alpha\alpha$$
$$\xi_2 = \alpha \otimes \alpha \otimes \beta = \alpha\alpha\beta$$

$$\xi_3 = \alpha \otimes \alpha \otimes \gamma = \alpha\alpha\gamma$$
$$\xi_4 = \alpha \otimes \beta \otimes \alpha = \alpha\beta\alpha$$
$$\xi_5 = \alpha \otimes \beta \otimes \beta = \alpha\beta\beta$$
$$\xi_6 = \alpha \otimes \beta \otimes \gamma = \alpha\beta\gamma$$
$$\xi_7 = \alpha \otimes \gamma \otimes \alpha = \alpha\gamma\alpha$$
$$\xi_8 = \alpha \otimes \gamma \otimes \beta = \alpha\gamma\beta$$
$$\xi_9 = \alpha \otimes \gamma \otimes \gamma = \alpha\gamma\gamma$$
$$\xi_{10} = \beta \otimes \alpha \otimes \alpha = \beta\alpha\alpha$$
$$\xi_{11} = \beta \otimes \alpha \otimes \beta = \beta\alpha\beta$$
$$\xi_{12} = \beta \otimes \alpha \otimes \gamma = \beta\alpha\gamma$$
$$\xi_{13} = \beta \otimes \beta \otimes \alpha = \beta\beta\alpha$$
$$\xi_{14} = \beta \otimes \beta \otimes \beta = \beta\beta\beta$$
$$\xi_{15} = \beta \otimes \beta \otimes \gamma = \beta\beta\gamma$$
$$\xi_{16} = \beta \otimes \gamma \otimes \alpha = \alpha\gamma\alpha$$
$$\xi_{17} = \beta \otimes \gamma \otimes \beta = \alpha\gamma\beta$$
$$\xi_{18} = \beta \otimes \gamma \otimes \gamma = \alpha\gamma\gamma$$
$$\xi_{19} = \gamma \otimes \alpha \otimes \alpha = \gamma\alpha\alpha$$
$$\xi_{20} = \gamma \otimes \alpha \otimes \beta = \gamma\alpha\beta$$
$$\xi_{21} = \gamma \otimes \alpha \otimes \gamma = \gamma\alpha\gamma$$
$$\xi_{22} = \gamma \otimes \beta \otimes \alpha = \gamma\beta\alpha$$
$$\xi_{23} = \gamma \otimes \beta \otimes \beta = \gamma\beta\beta$$
$$\xi_{24} = \gamma \otimes \beta \otimes \gamma = \gamma\beta\gamma$$
$$\xi_{25} = \gamma \otimes \gamma \otimes \alpha = \gamma\gamma\alpha$$
$$\xi_{26} = \gamma \otimes \gamma \otimes \beta = \gamma\gamma\beta$$
$$\xi_{18} = \gamma \otimes \gamma \otimes \gamma = \gamma\gamma\gamma$$

We now consider the state of two particles with the angular momentum *ħ*.

$$D_1 \times D_1 \times D_1 = (D_2 + D_1 + D_0) \times D_1 = D_3 + 2D_2 + 3D_1 + D_0$$

$$
\begin{aligned}
\boldsymbol{M}^2 = {} & 2(L_x \otimes L_x \otimes I_3 + L_y \otimes L_y \otimes I_3 + L_z \otimes L_z \otimes I_3 + L_x \otimes I_3 \otimes L_x \\
& + L_y \otimes I_3 \otimes L_y + L_z \otimes I_3 \otimes L_z + I_3 \otimes L_x \otimes L_x + I_3 \otimes L_y \otimes L_y + I_3 \otimes L_z \otimes L_z) \\
& + 3L(L+1) I_3 \otimes I_3 \\
& =
\end{aligned}
$$

$$
\begin{pmatrix}
12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 8 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 6 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 6 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 6 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12
\end{pmatrix}
$$

$$M_z = L_z \otimes I_3 \otimes I_3 + I_3 \otimes L_z \otimes I_3 + I_3 \otimes I_3 \otimes L_z$$

$$
\begin{pmatrix}
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3
\end{pmatrix}
$$

with $L = 1$ and I_3 is the identity matrix (3x3),where $\hbar = 1$.

$\hat{U}^+ \boldsymbol{S}_{tot}{}^2 \hat{U}$

$$
\begin{pmatrix}
12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$\hat{U}^{+} M_z \hat{U}$

$$
\begin{pmatrix}
3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
$$

$$\left| j=3, m=3 \right\rangle = U\xi_1 = \alpha\alpha\alpha$$

$$\left| j=2, m=2 \right\rangle = U\xi_2 = \frac{1}{\sqrt{3}}(\alpha\alpha\beta + \alpha\beta\alpha + \beta\alpha\alpha)$$

$$\left| j=3, m=1 \right\rangle = U\xi_3 = \frac{1}{\sqrt{15}}(\alpha\alpha\gamma + 2\alpha\beta\beta + \alpha\gamma\alpha + 2\beta\alpha\beta + 2\beta\beta\alpha + \gamma\alpha\alpha)$$

$$\left| j=3, m=0 \right\rangle = U\xi_4 = \frac{1}{\sqrt{10}}(\alpha\beta\gamma + \alpha\gamma\beta + \beta\alpha\gamma + 2\beta\beta\beta + \beta\gamma\alpha + \gamma\alpha\beta + \gamma\beta\alpha)$$

$$\left| j=3, m=-1 \right\rangle = U\xi_5 = \frac{1}{\sqrt{15}}(\alpha\gamma\gamma + 2\beta\beta\gamma + 2\beta\gamma\beta + \gamma\alpha\gamma + 2\gamma\beta\beta + \gamma\gamma\alpha)$$

$$\left| j=3, m=-2 \right\rangle = U\xi_6 = \frac{1}{\sqrt{3}}(\beta\gamma\gamma + \gamma\beta\gamma + \gamma\gamma\beta)$$

$$\left| j=3, m=-3 \right\rangle = U\xi_7 = \gamma\gamma\gamma$$

$$\left| j=2, m=2 \right\rangle = U\xi_8 = \frac{1}{\sqrt{6}}(\alpha\alpha\beta - 2\alpha\beta\alpha + \beta\alpha\alpha)$$

$$\left| j=2, m=2 \right\rangle = U\xi_9 = \frac{1}{\sqrt{2}}(\alpha\alpha\beta - \beta\alpha\alpha)$$

$$\left| j=2, m=1 \right\rangle = U\xi_{10} = \frac{1}{2\sqrt{3}}(2\alpha\alpha\gamma + \alpha\beta\beta - \alpha\gamma\alpha + \beta\alpha\beta - 2\beta\beta\alpha - \gamma\alpha\alpha)$$

$$\left| j=2, m=1 \right\rangle = U\xi_{11} = \frac{1}{2}(\alpha\beta\beta + \alpha\gamma\alpha - \beta\alpha\beta - \gamma\alpha\alpha)$$

$$\left| j=2, m=0 \right\rangle = U\xi_{12} = \frac{1}{2\sqrt{2}3}(\alpha\beta\gamma + 2\alpha\gamma\beta - \beta\alpha\gamma + \beta\gamma\alpha - 2\gamma\alpha\beta - \gamma\beta\alpha)$$

$$\left| j=2, m=0 \right\rangle = U\xi_{13} = \frac{1}{2}(\alpha\beta\gamma + \beta\alpha\gamma - \beta\gamma\alpha - \gamma\beta\alpha)$$

$$\left| j=2, m=-1 \right\rangle = U\xi_{14} = \frac{1}{2\sqrt{3}}(2\alpha\gamma\gamma + \beta\beta\gamma + \beta\gamma\beta - \gamma\alpha\gamma - 2\gamma\beta\beta - \gamma\gamma\alpha)$$

$$\left| j=2, m=-1 \right\rangle = U\xi_{15} = \frac{1}{2}(\beta\beta\gamma - \beta\gamma\beta + \gamma\alpha\gamma - \gamma\gamma\alpha)$$

$$\left| j=2, m=-2 \right\rangle = U\xi_{16} = \frac{1}{\sqrt{6}}(\beta\gamma\gamma - 2\gamma\beta\gamma + \gamma\gamma\beta)$$

$$\left| j=2, m=-2 \right\rangle = U\xi_{17} = \frac{1}{\sqrt{2}}(\beta\gamma\gamma - \gamma\gamma\beta)$$

$$\left| j=1, m=1 \right\rangle = U\xi_{18} = \frac{1}{2\sqrt{15}}(\alpha\alpha\gamma - 3\alpha\beta\beta + 6\alpha\gamma\alpha + 2\beta\alpha\beta - 3\beta\beta\alpha + \gamma\alpha\alpha)$$

$$\left| j=1,m=1\right\rangle = U\xi_{19} = \frac{1}{2}(\alpha\alpha\gamma - \alpha\beta\beta + \beta\beta\alpha - \gamma\alpha\alpha$$

$$\left| j=1,m=1\right\rangle = U\xi_{20} = \frac{1}{\sqrt{3}}(\alpha\alpha\gamma - \beta\alpha\beta + \gamma\alpha\alpha)$$

$$\left| j=1,m=0\right\rangle = U\xi_{21} = \frac{1}{2\sqrt{10}}(\alpha\beta\gamma + \alpha\gamma\beta - 4\beta\alpha\gamma + 2\beta\beta\beta - 4\beta\gamma\alpha + \gamma\alpha\beta + \gamma\beta\alpha)$$

$$\left| j=1,m=0\right\rangle = U\xi_{22} = \frac{1}{2\sqrt{6}}(\alpha\beta\gamma - 3\alpha\gamma\beta + 2\beta\beta\beta - 3\gamma\alpha\beta + \gamma\beta\alpha)$$

$$\left| j=1,m=0\right\rangle = U\xi_{23} = \frac{1}{\sqrt{3}}(\alpha\beta\gamma - \beta\beta\beta + \gamma\beta\alpha)$$

$$\left| j=1,m=-1\right\rangle = U\xi_{24} = \frac{1}{2\sqrt{15}}(\alpha\gamma\gamma - 3\beta\beta\gamma + 2\beta\gamma\beta + 6\gamma\alpha\gamma - 3\gamma\beta\beta + \gamma\gamma\alpha)$$

$$\left| j=1,m=-1\right\rangle = U\xi_{25} = \frac{1}{2}(\alpha\gamma\gamma - \beta\beta\gamma + \gamma\beta\beta - \gamma\gamma\alpha)$$

$$\left| j=1,m=-1\right\rangle = U\xi_{26} = \frac{1}{\sqrt{3}}(\alpha\gamma\gamma - \beta\gamma\beta + \gamma\gamma\alpha)$$

$$\left| j=0,m=0\right\rangle = U\xi_{27} = \frac{1}{\sqrt{6}}(\alpha\beta\gamma - \alpha\gamma\beta - \beta\alpha\gamma + \beta\gamma\alpha + \gamma\alpha\beta - \gamma\beta\alpha)$$

COCLUSION

The eigenvalue problems for spin states with *n* spin systems (typically, n = 2, 3, 4, 5, 6, 7,...) can be solved with the use of the KroneckerProduct and Eigensystem of the Mathematica. This method provides the most reliable results, although we need to use Mathematica. The dimensions of the matrices for *n* spin systems is (2^n x 2^n) for spin 1/2 and (3^n x 3^n) for spin 1. For n = 5, for example, we need to solve the eigenvalue problems of matrix (S = 1/2) with 32 x 32. Here we only discuss the spin with 1/2 and 1. We can also discuss the case for many spin systems with higher spins. The method of the Clebsch-Gordan coefficient is useful for two spins with the same or different spins. However, it seems that this method becomes much more complicated as the number of spins increases.

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APPENDIX-I Matrix representation of KroneckerProduct: Mathematica programs

(a) Three spins

```
Clear["Global`*"]; S = 1/2; ℏ = 1;
exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]};
ψ1 = (1
      0);
ψ2 = (0
      1);
Sx = ℏ/2 (0 1
          1 0);
Sy = ℏ/2 (0 -i
          i  0);
Sz = ℏ/2 (1  0
          0 -1);
I2 = IdentityMatrix[2];
I8 = IdentityMatrix[8]

{{1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0},
 {0, 0, 1, 0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0},
 {0, 0, 0, 0, 1, 0, 0, 0}, {0, 0, 0, 0, 0, 1, 0, 0},
 {0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 1}}

ST1 =
  2 (KroneckerProduct[Sx, Sx, I2] +
      KroneckerProduct[Sy, Sy, I2]
      + KroneckerProduct[Sz, Sz, I2] +
      KroneckerProduct[I2, Sx, Sx] +
      KroneckerProduct[I2, Sy, Sy]
      + KroneckerProduct[I2, Sz, Sz] +
      KroneckerProduct[Sx, I2, Sx] +
      KroneckerProduct[Sy, I2, Sy]
      + KroneckerProduct[Sz, I2, Sz]) +
   3 S (S + 1) KroneckerProduct[I2, I2, I2];
```

```
ST1 // MatrixForm
```

$$\begin{pmatrix} \frac{15}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{7}{4} & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{7}{4} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{7}{4} & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & \frac{7}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \frac{7}{4} & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & \frac{7}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{15}{4} \end{pmatrix}$$

```
eq1 = Eigensystem[ST1] // Simplify
```

$$\left\{\left\{\frac{15}{4}, \frac{15}{4}, \frac{15}{4}, \frac{15}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}\right\},\right.$$

```
 {{0, 0, 0, 0, 0, 0, 0, 1}, {0, 0, 0, 1, 0, 1, 1, 0},
  {0, 1, 1, 0, 1, 0, 0, 0}, {1, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, -1, 0, 0, 1, 0}, {0, 0, 0, -1, 0, 1, 0, 0},
  {0, -1, 0, 0, 1, 0, 0, 0}, {0, -1, 1, 0, 0, 0, 0, 0}}}
```

(b) Four spins

```
Clear["Global`*"]; ℏ = 1; S = 1 / 2;
exp_* := exp /. {Complex[re_, im_] ⧴ Complex[re, -im]};
σx = (0 1
      1 0);
σy = (0 -ⅈ
      ⅈ  0);
σz = (1  0
      0 -1);
Sx = ℏ/2 σx; Sy = ℏ/2 σy;
Sz = ℏ/2 σz;
ψ1 = (1
      0);
ψ2 = (0
      1);
I2 = IdentityMatrix[2];

ST1 =
  2 (KroneckerProduct[Sx, Sx, I2, I2] +
      KroneckerProduct[Sy, Sy, I2, I2]
      + KroneckerProduct[Sz, Sz, I2, I2] +
      KroneckerProduct[Sx, I2, Sx, I2] +
      KroneckerProduct[Sy, I2, Sy, I2] +
      KroneckerProduct[Sz, I2, Sz, I2] +
      KroneckerProduct[Sx, I2, I2, Sx] +
      KroneckerProduct[Sy, I2, I2, Sy] +
      KroneckerProduct[Sz, I2, I2, Sz] +
      KroneckerProduct[I2, Sx, Sx, I2] +
      KroneckerProduct[I2, Sy, Sy, I2] +
      KroneckerProduct[I2, Sz, Sz, I2] +
      KroneckerProduct[I2, I2, Sx, Sx] +
      KroneckerProduct[I2, I2, Sy, Sy] +
      KroneckerProduct[I2, I2, Sz, Sz] +
      KroneckerProduct[I2, Sx, I2, Sx] +
      KroneckerProduct[ I2, Sy, I2, Sy] +
      KroneckerProduct[I2, Sz, I2, Sz]) +
   4 S (S + 1) KroneckerProduct[I2, I2, I2, I2];
```

```
ST1 // MatrixForm
```

6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	3	1	0	1	0	0	0	1	0	0	0	0	0	0	0
0	1	3	0	1	0	0	0	1	0	0	0	0	0	0	0
0	0	0	2	0	1	1	0	0	1	1	0	0	0	0	0
0	1	1	0	3	0	0	0	1	0	0	0	0	0	0	0
0	0	0	1	0	2	1	0	0	1	0	0	1	0	0	0
0	0	0	1	0	1	2	0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	0	3	0	0	0	1	0	1	1	0
0	1	1	0	1	0	0	0	3	0	0	0	0	0	0	0
0	0	0	1	0	1	0	0	0	2	1	0	1	0	0	0
0	0	0	1	0	0	1	0	0	1	2	0	1	0	0	0
0	0	0	0	0	0	0	1	0	0	0	3	0	1	1	0
0	0	0	0	0	1	1	0	0	1	1	0	2	0	0	0
0	0	0	0	0	0	0	1	0	0	0	1	0	3	1	0
0	0	0	0	0	0	0	1	0	0	0	1	0	1	3	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	6

```
eq1 = Eigensystem[ST1] // Simplify
```

```
{{6, 6, 6, 6, 6, 2, 2, 2, 2, 2, 2, 2, 2, 2, 0, 0},
 {{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1},
  {0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 1, 0},
  {0, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 0},
  {0, 1, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0},
  {1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 1, 0},
  {0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 1, 0, 0},
  {0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 1, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, -1, 0, 0, 1, 0, 0, 0, 0, 0, 0},
  {0, -1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0},
  {0, -1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 1, 0, 0, -1, 0, 0, -1, 0, 0, 1, 0, 0, 0},
  {0, 0, 0, 0, 0, 1, -1, 0, 0, -1, 1, 0, 0, 0, 0, 0}}}
```

(e) Five spins

```
Clear["Global`*"]; S = 1 / 2; ℏ = 1;
exp_* := exp /. {Complex[re_, im_] ⧴ Complex[re, -im]};
Sx = ℏ/2 ( 0 1
           1 0 );
Sy = ℏ/2 ( 0 -ⅈ
           ⅈ  0 );
Sz = ℏ/2 ( 1  0
           0 -1 );
I2 = IdentityMatrix[2];
ψ1 = ( 1
       0 );
ψ2 = ( 0
       1 );
```

D1/2 x D1/2 x D1/2 x D1/2 x D1/2=D5/2 +4 D3/2 + 5 D1/2

6 + 16 + 10=32

```
ST1 = 2 (KroneckerProduct[Sx, Sx, I2, I2, I2] + KroneckerProduct[Sy, Sy, I2, I2, I2] +
      KroneckerProduct[Sz, Sz, I2, I2, I2] + KroneckerProduct[Sx, I2, Sx, I2, I2] +
      KroneckerProduct[Sy, I2, Sy, I2, I2] + KroneckerProduct[Sz, I2, Sz, I2, I2] +
      KroneckerProduct[Sx, I2, I2, Sx, I2] + KroneckerProduct[Sy, I2, I2, Sy, I2] +
      KroneckerProduct[Sz, I2, I2, Sz, I2] + KroneckerProduct[Sx, I2, I2, I2, Sx] +
      KroneckerProduct[Sy, I2, I2, I2, Sy] +
      KroneckerProduct[Sz, I2, I2, I2, Sz] + KroneckerProduct[I2, Sx, Sx, I2, I2] +
      KroneckerProduct[I2, Sy, Sy, I2, I2] + KroneckerProduct[I2, Sz, Sz, I2, I2] +
      KroneckerProduct[I2, Sx, I2, Sx, I2] + KroneckerProduct[ I2, Sy, I2, Sy, I2] +
      KroneckerProduct[I2, Sz, I2, Sz, I2] +
      KroneckerProduct[I2, Sx, I2, I2, Sx] +
      KroneckerProduct[I2, Sy, I2, I2, Sy] +
      KroneckerProduct[I2, Sz, I2, I2, Sz] + KroneckerProduct[I2, I2, Sx, Sx, I2] +
      KroneckerProduct[I2, I2, Sy, Sy, I2] + KroneckerProduct[I2, I2, Sz, Sz, I2] +
      KroneckerProduct[I2, I2, Sx, I2, Sx] + KroneckerProduct[I2, I2, Sy, I2, Sy] +
      KroneckerProduct[I2, I2, Sz, I2, Sz] + KroneckerProduct[I2, I2, I2, Sx, Sx] +
      KroneckerProduct[I2, I2, I2, Sy, Sy] + KroneckerProduct[I2, I2, I2, Sz, Sz]) +
   5 S (S + 1) KroneckerProduct[I2, I2, I2, I2, I2];
```

ST1 // MatrixForm

$$
\begin{pmatrix}
\frac{35}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{19}{4} & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & \frac{19}{4} & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{11}{4} & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & \frac{19}{4} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & \frac{11}{4} & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & \frac{11}{4} & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{11}{4} & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & \frac{19}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \frac{11}{4} & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & \frac{11}{4} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{11}{4} & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & \frac{11}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & \frac{11}{4} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & \frac{11}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{19}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{19}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{11}{4} & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{11}{4} & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{11}{4} & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & \frac{11}{4} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{11}{4} & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & \frac{11}{4} & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{19}{4} & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & \frac{11}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \frac{11}{4} & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & \frac{11}{4} & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{19}{4} & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & \frac{11}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & \frac{19}{4} & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & \frac{19}{4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{35}{4}
\end{pmatrix}
$$