Matrix representation of angular momentum with *J*

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Here we summarize the matrix representation of the angular momentum with

j = 1/2, 1, 3/2. 2, 5/2, 3, and so on.

We also show the eigenkets and the corresponding unitary operators. The Mathematica programs are very useful for the derivation of these forms. We solve the eigenvalue problem for the angular momentum ($\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$). The normalized eigenkets are obtained. The corresponding unitary operators are also discussed.

1. $j = 1/2$

2x2 matrices

$$\hat{J}_x = \hbar\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} = \frac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}\hbar\hat{\sigma}_x,$$

$$\hat{J}_y = \hbar\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} = \frac{1}{2}\hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{2}\hbar\hat{\sigma}_y,$$

$$\hat{J}_z = \hbar\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} = \frac{1}{2}\hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}\hbar\hat{\sigma}_z$$

under the basis of $\{\left|+z\right\rangle, \left|-z\right\rangle\}$, where $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$ are the Pauli matrices.

where

$$\left|+z\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \left|-z\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

(a) The eigenkets of $\hat{J}_x$:

$$|+x\rangle = \hat{U}_x|+z\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \qquad |-x\rangle = \hat{U}_x|-z\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix},$$

with the unitary operator $\hat{U}_x$,

$$\hat{U}_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}.$$

(b) The eigenkets of $\hat{J}_y$:

$$|+y\rangle = \hat{U}_y|+z\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}, \qquad |-y\rangle = \hat{U}_y|-z\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix},$$

with

$$\hat{U}_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\i & -i\end{pmatrix}, \qquad \hat{U}_y^{+} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\1 & i\end{pmatrix}$$

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2. $j = 1$

3x3 matrices

$$\hat{J}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\1 & 0 & 1\\0 & 1 & 0\end{pmatrix},$$

$$\hat{J}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\i & 0 & -i\\0 & i & 0\end{pmatrix},$$

$$\hat{J}_z = \hbar\begin{pmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & -1\end{pmatrix}.$$

where

$$|1,1\rangle_z = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |1,0\rangle_z = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |1,-1\rangle_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(a) The eigenkets of $\hat{J}_x$:

$$|1,1\rangle_x = \hat{U}_x|1,1\rangle_z = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \qquad |1,0\rangle_x = \hat{U}_x|1,0\rangle_z = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

$$|1,-1\rangle_x = \hat{U}_x|1,-1\rangle_z = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}.$$

with the unitary operator,

$$\hat{U}_x = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}, \qquad \hat{U}_x^{+} = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}.$$

(b) The eigenkets of $\hat{J}_y$:

$$|1,1\rangle_y = \hat{U}_y|1,1\rangle_z = \frac{1}{2}\begin{pmatrix} 1 \\ i\sqrt{2} \\ -1 \end{pmatrix}, \quad |1,0\rangle_y = \hat{U}_y|1,0\rangle_z = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

$$|1,-1\rangle_y = \hat{U}_y|1,-1\rangle_z = \frac{1}{2}\begin{pmatrix} 1 \\ -i\sqrt{2} \\ -1 \end{pmatrix}$$

with the unitary operator,

$$\hat{U}_y = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix}, \qquad \hat{U}_y^{+} = \begin{pmatrix} \frac{1}{2} & -\frac{i}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{i}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix}$$

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3. $j = 3/2$

4 x 4 matrices

$$\hat{J}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix},$$

$$\hat{J}_y = \frac{i\hbar}{2}\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix},$$

$$\hat{J}_z = \frac{\hbar}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix},$$

$$\left|\frac{3}{2},\frac{3}{2}\right\rangle_z = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \left|\frac{3}{2},\frac{1}{2}\right\rangle_z = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\left|\frac{3}{2},-\frac{1}{2}\right\rangle_z = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \left|\frac{3}{2},-\frac{3}{2}\right\rangle_z = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(a) The eigenket of $\hat{J}_x$;

$$\left|\frac{3}{2},\frac{3}{2}\right\rangle_x = \hat{U}_x\left|\frac{3}{2},\frac{3}{2}\right\rangle_z = \frac{1}{2\sqrt{2}}\begin{pmatrix}1\\ \sqrt{3}\\ \sqrt{3}\\ 1\end{pmatrix}, \qquad \left|\frac{3}{2},\frac{1}{2}\right\rangle_x = \hat{U}_x\left|\frac{3}{2},\frac{1}{2}\right\rangle_z = \frac{1}{2\sqrt{2}}\begin{pmatrix}\sqrt{3}\\ 1\\ -1\\ -\sqrt{3}\end{pmatrix}$$

$$\left|\frac{3}{2},-\frac{1}{2}\right\rangle_x = \hat{U}_x\left|\frac{3}{2},-\frac{1}{2}\right\rangle_z = \frac{1}{2\sqrt{2}}\begin{pmatrix}\sqrt{3}\\ -1\\ -1\\ \sqrt{3}\end{pmatrix}, \qquad \left|\frac{3}{2},-\frac{3}{2}\right\rangle_x = \hat{U}_x\left|\frac{3}{2},-\frac{3}{2}\right\rangle_z = \frac{1}{2\sqrt{2}}\begin{pmatrix}1\\ -\sqrt{3}\\ \sqrt{3}\\ -1\end{pmatrix}$$

with the unitary operator

$$\hat{U}_x = \frac{1}{2\sqrt{2}}\begin{pmatrix}1 & \sqrt{3} & \sqrt{3} & 1\\ \sqrt{3} & 1 & -1 & -\sqrt{3}\\ \sqrt{3} & -1 & -1 & \sqrt{3}\\ 1 & -\sqrt{3} & \sqrt{3} & -1\end{pmatrix}$$

(b) The eigenket of $\hat{J}_y$;

$$\left|\frac{3}{2},\frac{3}{2}\right\rangle_y = \hat{U}_y\left|\frac{3}{2},\frac{3}{2}\right\rangle_z = \frac{1}{2\sqrt{2}}\begin{pmatrix}1\\ i\sqrt{3}\\ -\sqrt{3}\\ -i\end{pmatrix}, \qquad \left|\frac{3}{2},\frac{1}{2}\right\rangle_y = \hat{U}_y\left|\frac{3}{2},\frac{1}{2}\right\rangle_z = \frac{1}{2\sqrt{2}}\begin{pmatrix}\sqrt{3}\\ i\\ 1\\ i\sqrt{3}\end{pmatrix}$$

$$\left|\frac{3}{2},-\frac{1}{2}\right\rangle_y = \hat{U}_y\left|\frac{3}{2},-\frac{1}{2}\right\rangle_z = \frac{1}{2\sqrt{2}}\begin{pmatrix}\sqrt{3}\\ -i\\ 1\\ -i\sqrt{3}\end{pmatrix}, \qquad \left|\frac{3}{2},-\frac{3}{2}\right\rangle_y = \hat{U}_y\left|\frac{3}{2},-\frac{3}{2}\right\rangle_z = \frac{1}{2\sqrt{2}}\begin{pmatrix}1\\ -i\sqrt{3}\\ -\sqrt{3}\\ i\end{pmatrix}$$

with

$$\hat{U}_y = \frac{1}{2\sqrt{2}}\begin{pmatrix}1 & \sqrt{3} & \sqrt{3} & 1\\ i\sqrt{3} & i & -i & -i\sqrt{3}\\ -\sqrt{3} & 1 & 1 & -\sqrt{3}\\ -i & i\sqrt{3} & -i\sqrt{3} & i\end{pmatrix}$$

4. $j = 2$

5x5 matrices

$$\hat{J}_x = \hbar \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\hat{J}_y = \hbar \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & -i\sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & i\sqrt{\frac{3}{2}} & 0 & -i\sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & i\sqrt{\frac{3}{2}} & 0 & -i \\ 0 & 0 & 0 & i & 0 \end{pmatrix},$$

$$\hat{J}_z = \hbar \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix},$$

with

$$|2,2\rangle_z = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad |2,1\rangle_z = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad |2,0\rangle_z = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$|2,-1\rangle_z = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |2,-2\rangle_z = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

(a) Eigenvectors of $\hat{J}_x$

$$|2,2\rangle_x = \hat{U}_x |2,2\rangle_z = \frac{1}{4}\begin{pmatrix} 1 \\ 2 \\ \sqrt{6} \\ 2 \\ 1 \end{pmatrix}, \qquad |2,1\rangle_x = \hat{U}_x |2,1\rangle_z = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ -1 \end{pmatrix}$$

$$|2,0\rangle_x = \hat{U}_x |2,0\rangle_z = \frac{1}{4}\begin{pmatrix} \sqrt{6} \\ 0 \\ -2 \\ 0 \\ \sqrt{6} \end{pmatrix}, \qquad |2,-1\rangle_x = \hat{U}_x |2,-1\rangle_z = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$|2,-2\rangle_x = \hat{U}_x |2,-2\rangle_z = \frac{1}{4}\begin{pmatrix} 1 \\ -2 \\ \sqrt{6} \\ -2 \\ 1 \end{pmatrix}$$

with the unitary operator,

$$\hat{U}_x = \frac{1}{4}\begin{pmatrix} 1 & 2 & \sqrt{6} & 2 & 1 \\ 2 & 2 & 0 & -2 & -2 \\ \sqrt{6} & 0 & -2 & 0 & \sqrt{6} \\ 2 & -2 & 0 & 2 & -2 \\ 1 & -2 & \sqrt{6} & -2 & 1 \end{pmatrix}$$

(b) Eigenvectors of $\hat{J}_y$

$$|2,2\rangle_y = \hat{U}_y |2,2\rangle_z = \frac{1}{4}\begin{pmatrix} 1 \\ 2i \\ -\sqrt{6} \\ -2i \\ 1 \end{pmatrix}, \qquad |2,1\rangle_y = \hat{U}_y |2,1\rangle_z = \frac{1}{2}\begin{pmatrix} 1 \\ i \\ 0 \\ i \\ -1 \end{pmatrix},$$

$$|2,0\rangle_y = \hat{U}_y |2,0\rangle_z = \frac{1}{4}\begin{pmatrix} \sqrt{6} \\ 0 \\ 2 \\ 0 \\ \sqrt{6} \end{pmatrix}, \quad |2,-1\rangle_y = \hat{U}_y |2,-1\rangle_z = \frac{1}{2}\begin{pmatrix} 1 \\ -i \\ 0 \\ -i \\ -1 \end{pmatrix},$$

$$|2,-2\rangle_y = \hat{U}_y |2,-2\rangle_z = \frac{1}{4}\begin{pmatrix} 1 \\ -2i \\ -\sqrt{6} \\ 2i \\ 1 \end{pmatrix},$$

with the unitary operator,

$$\hat{U}_y = \frac{1}{4}\begin{pmatrix} 1 & 2 & \sqrt{6} & 2 & 1 \\ 2i & 2i & 0 & -2i & -2i \\ -\sqrt{6} & 0 & 2 & 0 & -\sqrt{6} \\ -2i & 2i & 0 & -2i & 2i \\ 1 & -2 & \sqrt{6} & -2 & 1 \end{pmatrix}$$

$$\hat{U}_y^{+} = \frac{1}{4}\begin{pmatrix} 1 & -2i & -\sqrt{6} & 2i & 1 \\ 2 & -2i & 0 & -2i & -2 \\ \sqrt{6} & 0 & 2 & 0 & \sqrt{6} \\ 2 & 2i & 0 & 2i & -2 \\ 1 & 2i & -\sqrt{6} & -2i & 1 \end{pmatrix}$$

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5. $j = 5/2$

6x6 matrices

$$\hat{J}_x = \hbar \begin{pmatrix} 0 & \frac{\sqrt{5}}{2} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{5}}{2} & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{3}{2} & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & \frac{\sqrt{5}}{2} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{5}}{2} & 0 \end{pmatrix},$$

$$\hat{J}_y = \hbar \begin{pmatrix} 0 & \frac{-i\sqrt{5}}{2} & 0 & 0 & 0 & 0 \\ \frac{i\sqrt{5}}{2} & 0 & -i\sqrt{2} & 0 & 0 & 0 \\ 0 & i\sqrt{2} & 0 & -\frac{3i}{2} & 0 & 0 \\ 0 & 0 & \frac{3i}{2} & 0 & -i\sqrt{2} & 0 \\ 0 & 0 & 0 & i\sqrt{2} & 0 & \frac{-i\sqrt{5}}{2} \\ 0 & 0 & 0 & 0 & \frac{i\sqrt{5}}{2} & 0 \end{pmatrix},$$

$$\hat{J}_z = \hbar \begin{pmatrix} \frac{5}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{5}{2} \end{pmatrix}.$$

with

$$\left|\frac{5}{2},\frac{5}{2}\right\rangle_z = \begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}, \qquad \left|\frac{5}{2},\frac{3}{2}\right\rangle_z = \begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}, \qquad \left|\frac{5}{2},\frac{1}{2}\right\rangle_z = \begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix}$$

$$\left|\frac{5}{2},-\frac{1}{2}\right\rangle_z = \begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}, \qquad \left|\frac{5}{2},-\frac{3}{2}\right\rangle_z = \begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix}, \qquad \left|\frac{5}{2},-\frac{5}{2}\right\rangle_z = \begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}$$

(a) Eigenvectors of $\hat{J}_x$

$$\left|\frac{5}{2},\frac{5}{2}\right\rangle_x = \hat{U}_x\left|\frac{5}{2},\frac{5}{2}\right\rangle_z = \frac{1}{4\sqrt{2}}\begin{pmatrix}1\\\sqrt{5}\\\sqrt{10}\\\sqrt{10}\\\sqrt{5}\\1\end{pmatrix}$$

$$\left|\frac{5}{2},\frac{3}{2}\right\rangle_x = \hat{U}_x\left|\frac{5}{2},\frac{3}{2}\right\rangle_z = \frac{1}{4\sqrt{2}}\begin{pmatrix}\sqrt{5}\\3\\\sqrt{2}\\-\sqrt{2}\\-3\\-\sqrt{5}\end{pmatrix}$$

$$\left|\frac{5}{2},\frac{1}{2}\right\rangle_x = \hat{U}_x\left|\frac{5}{2},\frac{1}{2}\right\rangle_z = \frac{1}{4\sqrt{2}}\begin{pmatrix}\sqrt{10}\\\sqrt{2}\\-2\\-2\\\sqrt{2}\\\sqrt{10}\end{pmatrix}$$

$$\left|\frac{5}{2},-\frac{1}{2}\right\rangle_x = \hat{U}_x\left|\frac{5}{2},-\frac{1}{2}\right\rangle_z = \frac{1}{4\sqrt{2}}\begin{pmatrix}\sqrt{10}\\ -\sqrt{2}\\ -2\\ 2\\ \sqrt{2}\\ -\sqrt{10}\end{pmatrix},$$

$$\left|\frac{5}{2},-\frac{3}{2}\right\rangle_x = \hat{U}_x\left|\frac{5}{2},-\frac{3}{2}\right\rangle_z = \frac{1}{4\sqrt{2}}\begin{pmatrix}\sqrt{5}\\ -3\\ \sqrt{2}\\ \sqrt{2}\\ -3\\ \sqrt{5}\end{pmatrix},$$

$$\left|\frac{5}{2},-\frac{5}{2}\right\rangle_x = \hat{U}_x\left|\frac{5}{2},-\frac{5}{2}\right\rangle_z = \frac{1}{4\sqrt{2}}\begin{pmatrix}1\\ -\sqrt{5}\\ \sqrt{10}\\ -\sqrt{10}\\ \sqrt{5}\\ -1\end{pmatrix},$$

with the unitary operator given by

$$\hat{U}_x = \frac{1}{4\sqrt{2}}\begin{pmatrix}1 & \sqrt{5} & \sqrt{10} & \sqrt{10} & \sqrt{5} & 1\\ \sqrt{5} & 3 & \sqrt{2} & -\sqrt{2} & -3 & -\sqrt{5}\\ \sqrt{10} & \sqrt{2} & -2 & -2 & \sqrt{2} & \sqrt{10}\\ \sqrt{10} & -\sqrt{2} & -2 & 2 & \sqrt{2} & -\sqrt{10}\\ \sqrt{5} & -3 & \sqrt{2} & \sqrt{2} & -3 & \sqrt{5}\\ 1 & -\sqrt{5} & \sqrt{10} & -\sqrt{10} & \sqrt{5} & -1\end{pmatrix}.$$

(b) Eigenvectors of $\hat{J}_y$

$$\left|\frac{5}{2},\frac{5}{2}\right\rangle_y = \hat{U}_y\left|\frac{5}{2},\frac{5}{2}\right\rangle_z = \frac{1}{4\sqrt{2}}\begin{pmatrix} 1 \\ i\sqrt{5} \\ -\sqrt{10} \\ -i\sqrt{10} \\ \sqrt{5} \\ i \end{pmatrix},$$

$$\left|\frac{5}{2},\frac{3}{2}\right\rangle_y = \hat{U}_y\left|\frac{5}{2},\frac{3}{2}\right\rangle_z = \frac{1}{4\sqrt{2}}\begin{pmatrix} \sqrt{5} \\ i3 \\ -\sqrt{2} \\ i\sqrt{2} \\ -i3 \\ -i\sqrt{5} \end{pmatrix},$$

$$\left|\frac{5}{2},\frac{1}{2}\right\rangle_y = \hat{U}_y\left|\frac{5}{2},\frac{1}{2}\right\rangle_z = \frac{1}{4\sqrt{2}}\begin{pmatrix} \sqrt{10} \\ i\sqrt{2} \\ 2 \\ i2 \\ \sqrt{2} \\ i\sqrt{10} \end{pmatrix},$$

$$\left|\frac{5}{2},-\frac{1}{2}\right\rangle_y = \hat{U}_y\left|\frac{5}{2},-\frac{1}{2}\right\rangle_z = \frac{1}{4\sqrt{2}}\begin{pmatrix} \sqrt{10} \\ -i\sqrt{2} \\ 2 \\ -i2 \\ \sqrt{2} \\ -i\sqrt{10} \end{pmatrix},$$

$$\left|\frac{5}{2},-\frac{3}{2}\right\rangle_y = \hat{U}_y\left|\frac{5}{2},-\frac{3}{2}\right\rangle_z = \frac{1}{4\sqrt{2}}\begin{pmatrix} \sqrt{5} \\ -i3 \\ -\sqrt{2} \\ -i\sqrt{2} \\ -3 \\ i\sqrt{5} \end{pmatrix},$$

$$\left|\frac{5}{2},-\frac{5}{2}\right\rangle_y = \hat{U}_y\left|\frac{5}{2},-\frac{5}{2}\right\rangle_z = \frac{1}{4\sqrt{2}}\begin{pmatrix} 1 \\ -i\sqrt{5} \\ -\sqrt{10} \\ i\sqrt{10} \\ \sqrt{5} \\ -i \end{pmatrix},$$

with the unitary operator given by

$$\hat{U}_y = \frac{1}{4\sqrt{2}}\begin{pmatrix} 1 & \sqrt{5} & \sqrt{10} & \sqrt{10} & \sqrt{5} & 1 \\ i\sqrt{5} & 3i & i\sqrt{2} & -i\sqrt{2} & -i3 & -i\sqrt{5} \\ -\sqrt{10} & -\sqrt{2} & 2 & 2 & -\sqrt{2} & -\sqrt{10} \\ -i\sqrt{10} & i\sqrt{2} & i2 & -i2 & -i\sqrt{2} & i\sqrt{10} \\ \sqrt{5} & -3 & \sqrt{2} & \sqrt{2} & -3 & \sqrt{5} \\ i & -i\sqrt{5} & i\sqrt{10} & -i\sqrt{10} & i\sqrt{5} & -i \end{pmatrix}.$$

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6. $j = 3$

7x7 matrices

$$\hat{J}_x = \hbar\begin{pmatrix} 0 & \sqrt{\frac{3}{2}} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\frac{3}{2}} & 0 & \sqrt{\frac{5}{2}} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{5}{2}} & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & \sqrt{\frac{5}{2}} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{5}{2}} & 0 & \sqrt{\frac{3}{2}} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{3}{2}} & 0 \end{pmatrix}$$

$$\hat{J}_y = \hbar\begin{pmatrix} 0 & -i\sqrt{\frac{3}{2}} & 0 & 0 & 0 & 0 & 0 \\ i\sqrt{\frac{3}{2}} & 0 & -i\sqrt{\frac{5}{2}} & 0 & 0 & 0 & 0 \\ 0 & i\sqrt{\frac{5}{2}} & 0 & -i\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & i\sqrt{3} & 0 & -i\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & i\sqrt{3} & 0 & -i\sqrt{\frac{5}{2}} & 0 \\ 0 & 0 & 0 & 0 & i\sqrt{\frac{5}{2}} & 0 & -i\sqrt{\frac{3}{2}} \\ 0 & 0 & 0 & 0 & 0 & i\sqrt{\frac{3}{2}} & 0 \end{pmatrix},$$

$$\hat{J}_z = \hbar\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{pmatrix}.$$

(a) Eigenvectors of $\hat{J}_x$

$$|3,3\rangle_x = \hat{U}_x|3,3\rangle_z = \frac{1}{8}\begin{pmatrix} 1 \\ \sqrt{6} \\ \sqrt{15} \\ 2\sqrt{5} \\ \sqrt{15} \\ \sqrt{6} \\ 1 \end{pmatrix},$$

$$\left|3,2\right\rangle_x = \hat{U}_x\left|3,2\right\rangle_z = \frac{1}{8}\begin{pmatrix} \sqrt{6} \\ 4 \\ \sqrt{10} \\ 0 \\ -\sqrt{10} \\ -4 \\ -\sqrt{6} \end{pmatrix},$$

$$\left|3,1\right\rangle_x = \hat{U}_x\left|3,1\right\rangle_z = \frac{1}{8}\begin{pmatrix} \sqrt{15} \\ \sqrt{10} \\ -1 \\ -2\sqrt{3} \\ -1 \\ \sqrt{10} \\ \sqrt{15} \end{pmatrix},$$

$$\left|3,0\right\rangle_x = \hat{U}_x\left|3,0\right\rangle_z = \frac{1}{4}\begin{pmatrix} \sqrt{5} \\ 0 \\ -\sqrt{3} \\ 0 \\ \sqrt{3} \\ 0 \\ -\sqrt{5} \end{pmatrix},$$

$$\left|3,-1\right\rangle_x = \hat{U}_x\left|3,-1\right\rangle_z = \frac{1}{8}\begin{pmatrix} \sqrt{15} \\ -\sqrt{10} \\ -1 \\ 2\sqrt{3} \\ -1 \\ -\sqrt{10} \\ \sqrt{15} \end{pmatrix},$$

$$\left|3,-2\right\rangle_x = \hat{U}_x\left|3,-2\right\rangle_z = \frac{1}{8}\begin{pmatrix} \sqrt{6} \\ -4 \\ \sqrt{10} \\ 0 \\ -\sqrt{10} \\ 4 \\ -\sqrt{6} \end{pmatrix},$$

$$\left|3,-3\right\rangle_x = \hat{U}_x\left|3,-3\right\rangle_z = \frac{1}{8}\begin{pmatrix} 1 \\ -\sqrt{6} \\ \sqrt{15} \\ -2\sqrt{5} \\ \sqrt{15} \\ -\sqrt{6} \\ 1 \end{pmatrix},$$

(b) Eigenvectors of $\hat{J}_y$

$$\left|3,3\right\rangle_y = \hat{U}_y\left|3,3\right\rangle_z = \frac{1}{8}\begin{pmatrix} 1 \\ i\sqrt{6} \\ -\sqrt{15} \\ -i2\sqrt{5} \\ \sqrt{15} \\ i\sqrt{6} \\ -1 \end{pmatrix},$$

$$\left|3,2\right\rangle_y = \hat{U}_y\left|3,2\right\rangle_z = \frac{1}{8}\begin{pmatrix} \sqrt{6} \\ i4 \\ -\sqrt{10} \\ 0 \\ -\sqrt{10} \\ -i4 \\ \sqrt{6} \end{pmatrix},$$

$$\left|3,1\right\rangle_y = \hat{U}_y\left|3,1\right\rangle_z = \frac{1}{8}\begin{pmatrix} \sqrt{15} \\ i\sqrt{10} \\ 1 \\ i2\sqrt{3} \\ -1 \\ i\sqrt{10} \\ -\sqrt{15} \end{pmatrix},$$

$$\left|3,0\right\rangle_y = \hat{U}_y\left|3,0\right\rangle_z = \frac{1}{4}\begin{pmatrix} \sqrt{5} \\ 0 \\ \sqrt{3} \\ 0 \\ \sqrt{3} \\ 0 \\ \sqrt{5} \end{pmatrix},$$

$$\left|3,-1\right\rangle_y = \hat{U}_y\left|3,-1\right\rangle_z = \frac{1}{8}\begin{pmatrix} \sqrt{15} \\ -i\sqrt{10} \\ 1 \\ -i2\sqrt{3} \\ -1 \\ -i\sqrt{10} \\ -\sqrt{15} \end{pmatrix},$$

$$\left|3,-2\right\rangle_y = \hat{U}_y\left|3,-2\right\rangle_z = \frac{1}{8}\begin{pmatrix} \sqrt{6} \\ -i4 \\ -\sqrt{10} \\ 0 \\ -\sqrt{10} \\ i4 \\ \sqrt{6} \end{pmatrix},$$

$$\left|3,-3\right\rangle_y = \hat{U}_y\left|3,-3\right\rangle_z = \frac{1}{8}\begin{pmatrix} 1 \\ -i\sqrt{6} \\ -\sqrt{15} \\ i2\sqrt{5} \\ \sqrt{15} \\ -i\sqrt{6} \\ -1 \end{pmatrix}.$$

7. *j* = 7/2

8x8 matrices

$$\hat{J}_x = \hbar\begin{pmatrix} 0 & \frac{\sqrt{7}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{7}}{2} & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & \frac{\sqrt{15}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{15}}{2} & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & \frac{\sqrt{15}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{15}}{2} & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 & \frac{\sqrt{7}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{7}}{2} & 0 \end{pmatrix}$$

$$
\hat{J}_x = \hbar \begin{pmatrix}
0 & \frac{-i\sqrt{7}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{i\sqrt{7}}{2} & 0 & -i\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\
0 & i\sqrt{3} & 0 & \frac{-i\sqrt{15}}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{i\sqrt{15}}{2} & 0 & -2i & 0 & 0 & 0 \\
0 & 0 & 0 & 2i & 0 & \frac{-i\sqrt{15}}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{i\sqrt{15}}{2} & 0 & -i\sqrt{3} & 0 \\
0 & 0 & 0 & 0 & 0 & i\sqrt{3} & 0 & \frac{-i\sqrt{7}}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{i\sqrt{7}}{2} & 0
\end{pmatrix}
$$

$$
\hat{J}_z = \hbar \begin{pmatrix}
\frac{7}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{5}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{3}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{3}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{5}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{7}{2}
\end{pmatrix}
$$

6. Eigenvalue problems for $j = 1/2$

We make a program for the matrix elements in Mathematica.

((**Mathematica-1**))

Matrices j = 1/2

```
Clear["Global`*"]; j = 1/2;
exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]};
Jx[j_, n_, m_] := ħ/2 Sqrt[(j - m) (j + m + 1)] KroneckerDelta[n, m + 1] +
   ħ/2 Sqrt[(j + m) (j - m + 1)] KroneckerDelta[n, m - 1];
Jy[j_, n_, m_] := -ħ/2 i Sqrt[(j - m) (j + m + 1)] KroneckerDelta[n, m + 1] +
   ħ/2 i Sqrt[(j + m) (j - m + 1)] KroneckerDelta[n, m - 1];
Jz[j_, n_, m_] := ħ m KroneckerDelta[n, m];
Jx = Table[Jx[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jy = Table[Jy[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jz = Table[Jz[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jx // MatrixForm
```

$$\begin{pmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{pmatrix}$$

```
Jy // MatrixForm
```

$$\begin{pmatrix} 0 & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & 0 \end{pmatrix}$$

```
Jz // MatrixForm
```

$$\begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix}$$

```
eq1 = Eigensystem[Jx]
```

$\{\{-\frac{\hbar}{2}, \frac{\hbar}{2}\}, \{\{-1, 1\}, \{1, 1\}\}\}$

```
ψ1x = Normalize[eq1[[2, 2]]]; ψ1x // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

```
ψ2x = -Normalize[eq1[[2, 1]]]; ψ2x // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

```
UxT = {ψ1x, ψ2x}; Ux = Transpose[UxT]; Ux // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

```
eq2 = Eigensystem[Jy]
```

$$\left\{\left\{-\frac{\hbar}{2}, \frac{\hbar}{2}\right\}, \{\{i, 1\}, \{-i, 1\}\}\right\}$$

```
ψ1y = i Normalize[eq2[[2, 2]]]; ψ1y // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$$

```
ψ2y = -i Normalize[eq2[[2, 1]]]; ψ2y // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$$

```
UyT = {ψ1y, ψ2y}; Uy = Transpose[UyT]; Uy // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix}$$

---

7. Eigenvalue problems for $j = 1$

((Mathematica))

Matrices j = 1

```
Clear["Global`*"]; j = 1; exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]};
Jx[j_, n_, m_] := ħ/2 Sqrt[(j - m) (j + m + 1)] KroneckerDelta[n, m + 1] +
   ħ/2 Sqrt[(j + m) (j - m + 1)] KroneckerDelta[n, m - 1];
Jy[j_, n_, m_] := -ħ/2 i Sqrt[(j - m) (j + m + 1)] KroneckerDelta[n, m + 1] +
   ħ/2 i Sqrt[(j + m) (j - m + 1)] KroneckerDelta[n, m - 1];
Jz[j_, n_, m_] := ħ m KroneckerDelta[n, m];
Jx = Table[Jx[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jy = Table[Jy[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jz = Table[Jz[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jx // MatrixForm
```

$$\begin{pmatrix} 0 & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & 0 & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & 0 \end{pmatrix}$$

```
Jy // MatrixForm
```

$$\begin{pmatrix} 0 & -\frac{i\hbar}{\sqrt{2}} & 0 \\ \frac{i\hbar}{\sqrt{2}} & 0 & -\frac{i\hbar}{\sqrt{2}} \\ 0 & \frac{i\hbar}{\sqrt{2}} & 0 \end{pmatrix}$$

```
Jz // MatrixForm
```

$$\begin{pmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{pmatrix}$$

```
eq1 = Eigensystem[Jx]
```

$$\{\{0, -\hbar, \hbar\}, \{\{-1, 0, 1\}, \{1, -\sqrt{2}, 1\}, \{1, \sqrt{2}, 1\}\}\}$$

```
ψ1x = Normalize[eq1[[2, 3]]]; ψ1x // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$$

```
ψ2x = -Normalize[eq1[[2, 1]]]; ψ2x // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

```
ψ3x = Normalize[eq1[[2, 2]]]; ψ3x // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$$

```
UxT = {ψ1x, ψ2x, ψ3x}; Ux = Transpose[UxT]; Ux // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

```
UxH = UxT*; UxH // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

```
UxH.Ux
```

{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}

```
eq2 = Eigensystem[Jy]
```

$\{\{0, -\hbar, \hbar\}, \{\{1, 0, 1\}, \{-1, i\sqrt{2}, 1\}, \{-1, -i\sqrt{2}, 1\}\}\}$

```
ψ1y = -Normalize[eq2[[2, 3]]]; ψ1y // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} \\ \frac{i}{\sqrt{2}} \\ -\frac{1}{2} \end{pmatrix}$$

```
ψ2y = Normalize[eq2[[2, 1]]]; ψ2y // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

```
ψ3y = - Normalize[eq2[[2, 2]]]; ψ3y // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} \\ -\frac{i}{\sqrt{2}} \\ -\frac{1}{2} \end{pmatrix}$$

```
UyT = {ψ1y, ψ2y, ψ3y}; Uy = Transpose[UyT]; Uy // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix}$$

```
UyH = UyT*; UyH // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} & -\frac{i}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{i}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix}$$

```
UyH.Uy // Simplify
```

```
{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}
```

8. Eigenvalue problems for $j = 3/2$

((Mathematica))

Matrices j = 3/2

```
Clear["Global`*"]; j = 3/2;
exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]};
Jx[j_, n_, m_] := ħ/2 Sqrt[(j - m) (j + m + 1)] KroneckerDelta[n, m + 1] +
  ħ/2 Sqrt[(j + m) (j - m + 1)] KroneckerDelta[n, m - 1];
Jy[j_, n_, m_] := -ħ/2 i Sqrt[(j - m) (j + m + 1)] KroneckerDelta[n, m + 1] +
  ħ/2 i Sqrt[(j + m) (j - m + 1)] KroneckerDelta[n, m - 1];
Jz[j_, n_, m_] := ħ m KroneckerDelta[n, m];
Jx = Table[Jx[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jy = Table[Jy[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jz = Table[Jz[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jx // MatrixForm
```

$$\begin{pmatrix} 0 & \frac{\sqrt{3}\,\hbar}{2} & 0 & 0 \\ \frac{\sqrt{3}\,\hbar}{2} & 0 & \hbar & 0 \\ 0 & \hbar & 0 & \frac{\sqrt{3}\,\hbar}{2} \\ 0 & 0 & \frac{\sqrt{3}\,\hbar}{2} & 0 \end{pmatrix}$$

```
Jy // MatrixForm
```

$$\begin{pmatrix} 0 & -\frac{1}{2} i \sqrt{3}\,\hbar & 0 & 0 \\ \frac{1}{2} i \sqrt{3}\,\hbar & 0 & -i\hbar & 0 \\ 0 & i\hbar & 0 & -\frac{1}{2} i \sqrt{3}\,\hbar \\ 0 & 0 & \frac{1}{2} i \sqrt{3}\,\hbar & 0 \end{pmatrix}$$

```
Jz // MatrixForm
```

$$\begin{pmatrix} \frac{3\hbar}{2} & 0 & 0 & 0 \\ 0 & \frac{\hbar}{2} & 0 & 0 \\ 0 & 0 & -\frac{\hbar}{2} & 0 \\ 0 & 0 & 0 & -\frac{3\hbar}{2} \end{pmatrix}$$

```
eq1 = Eigensystem[Jx]
```

$$\left\{\left\{-\frac{3\hbar}{2}, -\frac{\hbar}{2}, \frac{\hbar}{2}, \frac{3\hbar}{2}\right\}, \left\{\left\{-1, \sqrt{3}, -\sqrt{3}, 1\right\}, \left\{1, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 1\right\}, \left\{-1, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 1\right\}, \left\{1, \sqrt{3}, \sqrt{3}, 1\right\}\right\}\right\}$$

```
ψ1x = Normalize[eq1[[2, 2]]] // Simplify; ψ1x // MatrixForm
```

$$\begin{pmatrix} \frac{\sqrt{\frac{3}{2}}}{2} \\ -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} \\ \frac{\sqrt{\frac{3}{2}}}{2} \end{pmatrix}$$

```
ψ2x = -Normalize[eq1[[2, 4]]] // Simplify; ψ2x // MatrixForm
```

$$\begin{pmatrix} -\frac{1}{2\sqrt{2}} \\ -\frac{\sqrt{\frac{3}{2}}}{2} \\ -\frac{\sqrt{\frac{3}{2}}}{2} \\ -\frac{1}{2\sqrt{2}} \end{pmatrix}$$

```
ψ3x = Normalize[eq1[[2, 3]]] // Simplify; ψ3x // MatrixForm
```

$$\begin{pmatrix} -\frac{\sqrt{\frac{3}{2}}}{2} \\ -\frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ \frac{\sqrt{\frac{3}{2}}}{2} \end{pmatrix}$$

```
ψ4x = -Normalize[eq1[[2, 1]]] // Simplify;
ψ4x // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2\sqrt{2}} \\ -\frac{\sqrt{\frac{3}{2}}}{2} \\ \frac{\sqrt{\frac{3}{2}}}{2} \\ -\frac{1}{2\sqrt{2}} \end{pmatrix}$$

```
UxT = {ψ1x, ψ2x, ψ3x, ψ4x}; Ux = Transpose[UxT];
Ux // MatrixForm
```

$$\begin{pmatrix} \frac{\sqrt{\frac{3}{2}}}{2} & -\frac{1}{2\sqrt{2}} & -\frac{\sqrt{\frac{3}{2}}}{2} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & -\frac{\sqrt{\frac{3}{2}}}{2} & -\frac{1}{2\sqrt{2}} & -\frac{\sqrt{\frac{3}{2}}}{2} \\ -\frac{1}{2\sqrt{2}} & -\frac{\sqrt{\frac{3}{2}}}{2} & \frac{1}{2\sqrt{2}} & \frac{\sqrt{\frac{3}{2}}}{2} \\ \frac{\sqrt{\frac{3}{2}}}{2} & -\frac{1}{2\sqrt{2}} & \frac{\sqrt{\frac{3}{2}}}{2} & -\frac{1}{2\sqrt{2}} \end{pmatrix}$$

```
UxH = UxT*; UxH.Ux
```

{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}

```
eq2 = Eigensystem[Jy]
```

$$\left\{\left\{-\frac{3\hbar}{2}, -\frac{\hbar}{2}, \frac{\hbar}{2}, \frac{3\hbar}{2}\right\}, \left\{\left\{-i, -\sqrt{3}, i\sqrt{3}, 1\right\}, \left\{i, \frac{1}{\sqrt{3}}, \frac{i}{\sqrt{3}}, 1\right\}, \left\{-i, \frac{1}{\sqrt{3}}, -\frac{i}{\sqrt{3}}, 1\right\}, \left\{i, -\sqrt{3}, -i\sqrt{3}, 1\right\}\right\}\right\}$$

```
ψ1y = - i Normalize[eq2[[2, 2]]] // Simplify;
ψ1y // MatrixForm
```

$$\begin{pmatrix} \frac{\sqrt{\frac{3}{2}}}{2} \\ -\frac{i}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ -\frac{1}{2}\, i\sqrt{\frac{3}{2}} \end{pmatrix}$$

```
ψ2y = i Normalize[eq2[[2, 4]]] // Simplify;
ψ2y // MatrixForm
```

$$\begin{pmatrix} -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2}\, i\, \sqrt{\frac{3}{2}} \\ \frac{\sqrt{\frac{3}{2}}}{2} \\ \frac{i}{2\sqrt{2}} \end{pmatrix}$$

```
ψ3y = -i Normalize[eq2[[2, 3]]] // Simplify;
ψ3y // MatrixForm
```

$$\begin{pmatrix} -\frac{\sqrt{\frac{3}{2}}}{2} \\ -\frac{i}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2}\, i\, \sqrt{\frac{3}{2}} \end{pmatrix}$$

```
ψ4y = ⅈ Normalize[eq2[[2, 1]]] // Simplify;
ψ4y // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2\sqrt{2}} \\ -\frac{1}{2}\, i \sqrt{\frac{3}{2}} \\ -\frac{\sqrt{\frac{3}{2}}}{2} \\ \frac{i}{2\sqrt{2}} \end{pmatrix}$$

```
UyT = {ψ1y, ψ2y, ψ3y, ψ4y}; Uy = Transpose[UyT];
Uy // MatrixForm
```

$$\begin{pmatrix} \frac{\sqrt{\frac{3}{2}}}{2} & -\frac{1}{2\sqrt{2}} & -\frac{\sqrt{\frac{3}{2}}}{2} & \frac{1}{2\sqrt{2}} \\ -\frac{i}{2\sqrt{2}} & -\frac{1}{2}\, i \sqrt{\frac{3}{2}} & -\frac{i}{2\sqrt{2}} & -\frac{1}{2}\, i \sqrt{\frac{3}{2}} \\ \frac{1}{2\sqrt{2}} & \frac{\sqrt{\frac{3}{2}}}{2} & -\frac{1}{2\sqrt{2}} & -\frac{\sqrt{\frac{3}{2}}}{2} \\ -\frac{1}{2}\, i \sqrt{\frac{3}{2}} & \frac{i}{2\sqrt{2}} & -\frac{1}{2}\, i \sqrt{\frac{3}{2}} & \frac{i}{2\sqrt{2}} \end{pmatrix}$$

```
UyH = UyT*; UyH.Uy
{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}
```

9. Eigenvalue problems for $j = 2$

Matrices j = 2

```
Clear["Global`*"]; j = 2;
```

$Jx[j\_, n\_, m\_] := \frac{\hbar}{2} \sqrt{(j-m)(j+m+1)}$ KroneckerDelta$[n, m+1] + \frac{\hbar}{2} \sqrt{(j+m)(j-m+1)}$ KroneckerDelta$[n, m-1]$;

$Jy[j\_, n\_, m\_] := -\frac{\hbar}{2} i \sqrt{(j-m)(j+m+1)}$ KroneckerDelta$[n, m+1] + \frac{\hbar}{2} i \sqrt{(j+m)(j-m+1)}$ KroneckerDelta$[n, m-1]$;

$Jz[j\_, n\_, m\_] := \hbar\, m$ KroneckerDelta$[n, m]$;

```
Jx = Table[Jx[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jy = Table[Jy[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jz = Table[Jz[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
```

```
Jx // MatrixForm
```

$$\begin{pmatrix} 0 & \hbar & 0 & 0 & 0 \\ \hbar & 0 & \sqrt{\frac{3}{2}}\,\hbar & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}}\,\hbar & 0 & \sqrt{\frac{3}{2}}\,\hbar & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}}\,\hbar & 0 & \hbar \\ 0 & 0 & 0 & \hbar & 0 \end{pmatrix}$$

```
Jy // MatrixForm
```

$$\begin{pmatrix} 0 & -i\,\hbar & 0 & 0 & 0 \\ i\,\hbar & 0 & -i\sqrt{\frac{3}{2}}\,\hbar & 0 & 0 \\ 0 & i\sqrt{\frac{3}{2}}\,\hbar & 0 & -i\sqrt{\frac{3}{2}}\,\hbar & 0 \\ 0 & 0 & i\sqrt{\frac{3}{2}}\,\hbar & 0 & -i\,\hbar \\ 0 & 0 & 0 & i\,\hbar & 0 \end{pmatrix}$$

```
Jz // MatrixForm
```

$$\begin{pmatrix} 2\hbar & 0 & 0 & 0 & 0 \\ 0 & \hbar & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\hbar & 0 \\ 0 & 0 & 0 & 0 & -2\hbar \end{pmatrix}$$

eq1 = Eigensystem[Jx]

$$\left\{\{0, -2\hbar, -\hbar, \hbar, 2\hbar\}, \left\{\left\{1, 0, -\sqrt{\frac{2}{3}}, 0, 1\right\}, \left\{1, -2, \sqrt{6}, -2, 1\right\}, \{-1, 1, 0, -1, 1\}, \{-1, -1, 0, 1, 1\}, \left\{1, 2, \sqrt{6}, 2, 1\right\}\right\}\right\}$$

ψ1x = Normalize[eq1[[2, 5]]] // Simplify; ψ1x // MatrixForm

$$\begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{\sqrt{\frac{3}{2}}}{2} \\ \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}$$

ψ2x = -Normalize[eq1[[2, 4]]] // Simplify; ψ2x // MatrixForm

$$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

ψ3x = Normalize[eq1[[2, 1]]] // Simplify; ψ3x // MatrixForm

$$\begin{pmatrix} \frac{\sqrt{\frac{3}{2}}}{2} \\ 0 \\ -\frac{1}{2} \\ 0 \\ \frac{\sqrt{\frac{3}{2}}}{2} \end{pmatrix}$$

ψ4x = -Normalize[eq1[[2, 3]]] // Simplify; ψ4x // MatrixForm

$$\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

```
ψ5x = Normalize[eq1[[2, 2]]] // Simplify; ψ5x // MatrixForm
```

$$\begin{pmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ \frac{\sqrt{\frac{3}{2}}}{2} \\ -\frac{1}{2} \\ \frac{1}{4} \end{pmatrix}$$

```
UxT = {ψ1x, ψ2x, ψ3x, ψ4x, ψ5x }; Ux = Transpose[UxT]; Ux // MatrixForm
```

$$\begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{\sqrt{\frac{3}{2}}}{2} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{\sqrt{\frac{3}{2}}}{2} & 0 & -\frac{1}{2} & 0 & \frac{\sqrt{\frac{3}{2}}}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} & \frac{\sqrt{\frac{3}{2}}}{2} & -\frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

```
eq2 = Eigensystem[Jy]
```

$$\left\{\{0, -2\hbar, -\hbar, \hbar, 2\hbar\}, \left\{\left\{1, 0, \sqrt{\frac{2}{3}}, 0, 1\right\}, \left\{1, -2i, -\sqrt{6}, 2i, 1\right\}, \{-1, i, 0, i, 1\}, \{-1, -i, 0, -i, 1\}, \left\{1, 2i, -\sqrt{6}, -2i, 1\right\}\right\}\right\}$$

```
ψ1y = Normalize[eq2[[2, 5]]] // Simplify; ψ1y // MatrixForm
```

$$\begin{pmatrix} \frac{1}{4} \\ -\frac{i}{2} \\ -\frac{\sqrt{\frac{3}{2}}}{2} \\ \frac{i}{2} \\ \frac{1}{4} \end{pmatrix}$$

```
UyT = {ψ1y, ψ2y, ψ3y, ψ4y, ψ5y }; Uy = Transpose[UyT]; UyH = UyT*;
Uy // MatrixForm
```

$$\begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{\sqrt{\frac{3}{2}}}{2} & \frac{1}{2} & \frac{1}{4} \\ \frac{i}{2} & \frac{i}{2} & 0 & -\frac{i}{2} & -\frac{i}{2} \\ -\frac{\sqrt{\frac{3}{2}}}{2} & 0 & \frac{1}{2} & 0 & -\frac{\sqrt{\frac{3}{2}}}{2} \\ -\frac{i}{2} & \frac{i}{2} & 0 & -\frac{i}{2} & \frac{i}{2} \\ \frac{1}{4} & -\frac{1}{2} & \frac{\sqrt{\frac{3}{2}}}{2} & -\frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

10. Eigenvalue problems for *j* = 5/2

Matrices j = 5/2

```
Clear["Global`*"]; j = 5/2;
Jx[j_, n_, m_] := ħ/2 √((j - m) (j + m + 1)) KroneckerDelta[n, m + 1] +
   ħ/2 √((j + m) (j - m + 1)) KroneckerDelta[n, m - 1];
Jy[j_, n_, m_] := -ħ/2 ⅈ √((j - m) (j + m + 1)) KroneckerDelta[n, m + 1] +
   ħ/2 ⅈ √((j + m) (j - m + 1)) KroneckerDelta[n, m - 1];
Jz[j_, n_, m_] := ħ m KroneckerDelta[n, m];
Jx = Table[Jx[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jy = Table[Jy[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
Jz = Table[Jz[j, n, m], {n, j, -j, -1}, {m, j, -j, -1}];
```

```
Jx // MatrixForm
```

$$\begin{pmatrix} 0 & \frac{\sqrt{5}\,\hbar}{2} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{5}\,\hbar}{2} & 0 & \sqrt{2}\,\hbar & 0 & 0 & 0 \\ 0 & \sqrt{2}\,\hbar & 0 & \frac{3\hbar}{2} & 0 & 0 \\ 0 & 0 & \frac{3\hbar}{2} & 0 & \sqrt{2}\,\hbar & 0 \\ 0 & 0 & 0 & \sqrt{2}\,\hbar & 0 & \frac{\sqrt{5}\,\hbar}{2} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{5}\,\hbar}{2} & 0 \end{pmatrix}$$

```
Jy // MatrixForm
```

$$\begin{pmatrix} 0 & -\frac{1}{2} i \sqrt{5}\,\hbar & 0 & 0 & 0 & 0 \\ \frac{1}{2} i \sqrt{5}\,\hbar & 0 & -i\sqrt{2}\,\hbar & 0 & 0 & 0 \\ 0 & i\sqrt{2}\,\hbar & 0 & -\frac{3 i \hbar}{2} & 0 & 0 \\ 0 & 0 & \frac{3 i \hbar}{2} & 0 & -i\sqrt{2}\,\hbar & 0 \\ 0 & 0 & 0 & i\sqrt{2}\,\hbar & 0 & -\frac{1}{2} i \sqrt{5}\,\hbar \\ 0 & 0 & 0 & 0 & \frac{1}{2} i \sqrt{5}\,\hbar & 0 \end{pmatrix}$$

```
Jz // MatrixForm
```

$$\begin{pmatrix} \frac{5\hbar}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3\hbar}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\hbar}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\hbar}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{3\hbar}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{5\hbar}{2} \end{pmatrix}$$

```
eq1 = Eigensystem[Jx]
```

$$\left\{\left\{-\frac{5\hbar}{2}, -\frac{3\hbar}{2}, -\frac{\hbar}{2}, \frac{\hbar}{2}, \frac{3\hbar}{2}, \frac{5\hbar}{2}\right\},\right.$$

$$\left\{\left\{-1, \sqrt{5}, -\sqrt{10}, \sqrt{10}, -\sqrt{5}, 1\right\}, \left\{1, -\frac{3}{\sqrt{5}}, \sqrt{\frac{2}{5}}, \sqrt{\frac{2}{5}}, -\frac{3}{\sqrt{5}}, 1\right\},\right.$$

$$\left\{-1, \frac{1}{\sqrt{5}}, \sqrt{\frac{2}{5}}, -\sqrt{\frac{2}{5}}, -\frac{1}{\sqrt{5}}, 1\right\}, \left\{1, \frac{1}{\sqrt{5}}, -\sqrt{\frac{2}{5}}, -\sqrt{\frac{2}{5}}, \frac{1}{\sqrt{5}}, 1\right\},$$

$$\left.\left.\left\{-1, -\frac{3}{\sqrt{5}}, -\sqrt{\frac{2}{5}}, \sqrt{\frac{2}{5}}, \frac{3}{\sqrt{5}}, 1\right\}, \left\{1, \sqrt{5}, \sqrt{10}, \sqrt{10}, \sqrt{5}, 1\right\}\right\}\right\}$$

```
ψ1x = Normalize[eq1[[2, 6]]]; ψ1x // MatrixForm
```

$$\begin{pmatrix} \frac{1}{4\sqrt{2}} \\ \frac{\sqrt{\frac{5}{2}}}{4} \\ \frac{\sqrt{5}}{4} \\ \frac{\sqrt{5}}{4} \\ \frac{\sqrt{\frac{5}{2}}}{4} \\ \frac{1}{4\sqrt{2}} \end{pmatrix}$$

```
ψ2x = -Normalize[eq1[[2, 5]]]; ψ2x // MatrixForm
```

$$\begin{pmatrix} \frac{\sqrt{\frac{5}{2}}}{4} \\ \frac{3}{4\sqrt{2}} \\ \frac{1}{4} \\ -\frac{1}{4} \\ -\frac{3}{4\sqrt{2}} \\ -\frac{\sqrt{\frac{5}{2}}}{4} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\sqrt{5}}{4} \\ \frac{1}{4} \\ -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} \\ \frac{1}{4} \\ \frac{\sqrt{5}}{4} \end{pmatrix}$$

```
ψ4x = -Normalize[eq1[[2, 3]]]; ψ4x // MatrixForm
```

$$\begin{pmatrix} \frac{\sqrt{5}}{4} \\ -\frac{1}{4} \\ -\frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{4} \\ -\frac{\sqrt{5}}{4} \end{pmatrix}$$

```
ψ5x = Normalize[eq1[[2, 2]]]; ψ5x // MatrixForm
```

$$\begin{pmatrix} \frac{\sqrt{\frac{5}{2}}}{4} \\ -\frac{3}{4\sqrt{2}} \\ \frac{1}{4} \\ \frac{1}{4} \\ -\frac{3}{4\sqrt{2}} \\ \frac{\sqrt{\frac{5}{2}}}{4} \end{pmatrix}$$

```
ψ6x = -Normalize[eq1[[2, 1]]]; ψ6x // MatrixForm
```

$$\begin{pmatrix} \frac{1}{4\sqrt{2}} \\ -\frac{\sqrt{\frac{5}{2}}}{4} \\ \frac{\sqrt{5}}{4} \\ -\frac{\sqrt{5}}{4} \\ \frac{\sqrt{\frac{5}{2}}}{4} \\ -\frac{1}{4\sqrt{2}} \end{pmatrix}$$

```
UxT = {ψ1x, ψ2x, ψ3x, ψ4x, ψ5x, ψ6x}; Ux = Transpose[UxT]; UxH = UxT*;
```

```
Ux // MatrixForm
```

$$\begin{pmatrix} \frac{1}{4\sqrt{2}} & \frac{\sqrt{\frac{5}{2}}}{4} & \frac{\sqrt{5}}{4} & \frac{\sqrt{5}}{4} & \frac{\sqrt{\frac{5}{2}}}{4} & \frac{1}{4\sqrt{2}} \\ \frac{\sqrt{\frac{5}{2}}}{4} & \frac{3}{4\sqrt{2}} & \frac{1}{4} & -\frac{1}{4} & -\frac{3}{4\sqrt{2}} & -\frac{\sqrt{\frac{5}{2}}}{4} \\ \frac{\sqrt{5}}{4} & \frac{1}{4} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{4} & \frac{\sqrt{5}}{4} \\ \frac{\sqrt{5}}{4} & -\frac{1}{4} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{4} & -\frac{\sqrt{5}}{4} \\ \frac{\sqrt{\frac{5}{2}}}{4} & -\frac{3}{4\sqrt{2}} & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4\sqrt{2}} & \frac{\sqrt{\frac{5}{2}}}{4} \\ \frac{1}{4\sqrt{2}} & -\frac{\sqrt{\frac{5}{2}}}{4} & \frac{\sqrt{5}}{4} & -\frac{\sqrt{5}}{4} & \frac{\sqrt{\frac{5}{2}}}{4} & -\frac{1}{4\sqrt{2}} \end{pmatrix}$$

```
UxH // MatrixForm
```

$$\begin{pmatrix} \frac{1}{4\sqrt{2}} & \frac{\sqrt{\frac{5}{2}}}{4} & \frac{\sqrt{5}}{4} & \frac{\sqrt{5}}{4} & \frac{\sqrt{\frac{5}{2}}}{4} & \frac{1}{4\sqrt{2}} \\ \frac{\sqrt{\frac{5}{2}}}{4} & \frac{3}{4\sqrt{2}} & \frac{1}{4} & -\frac{1}{4} & -\frac{3}{4\sqrt{2}} & -\frac{\sqrt{\frac{5}{2}}}{4} \\ \frac{\sqrt{5}}{4} & \frac{1}{4} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{4} & \frac{\sqrt{5}}{4} \\ \frac{\sqrt{5}}{4} & -\frac{1}{4} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{4} & -\frac{\sqrt{5}}{4} \\ \frac{\sqrt{\frac{5}{2}}}{4} & -\frac{3}{4\sqrt{2}} & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4\sqrt{2}} & \frac{\sqrt{\frac{5}{2}}}{4} \\ \frac{1}{4\sqrt{2}} & -\frac{\sqrt{\frac{5}{2}}}{4} & \frac{\sqrt{5}}{4} & -\frac{\sqrt{5}}{4} & \frac{\sqrt{\frac{5}{2}}}{4} & -\frac{1}{4\sqrt{2}} \end{pmatrix}$$

```
UxH.Ux
```

```
{{1, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0},
 {0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 1}}
```

```
eq2 = Eigensystem[Jy]
```

$$\left\{\left\{-\frac{5\hbar}{2}, -\frac{3\hbar}{2}, -\frac{\hbar}{2}, \frac{\hbar}{2}, \frac{3\hbar}{2}, \frac{5\hbar}{2}\right\},\right.$$

$$\left\{\left\{i, \sqrt{5}, -i\sqrt{10}, -\sqrt{10}, i\sqrt{5}, 1\right\}, \left\{-i, -\frac{3}{\sqrt{5}}, i\sqrt{\frac{2}{5}}, -\sqrt{\frac{2}{5}}, \frac{3i}{\sqrt{5}}, 1\right\},\right.$$

$$\left\{i, \frac{1}{\sqrt{5}}, i\sqrt{\frac{2}{5}}, \sqrt{\frac{2}{5}}, \frac{i}{\sqrt{5}}, 1\right\}, \left\{-i, \frac{1}{\sqrt{5}}, -i\sqrt{\frac{2}{5}}, \sqrt{\frac{2}{5}}, -\frac{i}{\sqrt{5}}, 1\right\},$$

$$\left.\left.\left\{i, -\frac{3}{\sqrt{5}}, -i\sqrt{\frac{2}{5}}, -\sqrt{\frac{2}{5}}, -\frac{3i}{\sqrt{5}}, 1\right\}, \left\{-i, \sqrt{5}, i\sqrt{10}, -\sqrt{10}, -i\sqrt{5}, 1\right\}\right\}\right\}$$

```
ψ1y = i Normalize[eq2[[2, 6]]]; ψ1y // MatrixForm
```

$$\begin{pmatrix} \frac{1}{4\sqrt{2}} \\ \frac{1}{4} i \sqrt{\frac{5}{2}} \\ -\frac{\sqrt{5}}{4} \\ -\frac{i\sqrt{5}}{4} \\ \frac{\sqrt{\frac{5}{2}}}{4} \\ \frac{i}{4\sqrt{2}} \end{pmatrix}$$

```
ψ2y = -ⅈ Normalize[eq2[[2, 5]]]; ψ2y // MatrixForm
```

$$\begin{pmatrix} \frac{\sqrt{\frac{5}{2}}}{4} \\ \frac{3\,i}{4\sqrt{2}} \\ -\frac{1}{4} \\ \frac{i}{4} \\ -\frac{3}{4\sqrt{2}} \\ -\frac{1}{4}\,i\sqrt{\frac{5}{2}} \end{pmatrix}$$

```
ψ3y = ⅈ Normalize[eq2[[2, 4]]]; ψ3y // MatrixForm
```

$$\begin{pmatrix} \frac{\sqrt{5}}{4} \\ \frac{i}{4} \\ \frac{1}{2\sqrt{2}} \\ \frac{i}{2\sqrt{2}} \\ \frac{1}{4} \\ \frac{i\sqrt{5}}{4} \end{pmatrix}$$

```
ψ4y = -ⅈ Normalize[eq2[[2, 3]]]; ψ4y // MatrixForm
```

$$\begin{pmatrix} \frac{\sqrt{5}}{4} \\ -\frac{i}{4} \\ \frac{1}{2\sqrt{2}} \\ -\frac{i}{2\sqrt{2}} \\ \frac{1}{4} \\ -\frac{i\sqrt{5}}{4} \end{pmatrix}$$

```
ψ5y = ⅈ Normalize[eq2[[2, 2]]]; ψ5y // MatrixForm
```

$$\begin{pmatrix} \frac{\sqrt{\frac{5}{2}}}{4} \\ -\frac{3 i}{4\sqrt{2}} \\ -\frac{1}{4} \\ -\frac{i}{4} \\ -\frac{3}{4\sqrt{2}} \\ \frac{1}{4} i \sqrt{\frac{5}{2}} \end{pmatrix}$$

ψ6y = -ⅈ Normalize[eq2[[2, 1]]]; ψ6y // MatrixForm

$$\begin{pmatrix} \frac{1}{4\sqrt{2}} \\ -\frac{1}{4} i \sqrt{\frac{5}{2}} \\ -\frac{\sqrt{5}}{4} \\ \frac{i\sqrt{5}}{4} \\ \frac{\sqrt{\frac{5}{2}}}{4} \\ -\frac{i}{4\sqrt{2}} \end{pmatrix}$$

UyT = {ψ1y, ψ2y, ψ3y, ψ4y, ψ5y, ψ6y}; Uy = Transpose[UyT]; UyH = UyT*;

Uy // MatrixForm

$$\begin{pmatrix} \frac{1}{4\sqrt{2}} & \frac{\sqrt{\frac{5}{2}}}{4} & \frac{\sqrt{5}}{4} & \frac{\sqrt{5}}{4} & \frac{\sqrt{\frac{5}{2}}}{4} & \frac{1}{4\sqrt{2}} \\ \frac{1}{4} i \sqrt{\frac{5}{2}} & \frac{3 i}{4\sqrt{2}} & \frac{i}{4} & -\frac{i}{4} & -\frac{3 i}{4\sqrt{2}} & -\frac{1}{4} i \sqrt{\frac{5}{2}} \\ -\frac{\sqrt{5}}{4} & -\frac{1}{4} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{4} & -\frac{\sqrt{5}}{4} \\ -\frac{i\sqrt{5}}{4} & \frac{i}{4} & \frac{i}{2\sqrt{2}} & -\frac{i}{2\sqrt{2}} & -\frac{i}{4} & \frac{i\sqrt{5}}{4} \\ \frac{\sqrt{\frac{5}{2}}}{4} & -\frac{3}{4\sqrt{2}} & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4\sqrt{2}} & \frac{\sqrt{\frac{5}{2}}}{4} \\ \frac{i}{4\sqrt{2}} & -\frac{1}{4} i \sqrt{\frac{5}{2}} & \frac{i\sqrt{5}}{4} & -\frac{i\sqrt{5}}{4} & \frac{1}{4} i \sqrt{\frac{5}{2}} & -\frac{i}{4\sqrt{2}} \end{pmatrix}$$

```
UyH // MatrixForm
```

$$\begin{pmatrix} \frac{1}{4\sqrt{2}} & -\frac{1}{4} i \sqrt{\frac{5}{2}} & -\frac{\sqrt{5}}{4} & \frac{i\sqrt{5}}{4} & \frac{\sqrt{\frac{5}{2}}}{4} & -\frac{i}{4\sqrt{2}} \\ \frac{\sqrt{\frac{5}{2}}}{4} & -\frac{3i}{4\sqrt{2}} & -\frac{1}{4} & -\frac{i}{4} & -\frac{3}{4\sqrt{2}} & \frac{1}{4} i \sqrt{\frac{5}{2}} \\ \frac{\sqrt{5}}{4} & -\frac{i}{4} & \frac{1}{2\sqrt{2}} & -\frac{i}{2\sqrt{2}} & \frac{1}{4} & -\frac{i\sqrt{5}}{4} \\ \frac{\sqrt{5}}{4} & \frac{i}{4} & \frac{1}{2\sqrt{2}} & \frac{i}{2\sqrt{2}} & \frac{1}{4} & \frac{i\sqrt{5}}{4} \\ \frac{\sqrt{\frac{5}{2}}}{4} & \frac{3i}{4\sqrt{2}} & -\frac{1}{4} & \frac{i}{4} & -\frac{3}{4\sqrt{2}} & -\frac{1}{4} i \sqrt{\frac{5}{2}} \\ \frac{1}{4\sqrt{2}} & \frac{1}{4} i \sqrt{\frac{5}{2}} & -\frac{\sqrt{5}}{4} & -\frac{i\sqrt{5}}{4} & \frac{\sqrt{\frac{5}{2}}}{4} & \frac{i}{4\sqrt{2}} \end{pmatrix}$$

```
UyH.Uy
```

```
{{1, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0},
 {0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 1}}
```

11. Angular momentum for $j = 3$

7x7 matrices

$\hat{J}_x =$

```
Jx // MatrixForm
```

$$\begin{pmatrix} 0 & \sqrt{\frac{3}{2}}\,\hbar & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\frac{3}{2}}\,\hbar & 0 & \sqrt{\frac{5}{2}}\,\hbar & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{5}{2}}\,\hbar & 0 & \sqrt{3}\,\hbar & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3}\,\hbar & 0 & \sqrt{3}\,\hbar & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3}\,\hbar & 0 & \sqrt{\frac{5}{2}}\,\hbar & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{5}{2}}\,\hbar & 0 & \sqrt{\frac{3}{2}}\,\hbar \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{3}{2}}\,\hbar & 0 \end{pmatrix}$$

$\hat{J}_y =:$

```
Jy // MatrixForm
```

$$\begin{pmatrix} 0 & -i\sqrt{\frac{3}{2}}\,\hbar & 0 & 0 & 0 & 0 & 0 \\ i\sqrt{\frac{3}{2}}\,\hbar & 0 & -i\sqrt{\frac{5}{2}}\,\hbar & 0 & 0 & 0 & 0 \\ 0 & i\sqrt{\frac{5}{2}}\,\hbar & 0 & -i\sqrt{3}\,\hbar & 0 & 0 & 0 \\ 0 & 0 & i\sqrt{3}\,\hbar & 0 & -i\sqrt{3}\,\hbar & 0 & 0 \\ 0 & 0 & 0 & i\sqrt{3}\,\hbar & 0 & -i\sqrt{\frac{5}{2}}\,\hbar & 0 \\ 0 & 0 & 0 & 0 & i\sqrt{\frac{5}{2}}\,\hbar & 0 & -i\sqrt{\frac{3}{2}}\,\hbar \\ 0 & 0 & 0 & 0 & 0 & i\sqrt{\frac{3}{2}}\,\hbar & 0 \end{pmatrix}$$

APPENDIX

Matrix of the angular momentum with $j = 4$ and 9/2.

$J = 4$

9 x 9 matrix

$$\hat{J}_x =$$

$$\begin{pmatrix} 0 & \sqrt{2}\,\hbar & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}\,\hbar & 0 & \sqrt{\frac{7}{2}}\,\hbar & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{7}{2}}\,\hbar & 0 & \frac{3\,\hbar}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3\,\hbar}{\sqrt{2}} & 0 & \sqrt{5}\,\hbar & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{5}\,\hbar & 0 & \sqrt{5}\,\hbar & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{5}\,\hbar & 0 & \frac{3\,\hbar}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3\,\hbar}{\sqrt{2}} & 0 & \sqrt{\frac{7}{2}}\,\hbar & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{7}{2}}\,\hbar & 0 & \sqrt{2}\,\hbar \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}\,\hbar & 0 \end{pmatrix}$$

$$\hat{J}_y =$$

$$\begin{pmatrix} 0 & -i\sqrt{2}\,\hbar & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ i\sqrt{2}\,\hbar & 0 & -i\sqrt{\frac{7}{2}}\,\hbar & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i\sqrt{\frac{7}{2}}\,\hbar & 0 & -\frac{3\,i\,\hbar}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3\,i\,\hbar}{\sqrt{2}} & 0 & -i\sqrt{5}\,\hbar & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\sqrt{5}\,\hbar & 0 & -i\sqrt{5}\,\hbar & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i\sqrt{5}\,\hbar & 0 & -\frac{3\,i\,\hbar}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3\,i\,\hbar}{\sqrt{2}} & 0 & -i\sqrt{\frac{7}{2}}\,\hbar & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i\sqrt{\frac{7}{2}}\,\hbar & 0 & -i\sqrt{2}\,\hbar \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i\sqrt{2}\,\hbar & 0 \end{pmatrix}$$

$$\hat{J}_z =$$

$$\begin{pmatrix} 4\hbar & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3\hbar & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\hbar & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \hbar & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\hbar & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2\hbar & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3\hbar & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4\hbar \end{pmatrix}$$

***J* = 9/2**

10 x 10 matrix

$$\hat{J}_x =$$

$$\begin{pmatrix} 0 & \frac{3\hbar}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3\hbar}{2} & 0 & 2\hbar & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\hbar & 0 & \frac{\sqrt{21}\,\hbar}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{21}\,\hbar}{2} & 0 & \sqrt{6}\,\hbar & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6}\,\hbar & 0 & \frac{5\hbar}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5\hbar}{2} & 0 & \sqrt{6}\,\hbar & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{6}\,\hbar & 0 & \frac{\sqrt{21}\,\hbar}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{21}\,\hbar}{2} & 0 & 2\hbar & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\hbar & 0 & \frac{3\hbar}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3\hbar}{2} & 0 \end{pmatrix}$$

$$\hat{J}_y =$$

$$\begin{pmatrix}
0 & -\frac{3 i \hbar}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{3 i \hbar}{2} & 0 & -2 i \hbar & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 i \hbar & 0 & -\frac{1}{2} i \sqrt{21} \hbar & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} i \sqrt{21} \hbar & 0 & -i \sqrt{6} \hbar & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & i \sqrt{6} \hbar & 0 & -\frac{5 i \hbar}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{5 i \hbar}{2} & 0 & -i \sqrt{6} \hbar & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & i \sqrt{6} \hbar & 0 & -\frac{1}{2} i \sqrt{21} \hbar & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} i \sqrt{21} \hbar & 0 & -2 i \hbar & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 i \hbar & 0 & -\frac{3 i \hbar}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3 i \hbar}{2} & 0
\end{pmatrix}$$

$$\hat{J}_z =$$

$$\begin{pmatrix}
\frac{9 \hbar}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{7 \hbar}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{5 \hbar}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{3 \hbar}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\hbar}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{\hbar}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{3 \hbar}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5 \hbar}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{7 \hbar}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{9 \hbar}{2}
\end{pmatrix}$$