Angular momentum of photon

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1. Angular momentum

The orbital angular momentum is defined as

$$\hat{\boldsymbol{L}} = \hat{\boldsymbol{r}} \times \hat{\boldsymbol{p}}$$

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$$

$$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z$$

We consider the commutation relation:

$$\hat{\boldsymbol{L}} \times \hat{\boldsymbol{L}} = i\hbar\hat{\boldsymbol{L}}$$

$$[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x, \qquad [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y, \qquad [\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$$

$$[\hat{L}_z, \hat{z}] = [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{z}] = 0,$$

$$[\hat{L}_z, \hat{x}] = [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{x}] = -[\hat{y}\hat{p}_x, \hat{x}] = -\hat{y}[\hat{p}_x, \hat{x}] = i\hbar\hat{y}$$

$$[\hat{L}_z, \hat{y}] = [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{y}] = [\hat{x}\hat{p}_y, \hat{y}] = -i\hbar\hat{x}$$

or

$$[\hat{L}_z, \hat{x} + i\hat{y}] = [\hat{L}_z \hat{x}] + i[\hat{L}_z, \hat{y}] = i\hbar\hat{y} + i(-i\hbar\hat{x}) = \hbar(\hat{x} + i\hat{y})$$

$$[\hat{L}_z, \hat{x} - i\hat{y}] = [\hat{L}_z \hat{x}] - i[\hat{L}_z, \hat{y}] = i\hbar\hat{y} - i(-i\hbar\hat{x}) = -\hbar(\hat{x} - i\hat{y})$$

We also note that

$$[\hat{\boldsymbol{L}}^2, [\hat{\boldsymbol{L}}^2, \hat{x}]] = 2\hbar^2 \{\hat{x}, \hat{\boldsymbol{L}}^2\}$$

where

$$\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$$

((**Mathematica**)) Proof

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Clear["Global`"];

ux = {1, 0, 0}; uy = {0, 1, 0}; uz = {0, 0, 1};
r = {x, y, z};

Lx := (ħ ux.(-ⅈ Cross[r, Grad[#, {x, y, z}]]) &) // Simplify;
Ly := (ħ uy.(-ⅈ Cross[r, Grad[#, {x, y, z}]]) &) // Simplify;
Lz := (ħ uz.(-ⅈ Cross[r, Grad[#, {x, y, z}]]) &) // Simplify;
Lsq := (Lx[Lx[#]] + Ly[Ly[#]] + Lz[Lz[#]] &);

eq2 = Lsq[Lsq[x ψ[x, y, z]]] - Lsq[x Lsq[ψ[x, y, z]]] -
    Lsq[x Lsq[ψ[x, y, z]]] + x Lsq[Lsq[ψ[x, y, z]]] //
   FullSimplify;

eq3 = 2 ħ^2 (x Lsq[ψ[x, y, z]] + Lsq[x ψ[x, y, z]]) //
   FullSimplify;

eq2 - eq3 // Simplify

0
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2. Eigenkets of angular momentum

$$\hat{\boldsymbol{L}}^2 |l, m\rangle = \hbar^2 l(l+1) |l, m\rangle$$

$$\hat{L}_z|l,m\rangle = \hbar m|l,m\rangle$$

$$\hat{L}_+|l,m\rangle = \hbar\sqrt{(l-m)(l+m+1)}|l,m+1\rangle$$

$$\hat{L}_-|l,m\rangle = \hbar\sqrt{(l+m)(l-m+1)}|l,m-1\rangle$$

where

$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y, \qquad \hat{L}_- = \hat{L}_x - i\hat{L}_y$$

3. Selection rule-I

Using the relation

$$\hat{L}_z|l,m\rangle = \hbar m|l,m\rangle$$

we have

$$\langle l',m'|[\hat{L}_z,\hat{z}]|l,m\rangle ==,$$

or

$$\langle l',m'|\hat{L}_z\hat{z} - \hat{z}\hat{L}_z|l,m\rangle = 0$$

or

$$(m'-m)\langle l',m'|\hat{z}|l,m\rangle = 0$$

Then we get the relation

$m' = m$, for the dipole ion the z direction.

4. Selection rule-II

Using the relation

$$\hat{L}_z|l,m\rangle = \hbar m|l,m\rangle$$

we have

$$\langle l',m'|[\hat{L}_z,\hat{x}+i\hat{y}]|l,m\rangle = \hbar\langle l',m'|\hat{x}+i\hat{y}|l,m\rangle,$$

or

$$\langle l',m'|\hat{L}_z(\hat{x}+i\hat{y})-(\hat{x}+i\hat{y})\hat{L}_z|l,m\rangle = \hbar\langle l',m'|\hat{x}+i\hat{y}|l,m\rangle$$

or

$$(m'-m-1)\langle l',m'|\hat{x}+i\hat{y}|l,m\rangle = 0$$

Then we get the relation

$m' = m+1$, for the dipole ion the *x*, *y* direction.

5. Selection rule-III

Using the relation

$$\hat{L}_z|l,m\rangle = \hbar m|l,m\rangle$$

we have

$$\langle l',m'|[\hat{L}_z,\hat{x}-i\hat{y}]|l,m\rangle = -\hbar\langle l',m'|\hat{x}-i\hat{y}|l,m\rangle,$$

or

$$\langle l',m'|\hat{L}_z(\hat{x}-i\hat{y})-(\hat{x}-i\hat{y})\hat{L}_z|l,m\rangle = -\hbar\langle l',m'|\hat{x}-i\hat{y}|l,m\rangle$$

or

$$(m'-m+1)\langle l',m'|\hat{x}-i\hat{y}|l,m\rangle = 0$$

Then we get the relation

$m' = m-1$, for the dipole ion the *x*, *y* direction.

6. Selection rule-IV

Using the commutation relation

$$[\hat{\boldsymbol{L}}^2,[\hat{\boldsymbol{L}}^2,\hat{x}]] = 2\hbar^2\{\hat{x},\hat{\boldsymbol{L}}^2\}$$

we get the following equation,

$$\langle l',m'|[\hat{\boldsymbol{L}}^2,[\hat{\boldsymbol{L}}^2,\hat{x}]]|l,m\rangle = 2\hbar^2\langle l',m'|\{\hat{x},\hat{\boldsymbol{L}}^2\}|l,m\rangle$$

or

$$\langle l',m'|\hat{\boldsymbol{L}}^2\hat{\boldsymbol{L}}^2\hat{x} - 2\hat{\boldsymbol{L}}^2\hat{x}\hat{\boldsymbol{L}}^2 + \hat{x}\hat{\boldsymbol{L}}^2\hat{\boldsymbol{L}}^2|l,m\rangle = 2\hbar^2\langle l',m'|\hat{x}\hat{\boldsymbol{L}}^2 + \hat{\boldsymbol{L}}^2\hat{x}|l,m\rangle$$

Here we use the relation

$$\hat{\boldsymbol{L}}^2|l,m\rangle = \hbar^2 l(l+1)|l,m\rangle, \qquad \text{and} \qquad \langle l,m|\hat{\boldsymbol{L}}^2 = \hbar^2 l(l+1)\langle l,m|$$

Then we have

$$\hbar^4[l'^2(l'+1)^2 - 2l'(l'+1)l(l+1) + l^2(l+1)^2 - 2l'(l'+1) - 2l(l+1)]\langle l',m'|\hat{x}|l,m\rangle = 0$$

or

$$(l'-l-1)(l'-l+1)(l'+l)(l'+l+2)\langle l',m'|\hat{x}|l,m\rangle = 0$$

The last factor yields the selection rule

$$l' = l \pm 1$$

((Mathemtica))

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g1 = a^2 (a + 1)^2 - 2 a (a + 1) b (b + 1) + b^2 (b + 1)^2 -
   2 a (a + 1) - 2 b (b + 1) // Factor

(-1 + a - b) (1 + a - b) (a + b) (2 + a + b)
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Since l' and l are both non-negative, the $(l'+l+2)$ term cannot vanish, and the $(l'+l)$ term can only vanish for $l' = l = 0$. However, this selection rule cannot be satisfied, since the states with $l' = l = 0$ are independent of direction, and therefore these matrix elements of $\hat{x}$ vanish. Formally, one easily shows this

$$\langle 0,0|\hat{x}|0,0\rangle = 0$$

using the parity operator.
((Proof))

$$\hat{\pi}\hat{x}\hat{\pi} = -\hat{x}$$

where the parity operator satisfies the relations,

$$\hat{\pi}^{+} = \hat{\pi}\,, \qquad \hat{\pi}^{2} = \hat{1}$$

$$\langle 0,0|\hat{\pi}\hat{x}\hat{\pi}|0,0\rangle = -\langle 0,0|\hat{x}|0,0\rangle$$

or

$$\langle 0,0|\hat{x}|0,0\rangle = 0$$

where

$$\hat{\pi}|l,m\rangle = (-1)^{l}|l,m\rangle\,,$$

and

$$\hat{\pi}|0,0\rangle = |0,0\rangle\,, \qquad \text{and} \qquad \langle 0,0|\hat{\pi} = \langle 0,0|$$

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7. Dipole selection rule

The dipole radiation is emitted if

$$M = \langle f|\boldsymbol{e}\cdot\hat{\boldsymbol{r}}|i\rangle = \boldsymbol{e}\cdot\langle f|\hat{\boldsymbol{r}}|i\rangle = \boldsymbol{e}\cdot\boldsymbol{D}_{fi}$$

does not vanish, where $\boldsymbol{e}$ is the electric field (the polarization vector), and

$$\boldsymbol{D}_{fi} = \langle f|\hat{\boldsymbol{r}}|i\rangle$$

We assume that $|i\rangle = |l, m\rangle$ and $|f\rangle = |l', m'\rangle$. Then we have

$$\boldsymbol{D}_{fi} = \langle l', m'|\hat{x}|l, m\rangle \boldsymbol{e}_x + \langle l', m'|\hat{y}|l, m\rangle \boldsymbol{e}_y + \langle l', m'|\hat{z}|l, m\rangle \boldsymbol{e}_z,$$

(i) For $m' = m$.

$$\langle l', m'|\hat{z}|l, m\rangle \neq 0, \qquad \langle l', m'|\hat{x}|l, m\rangle = 0, \qquad \langle l', m'|\hat{y}|l, m\rangle = 0$$

$$\begin{aligned} \boldsymbol{D}_{fi} &= \langle l', m'|\hat{x}|l, m\rangle \boldsymbol{e}_x + \langle l', m'|\hat{y}|l, m\rangle \boldsymbol{e}_y + \langle l', m'|\hat{z}|l, m\rangle \boldsymbol{e}_z \\ &= \langle l', m'|\hat{z}|l, m\rangle \boldsymbol{e}_z \end{aligned}$$

$\boldsymbol{D}_{fi}$ is directed along the z axis.

(a) Suppose that the wavevector $\boldsymbol{k}$ of the emitted photon is along the z axis. There is no radiation in the z-direction since the polarization vector $\boldsymbol{\varepsilon}$ is perpendicular to $\boldsymbol{D}_{fi}$ (the z axis).

(b) For example, we consider light going in the x direction. It can have two directions of polarization, either in the z or in the y direction. A transition in which $\Delta m = 0$, can produce only light which is polarized in the z direction.

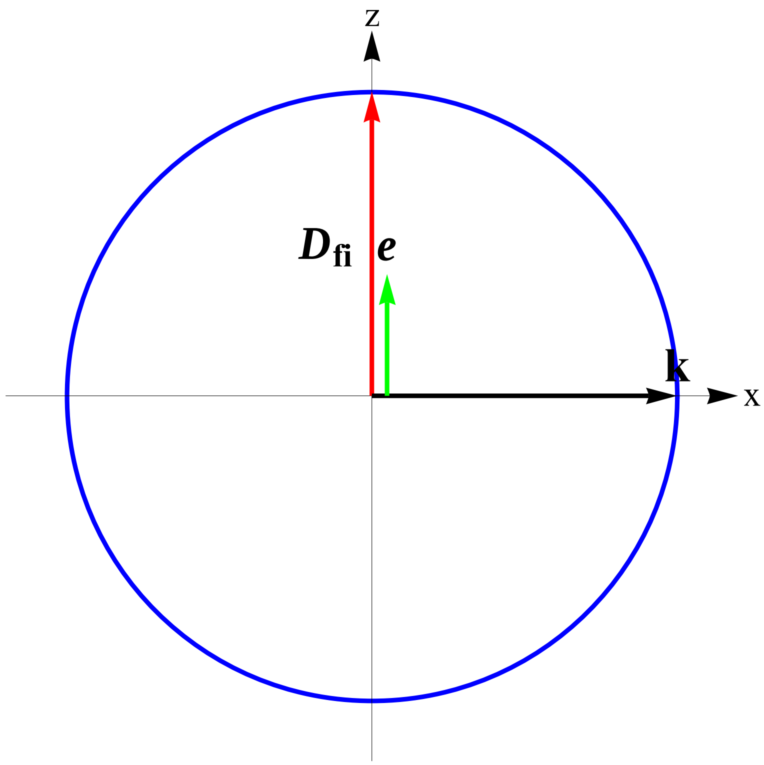

Fig. $m' = m$. $\boldsymbol{D}_{fi} // z$. The light propagating along the x direction. It is a linearly polarized wave (along the z axis).

(ii) For $m' = m + 1$

$$\langle l', m' | \hat{r}_+ | l, m \rangle = 0 , \qquad \langle l', m' | \hat{z} | l, m \rangle = 0 .$$

where

$$\hat{r}_+ = \frac{\hat{x} + i\hat{y}}{\sqrt{2}} .$$

(iii) For $m' = m - 1$

$$\langle l', m' | \hat{r}_- | l, m \rangle = 0 , \qquad \langle l', m' | \hat{z} | l, m \rangle = 0 .$$

where

$$\hat{r}_- = \frac{\hat{x} - i\hat{y}}{\sqrt{2}}$$

We now consider the matrix element with $m' = m \pm 1$.

$$\begin{aligned}
\boldsymbol{D}_{fi} &= \langle l', m' | \hat{x}\boldsymbol{e}_x + \hat{y}\boldsymbol{e}_y + \hat{z}\boldsymbol{e}_z | l, m \rangle \\
&= \langle l', m' | \hat{x}\boldsymbol{e}_x + \hat{y}\boldsymbol{e}_y | l, m \rangle \\
&= \langle l', m' | \hat{r}_+\boldsymbol{e}_- + \hat{r}_-\boldsymbol{e}_+ | l, m \rangle \\
&= \boldsymbol{e}_- \cdot \langle l', m' | \hat{r}_+ | l, m \rangle + \boldsymbol{e}_+ \cdot \langle l', m' | \hat{r}_- | l, m \rangle
\end{aligned}$$

where

$$\boldsymbol{e}_+ = \frac{\boldsymbol{e}_x + i\boldsymbol{e}_y}{\sqrt{2}}, \qquad \boldsymbol{e}_- = \frac{\boldsymbol{e}_x - i\boldsymbol{e}_y}{\sqrt{2}}. \qquad \text{(Jones vector notation)}$$

or

$$\boldsymbol{e}_x = \frac{1}{\sqrt{2}}(\boldsymbol{e}_+ + \boldsymbol{e}_-), \qquad \boldsymbol{e}_y = \frac{1}{\sqrt{2}i}(\boldsymbol{e}_+ - \boldsymbol{e}_-)$$

and

$$\hat{x}\boldsymbol{e}_x + \hat{y}\boldsymbol{e}_y = \hat{r}_+\boldsymbol{e}_- + \hat{r}_-\boldsymbol{e}_+$$

Note that

$$\boldsymbol{e}_+ \cdot \boldsymbol{e}_+ = 0. \qquad \boldsymbol{e}_- \cdot \boldsymbol{e}_- = 0$$

$$\boldsymbol{e}_+ \cdot \boldsymbol{e}_- = 1. \qquad \boldsymbol{e}_- \cdot \boldsymbol{e}_+ = 1$$

(a) When $m' = m + 1$,

$$\boldsymbol{D}_{fi} = \langle l', m' | \hat{x}\boldsymbol{e}_x + \hat{y}\boldsymbol{e}_y | l, m \rangle = \boldsymbol{e}_- \langle l', m' | \hat{r}_+ | l, m \rangle]$$

has the same direction of the left circularly polarization vector ($\boldsymbol{e}_-$). Then the emitted photon which is right circularly polarized ($\boldsymbol{e}_+$), can propagate along the z axis. A photon with right-hand circular polarization carries a spin $+\hbar$ in the z direction (the propagation direction).

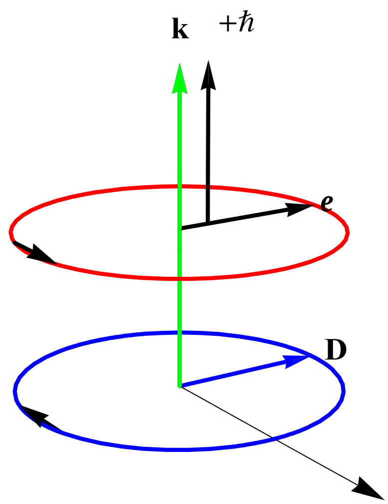

Fig. The case of $m' = m+1$ (right circularly polarization). A right circularly polarized photon ($\boldsymbol{e}_+$) propagates with a wavevector $\boldsymbol{k}$ in the z direction. Note that the electric field is denoted by $\cos(kz-\omega t)\boldsymbol{e}_x + \sin(kz-\omega t)\boldsymbol{e}_y$. This electric field rotates in clock-wise sense with time t, and rotates in counter clock-wise sense with z (as the wave propagates forward). The corresponding spin of the photon is directed in the positive z direction ($\hbar$). $\boldsymbol{D}_{fi}(\approx \boldsymbol{e}_-)$. $\boldsymbol{E}(\approx \boldsymbol{e}_+)$. ($\boldsymbol{e}_+ \cdot \boldsymbol{e}_- = 1, \boldsymbol{e}_- \cdot \boldsymbol{e}_- = 0$).

(b) When $m' = m-1$

$$\boldsymbol{D}_{fi} = \langle l', m' | \hat{x}\boldsymbol{e}_x + \hat{y}\boldsymbol{e}_y | l, m \rangle = \boldsymbol{e}_+ \langle l', m' | \hat{r}_- | l, m \rangle]$$

is parallel to the right circularly polarization vector $\boldsymbol{e}_+$. The emitted photon with left circularly polarization ($\boldsymbol{e}_-$) can propagate along the z axis. A photon with the left-hand polarization carries a spin (-$\hbar$), that is, a spin direction opposite to the z direction.

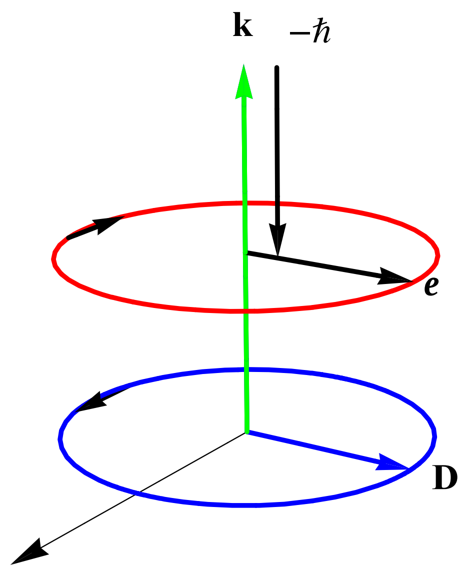

Fig. The case of $m' = m - 1$ (left circularly polarization). A left circularly polarized photon ($\boldsymbol{e}_-$) propagates with a wavevector $\boldsymbol{k}$ in the z direction. Note that the electric field is given by $\cos(kz - \omega t)\boldsymbol{e}_x - \sin(kz - \omega t)\boldsymbol{e}_y$. This electric field rotates in counter clock-wise sense with time t, and rotates in clock-wise sense with z (as the wave propagates forward). The corresponding spin of the photon is directed in the negative z direction, as -$\hbar$. $\boldsymbol{D}_{fi}(\approx \boldsymbol{e}_+)$. $\boldsymbol{E}(\approx \boldsymbol{e}_-)$. ($\boldsymbol{e}_+ \cdot \boldsymbol{e}_- = 1, \boldsymbol{e}_+ \cdot \boldsymbol{e}_+ = 0$).

The rules on Δm can be understood by realizing that σ^+ and σ^- circularly polarized photons carry angular momenta of $+\hbar$ ans $-\hbar$, respectively, along the z axis, and hence m must change by one unit to conserve angular momentum. For linearly polarized light along the z axis, the photons carry no z-component of momentum, implying $\Delta m = 0$, while x or y-polarized light can be considered as a equal combination of σ^+ and σ^- photons, giving $\Delta m = \pm 1$.

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