The appearance of the Berry phase for the precession of nuclear spin with spin 1/2

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Here we use the spin-echo method which is used for the r.f. spin echo of nuclear magnetic resonance. We learned this method from a book of M.H. Levitt, Spin Dynamics Basic of Nuclear Magnetic Resonance). We present an exact solution the time dependence of the spin state of the nuclear spin, and discuss the adiabatic change of the nuclear spin. We solve the example 10.1 and problem 10.2 of the textbook of D.J. Griffiths (Introduction to Quantum Mechanics). We use the Mathematica for solving the problem.

1. Introduction

In order to understand the adiabatic approximation, we consider the time dependent behavior of a nuclear spin (spin 1/2) in the presence of a magnetic field whose magnitude (B_0) is constant, but whose direction sweeps out a cone, of opening angle θ of the magnetic field at constant angular frequency ω. Note that the angle θ is fixed. The magnetic field is expressed by

$$\boldsymbol{B} = B_0(\sin\theta\cos(\omega t+\phi_p), \sin\theta\sin(\omega t+\phi_p), \cos\theta),$$

where B_0 is the magnitude of the magnetic field and ϕ_p is the phase.

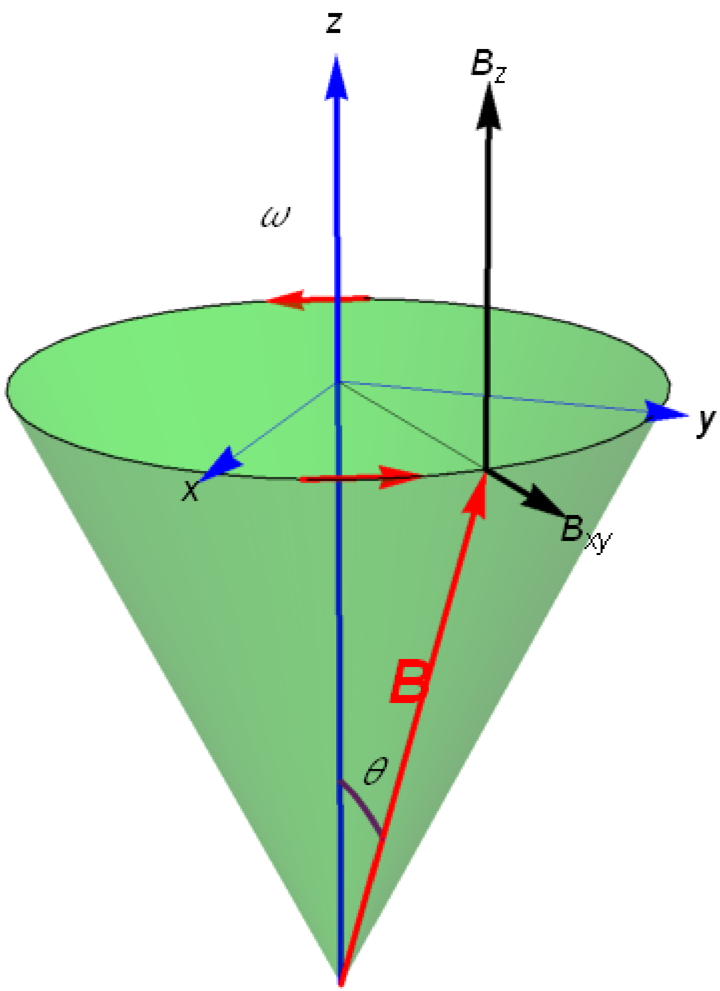


Fig. Magnetic field sweeps around in a cone, at the constant angular frequency ω. The case with $-\gamma > 0$ (rotation of nuclear spin around the z axis in counter clockwise.

((Gyromagnetic ratio of nuclear spin))

Proton (neutron); nuclear spin

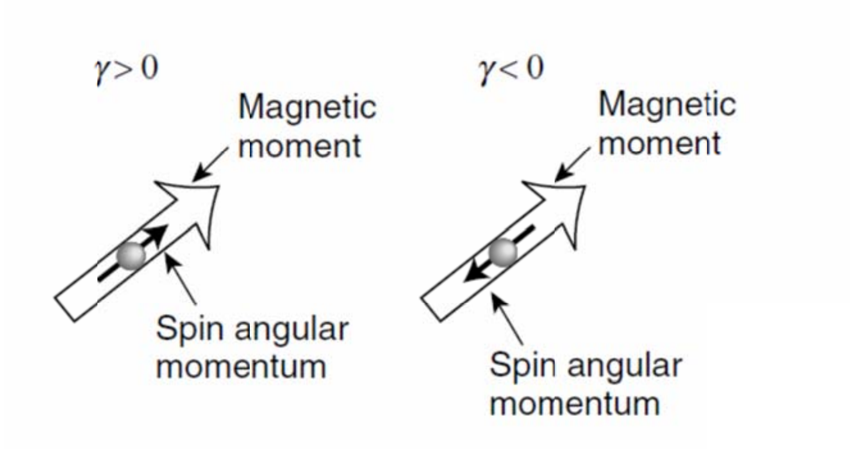


Fig. The definition of the sign for the gyromagnetic ratio γ. (Levitt, 2008)

The gyromagnetic ratio is defined by

$$\gamma = \frac{\boldsymbol{\mu}_I}{\hbar \boldsymbol{I}} = \frac{g_n \mu_N \boldsymbol{I}}{\hbar \boldsymbol{I}} = \frac{g_n \mu_N}{\hbar} = g_n \frac{e}{2m_p c}$$

where $\hbar \boldsymbol{I}$ is the angular momentum. So the magnetic moment of the nuclear spin is given by

$$\hat{\boldsymbol{\mu}}_I = \gamma \hbar \hat{\boldsymbol{I}}, \qquad \hat{\boldsymbol{I}} = \frac{1}{2}\hat{\boldsymbol{\sigma}}$$

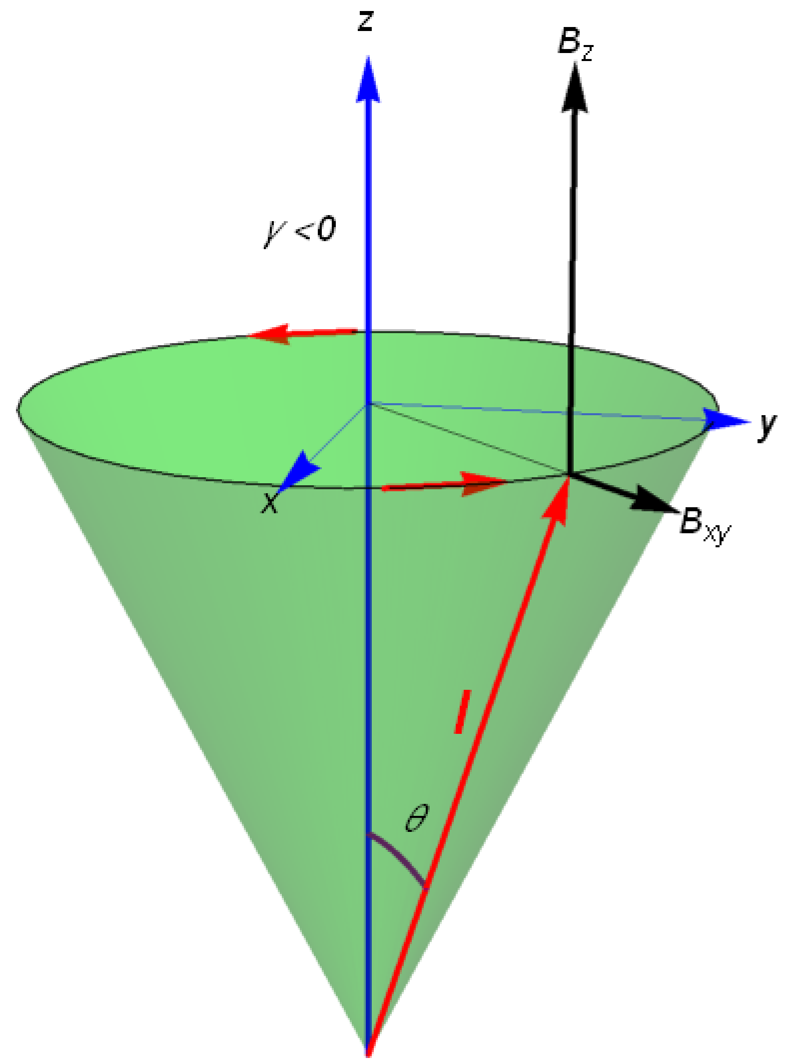
where $\hat{\boldsymbol{\sigma}}$ is the Pauli operator. The Hamiltonian is given by the Zeeman energy as

$$\begin{aligned}
\hat{H} &= -\hat{\boldsymbol{\mu}}_I \cdot \boldsymbol{B} \\
&= -\gamma \hbar \hat{\boldsymbol{I}} \cdot \boldsymbol{B} \\
&= -\gamma \hbar B_0 [\sin\theta \cos(\omega t + \phi_p)\hat{I}_x + \sin\theta \sin(\omega t + \phi_p)\hat{I}_y + \cos\theta \hat{I}_z] \\
&= \hbar \omega_1 \hat{I}_z + \hbar \omega_2 [\cos(\omega t + \phi_p)\hat{I}_x + \sin(\omega t + \phi_p)\hat{I}_y] \\
&= \frac{1}{2}\hbar \omega_1 \hat{\sigma}_z + \frac{1}{2}\hbar \omega_2 [\cos(\omega t + \phi_p)\hat{\sigma}_x + \sin(\omega t + \phi_p)\hat{\sigma}_y]
\end{aligned}$$

where

$$\omega_1 = -\gamma B_1 \cos\theta, \qquad \omega_2 = -\gamma B_0 \sin\theta$$

2. Spin recession (Classical theory)



The magnetic moment of nuclear spin with angular momentum $\hbar I_z$ -s given by

$$\mu_I = \gamma\hbar I_z$$

When a static magnetic field is applied along the *z* axis, the Hamiltonian has the form of the Zeeman energy, given by

$$H = -\boldsymbol{\mu}_I \cdot B = -\gamma\hbar \boldsymbol{I} \cdot \boldsymbol{B},$$

We consider the equation of motion for the angular momentum (Newton's second law for rotation)

$$\hbar \frac{d\boldsymbol{L}}{dt} = \boldsymbol{\tau} = \boldsymbol{\mu} \times \boldsymbol{B} = \gamma\hbar \boldsymbol{I} \times B,$$

$$\frac{d\boldsymbol{I}}{dt} = \gamma(\boldsymbol{I} \times \boldsymbol{B}).$$

When the magnetic field is applied along the +z axis,

$$\dot{I}_z = 0, \qquad \dot{I}_x = \gamma B I_y, \qquad \dot{I}_y = -\gamma B I_x$$

leading to the equation

$$\frac{d}{dt}(I_x + iI_y) = \gamma B I_y - i\gamma B I_x = -i\gamma B(I_x + iI_y)$$

or

$$I_x + iI_y = I_0 \exp(-i\gamma B t) = I_0 e^{i\omega_1 t}$$

where

$\omega_1 = -\gamma B \cos\theta$ (counter clockwise for $\gamma < 0$).

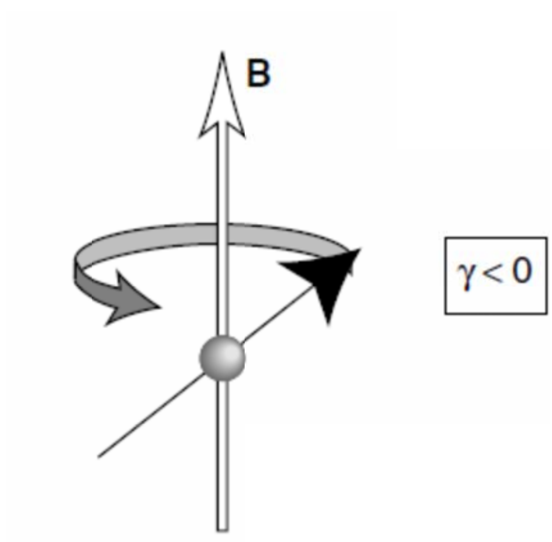


Fig. Precession of nuclear spin with the angular frequency($-\gamma B > 0$) for $\gamma > 0$. (Clockwise direction). (Levitt, 2008).

3. Spin precession using quantum mechanics

The time evolution operator is defined by $\hat{T} = \exp(-\frac{i}{\hbar}\hat{H}t)$. When a static magnetic field is applied along the z axis (in the absence of the r.f. field), the Hamiltonian is given by

$$\hat{H}_0 = -\gamma\hbar\hat{I}_z B_0 \sin\theta = \hbar\omega_1\hat{I}_z$$

where

$$\omega_1 = -\gamma B_0 \sin\theta \qquad \text{and} \qquad \hat{I}_z = \frac{1}{2}\hat{\sigma}_z .$$

The state vector at the time t is related to the state vector at the time $t = 0$ through the time evolution operator as

$$\begin{aligned}
|\psi(t)\rangle &= \exp(-\frac{i}{\hbar}\hat{H}_0 t)|\psi(t=0)\rangle \\
&= \exp(-\frac{i}{\hbar}\hbar\omega_1\hat{I}_z t)|\psi(t=0)\rangle \\
&= \exp(-i\omega_1 t\hat{I}_z)|\psi(t_1)\rangle \\
&= \exp(-i\frac{\omega_1 t}{2}\hat{\sigma}_z)|\psi(t_1)\rangle
\end{aligned}$$

Here we define the rotation operator as

$$\hat{R}_z(\phi) = \exp(-\frac{i}{2}\phi\hat{\sigma}_z) = \begin{pmatrix} \exp(-\frac{i\phi}{2}) & 0 \\ 0 & \exp(\frac{i\phi}{2}) \end{pmatrix}.$$

This means that spin rotates around the z axis through the angle $\omega_1 t$. The r.f. field is given by

$$\boldsymbol{B}_{RF}(t) = B_0 \sin\theta[\boldsymbol{e}_x \cos(\omega t + \phi_p) + \boldsymbol{e}_y \sin(\omega t + \phi_p),$$

(counter-clockwise for $\omega > 0$)

$$\hat{H}_L = -\gamma\hbar\hat{\boldsymbol{I}} \cdot \boldsymbol{B}_{RF}(t)$$
$$= \hbar\omega_2[\hat{I}_x \cos(\omega t + \phi_p) + \hat{I}_y \sin(\omega t + \phi_p)]$$

Note that

$$\omega_2 = -\gamma B_0 \sin\theta ,$$

is the nutation angular frequency.

4. State vector in the rotation frame and Laboratory frame

$$\left|+x'\right\rangle = \hat{R}_z(\Phi)\left|+x\right\rangle ,$$

$$\left|+x\right\rangle = \hat{R}_z(-\Phi)\left|+x'\right\rangle .$$

Here we use the notations

$\left|+x\right\rangle = \left|+x'\right\rangle_R$ Rotating frame

$\left|+x'\right\rangle = \left|+x\right\rangle_L$ Laboratory frame

Then we have

$$\left|+x'\right\rangle_R = \hat{R}_z(-\Phi)\left|+x\right\rangle_L$$

This relationship may be generalized. Any spin state, viewed from the rotating frame, is related to the spin state, viewed from the fixed frame through

$$\left|\psi\right\rangle_R = \hat{R}_z(-\Phi)\left|\psi\right\rangle, \qquad \left|\psi\right\rangle = \hat{R}_z(\Phi)\left|\psi\right\rangle_R$$

Note that for simplicity we use the notation of $\left|\psi\right\rangle_L = \left|\psi\right\rangle$.

Now we consider the Schrödinger equation in the laboratory frame,

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle$$

or

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle_R = i\hbar\frac{\partial}{\partial t}\hat{R}_z(-\Phi)|\psi\rangle$$
$$= i\hbar[\frac{\partial}{\partial t}\hat{R}_z(-\Phi)]|\psi\rangle + \hat{R}_z(-\Phi)i\hbar\frac{\partial}{\partial t}|\psi\rangle$$

Here we note that

$$\hat{R}_z(-\Phi) = \exp(\frac{i}{\hbar}\hat{J}_z\Phi) = \exp(i\Phi\hat{I}_z)$$

where

$$\hat{J}_z = \hbar\hat{I}_z = \frac{\hbar}{2}\hat{\sigma}_z\,,$$

The derivative of $\hat{R}_z(-\Phi)$ with respect to *t*;

$$\frac{\partial}{\partial t}\hat{R}_z(-\Phi) = \frac{\partial}{\partial t}\exp(i\Phi\hat{I}_z)$$
$$= i\hat{I}_z\frac{d\Phi}{dt}\exp(i\Phi\hat{I}_z)$$
$$= i\omega\hat{I}_z\hat{R}_z(-\Phi)$$

where

$$\frac{d\Phi}{dt} = \omega, \qquad \Phi = \omega t$$

Thus we have the Schrödinger equation in the rotating frame

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}|\psi\rangle_R &= -\hbar\omega\hat{I}_z\hat{R}_z(-\Phi)|\psi\rangle + \hat{R}_z(-\Phi)i\hbar\frac{\partial}{\partial t}|\psi\rangle \\
&= -\hbar\omega\hat{I}_z\hat{R}_z(-\Phi)|\psi\rangle + \hat{R}_z(-\Phi)\hat{H}_L|\psi\rangle \\
&= -\hbar\omega\hat{I}_z|\psi\rangle_R + \hat{R}_z(-\Phi)\hat{H}_L\hat{R}_z(\Phi)|\psi\rangle_R \\
&= [-\hbar\omega\hat{I}_z + \hat{R}_z(-\Phi)\hat{H}_L\hat{R}_z(\Phi)]|\psi\rangle_R \\
&= \hat{H}_R|\psi\rangle_R
\end{aligned}
$$

The Schrödinger equation in the rotating frame is given by

$$
i\hbar\frac{\partial}{\partial t}|\psi\rangle_R = \hat{H}_R|\psi\rangle_R
$$

with the Hamiltonian in the rotating frame,

$$
\hat{H}_R = -\hbar\omega\hat{I}_z + \hat{R}_z(-\Phi)\hat{H}_L\hat{R}_z(\Phi)
$$

5. Rotating-frame Hamiltonian

In the presence of a static field along the *z* axis, the Hamiltonian is given by

$$
\hat{H}_0 = \hbar\omega_1\hat{I}_z = \frac{\hbar\omega_1}{2}\hat{\sigma}_z
$$

with

$$
\hat{I}_z = \frac{1}{2}\hat{\sigma}_z
$$

$$
\omega_1 = -\gamma B\cos\theta
$$

The Hamiltonian due to the AC (r.f.) magnetic field in the *x*-*y* plane is

$$
\begin{aligned}
\hat{H}_{RF} &= \hbar\omega_2[\hat{I}_x\cos(\omega t + \phi_p) + \hat{I}_y\sin(\omega t + \phi_p)] \\
&= \hbar\omega_2(\hat{I}_x\cos\Phi_p + \hat{I}_y\sin\Phi_p)
\end{aligned}
$$

where

$$
\Phi_P = \omega t + \phi_P ,
$$

Then the resulting Hamiltonian is

$$\begin{aligned}\hat{H}_L &= \hat{H}_0 + \hat{H}_{RF} \\ &= \hbar\omega_1 \hat{I}_z + \hbar\omega_2 [\hat{I}_x \cos(\Phi_p) + \hat{I}_y \sin(\Phi_p)]\end{aligned}$$

The Hamiltonian in the rotating frame is

$$\begin{aligned}\hat{H}_R &= -\hbar\omega \hat{I}_z + \hat{R}_z(-\Phi)\hat{H}_L \hat{R}_z(\Phi) \\ &= -\hbar\omega \hat{I}_z + \hbar\omega_1 \hat{R}_z(-\Phi)\hat{I}_z \hat{R}_z(\Phi) + \hbar\omega_2 \hat{R}_z(-\Phi)[\hat{I}_x \cos\Phi_P + \hat{I}_y \sin\Phi_P]\hat{R}_z(\Phi) \\ &= \hbar(\omega_1 - \omega)\hat{I}_z + \hbar\omega_2 \hat{R}_z(-\Phi + \Phi_P)\hat{R}_z(-\Phi_P)[\hat{I}_x \cos\Phi_P + \hat{I}_y \sin\Phi_P]\hat{R}_z(\Phi_P)\hat{R}_z(\Phi - \Phi_P) \\ &= \hbar(\omega_1 - \omega)\hat{I}_z + \hbar\omega_2 \hat{R}_z(-\Phi + \Phi_P)\hat{I}_x \hat{R}_z(\Phi - \Phi_P)\end{aligned}$$

where

$$\hat{R}_z(-\Phi_p)[\hat{I}_x \cos\Phi_p + \hat{I}_y \sin\Phi_p]\hat{R}_z(\Phi_p) = \hat{I}_x$$

$$\hat{R}_z(-\Phi + \Phi_P)\hat{R}_z(-\Phi_P) = \hat{R}_z(-\Phi)$$

$$\hat{R}_z(\Phi_P)\hat{R}_z(\Phi - \Phi_P) = \hat{R}_z(\Phi)$$

((**Formula**))

$$\hat{R}_z(\Phi_p)\hat{I}_x \hat{R}_z(-\Phi_p) = \cos(\Phi_p)\hat{I}_x + \sin(\Phi_p)\hat{I}_y$$

$$\hat{R}_z(\Phi_p)\hat{I}_y \hat{R}_z(-\Phi_p) = -\sin(\Phi_p)\hat{I}_x + \cos(\Phi_p)\hat{I}_y$$

or

$$\hat{I}_x = \hat{R}_z(-\Phi_p)[\cos(\Phi_p)\hat{I}_x + \sin(\Phi_p)\hat{I}_y]\hat{R}_z(\Phi_p)$$

$$\hat{I}_y = \hat{R}_z(-\Phi_p)[-\sin(\Phi_p)\hat{I}_x + \cos(\Phi_p)\hat{I}_y]\hat{R}_z(\Phi_p)$$

Note that these relations can be checked using the Mathematica.

Thus we have

$$\hat{H}_R = \hbar\Omega_0\hat{I}_z + \hbar\omega_2\hat{R}_z(\phi_p)\hat{I}_x\hat{R}_z(-\phi_P)$$

Note that

$$\Omega_0 = \omega_1 - \omega ,$$

$$\Phi = \omega t , \qquad \Phi_p = \omega t + \phi_p ,$$

$$-\Phi + \Phi_p = \phi_p .$$

Ω_0 is the resonance offset. Note that the time dependence has vanished from this expression. This is the point of the rotating frame. It transforms a time-dependent quantum-mechanical problem into a time-independent one. The Hamiltonian in the rotation-frame is

$$\hat{H}_R = \hbar\Omega_0\hat{I}_z + \hbar\omega_2(\cos\phi_p\hat{I}_x + \sin\phi_p\hat{I}_y) .$$

6. Off-Resonance effect

For $\Omega_0 = \omega_1 - \omega \neq 0$ (resonance condition), the rotating-frame Hamiltonian is given by

$$\hat{H}_R = \hbar\Omega_0\hat{I}_z + \hbar\omega_2(\cos\phi_p\hat{I}_x + \sin\phi_p\hat{I}_y)$$

with $\phi_p \neq 0$. The state vector in the rotating-frame is

$$|\psi(t)\rangle_R = \hat{R}_{off}(\omega_{eff}t)|\psi(0)\rangle_R$$

where

$$\hat{R}_{off}(\omega_{eff}t) = \exp[-i\omega_2 t(\cos\phi_p\hat{I}_x + \sin\phi_p\hat{I}_y) - i\Omega_0 t\hat{I}_z]$$

$$= \begin{pmatrix} \cos(\frac{t\omega_{eff}}{2}) - i\Omega_0\frac{\sin(\frac{t\omega_{eff}}{2})}{\omega_{eff}} & \frac{-i\omega_2 e^{-i\phi_p}}{\omega_{eff}}\sin(\frac{t\omega_{eff}}{2}) \\ \frac{-i\omega_2 e^{i\phi_p}}{\omega_{eff}}\sin(\frac{t\omega_{eff}}{2}) & \cos(\frac{t\omega_{eff}}{2}) + i\Omega_0\frac{\sin(\frac{t\omega_{eff}}{2})}{\omega_{eff}} \end{pmatrix}$$

where

$$\omega_{eff} = \sqrt{\omega_2^{\ 2} + \Omega_0^{\ 2}} = \sqrt{\omega_2^{\ 2} + (\omega_1 - \omega)^2}$$

Now we start with

$$|\psi(t)\rangle_R = \hat{R}_{\phi_p}(\omega_{eff}t)|\psi(0)\rangle_R, \quad \text{and} \qquad |\psi(t)\rangle_R = \hat{R}_z(-\Phi(t))|\psi(t)\rangle$$

Thus we get

$$\hat{R}_z[-\Phi(t)]|\psi(t)\rangle = \hat{R}_{\phi_p}(\omega_{eff}t)\hat{R}_z[-\Phi(0)]|\psi(0)\rangle$$

The state vector in the laboratory frame is given by

$$|\psi(t)\rangle = \hat{R}_z[\Phi(t)]\hat{R}_{\phi_p}(\omega_{eff}t)\hat{R}_z[-\Phi(0)]|\psi(0)\rangle$$

Note that

$$\hat{R}_z(\Phi) = \exp(-i\Phi\hat{I}_z)$$

where

$$\Phi = \omega t$$

$$\hat{R}_z[\Phi(t)]\hat{R}_{\phi_p}(\omega_{eff})\hat{R}_z[-\Phi(0)]$$
$$= \begin{pmatrix} [\cos(\frac{\omega_{eff}t}{2}) - i\frac{\Omega_0}{\omega_{eff}}\sin(\frac{\omega_{eff}t}{2})]e^{-\frac{i}{2}\omega t} & -i\frac{\omega_2}{\omega_{eff}}\sin(\frac{\omega_{eff}t}{2})e^{-\frac{i}{2}(2\phi_p+\omega t]} \\ -i\frac{\omega_2}{\omega_{eff}}\sin(\frac{\omega_{eff}t}{2})e^{\frac{i}{2}(2\phi_p+\omega t)} & [\cos(\frac{\omega_{eff}t}{2}) + i\frac{\Omega_0}{\omega_{eff}}\sin(\frac{\omega_{eff}t}{2})]e^{\frac{i}{2}\omega t} \end{pmatrix}$$

When $\omega_1 = \omega$ (on resonance), $\omega_{eff} = \omega_2$ and $\Omega_0 = 0$

$$\hat{R}_z[\Phi(t)]\hat{R}_{\phi_p}(\omega_{eff})\hat{R}_z[-\Phi(0)]$$
$$=\begin{pmatrix} \cos(\frac{\omega_2 t}{2})e^{-\frac{i}{2}\omega_1 t} & -i\sin(\frac{\omega_2 t}{2})e^{-\frac{i}{2}(2\phi_p+\omega_1 t)} \\ -i\sin(\frac{\omega_2 t}{2})e^{\frac{i}{2}(2\phi_p+\omega_2 t)} & \cos(\frac{\omega_2 t}{2})e^{\frac{i}{2}\omega_2 t} \end{pmatrix}$$

((**Mathematica**))

```
Clear["Global`*"];
σx = PauliMatrix[1];
σy = PauliMatrix[2];
σz = PauliMatrix[3];
Rx[θ_] := MatrixExp[(-ⅈ θ)/2 σx];
Ry[θ_] := MatrixExp[(-ⅈ θ)/2 σy];
Rz[θ_] := MatrixExp[(-ⅈ θ)/2 σz];
Roff[θ_] := MatrixExp[-ⅈ ω2 t (Cos[θ]/2 σx + Sin[θ]/2 σy) - ⅈ (Ω0 t)/2 σz];
s1 = Roff[ϕp] /. {Sqrt[ω2^2 + Ω0^2] → ωeff, 1/Sqrt[ω2^2 + Ω0^2] → 1/ωeff} //
  FullSimplify;
s1 // MatrixForm
Clear["Global`*"];
σx = PauliMatrix[1];
σy = PauliMatrix[2];
σz = PauliMatrix[3];
Rx[θ_] := MatrixExp[(-ⅈ θ)/2 σx];
Ry[θ_] := MatrixExp[(-ⅈ θ)/2 σy];
Rz[θ_] := MatrixExp[(-ⅈ θ)/2 σz];
Roff[θ_] := MatrixExp[-ⅈ ω1 t (Cos[θ]/2 σx + Sin[θ]/2 σy) - ⅈ (Ω0 t)/2 σz];
s1 = Roff[ϕp] /. {Sqrt[ω1^2 + Ω0^2] → ωeff, 1/Sqrt[ω1^2 + Ω0^2] → 1/ωeff} //
  FullSimplify;
s1 // MatrixForm
```

$$\begin{pmatrix} \text{Cos}\left[\frac{t\,\omega_{eff}}{2}\right] - \frac{i\,\Omega 0\,\text{Sin}\left[\frac{t\,\omega_{eff}}{2}\right]}{\omega_{eff}} & \frac{\omega 2\,(-i\,\text{Cos}[\phi p]-\text{Sin}[\phi p])\,\text{Sin}\left[\frac{t\,\omega_{eff}}{2}\right]}{\omega_{eff}} \\ \frac{\omega 2\,(-i\,\text{Cos}[\phi p]+\text{Sin}[\phi p])\,\text{Sin}\left[\frac{t\,\omega_{eff}}{2}\right]}{\omega_{eff}} & \text{Cos}\left[\frac{t\,\omega_{eff}}{2}\right] + \frac{i\,\Omega 0\,\text{Sin}\left[\frac{t\,\omega_{eff}}{2}\right]}{\omega_{eff}} \end{pmatrix}$$

```
s2 = Rz[ω t].s1.Rz[0] // FullSimplify; s2 // MatrixForm
```

$$\begin{pmatrix} e^{-\frac{1}{2}i t \omega}\left(\text{Cos}\left[\frac{t\,\omega_{eff}}{2}\right] - \frac{i\,\Omega 0\,\text{Sin}\left[\frac{t\,\omega_{eff}}{2}\right]}{\omega_{eff}}\right) & \frac{e^{-\frac{1}{2}i t \omega}\,\omega 2\,(-i\,\text{Cos}[\phi p]-\text{Sin}[\phi p])\,\text{Sin}\left[\frac{t\,\omega_{eff}}{2}\right]}{\omega_{eff}} \\ \frac{e^{\frac{i t \omega}{2}}\,\omega 2\,(-i\,\text{Cos}[\phi p]+\text{Sin}[\phi p])\,\text{Sin}\left[\frac{t\,\omega_{eff}}{2}\right]}{\omega_{eff}} & e^{\frac{i t \omega}{2}}\left(\text{Cos}\left[\frac{t\,\omega_{eff}}{2}\right] + \frac{i\,\Omega 0\,\text{Sin}\left[\frac{t\,\omega_{eff}}{2}\right]}{\omega_{eff}}\right) \end{pmatrix}$$

7. The form of $|\psi(t)\rangle$ with initial condition $|\psi(0)\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$

Suppose that the initial condition is given by

$$|\psi(0)\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

Then we have

$$|\psi(t)\rangle = \hat{R}_z[\Phi(t)]\hat{R}_{\phi_p}(\omega_{eff}t)\hat{R}_z[-\Phi(0)]|\psi(0)\rangle$$

$$= \begin{pmatrix} [\cos(\frac{\omega_{eff}t}{2}) - i\frac{\Omega_0}{\omega_{eff}}\sin(\frac{\omega_{eff}t}{2})]e^{-\frac{i}{2}\omega t} & -i\frac{\omega_2}{\omega_{eff}}\sin(\frac{\omega_{eff}t}{2})e^{-\frac{i}{2}(2\phi_p+\omega t)} \\ -i\frac{\omega_2}{\omega_{eff}}\sin(\frac{\omega_{eff}t}{2})e^{\frac{i}{2}(2\phi_p+\omega t)} & [\cos(\frac{\omega_{eff}t}{2}) + i\frac{\Omega_0}{\omega_{eff}}\sin(\frac{\omega_{eff}t}{2})]e^{\frac{i}{2}\omega t} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} [\cos(\frac{\omega_{eff}t}{2}) - i\frac{\Omega_0}{\omega_{eff}}\sin(\frac{\omega_{eff}t}{2})]\cos\frac{\theta}{2}e^{-\frac{i}{2}\omega t} - i\frac{\omega_2}{\omega_{eff}}\sin(\frac{\omega_{eff}t}{2})\sin\frac{\theta}{2}e^{-\frac{i}{2}(2\phi_p+\omega t]} \\ -i\frac{\omega_2}{\omega_{eff}}\sin(\frac{\omega_{eff}t}{2})\cos\frac{\theta}{2}e^{\frac{i}{2}(2\phi_p+\omega t)} + [\cos(\frac{\omega_{eff}t}{2}) + i\frac{\Omega_0}{\omega_{eff}}\sin(\frac{\omega_{eff}t}{2})]\sin\frac{\theta}{2}e^{\frac{i}{2}\omega t} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

where

$$\omega_{eff} = \sqrt{\omega_2^{\ 2} + \Omega_0^{\ 2}} = \sqrt{\omega_0^{\ 2} + \omega^2 - 2\omega_1\omega} \,,$$

$$\Omega_0 = \omega_1 - \omega = \omega_0 \cos\theta - \omega$$

$$\omega_1 = -\gamma B_0 \cos\theta = \omega_0 \cos\theta \,, \qquad \omega_2 = -\gamma B_0 \sin\theta = \omega_0 \sin\theta$$

$$\omega_0 = -\gamma B_0$$

Suppose that $\phi_P = 0$. The matrix elements can be simplified as

$$\begin{aligned}
\alpha_1 &= [\cos(\frac{\omega_{eff} t}{2}) - i\frac{\Omega_0}{\omega_{eff}}\sin(\frac{\omega_{eff} t}{2})]\cos\frac{\theta}{2} e^{-\frac{i}{2}\omega t} - i\frac{\omega_2}{\omega_{eff}}\sin(\frac{\omega_{eff} t}{2})\sin\frac{\theta}{2} e^{-\frac{i}{2}\omega t} \\
&= \{[\cos(\frac{\omega_{eff} t}{2}) - i\frac{\Omega_0}{\omega_{eff}}\sin(\frac{\omega_{eff} t}{2})]\cos\frac{\theta}{2} - i\frac{\omega_0}{\omega_{eff}}\sin(\frac{\omega_{eff} t}{2})\sin\theta\sin\frac{\theta}{2}\} e^{-\frac{i}{2}\omega t} \\
&= [\cos(\frac{\omega_{eff} t}{2}) - i\frac{\Omega_0 + \omega_0(1-\cos\theta)}{\omega_{eff}}\sin(\frac{\omega_{eff} t}{2})]\cos\frac{\theta}{2} e^{-\frac{i}{2}\omega t} \\
&= [\cos(\frac{\omega_{eff} t}{2}) - i(\frac{\omega_0 - \omega}{\omega_{eff}})\sin(\frac{\omega_{eff} t}{2})]\cos\frac{\theta}{2} e^{-\frac{i}{2}\omega t}
\end{aligned}$$

$$\begin{aligned}
\alpha_2 &= -i\frac{\omega_1}{\omega_{eff}}\sin(\frac{\omega_{eff} t}{2})\cos\frac{\theta}{2} e^{\frac{i}{2}\omega t} + [\cos(\frac{\omega_{eff} t}{2}) + i\frac{\Omega_0}{\omega_{eff}}\sin(\frac{\omega_{eff} t}{2})]\sin\frac{\theta}{2} e^{\frac{i}{2}\omega t} \\
&= -i\frac{\omega_0}{\omega_{eff}}\sin(\frac{\omega_{eff} t}{2})\sin\theta\cos\frac{\theta}{2} e^{\frac{i}{2}\omega t} + [\cos(\frac{\omega_{eff} t}{2})\sin\frac{\theta}{2} + i\frac{\Omega_0}{\omega_{eff}}\sin(\frac{\omega_{eff} t}{2})\sin\frac{\theta}{2}] e^{\frac{i}{2}\omega t} \\
&= [-i\frac{2\omega_0}{\omega_{eff}}\sin(\frac{\omega_{eff} t}{2})\sin\frac{\theta}{2}\cos^2\frac{\theta}{2} + \cos(\frac{\omega_{eff} t}{2})\sin\frac{\theta}{2} + i\frac{\Omega_0}{\omega_{eff}}\sin(\frac{\omega_{eff} t}{2})\sin\frac{\theta}{2}] e^{\frac{i}{2}\omega t} \\
&= [\cos(\frac{\omega_{eff} t}{2}) + i\frac{\Omega_0 - \omega_0(1+\cos\theta)}{\omega_{eff}}\sin(\frac{\omega_{eff} t}{2})]\sin\frac{\theta}{2} e^{\frac{i}{2}\omega t} \\
&= [\cos(\frac{\omega_{eff} t}{2}) - i(\frac{\omega_0 + \omega}{\omega_{eff}})\sin(\frac{\omega_{eff} t}{2})]\sin\frac{\theta}{2} e^{\frac{i}{2}\omega t}
\end{aligned}$$

Then we have

$$|\psi(t)\rangle = \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \begin{pmatrix} [\cos(\frac{\omega_{eff}t}{2}) - i(\frac{\omega_0-\omega}{\omega_{eff}})\sin(\frac{\omega_{eff}t}{2})]\cos\frac{\theta}{2}e^{-\frac{i}{2}\omega t} \\ [-i(\frac{\omega_0+\omega}{\omega_{eff}})\sin(\frac{\omega_{eff}t}{2}) + \cos(\frac{\omega_{eff}t}{2})]\sin\frac{\theta}{2}e^{\frac{i}{2}\omega t}\end{pmatrix}$$

Note that

$$\alpha_1\alpha_1^* + \alpha_2\alpha_2^* = 1$$

((**Proof**))

$$\chi_1\chi_1^* = \cos^2\frac{\theta}{2}[\cos^2\left(\frac{\omega_{eff}t}{2}\right) + \frac{(\omega_0+\Omega_0-\omega_0\cos\theta)^2}{\omega_{eff}^2}\sin^2\left(\frac{\omega_{eff}t}{2}\right)]$$

$$\chi_2\chi_2^* = \sin^2\frac{\theta}{2}[\cos^2\left(\frac{\omega_{eff}t}{2}\right) + \frac{(\omega_0-\Omega_0+\omega_0\cos\theta)^2}{\omega_{eff}^2}\sin^2\left(\frac{\omega_{eff}t}{2}\right)]$$

$$\begin{aligned}\chi_1\chi_1^* + \chi_2\chi_2^* &= \cos^2\frac{\theta}{2}[\cos^2\left(\frac{\omega_{eff}t}{2}\right) + \frac{(\omega_0+\Omega_0-\omega_0\cos\theta)^2}{\omega_{eff}^2}\sin^2\left(\frac{\omega_{eff}t}{2}\right)] \\ &\quad + \sin^2\frac{\theta}{2}[\cos^2\left(\frac{\omega_{eff}t}{2}\right) + \frac{(\omega_0-\Omega_0+\omega_0\cos\theta)^2}{\omega_{eff}^2}\sin^2\left(\frac{\omega_{eff}t}{2}\right)] \\ &= \cos^2\left(\frac{\omega_{eff}t}{2}\right) + [\frac{(\omega_0+\Omega_0-\omega_0\cos\theta)^2}{\omega_{eff}^2}\cos^2\frac{\theta}{2} \\ &\quad + \frac{(\omega_0-\Omega_0+\omega_0\cos\theta)^2}{\omega_{eff}^2}\sin^2\frac{\theta}{2}]\sin^2\left(\frac{\omega_{eff}t}{2}\right)\end{aligned}$$

where

$$\Omega_0 = \omega_1 - \omega, \qquad \omega_{eff}^2 = \omega_0^2 - 2\omega_1\omega + \omega^2, \qquad \omega_1 = \omega_0\cos\theta$$

Her we get

$$(\omega_0+\Omega_0-\omega_0\cos\theta)^2 = (\omega_0-\omega)^2, \qquad (\omega_0-\Omega_0+\omega_0\cos\theta)^2 = (\omega_0+\omega)^2$$

Thus we have

$$(\omega_0 + \Omega_0 - \omega_0 \cos\theta)^2 \cos^2\frac{\theta}{2} + (\omega_0 - \Omega_0 + \omega_0 \cos\theta)^2 \sin^2\frac{\theta}{2}$$
$$= (\omega_0 - \omega)^2 \cos^2\frac{\theta}{2} + (\omega_0 + \omega)^2 \sin^2\frac{\theta}{2}$$
$$= \omega_0{}^2 + \omega^2 - 2\omega_0\omega(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2})$$
$$= \omega_0{}^2 + \omega^2 - 2\omega_0\omega\cos\theta$$
$$= \omega_{eff}{}^2$$

This leads to the relation

$$\chi_1\chi_1{}^* + \chi_2\chi_2{}^* = \cos^2\left(\frac{\omega_{eff}t}{2}\right) + \sin^2\left(\frac{\omega_{eff}t}{2}\right) = 1$$

8 Calculation of α_1 and α_2

We note that

$$\alpha_1 = [\cos(\frac{\omega_{eff}t}{2})\cos\frac{\theta}{2} - i\frac{\omega_0}{\omega_{eff}}\cos\frac{\theta}{2}\sin(\frac{\omega_{eff}t}{2}) + i\frac{\omega}{\omega_{eff}}(\cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2})\sin(\frac{\omega_{eff}t}{2})]e^{-\frac{i}{2}\omega t}$$
$$= [\cos(\frac{\omega_{eff}t}{2})\cos\frac{\theta}{2} - i\frac{\omega_0}{\omega_{eff}}\cos\frac{\theta}{2}\sin(\frac{\omega_{eff}t}{2}) + i\frac{\omega}{\omega_{eff}}\cos\theta\cos\frac{\theta}{2}\sin(\frac{\omega_{eff}t}{2}) + i\frac{\omega}{\omega_{eff}}\sin\theta\sin\frac{\theta}{2}\sin(\frac{\omega_{eff}t}{2})]e^{-\frac{i}{2}\omega t}$$
$$= [\cos(\frac{\omega_{eff}t}{2})\cos\frac{\theta}{2} - i\frac{(\omega_0 - \omega\cos\theta)}{\omega_{eff}}\cos\frac{\theta}{2}\sin(\frac{\omega_{eff}t}{2}) - i\frac{\omega}{\omega_{eff}}\sin\theta\sin\frac{\theta}{2}\sin(\frac{\omega_{eff}t}{2})]e^{-\frac{i}{2}\omega t}$$

$$\alpha_2 = [\cos(\frac{\omega_{eff}t}{2})\sin\frac{\theta}{2} - i\frac{\omega_0}{\omega_{eff}}\sin\frac{\theta}{2}\sin(\frac{\omega_{eff}t}{2}) - i\frac{\omega}{\omega_{eff}}(\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2})\sin(\frac{\omega_{eff}t}{2})]e^{\frac{i}{2}\omega t}$$
$$= [\cos(\frac{\omega_{eff}t}{2})\sin\frac{\theta}{2} - i\frac{\omega_0}{\omega_{eff}}\sin\frac{\theta}{2}\sin(\frac{\omega_{eff}t}{2}) - i\frac{\omega}{\omega_{eff}}\sin\theta\cos\frac{\theta}{2}\sin(\frac{\omega_{eff}t}{2}) + i\frac{\omega}{\omega_{eff}}\cos\theta\sin\frac{\theta}{2}\sin(\frac{\omega_{eff}t}{2})]e^{\frac{i}{2}\omega t}$$
$$= [\cos(\frac{\omega_{eff}t}{2})\sin\frac{\theta}{2} - i\frac{(\omega_0 - \omega\cos\theta)}{\omega_{eff}}\sin\frac{\theta}{2}\sin(\frac{\omega_{eff}t}{2}) - i\frac{\omega}{\omega_{eff}}\sin\theta\cos\frac{\theta}{2}\sin(\frac{\omega_{eff}t}{2})]e^{\frac{i}{2}\omega t}$$

where we use the trigonometry law

$$\cos\frac{\theta}{2} = \cos(\theta - \frac{\theta}{2}) = \cos\theta\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\sin\frac{\theta}{2}$$

and

$$\sin\frac{\theta}{2} = \sin(\theta - \frac{\theta}{2}) = \sin\theta\cos\frac{\theta}{2} - \cos\frac{\theta}{2}\sin\frac{\theta}{2}$$

Then we have

$$\left|\psi(t)\right\rangle = [\cos(\frac{\omega_{eff}t}{2}) - i\frac{(\omega_0 - \omega\cos\theta)}{\omega_{eff}}\sin(\frac{\omega_{eff}t}{2})]e^{-i\frac{\omega t}{2}}\left|\chi_+(t)\right\rangle$$
$$+ i\frac{\omega\sin\theta}{\omega_{eff}}\sin(\frac{\omega_{eff}t}{2})e^{i\frac{\omega t}{2}}\left|\chi_-(t)\right\rangle$$

where

$$\left|\chi_+(t)\right\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\omega t}\sin\frac{\theta}{2} \end{pmatrix}, \qquad \left|\chi_-(t)\right\rangle = \begin{pmatrix} e^{-i\omega t}\sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix}$$

The equation is exactly the same as Eq.(10.33) of Griffiths book. The probabilities of finding the system in the states $\left|\chi_+(t)\right\rangle$ and $\left|\chi_-(t)\right\rangle$ are evaluated as

$$\begin{aligned}
P_+ &= \left|\left\langle\chi_+(t)\middle|\psi(t)\right\rangle\right|^2 \\
&= \left|[\cos(\frac{\omega_{eff}t}{2}) - i\frac{(\omega_0 - \omega\cos\theta)}{\omega_{eff}}\sin(\frac{\omega_{eff}t}{2})]\right|^2 \\
&= \cos^2(\frac{\omega_{eff}t}{2}) + \frac{(\omega_0 - \omega\cos\theta)^2}{\omega_{eff}^2}\sin^2(\frac{\omega_{eff}t}{2}) \\
&= \frac{1}{2}[1 + \frac{(\omega_0 - \omega\cos\theta)^2}{\omega_{eff}^2}] + \frac{1}{2}[1 - \frac{(\omega_0 - \omega\cos\theta)^2}{\omega_{eff}^2}]\cos(\omega_{eff}t)
\end{aligned}$$

$$P_- = \left|\left\langle\chi_+(t)\middle|\psi(t)\right\rangle\right|^2 = (\frac{\omega\sin\theta}{\omega_{eff}})^2\sin^2(\frac{\omega_{eff}t}{2})$$

respectively. In the adiabatic region $\omega_{eff} = \omega_0 >> \omega$, P_- reduced to zero. Then we have

$$|\psi(t)\rangle = [\cos(\frac{\omega_{eff}t}{2}) - i\frac{(\omega_0 - \omega\cos\theta)}{\omega_{eff}}\sin(\frac{\omega_{eff}t}{2})]e^{-i\frac{\omega t}{2}}|\chi_+(t)\rangle$$

In the adiabatic process ($\omega << \omega_0$),

$$\omega_{eff} \approx \omega_0 \,,$$

we have

$$\begin{aligned}|\psi(t)\rangle &\approx [\cos(\frac{\omega_{eff}t}{2}) - i\sin(\frac{\omega_{eff}t}{2})]e^{-i\frac{\omega t}{2}}|\chi_+(t)\rangle \\ &= \exp[-i\frac{(\omega_{eff}+\omega)}{2}t] \\ &= \exp\{-\frac{i}{2}[\omega_0 + \omega(1-\cos\theta)]t\}\end{aligned}$$

where

$$\begin{aligned}\omega_{eff} &= \omega_0\sqrt{1+\frac{\omega^2}{\omega_0^{\ 2}} - 2\frac{\omega}{\omega_0}\cos\theta} \\ &\approx \omega_0\sqrt{1-2\frac{\omega}{\omega_0}\cos\theta} \\ &= \omega_0(1-\frac{\omega}{\omega_0}\cos\theta) \\ &= \omega_0 - \omega\cos\theta\end{aligned}$$

The first factor $\exp(-\frac{i}{2}\omega_0 t)$ is the dynamical phase. The remaining factor is

$$\exp(i\gamma) = \exp[-\frac{i}{2}\omega(1-\cos\theta)t] \,,$$

with the Berry's phase,

$$\gamma = -\frac{1}{2}\omega(1-\cos\theta) \,.$$

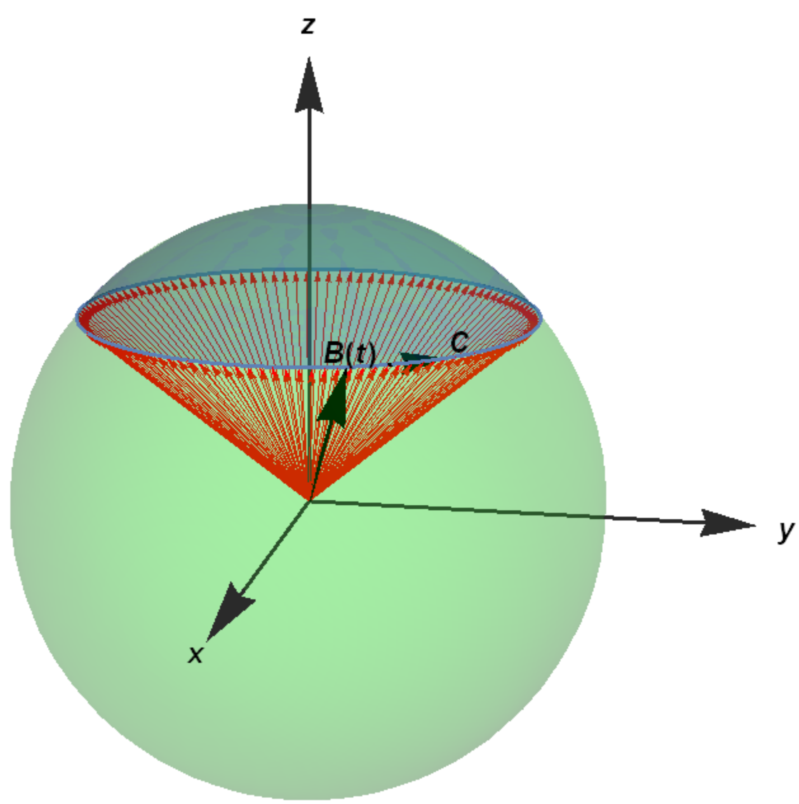
The solid angle Ω is the surface area of the unit sphere, swept by the magnetic field is

$$\Omega = \int_0^{\theta} \sin\theta d\theta \int_0^{2\pi} d\phi = 2\pi(1-\cos\theta)$$

The phase of change during the period $T = \frac{2\pi}{\omega}$ corresponds to the Berry phase and is given by

$$-\frac{1}{2}\omega T(1-\cos\theta) = -\pi(1-\cos\theta) = -\frac{\Omega}{2}$$

The factor 1/2 is related to the value of spin (1/2).



9. Average

Now we calculate the expectation value

$$\langle \hat{I}_x \rangle = \frac{1}{2}\langle \psi(t)|\hat{\sigma}|\psi(t)\rangle$$

$$= \frac{1}{2\omega_{eff}^{\ 2}} \sin\theta[\omega_{eff}^{\ 2}\cos(\omega t)\cos^2(\frac{1}{2}\omega_{eff}t) - (\Omega_0 + \omega_0 - \omega_0\cos\theta)$$

$$\times(\Omega_0 - \omega_0 - \omega_0\cos\theta)\cos(\omega t)\sin^2(\frac{1}{2}\omega_{eff}t) - \omega_{eff}(\Omega_0 - \omega_0\cos\theta)\sin(\omega t)\sin(\omega_{eff}t)]$$

$$\langle \hat{I}_y \rangle = \frac{1}{2}\langle \psi(t)|\hat{\sigma}_y|\psi(t)\rangle$$

$$= \frac{1}{2\omega_{eff}^{\ 2}} \sin\theta[\omega_{eff}^{\ 2}\sin(\omega t)\cos^2(\frac{1}{2}\omega_{eff}t) - (\Omega_0 + \omega_0 - \omega_0\cos\theta)$$

$$\times(\Omega_0 - \omega_0 - \omega_0\cos\theta)\sin(\omega t)\sin^2(\frac{1}{2}\omega_{eff}t) + \omega_{eff}(\Omega_0 - \omega_0\cos\theta)\cos(\omega t)\sin(\omega_{eff}t)$$

$$\langle \hat{I}_z \rangle = \frac{1}{2}\langle \psi(t)|\hat{\sigma}_z|\psi(t)\rangle$$

$$= \frac{1}{16\omega_{eff}^{\ 2}}\{-2\omega_0[4\Omega_0 - 4\Omega_0\cos(2\theta) + \omega_0\cos(3\theta)]\cos^2(\frac{1}{2}\omega_{eff}t)$$

$$+\cos\theta[-\omega_0^{\ 2} + 4(\Omega_0^{\ 2} + \omega_{eff}^{\ 2}) + (\omega_0^{\ 2} - 4\Omega_0^{\ 2} + 4\omega_{eff}^{\ 2})\cos(\omega_{eff}t)]\}$$

Note that

$$\langle \hat{I}_x \rangle_{t=0} = \sin\theta$$

$$\langle \hat{I}_y \rangle_{t=0} = 0$$

$$\langle \hat{I}_z \rangle_{t=0} = \cos\theta$$

10. The form of $|\psi(t)\rangle$ with initial condition $|\psi(0)\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix}$

Suppose that the initial condition is given by

$$|\psi(0)\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix}$$

We note that this state can be rewritten as

$$|\psi(0)\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix} = -\begin{pmatrix} \cos(\frac{\pi+\theta}{2}) \\ \sin(\frac{\pi+\theta}{2}) \end{pmatrix}$$

Then we have

$$\begin{aligned} |\psi(t)\rangle &= \hat{R}_z[\Phi(t)]\hat{R}_{\phi_p}(\omega_{eff}t)\hat{R}_z[-\Phi(0)]|\psi(0)\rangle \\ &= -\begin{pmatrix} [\cos(\frac{\omega_{eff}t}{2}) - i\frac{\Omega_0}{\omega_{eff}}\sin(\frac{\omega_{eff}t}{2})]e^{-\frac{i}{2}\omega t} & -i\frac{\omega_2}{\omega_{eff}}\sin(\frac{\omega_{eff}t}{2})e^{-\frac{i}{2}(2\phi_p+\omega t)} \\ -i\frac{\omega_2}{\omega_{eff}}\sin(\frac{\omega_{eff}t}{2})e^{\frac{i}{2}(2\phi_p+\omega t)} & [\cos(\frac{\omega_{eff}t}{2}) + i\frac{\Omega_0}{\omega_{eff}}\sin(\frac{\omega_{eff}t}{2})]e^{\frac{i}{2}\omega t} \end{pmatrix} \begin{pmatrix} \cos(\frac{\pi+\theta}{2}) \\ \sin(\frac{\pi+\theta}{2}) \end{pmatrix} \end{aligned}$$

The form of $|\psi(t)\rangle$ with initial condition $|\psi(0)\rangle = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix}$ can be obtained from the form of $|\psi(t)\rangle$ with initial condition $|\psi(0)\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$ by replacing θ by $\pi+\theta$. Thus we have

$$|\psi(t)\rangle = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = -\begin{pmatrix} -[\cos(\frac{\omega_{eff}t}{2}) - i(\frac{\omega_0-\omega}{\omega_{eff}})\sin(\frac{\omega_{eff}t}{2})]\sin\frac{\theta}{2}e^{-\frac{i}{2}\omega t} \\ [i(\frac{\omega_0+\omega}{\omega_{eff}})\sin(\frac{\omega_{eff}t}{2}) - \cos(\frac{\omega_{eff}t}{2})]\cos\frac{\theta}{2}e^{\frac{i}{2}\omega t} \end{pmatrix}$$

Here we note that

$$\beta_1 = [\cos(\frac{\omega_{eff}t}{2})\sin\frac{\theta}{2} - i\frac{\omega_0}{\omega_{eff}}\sin\frac{\theta}{2}\sin(\frac{\omega_{eff}t}{2}) + i\frac{\omega}{\omega_{eff}}(\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2})\sin(\frac{\omega_{eff}t}{2})]e^{-\frac{i}{2}\omega t}$$

$$= [\cos(\frac{\omega_{eff}t}{2})\sin\frac{\theta}{2} - i\frac{\omega_0}{\omega_{eff}}\sin\frac{\theta}{2}\sin(\frac{\omega_{eff}t}{2}) + i\frac{\omega}{\omega_{eff}}\sin\theta\cos\frac{\theta}{2}\sin(\frac{\omega_{eff}t}{2}) - i\frac{\omega}{\omega_{eff}}\cos\theta\sin\frac{\theta}{2}\sin(\frac{\omega_{eff}t}{2})]e^{-\frac{i}{2}\omega t}$$

$$= [\cos(\frac{\omega_{eff}t}{2})\sin\frac{\theta}{2} - i\frac{(\omega_0 + \omega\cos\theta)}{\omega_{eff}}\sin\frac{\theta}{2}\sin(\frac{\omega_{eff}t}{2}) + i\frac{\omega}{\omega_{eff}}\sin\theta\cos\frac{\theta}{2}\sin(\frac{\omega_{eff}t}{2})]e^{-\frac{i}{2}\omega t}$$

$$\beta_2 = -[\cos(\frac{\omega_{eff}t}{2})\cos\frac{\theta}{2} - i\frac{\omega_0}{\omega_{eff}}\cos\frac{\theta}{2}\sin(\frac{\omega_{eff}t}{2}) - i\frac{\omega}{\omega_{eff}}(\cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2})\sin(\frac{\omega_{eff}t}{2})]e^{\frac{i}{2}\omega t}$$

$$= [-\cos(\frac{\omega_{eff}t}{2})\cos\frac{\theta}{2} + i\frac{\omega_0}{\omega_{eff}}\cos\frac{\theta}{2}\sin(\frac{\omega_{eff}t}{2}) + i\frac{\omega}{\omega_{eff}}\cos\theta\cos\frac{\theta}{2}\sin(\frac{\omega_{eff}t}{2}) + i\frac{\omega}{\omega_{eff}}\sin\theta\sin\frac{\theta}{2}\sin(\frac{\omega_{eff}t}{2})]e^{\frac{i}{2}\omega t}$$

$$= [-\cos(\frac{\omega_{eff}t}{2})\cos\frac{\theta}{2} + i\frac{(\omega_0 + \omega\cos\theta)}{\omega_{eff}}\cos\frac{\theta}{2}\sin(\frac{\omega_{eff}t}{2}) + i\frac{\omega}{\omega_{eff}}\sin\theta\sin\frac{\theta}{2}\sin(\frac{\omega_{eff}t}{2})]e^{\frac{i}{2}\omega t}$$

Using the basis given by

$$\left|\chi_+(t)\right\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\omega t}\sin\frac{\theta}{2} \end{pmatrix}, \qquad \left|\chi_-(t)\right\rangle = \begin{pmatrix} e^{-i\omega t}\sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix}$$

$\left|\psi(t)\right\rangle$ can be rewritten as

$$\left|\psi(t)\right\rangle = [\cos(\frac{\omega_{eff}t}{2}) - i\frac{(\omega_0 + \omega\cos\theta)}{\omega_{eff}}\sin(\frac{\omega_{eff}t}{2})]e^{i\frac{\omega t}{2}}\left|\chi_-(t)\right\rangle$$
$$+ i\frac{\omega\sin\theta}{\omega_{eff}}\sin(\frac{\omega_{eff}t}{2})e^{-i\frac{\omega t}{2}}\left|\chi_+(t)\right\rangle$$

In the adiabatic process ($\omega << \omega_0$)

$$\omega_{eff} \approx \omega_0 - \omega\cos\theta \approx \omega_0 ,$$

$$|\psi(t)\rangle \approx [\cos(\frac{\omega_{eff}t}{2}) - i\sin(\frac{\omega_{eff}t}{2})]e^{i\frac{\omega t}{2}}|\chi_+(t)\rangle$$
$$= \exp[i\frac{(-\omega_{eff}+\omega)}{2}t]$$
$$= \exp\{\frac{i}{2}[-\omega_0 + \omega(1+\cos\theta)]t\}$$

where $\omega_{eff} = \omega_0 - \omega\cos\theta$ and θ is the angle of the direction of the magnetic field from the *z* axis. The first factor $\exp(-\frac{i}{2}\omega_0 t)$ is the dynamical phase. The remaining factor is

$$\exp(i\gamma) = \exp[\frac{i}{2}\omega(1+\cos\theta)]t]$$

with the Berry's phase,

$$\gamma = \frac{1}{2}\omega(1+\cos\theta)$$

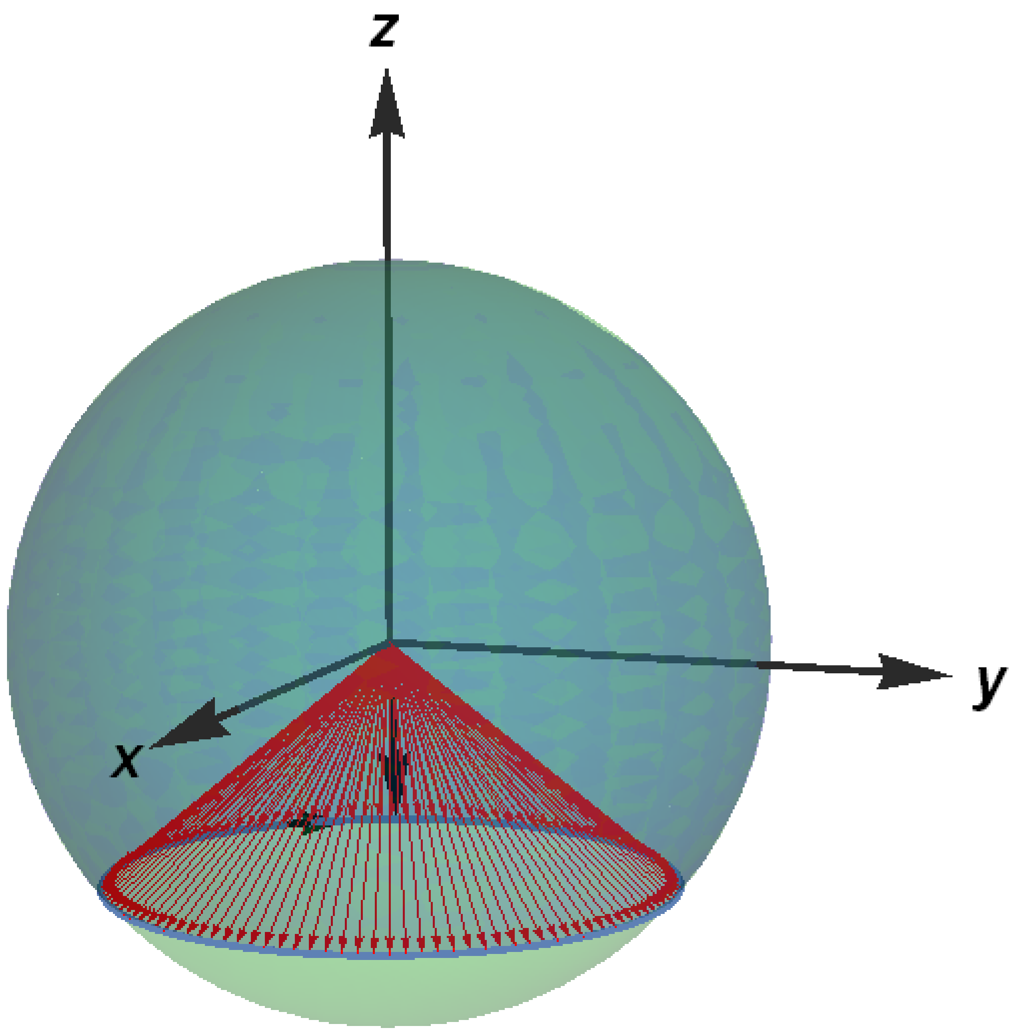
The solid angle Ω is the surface area of the unit sphere, swept by the magnetic field is

$$\Omega = \int_0^{\pi-\theta}\sin\theta d\theta\int_0^{2\pi}d\phi = 2\pi(1+\cos\theta)$$

The phase of change during the period $T = \frac{2\pi}{\omega}$ corresponds to the Berry phase and is

$$\frac{1}{2}\omega T(1+\cos\theta) = \pi(1+\cos\theta) = \frac{\Omega}{2}$$

The factor 1/2 is related to the value of spin (1/2).



11. Summary

The Berry phase for a closed loop increases by the same amount no matter how one goes around it, provided only that one never go so fast as to invalidate the adiabatic approximation. For this reason, it is called a geometric phase.

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