Casimir effect

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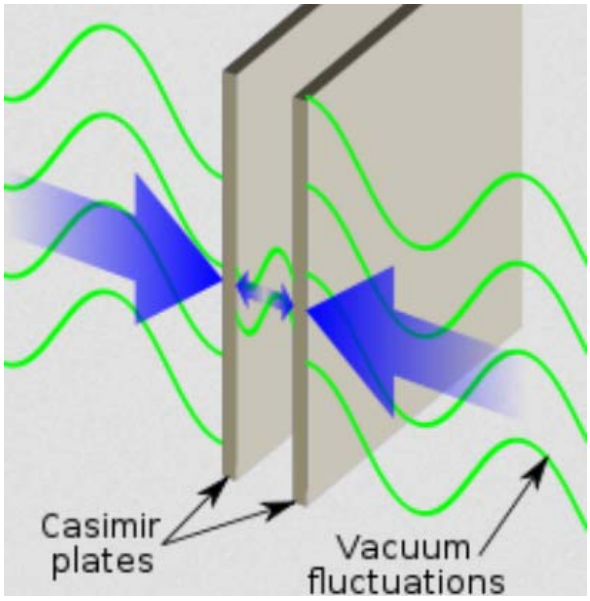
"I mentioned my results to Niels Bohr, during a walk. That is nice, he said, that something new. I told him I was puzzled by the extremely simple form of the expression for the interaction at very large distances and he mumbled something about zero-point energy. That was all, but it put me on a new track." (H.B.G. Casimir (private communication, March 1992).



In quantum field theory, the **Casimir effect** and the **Casimir–Polder force** are physical forces arising from a quantized field. They are named after the Dutch physicist Hendrik Casimir. The typical example is of two uncharged metallic plates in a vacuum, placed a few nanometers apart. In a classical description, the lack of an external field also means that there is no field between the plates, and no force would be measured between them. When this field is instead studied using the QED vacuum of quantum electrodynamics, it is seen that the plates do affect the virtual photons which constitute the field, and generate a net force—either an attraction or a repulsion depending on the specific arrangement of the two plates. Although the Casimir effect can be expressed in terms of virtual particles interacting with the objects, it is best described and more easily calculated in terms of the zero-point energy of a quantized field in the intervening space between the objects. This force has been measured, and is a striking example of an effect captured formally by second quantization. However, the treatment of boundary conditions in these calculations has led to some controversy. In fact "Casimir's original goal was to compute the van der Waals force between polarizable molecules" of the metallic plates. Thus it can be interpreted without any reference to the zero-point energy (vacuum energy) of quantum fields
http://en.wikipedia.org/wiki/Casimir_effect

Hendrik Brugt Gerhard Casimir (July 15, 1909 – May 4, 2000) was a Dutch physicist best known for his research on the two-fluid model of superconductors (together with C. J. Gorter) in 1934 and the Casimir effect (together with D. Polder) in 1948.

1 Electric field and magnetic field in the vacuum

We start with the Maxwell's equation in vacuum

$$\nabla \cdot \boldsymbol{E} = 0$$
$$\nabla \cdot \boldsymbol{B} = 0$$
$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}$$
$$\nabla \times \boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}$$

We assume that

$$\boldsymbol{E} = \mathrm{Re}[\widetilde{\boldsymbol{E}}_0 e^{-i\omega t}]$$
$$\boldsymbol{B} = \mathrm{Re}[\widetilde{\boldsymbol{B}}_0 e^{-i\omega t}]$$

$$\nabla \cdot \widetilde{\boldsymbol{E}}_0 = 0$$
$$\nabla \cdot \widetilde{\boldsymbol{B}}_0 = 0$$
$$\nabla \times \widetilde{\boldsymbol{E}}_0 = -\frac{i\omega}{c}\widetilde{\boldsymbol{B}}_0$$
$$\nabla \times \widetilde{\boldsymbol{B}}_0 = i\frac{\omega}{c}\widetilde{\boldsymbol{E}}_0$$

$$\nabla\times(\nabla\times\boldsymbol{E})=\nabla(\nabla\cdot\boldsymbol{E})-\nabla^2\boldsymbol{E}=-\nabla^2\boldsymbol{E}=-\frac{1}{c}\frac{\partial}{\partial t}(\nabla\times\boldsymbol{B})=-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{E}$$

or

$$\nabla^2\boldsymbol{E}=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{E}$$

or

$$\nabla^2\tilde{\boldsymbol{E}}_0+k^2\tilde{\boldsymbol{E}}_0=0$$

with $\omega=ck$. Similarly, we have

$$\nabla^2\boldsymbol{B}=\frac{1}{c^2}\frac{\partial\boldsymbol{B}}{\partial t}$$

$$\nabla^2\tilde{\boldsymbol{B}}_0+k^2\tilde{\boldsymbol{B}}_0=0$$

We now consider an electromagnetic wave in the closed cube with side *L*.

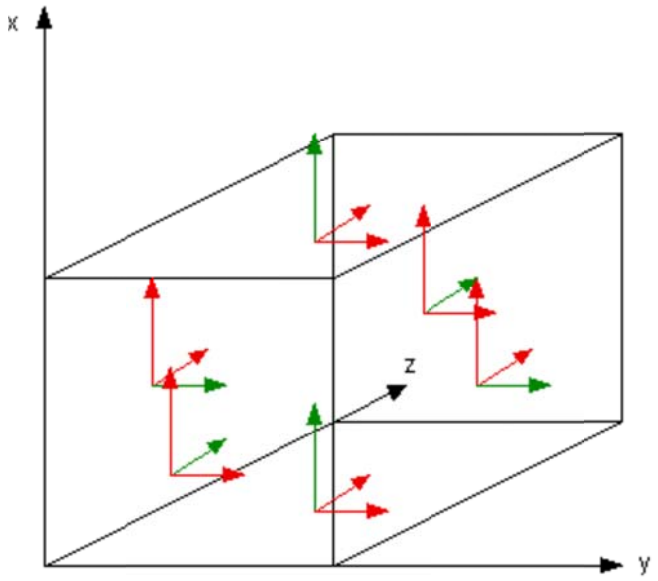


Fig. Boundary condition for the electric field (red) (tangential component continuous)) and the magnetic field (green) (normal component continuous).

From the boundary conditions we have

$$E_x = E_1 \begin{pmatrix} \sin(k_1 x) \\ \cos(k_1 x) \end{pmatrix} \begin{pmatrix} \sin(k_2 y) \\ \end{pmatrix} \begin{pmatrix} \sin(k_3 z) \\ \end{pmatrix}$$

$$E_y = E_2 \begin{pmatrix} \sin(k_1 x) \\ \end{pmatrix} \begin{pmatrix} \sin(k_2 y) \\ \cos(k_2 y) \end{pmatrix} \begin{pmatrix} \sin(k_3 z) \\ \end{pmatrix}$$

$$E_z = E_3 \begin{pmatrix} \sin(k_1 x) \\ \end{pmatrix} \begin{pmatrix} \sin(k_2 y) \\ \end{pmatrix} \begin{pmatrix} \sin(k_3 z) \\ \cos(k_3 y) \end{pmatrix}$$

where

$$k_1 = k_x = \frac{\pi}{L} n_x, \qquad k_2 = k_y = \frac{\pi}{L} n_y, \qquad k_3 = k_z = \frac{\pi}{a} n_z$$

$$(n_x, n_y, n_z = 1, 2, 3, \ldots)$$

$$\omega(n_x, n_y, n_z) = c\sqrt{k_x^{\,2} + k_y^{\,2} + k_z^{\,2}}$$

Note that

$E_x = 0$	for $y = 0$ and $y = L$ planes and $z = 0$ and $z = L$ planes.
$E_y = 0$	for $z = 0$ and $z = L$ planes and $x = 0$ and $x = L$ planes.
$E_z = 0$	for $x = 0$ and $x = L$ planes and $y = 0$ and $y = L$ planes.

From the condition

$$\nabla \cdot \widetilde{\boldsymbol{E}} = 0$$

we have

$$E_x = E_1 \cos(k_1 x) \sin(k_2 y) \sin(k_3 z),$$

$$E_y = E_2 \sin(k_1 x) \cos(k_2 y) \sin(k_3 z),$$

$$E_z = E_3 \sin(k_1 x) \sin(k_2 y) \cos(k_3 y)$$

From the condition

$$\nabla \times \widetilde{\boldsymbol{E}}_0 = -\frac{i\omega}{c} \widetilde{\boldsymbol{B}}_0,$$

we have

$$B_x = B_1 \sin(k_1 x)\cos(k_2 y)\cos(k_3 z)\,,$$

$$B_y = B_2 \cos(k_1 x)\sin(k_2 y)\cos(k_3 z)\,,$$

$$B_z = B_3 \cos(k_1 x)\cos(k_2 y)\sin(k_3 z)$$

where

$B_x = 0$ for $x = 0$ and $x = L$ planes
$B_y = 0$ for $y = 0$ and $y = L$ planes.
$B_z = 0$ for $z = 0$ and $z = L$ planes.

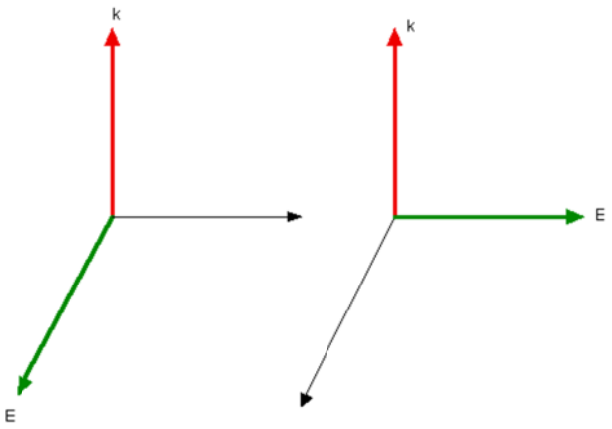
We note that

$$\nabla \cdot \boldsymbol{E} = (E_1 k_1 + E_2 k_2 + E_3 k_3)\sin(k_1 x)\sin(k_2 y)\sin(k_3 z) = 0$$

or

$$(E_1 k_1 + E_2 k_2 + E_3 k_3) = \frac{\pi}{L}(E_1 n_x + E_2 n_y + E_3 n_z) = 0$$

This means that the vector (E_1, E_2, E_3) is perpendicular to the wave vector $\boldsymbol{k} = (k_1, k_2, k_3)$. For each $\boldsymbol{k}$, there are two independent directions for (E_1, E_2, E_3); polarization.



2. Modes

$$E_x = E_1 \cos(k_1 x)\sin(k_2 y)\sin(k_3 z)\,,$$

$$E_y = E_2 \sin(k_1 x)\cos(k_2 y)\sin(k_3 z)\,,$$

$$E_z = E_3 \sin(k_1 x)\sin(k_2 y)\cos(k_3 y)$$

with

$$(E_1 k_1 + E_2 k_2 + E_3 k_3) = \frac{\pi}{L}(E_1 n_x + E_2 n_y + E_3 n_z) = 0$$

(a) $k_1 = \frac{\pi}{L}, \qquad k_2 = 0, \qquad k_3 = 0$

$E_x = 0, \qquad E_y = 0, \qquad E_z = 0$

So there is no electric field. In other words, there is no polarization vector

(b) $k_1 = \frac{\pi}{L}, \qquad k_2 = \frac{\pi}{L}, \qquad k_3 = 0$

$E_1 + E_2 = 0$

$E_x = 0, \qquad E_y = 0,$

$E_z = E_3 \sin(\frac{\pi}{L}x)\sin(\frac{\pi}{L}y)$

There is only one polarization vector (one mode).

(c) $k_1 = \frac{\pi}{L}, \qquad k_2 = \frac{\pi}{L}, \qquad k_3 = \frac{\pi}{L}$

$E_x = E_1 \cos(\frac{\pi x}{L})\sin(\frac{\pi y}{L})\sin(\frac{\pi z}{L}),$

$E_y = E_2 \sin(\frac{\pi x}{L})\cos(\frac{\pi y}{L})\sin(\frac{\pi z}{L}),$

$E_z = E_3 \sin(\frac{\pi x}{L})\sin(\frac{\pi y}{L})\cos(\frac{\pi z}{L}),$

with

$$E_1 + E_2 + E_3 = 0 .$$

There are two polarization vectors (two modes)

In conclusion,

(a) There are two modes for $n_x \neq 0$, $n_y \neq 0$, $n_z \neq 0$

(b) There is one mode for one of three indexes n_x, n_y, n_z is equal to zero.

(c) There is no mode for two of three indexes n_x, n_y, n_z are equal to zero.

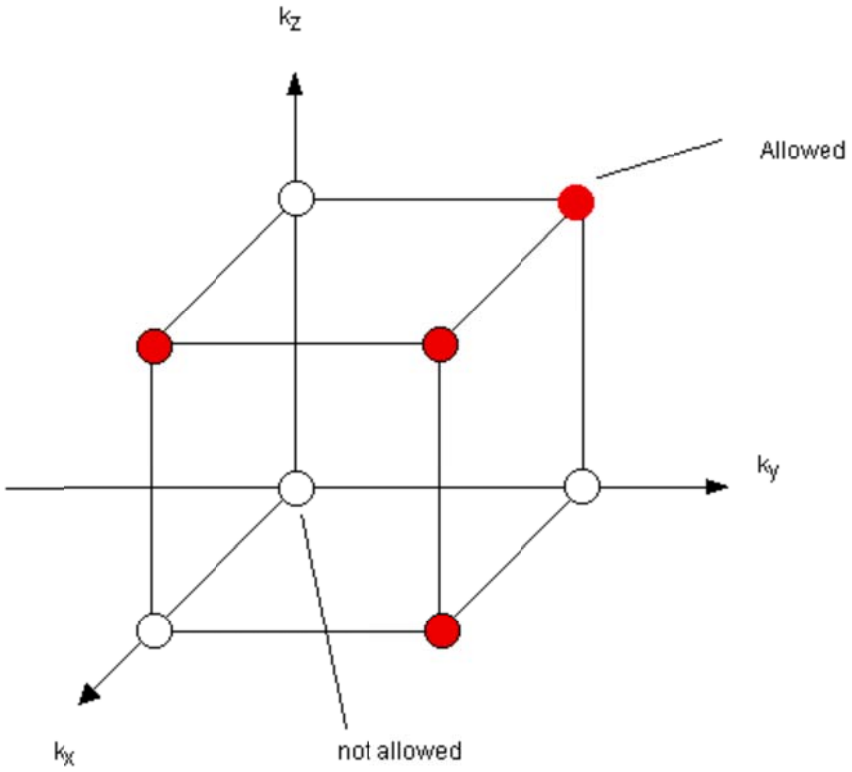


Fig. The number of modes. Open circles for states which are not allowed): Red points for one mode (one of three n_x, n_y, and n_z). Two modes for n_x, n_y, and n_z (one of them are zero).

3. Zero-point energy

The corresponding vector potential can be expressed by

$$A_x = A_1 \cos(k_1 x)\sin(k_2 y)\sin(k_3 z),$$

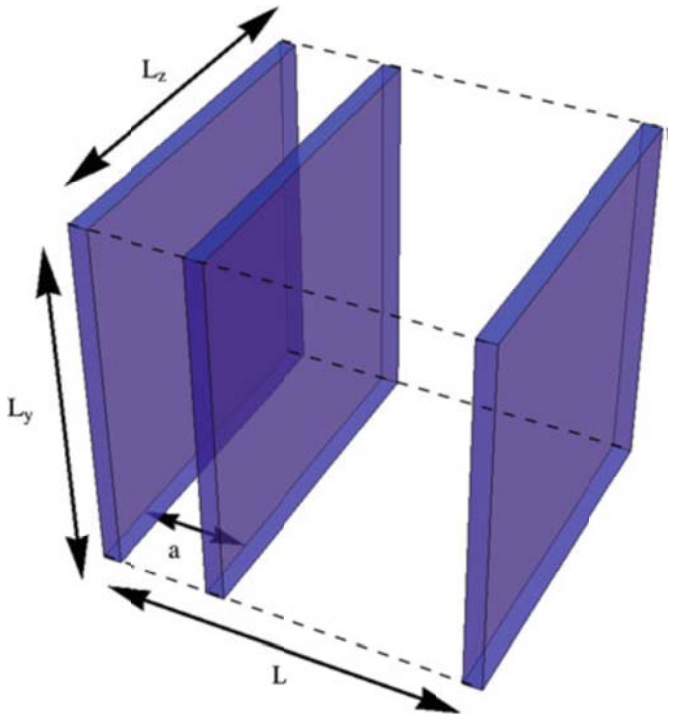
$$A_y = A_2 \sin(k_1 x)\cos(k_2 y)\sin(k_3 z),$$

$$A_z = A_3 \sin(k_1 x)\sin(k_2 y)\cos(k_3 y)$$

The Coulomb gauge

$$\nabla \cdot \boldsymbol{A} = -(k_1 A_1 + k_2 A_2 + k_3 A_3)\sin(k_1 x)\sin(k_2 y)\sin(k_3 y) = 0$$

We consider the 3D system with the sizes of L x L x a.



Here we have

$$k_1 = k_x = \frac{\pi}{L} n_x, \qquad k_2 = k_y = \frac{\pi}{L} n_y, \qquad k_3 = k_z = \frac{\pi}{a} n_z$$

The zero-point energy of this system can be evaluated as follows.

$$\varepsilon_0 = \frac{\hbar}{2}\sum_{n_x,n_y}\omega(n_x,n_y,0)+\frac{\hbar}{2}\sum_{n_y,n_z}\omega(0,n_y,n_z)+\frac{\hbar}{2}\sum_{n_z,n_x}\omega(n_x,0,n_z)+\frac{\hbar}{2}2\sum_{n_x,n_y,n_z}\omega(n_x,n_y,n_z)$$

$$= \frac{\hbar c}{2}\sum_{n_x=1}^{\infty}\sum_{n_y=1}^{\infty}\sqrt{k_x{}^2+k_y{}^2}+\frac{\hbar c}{2}\sum_{n_y=1}^{\infty}\sum_{n_z=1}^{\infty}\sqrt{k_y{}^2+k_z{}^2}+\frac{\hbar c}{2}\sum_{n_x=1}^{\infty}\sum_{n_z=1}^{\infty}\sqrt{k_x{}^2+k_z{}^2}$$

$$+\frac{\hbar c}{2}2\sum_{n_x=1}^{\infty}\sum_{n_y=1}^{\infty}\sum_{n_z=1}^{\infty}\sqrt{k_x{}^2+k_y{}^2+k_z{}^2}$$

$$\approx \hbar c\int_{n_x=0}^{\infty}dn_x\int_{n_y=0}^{\infty}dn_y[\frac{1}{2}\sqrt{k_x{}^2+k_y{}^2}+\sum_{n_z=1}^{\infty}\sqrt{k_x{}^2+k_y{}^2+k_z{}^2}]$$

$$= \frac{L^2\hbar c}{(2\pi)^2}\int_{-\infty}^{\infty}dk_x\int_{-\infty}^{\infty}dk_y[\frac{1}{2}\sqrt{k_x{}^2+k_y{}^2}+\sum_{n_z=1}^{\infty}\sqrt{k_x{}^2+k_y{}^2+k_z{}^2}]$$

$$= \frac{L^2\hbar c}{2\pi}[\int_0^{\infty}\frac{k_{//}{}^2}{2}dk_{//}+\sum_{n_z=1}^{\infty}\int_0^{\infty}k_{//}dk_{//}\sqrt{k_{//}{}^2+\frac{n_z{}^2\pi^2}{a^2}}]$$

with

$$\omega = ck = c\sqrt{k_x{}^2+k_y{}^2+k_z{}^2}\ .$$

$$k_{//} = \sqrt{k_x{}^2+k_y{}^2}$$

$n_z = n$ (we use this for simplicity.

Here we note that

$$\int_0^{\infty}\frac{k_{//}{}^2}{2}dk_{//} = \int_0^{\infty}\frac{\kappa^2}{2}d\kappa$$

where $k_{//} = \kappa$

$$\int_0^{\infty}k_{//}dk_{//}\sqrt{k_{//}{}^2+\frac{n^2\pi^2}{a^2}} = \int_{n\pi/a}^{\infty}\kappa^2 d\kappa\,,$$

where

$$\kappa = \sqrt{k_{//}{}^2 + \frac{n_z{}^2 \pi^2}{a^2}}, \qquad \kappa d\kappa = k_{//} dk_{//}.$$

Thus we get

$$\frac{\varepsilon_0}{L^2 \hbar c} = \frac{1}{2\pi}[\int_0^\infty \frac{\kappa^2}{2} d\kappa + \sum_{n=1}^{\infty} \int_{n\pi/a}^{\infty} \kappa^2 d\kappa]. \qquad (1)$$

If we choose an appropriate function $F(\kappa)$ such that

$$F(\kappa) = e^{-\kappa/\kappa_c}, \qquad \text{or} \qquad F(\kappa) = \begin{cases} 1 & \kappa < \kappa_c \\ 0 & \kappa > \kappa_c \end{cases},$$

the sum in Eq.(1) can converge. Here we use $F(\kappa) = e^{-\kappa/\kappa_c}$, where κ_c is the cut-off wave number. ω_c is the cut-off angular frequency and is defined by

$$\omega_c = c\kappa_c .$$

and

$$F(\omega) = F(\kappa) = e^{-c\kappa/c\kappa_c} = e^{-\omega/\omega_c} .$$

Thus

$$\begin{aligned} \frac{\varepsilon}{L^2 \hbar c} &= \frac{1}{2\pi}[\frac{1}{2}\int_0^\infty \kappa^2 F(\kappa) d\kappa + \sum_{n=1}^{\infty} \int_{n\pi/a}^{\infty} \kappa^2 F(\kappa) d\kappa] \\ &= \frac{1}{2\pi}[\frac{1}{2}\int_0^\infty \kappa^2 e^{-\kappa/\kappa_c} d\kappa + \sum_{n=1}^{\infty} \int_{n\pi/a}^{\infty} \kappa^2 e^{-\kappa/\kappa_c} d\kappa] \\ &= \frac{1}{2\pi}\frac{d^2}{dx^2}[\frac{1}{2}\int_0^\infty e^{-x\kappa} d\kappa + \sum_{n=1}^{\infty} \int_{n\pi/a}^{\infty} e^{-x\kappa_c} d\kappa]_{x=1/\kappa_c} \\ &= \frac{1}{2\pi}\frac{d^2}{dx^2}[\frac{1}{2x} + \frac{1}{x}\sum_{n=1}^{\infty} e^{-n\pi x/a}]_{x=1/\kappa_c} \end{aligned}$$

Then we get

$$\frac{\varepsilon}{L^2\hbar c}=\frac{1}{2\pi}\frac{d^2}{dx^2}[\frac{1}{2x}+\frac{1}{x}\frac{1}{e^{\pi x/a}-1}]_{x=1/\kappa_c}$$

We use the Mathematica of this calculation. The zero-point energy is evaluated as

$$\begin{aligned}\frac{\varepsilon}{\hbar cL^2}&=\frac{1}{2\pi}\frac{d^2}{dx^2}(\frac{1}{2x}+\frac{1}{x}\frac{1}{e^{\pi x/a}-1})\\&=\frac{1}{8a^2\pi x^3}[4a^2\coth(\frac{\pi x}{2a})+\pi x(2a+\pi x\coth(\frac{\pi x}{2a}))\frac{1}{\sinh^2(\frac{\pi x}{2a})}]\\&=\frac{3a}{\pi^2x^4}-\frac{\pi^2}{720a^3}+\frac{\pi^4x^2}{5040a^5}-\frac{\pi^6x^4}{80640a^7}+\frac{\pi^8x^6}{1710720a^9}+\ldots\end{aligned}$$

In the large limit of a, we have

$$\frac{\varepsilon_0}{\hbar cL^2}=\frac{3a}{\pi^2x^4}$$

Thus

$$\frac{\varepsilon-\varepsilon_0}{\hbar cL^2}\approx-\frac{\pi^2}{720a^3},$$

where $x=\frac{1}{\kappa_c}=\frac{c}{\omega_c}$ and ω_c is the cut-off angular frequency. We note that this quantity is independent of x. The pressure is defined by

$$P_z=-\hbar c\frac{d}{da}(\frac{\varepsilon-\varepsilon_0}{L^2})\approx-\frac{d}{da}(-\frac{\hbar c\pi^2}{720a^3})=-\frac{\hbar c\pi^2}{240a^4}.$$

The vacuum fluctuations of the electromagnetic field cause two parallel conducting plates to move toward each other.
((Example))

When a = 1 cm, P_z = 1.300 x 10^{-18} dyne/cm^2.
When a = 1 μm, P_z = 1.300 x 10^{-2} dyne/cm^2.
When a = 1 Å, P_z = 1.300 x 10^{14} dyne/cm^2.

5. Calculation of the zero-point energy for three dimensional system with sides *L*x*L*x*L*

First we discuss the density of states of the three-dimensional system with size L x L x L. We note that there are 2 states (modes) per $\left(\frac{\pi}{L}\right)^3$.

$$\omega = ck = c\sqrt{k_x^{\,2} + k_y^{\,2} + k_z^{\,2}}$$

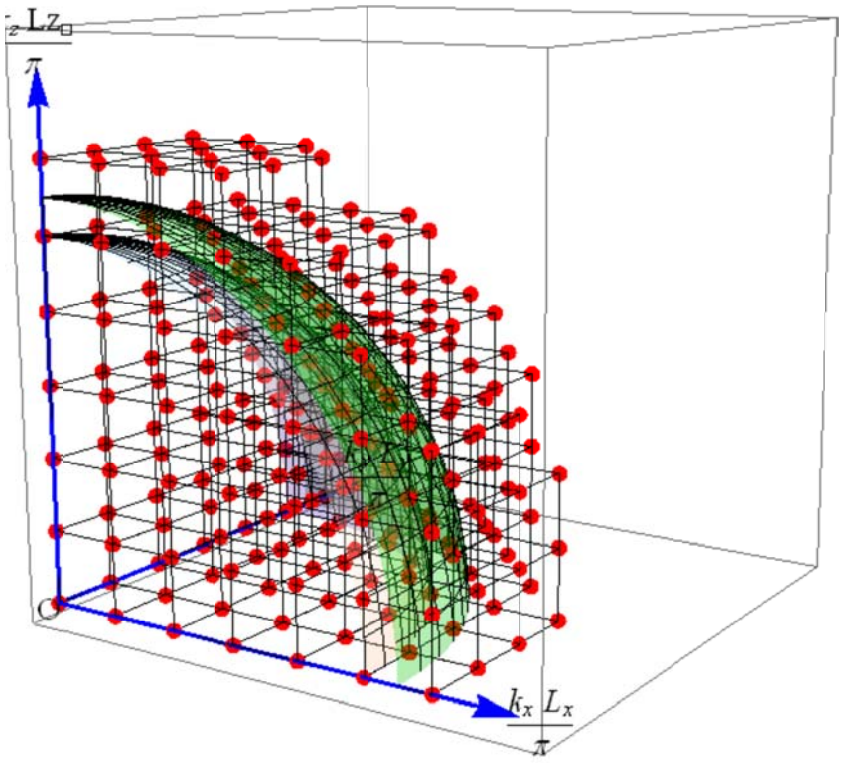
or

$$\frac{\omega^2}{c^2} = k_x^{\,2} + k_y^{\,2} + k_z^{\,2}$$

The density of states (k to k +dk)

$$\rho_k dk = \frac{1}{8}\frac{4\pi k^2 dk}{\left(\frac{\pi}{L}\right)^3} 2 = \frac{Vk^2 dk}{\pi^2}$$

where $V = L^3$, and the factor 2 is the number of independent polarization vector for each mode with $\boldsymbol{k}$.



Since $\omega = ck$,

$$\rho_\omega d\omega = \frac{V\left(\frac{\omega}{c}\right)^2 d\frac{\omega}{c}}{\pi^2} = \frac{V\omega^2 d\omega}{\pi^2 c^3}$$

of modes having their frequencies between ω and ω+dω.

$$\rho_\omega = \frac{V\omega^2}{\pi^2 c^3} = VD(\omega) \quad \text{(density of modes)}$$

where c is the velocity of light and

$$D(\omega) = \frac{\omega^2}{\pi^2 c^3}$$

We have the following formula;

$$\sum_k \rightarrow \int \rho_k dk$$

or

$$\sum_k \rightarrow \int \rho_\omega d\omega = V \int D(\omega) d\omega$$

For single mode $|\mathbf{k}\rangle$, the energy is given by

$$E_{n.\mathbf{k}} = (n_k + \frac{1}{2})\hbar\omega_{\mathbf{k}} .$$

The zero-point energy is given by

$$\varepsilon_0 = \sum_{\boldsymbol{k}} \frac{1}{2}\hbar\omega_{\mathbf{k}} = \int \frac{V\omega^2}{\pi^2 c^3} d\omega \frac{\hbar\omega}{2} = \frac{\hbar V}{2\pi^2 c^3} \int_0^\infty \omega^3 F(\omega) d\omega$$

where we use

$$F(\omega) = e^{-\omega/\omega_c}$$

in order to avoid the divergence of the integral. Then we have

$$\frac{\varepsilon_0}{c\hbar V} = \frac{1}{2\pi^2 c^4}\int_0^\infty \omega^3 e^{-\omega/\omega_c} d\omega$$
$$= \frac{1}{2\pi^2 c^4} 6\omega_c{}^4$$
$$= \frac{3}{\pi^2 c^4}\omega_c{}^4$$
$$= \frac{3}{\pi^2}\kappa_c{}^4$$

or

$$\varepsilon_0 = \frac{3c\hbar V}{\pi^2}\kappa_c{}^4.$$

Suppose that $V = aL^2$, then we get

$$\varepsilon_0 = \frac{3c\hbar aL^2}{\pi^2}\kappa_c{}^4, \qquad \text{or} \qquad \frac{\varepsilon_0}{c\hbar L^2} = \frac{3a}{\pi^2}\kappa_c{}^4$$

6. Intuitive method for the derivation of Casimir effect.

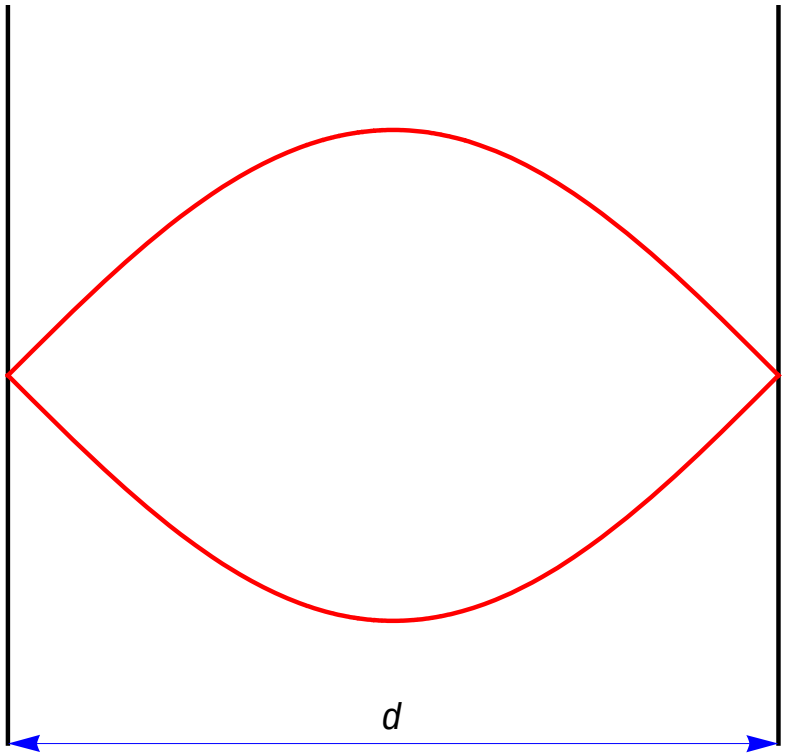


Fig. Two conducting plates with the separation distance *d*. Each mode has a wavelength satisfying the condition $d > \frac{\lambda}{2}$.

We consider a pair of conducting plate separated by a distance *d*. This separation distance should be larger than a half of the wavelength for each mode;

$$d > \frac{\lambda}{2}.$$

Since $\omega = ck = \frac{2\pi c}{\lambda} > \frac{c\pi}{d}$. The zero-point energy for the 3D system (with volume *Ad*) may be written as

$$\begin{aligned}\varepsilon &= Ad \int_{\pi c\beta/d}^{\infty} \frac{\hbar}{2}\omega D(\omega)d\omega \\ &= Ad \int_{\pi c\beta/d}^{\infty} \frac{\hbar\omega^3}{2\pi^2c^3}d\omega \\ &= Ad\int_0^{\infty} \frac{\hbar\omega^3}{2\pi^2c^3}d\omega - Ad \int_0^{\pi c\beta/d} \frac{\hbar\omega^3}{2\pi^2c^3}d\omega\end{aligned}$$

where β is a constant and is nearly equal to 1/3, and

$$D(\omega) = \frac{\omega^2}{\pi^2c^3}$$

We neglect the first term (volume energy). Then we have

$$\Delta\varepsilon = -Ad \int_0^{\pi c\beta/d} \frac{\hbar\omega^3}{2\pi^2c^3}d\omega = -Ad\frac{\hbar}{8\pi^2c^3}\left(\frac{\pi c\beta}{d}\right)^4$$

$$\frac{\Delta\varepsilon}{A} == -d\frac{\hbar}{8\pi^2c^3}\frac{\pi^4c^4\beta^4}{d^4} = -\frac{1}{8}\frac{\pi^2\hbar c\beta^4}{d^3}$$

The pressure *P* is given by

$$P = -\frac{\partial(\Delta\varepsilon / A)}{\partial d} = -\frac{3\beta^4}{8}\frac{\pi^2\hbar c}{d^4}$$

which should be equal to

$$P = -\frac{\hbar c\pi^2}{240a^4}.$$

Then β may be equal to

$$\beta = \left(\frac{1}{90}\right)^{1/4} = 0.324668 \approx \frac{1}{3}.$$

7. Experiment with atomic force microscopy (AFM)

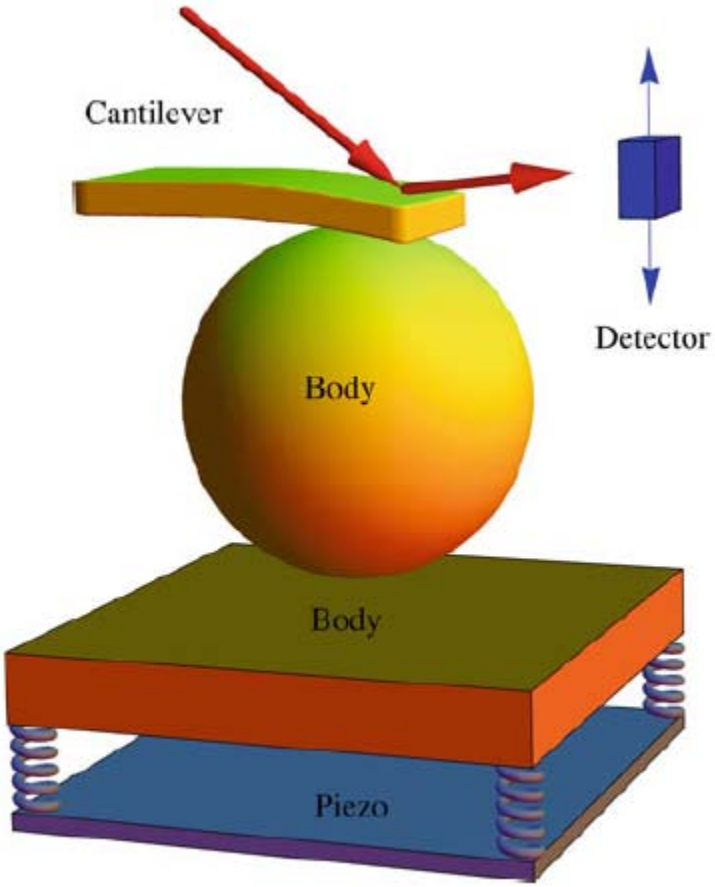
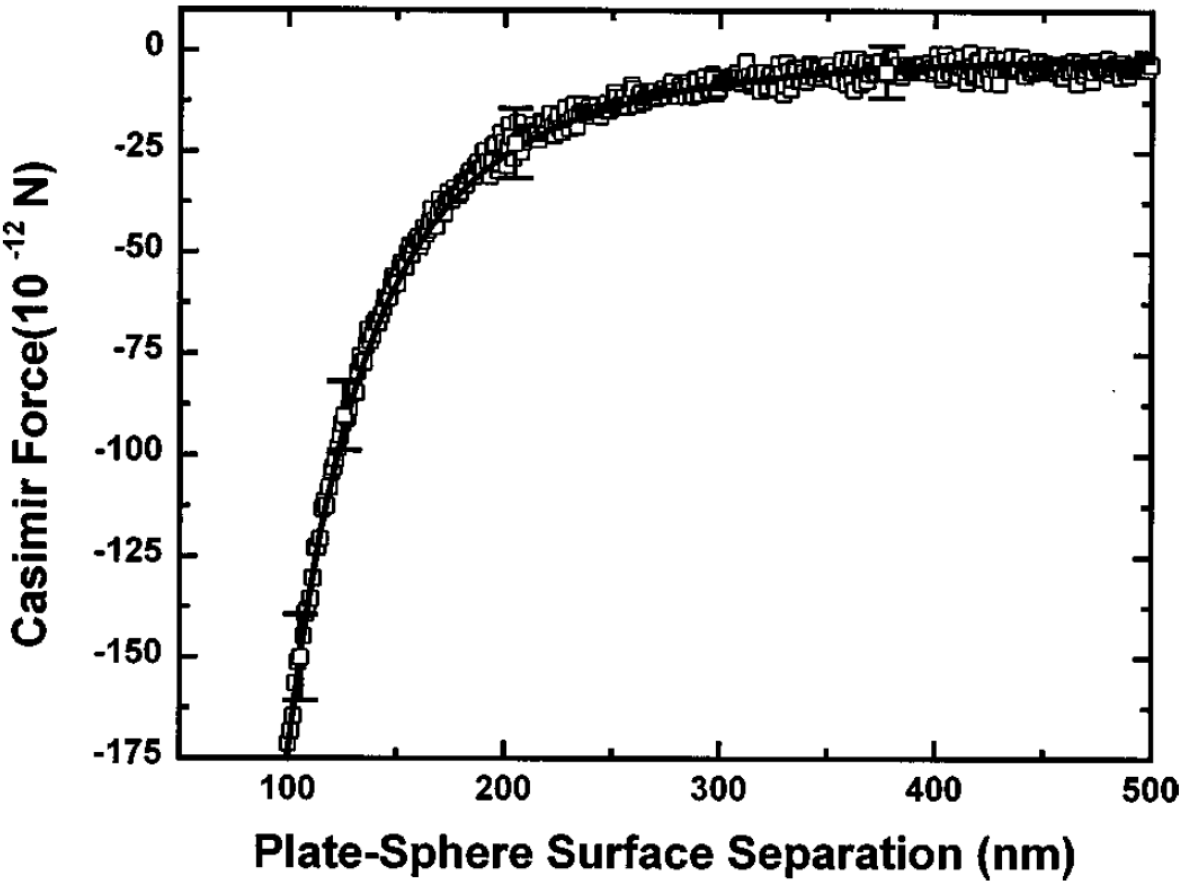


Fig. The measurement of Casimir effect using atomic force microscopy (AFM) [Simpson, 2015].

Including the Lamoreaux's ground-breaking experiment, a number of Casimir force measurements were made using variations on the atomic force microscope apparatus. A metal plate mounted on a piezoelectric translator interacts with a small metal sphere attached to a sensitive cantilever. As the two bodies are brought into proximity, the bending of the cantilever is detected by a laser beam reflected off the back of the cantilever, and observed as a change in the signal of a detector monitoring the difference in light intensity between the top and bottoms halves of the detector..



The data were obtained by Roy et al. using the AFM. The Casimir force between the sphere and the plate is measured as a function of the separation distance.

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