

**Deuteron**  
**Masatsugu Sei Suzuki**  
**Department of Physics, SUNY at Binghamton**  
**(Date: April 06, 2017)**

The deuteron was studied extensively a long time ago. Nevertheless, it continues to provide a very instructive illustration of how scattering and bound state data can be used to gain information about an unknown interaction. The basic facts about the deuteron ( $d$ ) are that it is the only bound state of the  $np$  system. It has binding energy 2.23 MeV, spin 1, mean square charge radius  $r = 2$  fm, magnetic moment  $0.857436 \mu_N$  (where  $\mu_N$  is the nucleon Bohr magneton), and quadrupole moment  $0.29 \text{ fm}^2 e$ .

This information is very revealing. The characteristic binding energy of nuclei is about 8 MeV per nucleon, so  $d$  is loosely bound by nuclear standards. As there is no  $J = 0$  bound state, the interaction must be spin-dependent. The sum of the neutron and proton magnetic moments is  $0.857436 \mu_N$ , which leaves but little room for any contribution to the magnetic moment from orbital motion of the proton. And the nonzero quadrupole moment is especially intriguing. (Gottfried and Yan, Quantum Mechanics)

Deuteron (nuclei of deuterium) consisting of one neutron and one proton

Spin angular momentum  $S = 1$  (in the units of  $\hbar$ )

Orbital angular momentum  $L = 0$

The total angular momentum:  $J = 1$ ,

The intrinsic parity of the deuteron is  $+1$ .

Isospin  $T = 0$ : the state is given by  $|0,0\rangle$

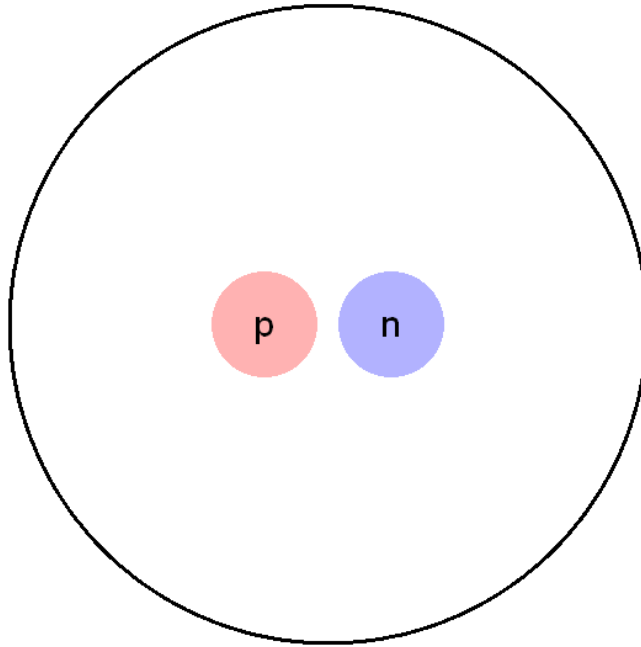
Magnetic moment:  $\mu_{\text{deuteron}} = 0.857436 \mu_N$

The nuclear binding energy:  $B_d = 2.224 \text{ MeV}$ .

## 1. Deuteron and deuterium

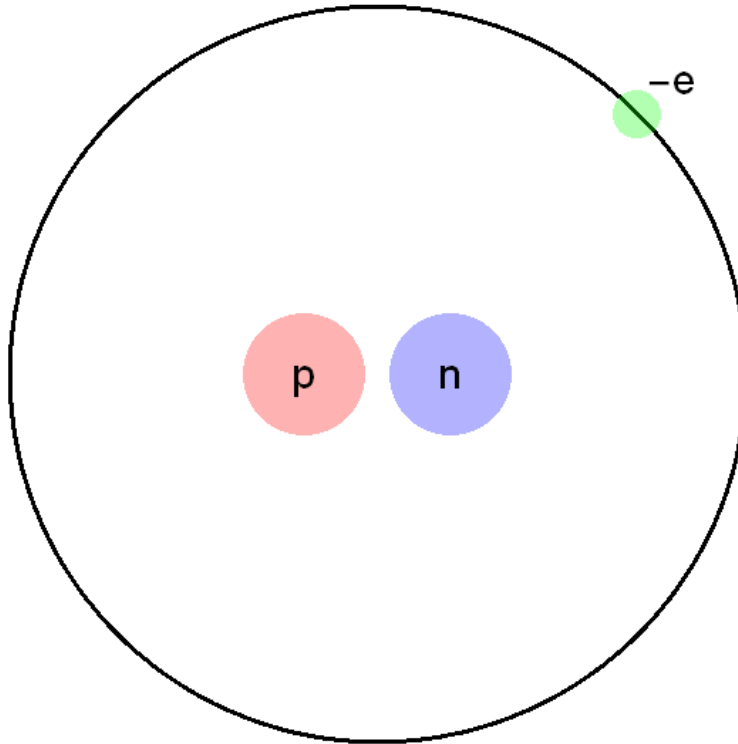
### Deuteron:

Nucleus of deuterium (heavy hydrogen) that consists of one proton and one neutron (the notation:  $d$  or  $D$ ). Deuterons are formed chiefly by ionizing deuterium (stripping the single electron away from the atom) and are used as projectiles to produce nuclear reactions after accumulating high energies in particle accelerators. A deuteron also results from the capture of a slow neutron by a proton, accompanied by the emission of a gamma photon.



**Fig.** Deuteron consisting of one proton and one neutron.

**Deuterium (D, heavy hydrogen,  $^2\text{H}$ ):** A heavy isotope of hydrogen that consists of one proton, one neutron, and one electron; The nuclei of deuterium, called *deuterons* (d), consist of one proton (p) and one neutron (n).



**Fig.** Deuterium (heavy hydrogen, D), consisting of one proton, one neutron, and one electron.

**((Note)) Deuteron and Big Bang model**

The stability of the deuteron is an important part of the story of the universe. In the Big Bang model it is presumed that in early stages there were equal numbers of neutrons and protons since the available energies were much higher than the 0.78 MeV required to convert a proton and electron to a neutron. When the temperature dropped to the point where neutrons could no longer be produced from protons, the decay of free neutrons began to diminish their population. Those which combined with protons to form deuterons were protected from further decay. This is fortunate for us because if all the neutrons had decayed, there would be no universe as we know it, and we wouldn't be here!

**2. Isotopic spin**

The charge independence of nuclear forces allows us to consider the proton and the neutron as different states of the same particles, or nucleon. Thus, in addition to position and spin variables, the state of nucleon requires for its specification a “charge” variable which can take two values. Therefore in the study of nucleon systems a formalism similar to the one developed for the eigenstates of spin 1/2 can be used. It is called the isotopic spin formalism.

The neutron and proton are very similar apart from a small mass difference (0.1%) and of course the difference in electric charge. Both play an equally important role in determining the properties of nuclei. Heisenberg (1932) postulates that  $n, p$  are two sub-states of a nucleon with iso-spin 1/2 by analogy with ordinary spin  $s = 1/2$ .

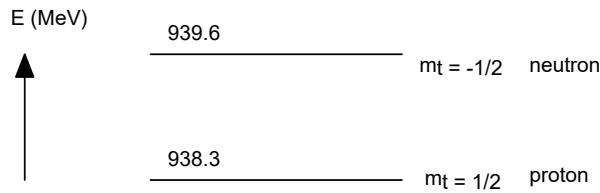
Briefly, the base vectors of a 2D isotopic spin space can be defined to be

$$|p\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |n\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and the operator representing isotopic spin and nucleon charge are then

$$\hat{T} = \frac{1}{2} \hat{\tau}, \quad \hat{Q} = \frac{1}{2}(1 + \hat{\tau}_3)$$

where the operator of  $\hat{\tau} = (\hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_3)$  on the base vector  $s$  can be represented by Pauli matrices.



Both neutrons and protons have spin  $s = 1/2$ .  $s$  and  $T$  are independent quantum number.  $T$  has no classical analog. It is a quantum mechanical vector, literally “like spin” (iso=like). So it follows the same addition as  $s, l, j$ , etc.

### 3. ${}^3S_1$ state and ${}^3D_1$ state in deuteron

Deuteron consists of one proton and one neutron where proton (fermion) has spin 1/2 and neutron (fermion) has spin 1/2. The intrinsic parity of deuteron is +1, since the intrinsic parity of proton and neutron is +1. For  $d \rightarrow n + p$ , the intrinsic parity of the deuteron is given by

$$\eta_d = \eta_p \eta_n (-1)^l = (+1)(+1)(-1)^l = +1$$

where  $l$  is the relative orbital angular momentum for the identical particles proton and neutron, the intrinsic parity of deuteron is  $\eta_d = +1$ , while the intrinsic parity of proton and neutron is  $\eta_p = \eta_n = +1$ . Thus we have

$$l = 0 \text{ (s), } 2 \text{ (d), } 4, 6, \dots$$

The total angular momentum of deuteron is given by

$$\mathbf{j} = \mathbf{l} + \mathbf{s}$$

where  $l$  is the relative angular momentum of the neutron and proton.  $s$  is the spin angular momentum. For the resultant spin,

$$D_{1/2} \times D_{1/2} = D_1 + D_0 \quad s = 1 \text{ (symm) or } s = 0 \text{ (antisymm)}$$

Of the possible quantum numbers,  $l = 0$  has the lowest energy. This value of orbital angular momentum  $l = 0$  combines with these spin values.

$$s = 1 \text{ (symm), } \quad l = 0 \text{ (symm):} \quad j = 1 \quad {}^3S_1$$

$$s = 0 \text{ (antisymm), } \quad l = 0 \text{ (symm):} \quad j = 0 \quad {}^1S_0$$

as well as

$$s = 1 \text{ (symm), } \quad l = 2 \text{ (symm):} \quad j = 3, 2, 1 \quad {}^3D_{1,2,3}$$

$$s = 0 \text{ (antisymm), } \quad l = 2 \text{ (symm):} \quad j = 2 \quad {}^1D_2$$

Note that isospin of the deuteron is  $T = 0$  (antisymmetric) or  $T = 1$  (symmetric).

$$D_{1/2} \times D_{1/2} = D_1 + D_0.$$

The total wave function for two identical fermions has to be antisymmetric with respect to particle exchange

$$|\psi\rangle = |\psi_{space}\rangle |\psi_{spin}\rangle |\chi_{isospin}\rangle$$

$$\underline{s = 1 \text{ (symm), } \quad l = 0 \text{ (symm):} \quad T = 0 \text{ (antisymm)} \quad \underline{{}^3S_1}$$

$$\underline{s = 1 \text{ (symm), } \quad l = 2 \text{ (symm):} \quad T = 0 \text{ (antisymm)} \quad \underline{{}^3D_1}$$

Thus the total wavefunction is antisymmetric under the exchange of neutron and proton. The isospin of deuteron is given by  $|0,0\rangle$ . Note that  $J = 1$ .

#### 4. Magnetic moment of deuteron

In fact the ground state is a mixture of  $\sqrt{0.96} \ ^3S_1$  and  $\sqrt{0.04} \ ^3D_1$ .

or

$$|\psi_G\rangle = a_S |^3S_1\rangle + a_D |^3D_1\rangle$$

with  $a_S = \sqrt{0.96} = 0.98$  and  $a_D = \sqrt{0.04} = 0.2$ .

As  $a_D \ll a_S$ , for the purpose of calculating the magnetic moment of the deuteron, it suffices to take  $|\psi_G\rangle$  to be entirely an  $S$  state ( $l = 0$ ).

The necessity of having both an  $S$  and  $D$  component in  $|\psi_G\rangle$  means that the  $np$  interaction does not conserve the orbital angular momentum  $L$ , only the total angular momentum  $J$ . But the small value of the quadrupole moment implies that to a first approximation the violation of  $L$  can be ignored, and for that reason we first ask what can be learned about the nuclear interaction from  $np$  scattering on the assumption that  $L$  is a constant of motion.

((Note))

$$\mu_{\text{proton}} = 2.79 \mu_N \quad (\text{I. Rabi})$$

$$\mu_{\text{deuteron}} = 0.8573 \mu_N$$

$$\mu_{\text{deuteron}} = \mu_{\text{neutron}} + \mu_{\text{proton}}$$

or

$$0.8573 \mu_N = 2.79 \mu_N + \mu_{\text{neutron}}$$

Then we have

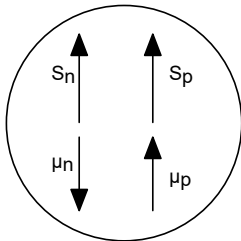
$$\mu_{\text{neutron}} = -1.93 \mu_N$$

The negative sign shows that the direction of the magnetic moment is opposite to that of spin. We note that the magnetic moments of neutron and proton are

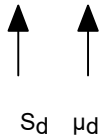
$$\mu_{neutron} = -1.9130427245 \mu_N,$$

$$\mu_{proton} = 2.7928473508 \mu_N.$$

Neutron and proton



Deuteron



## 5. Binding energy of deuteron

- ${}^1\text{H}$  Hydrogen
- ${}^2\text{H}$  Deuterium
- d: Deuteron
- p: proton
- n: neutron

The binding energy of deuteron,  $B_d$

$$M({}^2\text{H}) = m_n + M({}^1\text{H}) - \frac{B_d}{c^2}$$

where

$$m_n = 1.008665u \quad \text{neutron mass}$$

$$M(^1H) = 1.007825u \quad \text{atomic hydrogen mass}$$

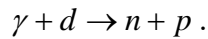
$$M(^2H) = 2.014102u \quad \text{atomic deuterium mass}$$

The nuclear binding energy is obtained as

$$B_d = 2.224 \text{ MeV.}$$

We find that the atomic electron binding energy of 13.6 eV is much smaller than the nuclear binding energy of 2.2 MeV.

This binding energy can be determined from the photodisintegration or a photonuclear reaction. We scatter gamma rays (photons) from deuterium gas and look for the breakup of a deuteron into a neutron and a proton;



The minimum energy required for the photodisintegration is given by

$$hf_{\min} = B_d \left[ 1 + \frac{B_d}{2M(^2H)c^2} \right] \approx B_d .$$

Experiment shows that a photon of energy less than 2.22 MeV cannot dissociate a deuteron. This energy is necessary to separate the two nucleons completely. No excited states of the deuteron have been found, and it is surmised that the deuteron has **only one bound state**.

## 6. Scattering cross section for neutron-proton

The total cross section is given by

$$\begin{aligned} \sigma_{\text{tot}} &= \frac{4\pi}{k^2} \sin^2 \delta_0 \\ &= \frac{4\pi}{k^2} \frac{1}{1 + \cot^2 \delta_0} \\ &= \frac{4\pi}{k^2 + \alpha^2} \\ &= \frac{2\pi\hbar^2}{m} \frac{1}{E + \frac{\hbar^2\alpha^2}{2m}} \end{aligned}$$



When  $E = 0$ , the total cross section is given by

$$\sigma_{tot} = \frac{4\pi}{\alpha^2} = \frac{2\pi\hbar^2}{m} \frac{1}{\frac{\hbar^2\alpha^2}{2m}} = \frac{2\pi\hbar^2}{m} \frac{1}{E_B}.$$

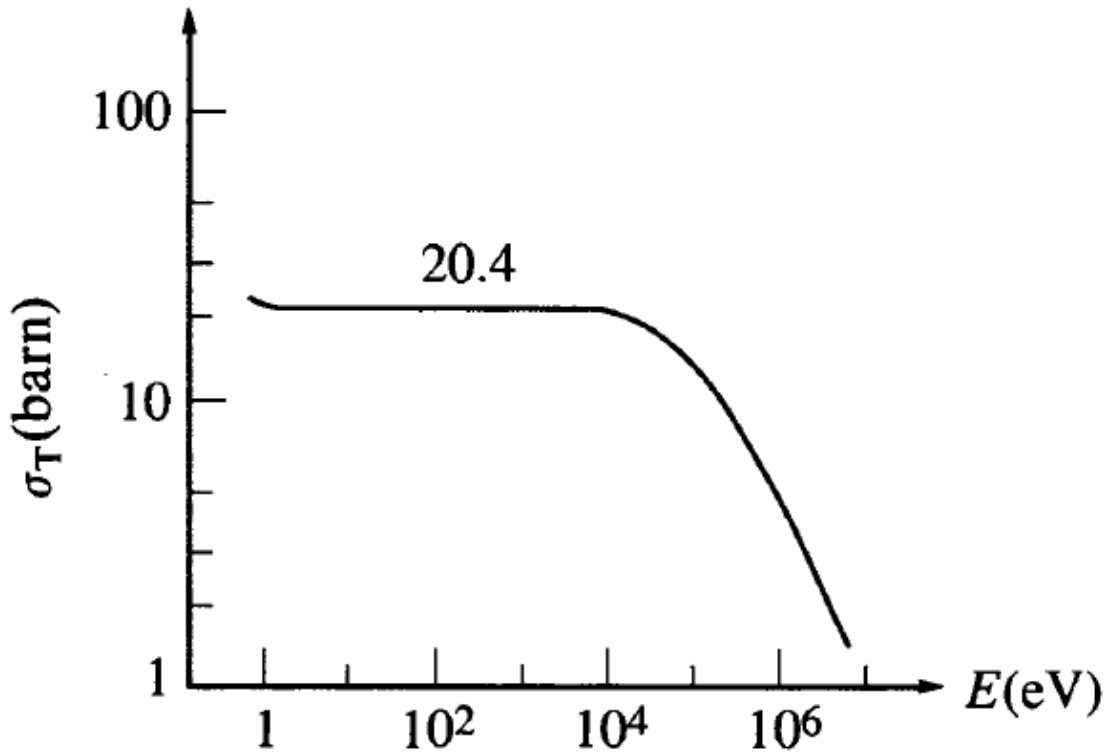
with  $m = M_n / 2$ .

$$\frac{2\pi\hbar^2}{m} = 5.20782 \text{ barns MeV}.$$

The binding energy of deuteron is  $E_B = 2.23$  MeV. Then  $\sigma_{tot}$  is obtained as

$$\sigma_{tot} = \frac{5.20782}{2.23} \text{ barns} = 2.3 \text{ barns}$$

The measurement, carried out with neutrons at thermal speeds, yield **21 barns**. The experimental result (below) is found in the book of Enrico Fermi (Nuclear Physics, Chicago).



**Fig.** Experimental neutron-proton total scattering cross section at very low energies of 1 eV to 10 keV is constant; however, it is much greater than the theoretical prediction. The answer is spin dependence of scattering. (Gasiorowicz, A. Goswami, Quantum Mechanics, 2003).

The explanation of this disagreement came with the realization that the spin of the neutron and the proton had not been taken into account. The observed neutron-proton scattering cross section at very low energy is about 20.3 barns. It seems that no manipulation of the triplet potential consistent with other data can possibly account for the observed scattering. It was first pointed out by Wigner that the large difference between the triplet scattering and the observed scattering offers a sensitive source of information concerning the singlet neutron-proton interaction. The total cross section is composite of the triplet cross section  $\sigma_3$  and the singlet cross section  $\sigma_1$ , namely

Spin dependent scattering

$$\sigma_{tot} = \frac{3}{4}\sigma_3 + \frac{1}{4}\sigma_1.$$

The statistical weights of the triplet and singlet states are 3 and 1, respectively, for random orientations. We note that the scattering lengths are given by

$$a_1 = -23.7 \text{ fm} \quad (\text{singlet})$$

$$a_3 = 5.38 \text{ fm} \quad (\text{triplet})$$

and the scattering cross sections as

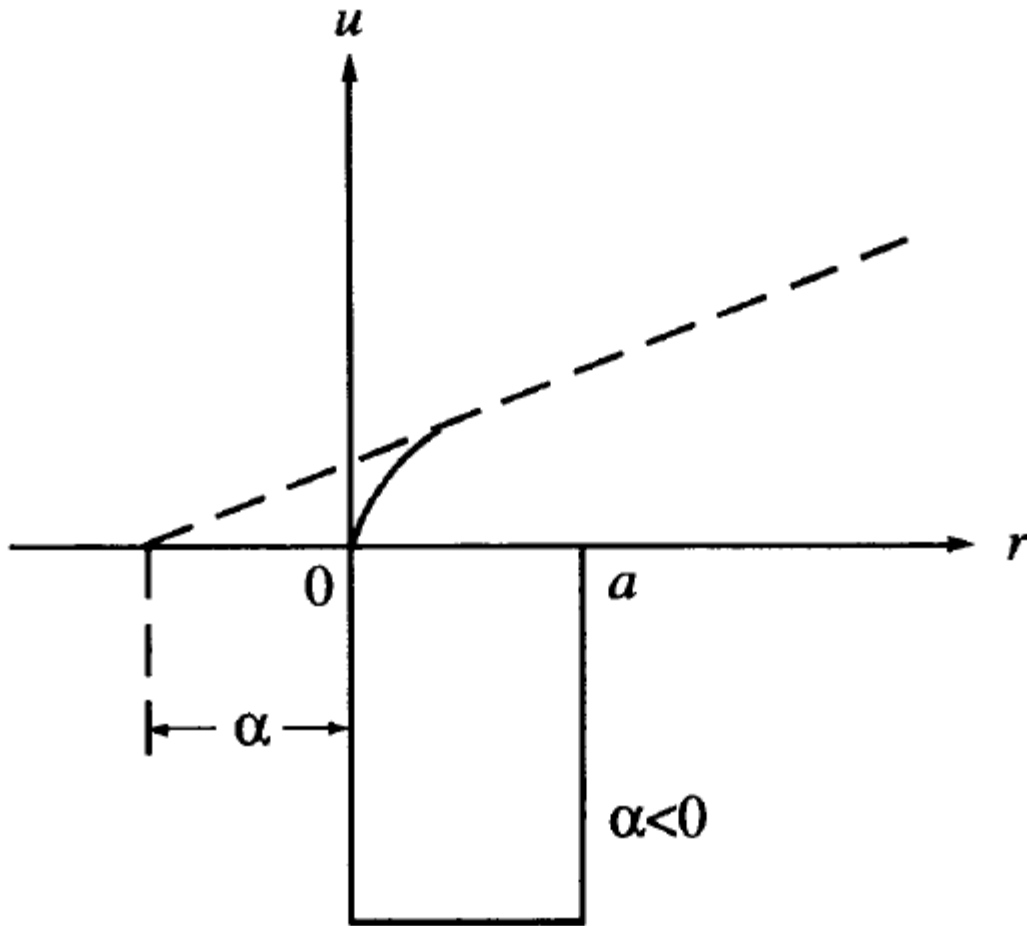
$$\sigma_1 = \sigma_s = 4\pi a_1^2 = 70.58 \text{ barns}, \quad \sigma_3 = \sigma_{tot} = 4\pi a_3^2 = 3.637 \text{ barns},$$

where

$$1 \text{ b} = 1 \text{ barn} = 10^{-24} \text{ cm}^2.$$

Thus we have

$$\sigma_{tot} = \frac{3}{4} 3.637 + \frac{1}{4} 70.58 = 2.72 + 17.65 = 20.37 \text{ barns}$$



**Fig.** The sign of  $a_1$  is found to be opposite to that of  $a_3$ . For a slow neutrons a negative  $a_1$  gives a wave function as shown in this figure.

### 6. Symmetric states of the nuclear motion of $D_2$

D has spin 1 (boson)

$s = 2$  and  $s = 0$  (symmetric)

$$\boxed{\quad} \boxed{\quad} : \boxed{1\ 1}, \boxed{1\ 2}, \boxed{1\ 3}, \boxed{2\ 2}, \boxed{2\ 3}, \boxed{3\ 3},$$

6 states ( $j = 2$  and  $0$ )

$$m=2 \quad m=1 \quad m=0 \quad m=0 \quad m=-1 \quad m=-2$$

$s = 1$  (anti-symmetric)

$$\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} : \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array},$$

$m=1 \quad m=0 \quad m=-1$

3 states ( $j=1$ )

The relative angular momentum

$$\hat{\pi}|l, m\rangle = \hat{P}|l, m\rangle = (-1)^l |l, m\rangle$$

$|l, m\rangle$ : symmetric for even  $l$  and anti-symmetric for odd  $l$ .

D is boson: the wave function should be symmetric.

$$|l_{\text{even}}, m_l\rangle |s = 2, m_s\rangle : \quad 5 \text{ states}$$

or

$$|l_{\text{even}}, m_l\rangle |s = 0, m_s = 0\rangle : \quad 1 \text{ state (ortho state)}$$

$$|l_{\text{odd}}, m_l\rangle |s = 1, m_s\rangle : \quad 3 \text{ states (para state)}$$

## 7. The Schrödinger equation for Deuteron

A deuteron ( $^2\text{H}$  nucleus) consists of neutron and a proton. It is the simplest bound state of nucleons and therefore gives us an ideal system of studying the nucleon-nucleon interaction. The binding energy of the deuteron is

$$2.23 \pm 0.03 \text{ MeV.}$$

What is the nuclear force holding neutron and proton together? This force cannot be electrical as the neutron is uncharged. So we must accept the nuclear force as a new type of force and try to find out more about it. We assume a central force. The interaction potential of neutron and proton is given by a function  $V(r)$ , where  $r$  is the distance between the particles. Second, the nuclear force has a short range.

$m$  is the reduced mass of proton and neutron.

$$m = \frac{M_n + M_p}{M_n + M_p} = \frac{M}{2}$$

The Schrödinger equation for the wave function  $\psi = RY_l^m$ :

$$-\frac{\hbar^2}{2m} \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} (rR) + \frac{\hbar^2 l(l+1)}{2mr^2} R + V(r)R = ER$$

Suppose that

$$u = rR$$

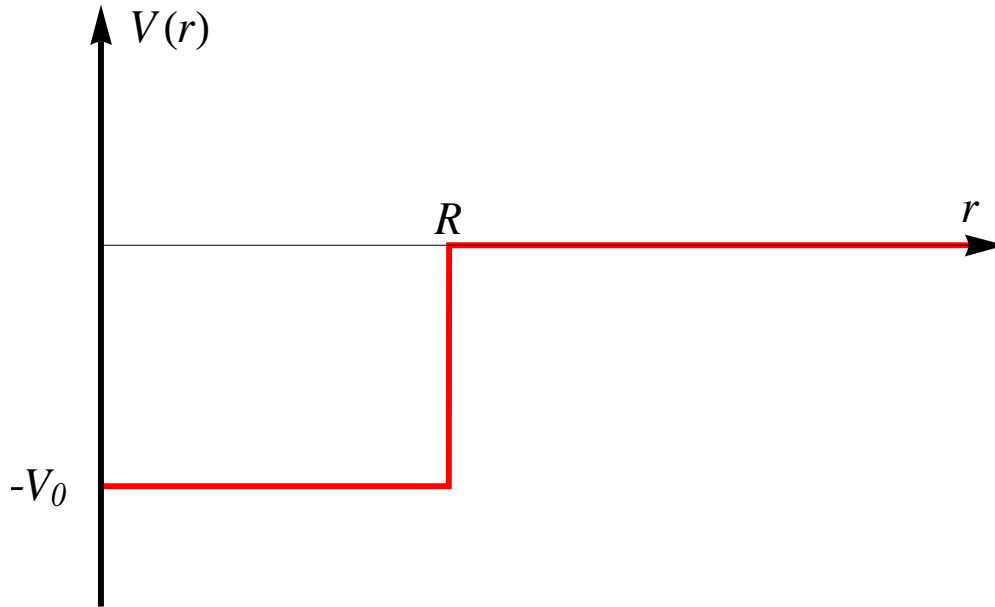
$$-\frac{\hbar^2}{2m} \frac{\partial^2 u}{\partial r^2} + \left[ \frac{\hbar^2 l(l+1)}{2mr^2} + V(r) \right] u = Eu$$

When  $l = 0$  (S-wave), we have

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + V(r)u = Eu$$

with

$$E = -\frac{\hbar^2 \alpha^2}{2m} \quad (\text{bound state})$$



**Fig.** Attractive square-well potential well for deuteron.  $V_0 = 35$  MeV. The binding energy is  $-2.2$  MeV.

We assume a square-well potential as  $V(r)$ . There are two parameters, width  $R$  and the depth of the well ( $-V_0$ ).

$$V(r) = \begin{cases} -V_0 & r < R \\ 0 & r > R \end{cases}$$

(i) Schrödinger equation for  $r < R$  ( $l = 0$ )

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} - V_0 u = E u = -\frac{\hbar^2 \alpha^2}{2m} u$$

or

$$\frac{d^2 u}{dr^2} + \frac{2mV_0}{\hbar^2} u = \alpha^2 u$$

or

$$u'' + (-\alpha^2 + k_0^2)u = 0$$

or

$$u'' + \kappa^2 u = 0$$

where

$$\frac{2mV_0}{\hbar^2} = U_0 = k_0^2,$$

$$\kappa^2 = -\alpha^2 + k_0^2$$

Then we get the solution as

$$u = B \sin(\kappa r)$$

with the boundary condition,  $u(r) = 0$ .

This means that  $u$  has the first peak at

$$r = r_{\max} = \frac{\pi}{2\kappa}.$$

The condition for the existence of the bound state is that

$$r_{\max} < R$$

or

$$\kappa R > \frac{\pi}{2}.$$

(ii) Schrödinger equation for  $r > R$

$$u'' - \alpha^2 u = 0,$$

or

$$u = Ae^{-\alpha r}$$



The boundary condition at  $r = R$ , leads to

$$B \sin(\kappa R) = A e^{-\alpha R}$$

$$B \kappa \cos(\kappa R) = -A \alpha e^{-\alpha R}$$

Then we have

$$\kappa R \cot(\kappa R) = -\alpha R$$

with

$$\kappa^2 + \alpha^2 = k_0^2$$

We determine the constants  $A$  and  $B$ .

$$A = \frac{e^{\alpha R} \sqrt{\kappa R \alpha R} \sin(\kappa R)}{\sqrt{\pi R} \sqrt{2\kappa R \alpha R + 2\kappa R \sin^2(\kappa R) - 2\alpha R \sin(2\kappa R)}}$$

$$B = \frac{\sqrt{\kappa R \alpha R}}{\sqrt{\pi R} \sqrt{2\kappa R \alpha R + 2\kappa R \sin^2(\kappa R) - 2\alpha R \sin(2\kappa R)}}$$

from the normalization of the wave function such that

$$4\pi \int R^2 r^2 dr = 4\pi \int u^2 dr = 1.$$

In the limit of  $\alpha R \rightarrow 0$ ,

$$A = \sqrt{\frac{\alpha}{2\pi}}, \quad B = \frac{\sqrt{\frac{\alpha}{2\pi}}}{|\sin(\kappa R)|}.$$

## 8. Numerical solution for the Schrödinger equation ( $l = 0$ )

We solve the problem using the Mathematica.

$$x = \kappa R, \quad y = \alpha R$$

In this case,

$$y = -x \cot x, \quad x^2 + y^2 = k_0^2 R^2 = a_0^2 \quad (\text{circle})$$

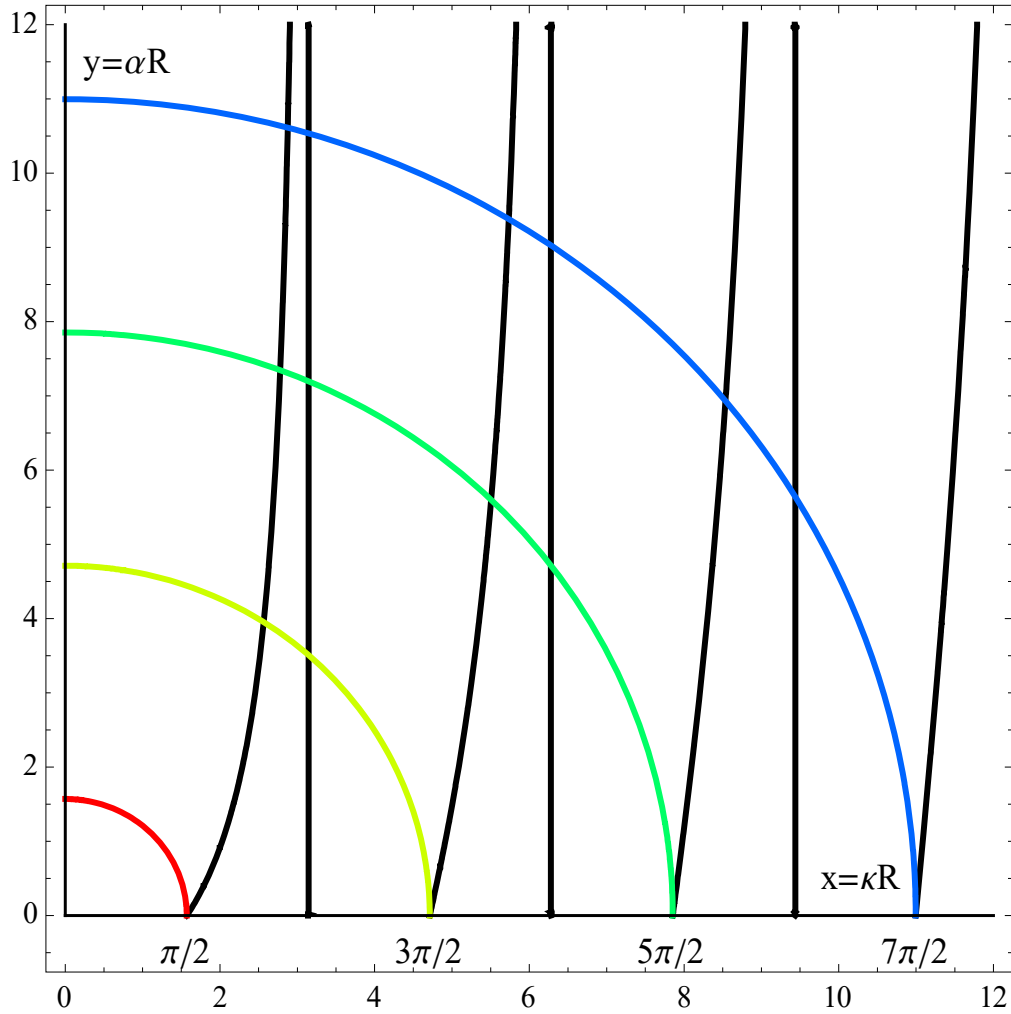
where

$$a_0 = k_0 R, \quad U_0 = \left( \frac{\pi}{2R} \right)^2,$$

$$\frac{2mV_0}{\hbar^2} = U_0 = k_0^2$$

We make a plot of these two curves, where  $a_0$  is varied as a parameter. The intersection of two curves is the solution of the values of  $x$  and  $y$  for the fixed value of  $a_0$ . The results are as follows.

- (i)  $\frac{\pi}{2} < a_0 < \frac{3\pi}{2}$       one bound state.
- (ii)  $\frac{3\pi}{2} < a_0 < \frac{5\pi}{2}$       two bound states.
- (iii)  $\frac{5\pi}{2} < a_0 < \frac{7\pi}{2}$       three bound states.



**Fig.** Plot of  $y = -x \cot x$  (red line) and  $x^2 + y^2 = a_0^2 = (k_0 R)^2$ .  $a_0 = \pi/2$ ,  $a_0 = 3\pi/2$  (light green line),  $a_0 = 5\pi/2$  (green line), and  $a_0 = 7\pi/2$  (blue line),  $x = \kappa R$ .  $y = \alpha R$ . One bound state for  $a_0 = \pi/2$ . Two bound states for  $a_0 = 3\pi/2$ . Three bound states for  $a_0 = 5\pi/2$ . As  $a_0$  increases, the potential well becomes deep.

**((Note))**

The simplest way to determine the condition for the existence for the bound state is as follows.

We start with

$$y = -x \cot x.$$

When  $y = 0$ , we have

$$x \cot x = 0$$

or

$$x = \pi/2, 3\pi/2, 5\pi/2, \dots$$

### 9. The form of wave function for $r > R$ ( $l = 0$ )

For  $r > R$ , we have the wave function as

$$u = Ae^{-\alpha r}$$

with

$$A = \frac{e^{\alpha R} \sqrt{\kappa R \alpha R} \sin(\kappa R)}{\sqrt{\pi R} \sqrt{2\kappa R \alpha R + 2\kappa R \sin^2(\kappa R) - 2\alpha R \sin(2\kappa R)}}$$

This function is a typical exponential decay. The quantity  $1/\alpha$  can be taken as a measure of the size of the deuteron. It was shown above that the radius of the deuteron is considerably larger than the range of nuclear forces, i.e.,

$$\frac{1}{\alpha} \gg R$$

Thus most of the area under  $u(r)$  occurs for  $r > R$ .

### 10. Schrödinger equation for higher $l = 1$ .

Here we show that the deuteron has no bound excited states for states of higher  $l$ .

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + \left[ \frac{\hbar^2 l(l+1)}{2mr^2} + V(r) \right] u = Eu$$

where

$$E = -\frac{\hbar^2}{2m} \alpha^2, \quad U_0 = \frac{2m}{\hbar^2} V_0 = k_0^2$$

The effective potential is given by

$$V_{eff} = \frac{\hbar^2 l(l+1)}{2mr^2} + V(r)$$

For  $r < R$

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + \left[ \frac{\hbar^2 l(l+1)}{2mr^2} - V_0 \right] u = -\frac{\hbar^2}{2m} \alpha^2 u$$

or

$$\frac{\partial^2 u}{\partial r^2} + \left[ -\frac{l(l+1)}{r^2} + U_0 \right] u = \alpha^2 u$$

### 11. The case for $l = 1$

We consider the case for  $l = 1$  ( $P$ -wave). Using the Mathematica, the solution for  $u$  is obtained as

$$u = B \left[ \frac{\sin(\kappa r)}{\kappa r} - \cos(\kappa r) \right]$$

with  $u(0) = 0$  (boundary condition)

$$\kappa^2 = U_0 - \alpha^2 = k_0^2 - \alpha^2$$

For  $r > R$

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + \frac{\hbar^2 l(l+1)}{2mr^2} u = -\frac{\hbar^2}{2m} \alpha^2 u$$

or

$$\frac{d^2 u}{dr^2} - \frac{l(l+1)}{r^2} u = \alpha^2 u$$

The solution of  $u$  is

$$u = A e^{-\alpha r} \left( \frac{1}{\alpha r} + 1 \right)$$

The function  $u$  and its derivative should be continuous at  $r = R$

$$\frac{\kappa^2 R^2 \sin(\kappa R)}{\sin(\kappa R) - \kappa R \cos(\kappa R)} = -\frac{\alpha^2 R^2}{1 + R\alpha}$$

We put

$$x = \kappa R, \quad y = \alpha R$$

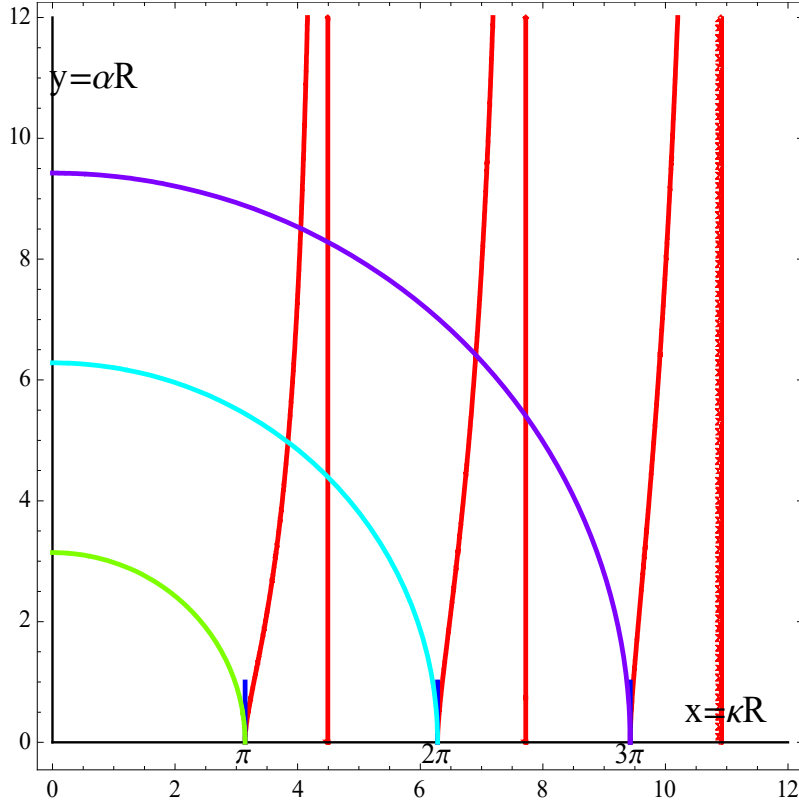
$$\frac{x^2 \sin(x)}{\sin(x) - x \cos(x)} = -\frac{y^2}{1 + y}$$

with

$$x^2 + y^2 = a_0^2 \quad (\text{circle})$$

We make a plot of these two curves, where  $a_0$  is varied as a parameter. The intersection of two curves is the solution of the values of  $x$  and  $y$  for the fixed value of  $a_0$ . The results are as follows.

- (i)  $\pi < a_0 < 2\pi$       one bound state.
- (ii)  $2\pi < a_0 < 3\pi$       two bound states.
- (iii)  $3\pi < a_0 < 4\pi$       three bound states.



From this figure, we have the bound state under the condition that  $x > \pi$ .

$$\frac{2m}{\hbar^2} V_0 R^2 = \frac{M_n}{\hbar^2} V_0 R^2 = a_0^2 = k_0^2 R^2 > \pi^2$$

**((Note-1))**

This required well depth  $V_0$  is almost four times as large as the actual well depth in the ground state

$$\frac{2m}{\hbar^2} V_0 R^2 = \frac{M_n}{\hbar^2} V_0 R^2 = a_0^2 = k_0^2 R^2 \approx \frac{\pi^2}{4}.$$

Note that  $a_0$  is slightly larger than  $\pi/2$  but definitely less than  $\pi$ .

**((Note-II))**

The simplest way to determine the condition for the existence for the bound state is as follows.

We start with

$$\frac{x^2 \sin(x)}{\sin(x) - x \cos(x)} = -\frac{y^2}{1+y}$$

When  $y = 0$ , we have

$$x \sin x = 0$$

or

$$x = \pi, 2\pi, 3\pi, \dots$$

## 12. The bound state of deuteron

There are three parameters to be determined,

- (i)  $V_0$  is the depth of the square-well potential.
- (ii)  $R$  is the width of the potential.
- (iii)  $E$  is the energy of bound state.

Here we assume that the values of  $E$  and  $R$  are given. We determine the value of  $V_0$ . We show that the depth of the potential as

$$V_0 = 33.73 \text{ MeV.}$$

which is close to the accepted value  $V_0 = 35 \text{ MeV}$ .

Suppose that the binding energy is  $E = -2.23 \text{ MeV}$  and the width of the potential is  $R = 2.1 \text{ fm}$ , where

$$\text{fm} = 10^{-13} \text{ cm (fm: Fermi).}$$

We use the following relations

$$y = -x \cot x, \quad x^2 + y^2 = k_0^2 R^2 = a_0^2$$

$$x = \kappa R, \quad y = \alpha R = \sqrt{\frac{2m|E|}{\hbar^2}} R = 0.231969 \quad R[\text{fm}]$$

where

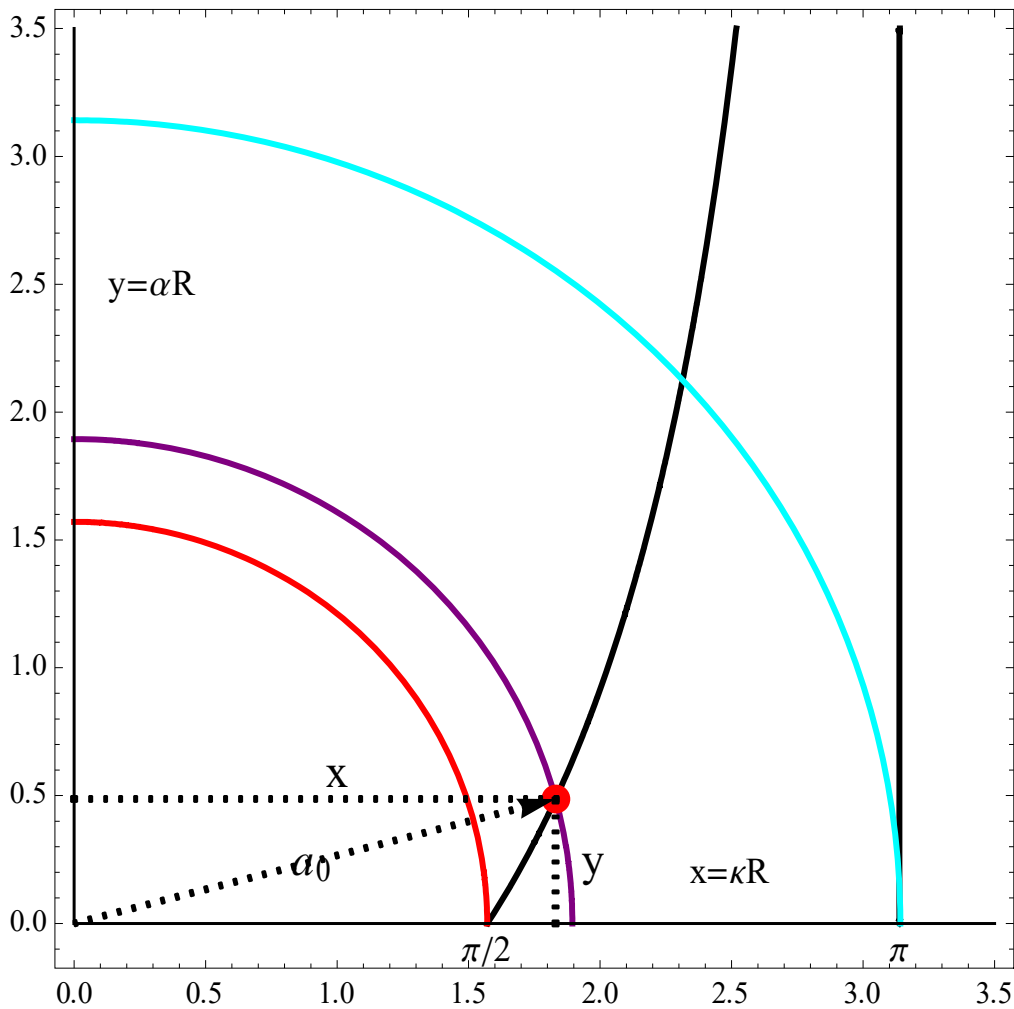


$$E = -\frac{\hbar^2 \alpha^2}{2m} = -2.24 \text{ MeV} \quad (\text{energy of bound state})$$

$$a_0^2 = k_0^2 R^2 = \frac{2m}{\hbar^2} V_0 R^2 = \frac{M_n}{\hbar^2} V_0 R^2$$

and  $m$  is the reduced mass,  $m = \frac{M_n}{2}$  ( $M_n$  is the mass of neutron).

((Mathematica))



**Fig.** Graphical solution ( $x = \kappa R$ ,  $y = \alpha R$ ) of the bound state of deuteron.  $|E| = 2.23$  MeV.  $R = 2.1$  fm. The value of  $V_0$  is obtained as 33.73 MeV. The red point

denote the solution of  $x$  and  $y$ . The black thick line is  $y = -x \cot x$ . The purple circle is denoted by  $x^2 + y^2 = a_0^2$  with  $a_0 = 1.89454$ .

For  $|E| = 2.23$  MeV.  $R = 2.1$  fm.

$$y = \alpha R = \sqrt{\frac{2m|E|}{\hbar^2}} R = 0.231969 R[\text{fm}] = 0.487134.$$

From the relations,  $y = -x \cot x$  and  $x^2 + y^2 = a_0^2$ , we get

$$x = 1.83084. a_0 = 1.89454,$$

Then we have

$$a_0 = \sqrt{\frac{2m}{\hbar^2} V_0 R^2} = 0.15534 \sqrt{V_0 (\text{MeV})} R (\text{fm}).$$

where  $m = M_n / 2$ . Since  $R = 2.1$  fm, we have

$$V_0 = 33.73 \text{ MeV}.$$

### 13. Wave function of the bound state for deuteron

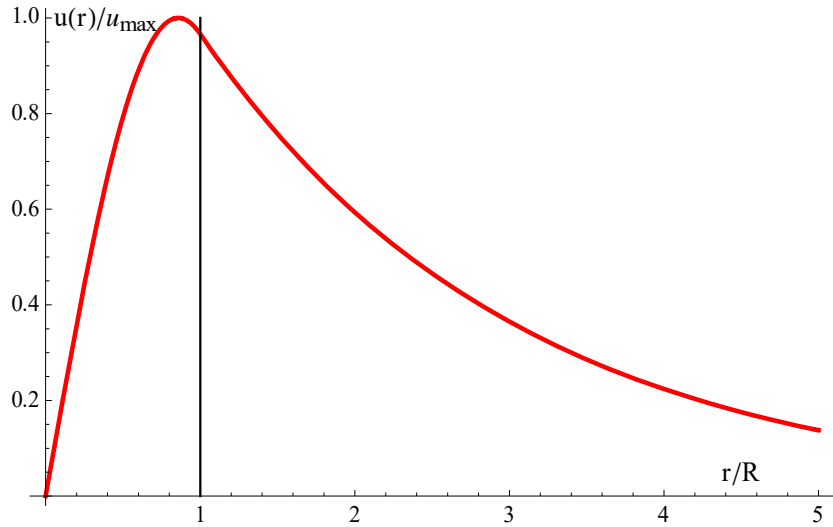
We make a plot of the wave function ( $u/u_{\text{max}}$ ) of the bound state for deuteron. Note that  $u$  is proportional to  $\sin(\kappa r)$  for  $r < R$ . The function  $u$  has a peak at

$$\frac{r}{R} = \frac{\pi/2}{\kappa R} = \frac{\pi/2}{x} = 0.858$$

where  $x = 1.83084$ . For  $r > R$ , the wave function is proportional to  $e^{-\alpha r}$ . The characteristic distance is

$$\frac{1}{\alpha} = \frac{R}{\alpha R} = \frac{R}{y} = 2.05R = 4.3 \text{ fm}.$$

where  $y = 0.487134$  and  $R = 2.1$  fm.



**Fig.** The plot of  $u(r)/u_{\max}$  as a function of  $r/R$ .  $R = 2.1$  fm.  $E = -2.23$  MeV

#### 14. Relation between $a_0$ and $R$

In order to discuss the relation between  $a_0$  and  $R$ , we use

$$a_0 = \sqrt{2.41298 \times 10^{-2} V_0 R^2} = 1.5534 \times 10^{-1} \sqrt{V_0 [\text{MeV}] R [\text{fm}]}$$

where  $V_0$  is in the unit of MeV and  $R$  is the unit of fm. We assume that  $V_0$  is given as  $V_0 = 35$  MeV. The plot of  $a_0$  vs  $R$  is shown below. When  $R_0 = 2.1$  fm,  $a_0$  is between  $\pi/2$  and  $\pi$ . This means that there is no bound state for  $l = 1, 2, \dots$ . There is only one bound state for  $l = 0$ .

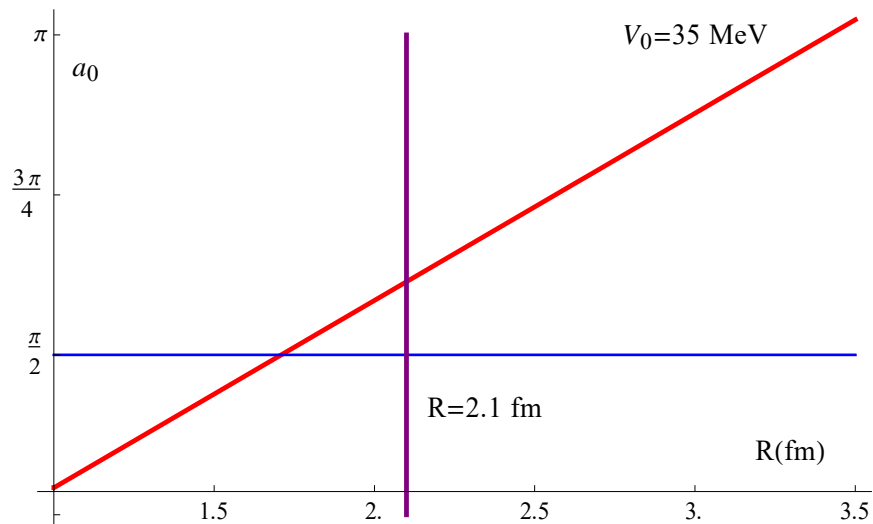


Fig. The relation between  $a_0$  and  $R$  [fm].  $V_0 = 35$  MeV.

## 14. Pion

### 14.1. Yukawa's idea ((Eisberg and Resnick))

In 1935, Yukawa proposed that a nucleon frequently emits a particle with an appreciable rest mass, now called a *n meson* or *pion*. This particle hovers near the nucleon in the so-called  $\pi$  *meson field* for a very short time, and then is absorbed by the nucleon. During the process the nucleon maintains its normal rest mass, and so while it is happening there is a violation of the law of mass-energy conservation because there is more rest mass present than there is before the  $\pi$  meson is emitted or after it is absorbed. The energy-time uncertainty principle shows, however, that such a violation is not impossible if it lasts for a sufficiently short time. Of course, the meson cannot permanently escape the nucleon because that would permanently violate the mass-energy conservation law. Such a pion is called a *virtual particle* because it has a very short existence limited by its violation of mass-energy conservation. If two nucleons are close enough for their meson fields to overlap, it is possible for a meson to leave one field and join the other, without permanently changing the total energy of the system of two nucleons. In the interaction, the momentum carried by the  $\pi$  meson is transferred from one field to the other, and therefore from one nucleon to the other. But if momentum is transferred, the effect is the same as if a force is acting between the nucleons. Thus the *exchange* of a virtual pion between two nucleons leads to the nucleon force acting between them, according to Yukawa. Yukawa was guided by two analogies available to him at the time. One is the covalent binding in the  $H_2^+$  molecule. In this process, a force arises from the sharing, or exchange, of an electron between two atoms.

### 14.2 Evaluation of mass of $\pi$ meson (Heisenberg's principle of uncertainty)

The range of the nucleon force is of the order of the radius  $r'$  of the  $n$ -meson field surrounding a nucleon, since two nucleons experience that force only when their meson fields overlap. To estimate the radius of the field, consider a process in which a nucleon emits a meson of rest mass  $m_\pi$ , which travels out to the limits of the field, and then returns to the nucleon where it is absorbed. In this process, the  $\pi$  meson travels a distance of the order of  $r'$ . While it is happening there is a violation of the conservation of mass-energy. The reason is that the total energy of the system equals one nucleon rest mass energy before and after the process, and one nucleon rest mass energy plus at least one  $\pi$ -

meson rest mass energy during the process. But the energy-time uncertainty principle shows that a violation of energy conservation by an amount

$$\Delta E \approx m_{\pi} c^2,$$

is not impossible if it does not happen for a time longer than  $\Delta t$ , where

$$\Delta E \Delta t \approx \hbar.$$

The reason is that such a violation could not be detected because the energy cannot be measured in a time  $\Delta t$  more accurately than  $\Delta E$ . Since the speed of the pion can be no greater than  $c$ , the time required for it to travel a distance of the order of  $r'$  is at least

$$\Delta t \approx \frac{r'}{c}.$$

These three relations give

$$m_{\pi} \approx \frac{\hbar}{r' c},$$

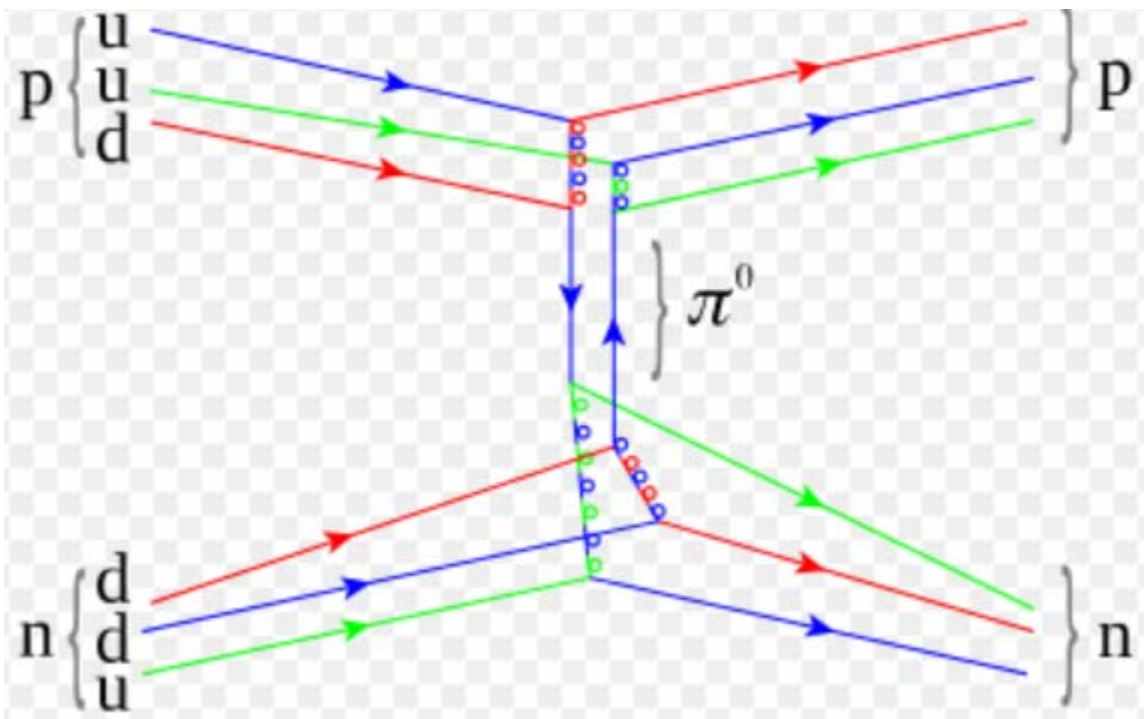
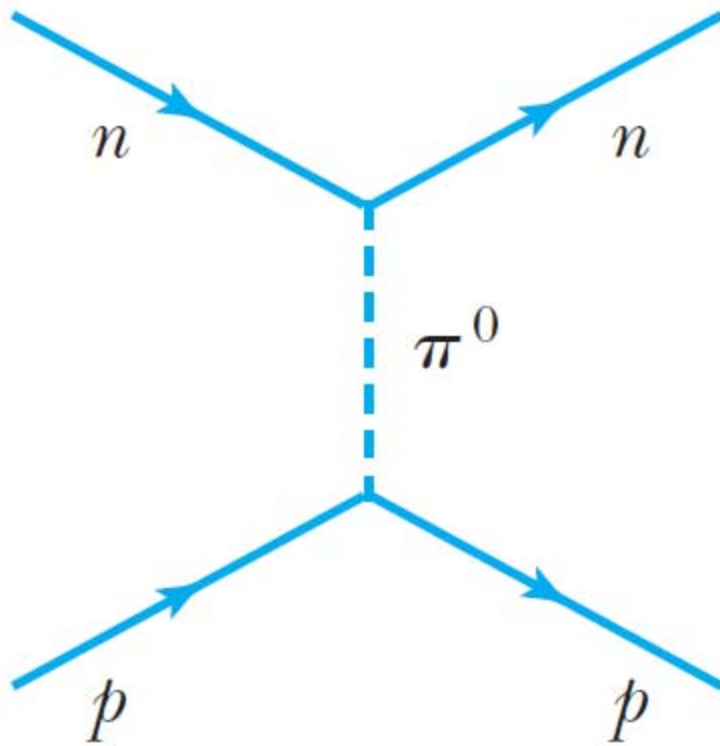
or

If we take  $r' = 2F = 2 \times 10^{-13}$  cm, the  $\pi$ -meson rest mass  $m_{\pi}$  can be evaluated as

$$m_{\pi} = 193.1 m_e.$$

or

$$m_{\pi} c^2 = 98.66 \text{ MeV}.$$

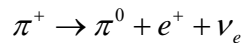
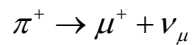
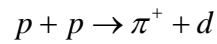


**15.  $\pi^+$ ,  $\pi^0$ , and  $\pi^-$  meson****15.1  $\pi^+$**  $u\bar{d}$ Rest mass:  $139.56995 \pm 0.00035$  MeVMean life:  $(2.6033 \pm 0.0005) \times 10^{-8}$  s.

Spin: 0 (boson)

Isospin:  $|1,1\rangle; T = 1, T_z = 1.$ Electric charge:  $+e$ 

Typical reactions

**15.2  $\pi^0$**  $u\bar{u}, d\bar{d}$ Rest mass:  $134.9766 \pm 0.0006$  MeV

Spin: 0 (boson)

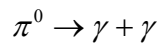
Isospin:  $|1,0\rangle; T = 1, T_z = 0.$ 

Intrinsic parity: odd

Electric charge: 0

Mean life:  $\tau = (8.4 \pm 0.6) \times 10^{-17}$  s

Typical reaction:

**15.3  $\pi^-$**  $d\bar{u}$ Rest mass:  $139.56995 \pm 0.00035$  MeV

The intrinsic parity: odd

Spin: 0 (boson)

Isospin:  $|1,-1\rangle; T = 1, T_z = -1.$ Charge:  $-e$ Typical reactions:  $\pi^- + d \rightarrow n + n$

$$\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$$

$$\pi^- \rightarrow \pi^0 + e^- + \bar{\nu}_e$$

where  $e > 0$ .

## REFERENCES

- H.A. Bethe, Elementary Nuclear Theory, A Short Course On Selected Topics (John Wiley & Sons, New York, 1947).
- E. Fermi, Nuclear Physics, Revised Edition (The University of Chicago Press, 1950).
- S.T. Thornton and A. Rex, Modern Physics for Scientists and Engineers, 4<sup>th</sup> edition (Cengage, 2013).
- R. Eisberg and R. Resnick, Quantum Physics of Atoms, Molecules, Solids, Nuclei, and Particles, 2nd edition (John Wiley and Sons, 1985).
- M.K. Sundaresan, Handbook of Particle Physics (CRC, 2001).

---

## APPENDIX

### Effective range theory and scattering cross section (Bethe)

- (a) The wave function of the bound state

$$\frac{d^2 u_b}{dr^2} + (-\alpha^2 + U_0)u_b = 0 \quad \text{for } 0 < r < R$$

$$u_b(r) = A_b \sin(\kappa_b r)$$

$$\frac{d^2 u_b}{dr^2} - \alpha^2 u_b = 0, \quad \text{for } r > R$$

$$u_b(r) = B_b e^{-\alpha r}$$

with

$$\kappa_b^2 = -\alpha^2 + U_0$$

- (b) The wave function of scattering wave;

$$\frac{d^2 u_s}{dr^2} + (k^2 + U_0)u_s = 0 \quad \text{for } 0 < r < R$$



$$u_s(r) = A_b \sin(\kappa_b r)$$

$$\frac{d^2 u_s}{dr^2} + k^2 u_s = 0 \quad \text{for } r > R$$

$$u_s(r) = B_b \sin(kr + \delta_0)$$

where

$$\kappa_b^2 = k^2 + U_0$$

Multiplying the first by  $u_s$ , the second by  $u_b$ , and subtracting, gives

$$\left[ u_s \frac{d^2 u_b}{dr^2} + (-\alpha^2 + U_0) u_s u_b \right] - \left[ u_b \frac{d^2 u_s}{dr^2} + (k^2 + U_0) u_b u_s \right] = 0$$

or

$$\left( u_s \frac{d^2 u_b}{dr^2} - u_b \frac{d^2 u_s}{dr^2} \right) - (\alpha^2 + k^2) u_s u_b = 0.$$

Integrating from 0 to  $R$ , we get

$$\int_0^R \left( u_s \frac{d^2 u_b}{dr^2} - u_b \frac{d^2 u_s}{dr^2} \right) dr - (\alpha^2 + k^2) \int_0^R u_s u_b dr = 0$$

or

$$\left( u_s \frac{du_b}{dr} - u_b \frac{du_s}{dr} \right) \Big|_0^R = (\alpha^2 + k^2) \int_0^R u_s u_b dr.$$

Since  $u_s(0) = u_b(0) = 0$ , we have

$$\left( u_s \frac{du_b}{dr} - u_b \frac{du_s}{dr} \right) \Big|_{r=R} = (\alpha^2 + k^2) \int_0^R u_s u_b dr.$$

Dividing by  $u_s(R)u_b(R)$ , we get

$$\left(\frac{1}{u_b} \frac{du_b}{dr} - \frac{1}{u_s} \frac{du_s}{dr}\right)_{r=R} = \frac{(\alpha^2 + k^2)}{u_s(R)u_b(R)} \int_0^R u_s u_b dr$$

We note that

$$\frac{1}{u_b} \frac{du_b}{dr} \Big|_{r=R-0} = \kappa_b \cot(\kappa_b R),$$

$$\frac{1}{u_b} \frac{du_b}{dr} \Big|_{r=R+0} = -\alpha$$

leading to the relation

$$\kappa_b \cot(\kappa_b R) = -\alpha$$

We also note that

$$\frac{1}{u_s} \frac{du_s}{dr} \Big|_{r=R-0} = \kappa_s \cot(\kappa_s R)$$

$$\frac{1}{u_s} \frac{du_s}{dr} \Big|_{r=R+0} = k \cot(kR + \delta_0)$$

leading to

$$\kappa_s \cot(\kappa_s R) = k \cot(kR + \delta_0)$$

We need to calculate the integral as

$$\begin{aligned} \frac{1}{u_s(R)u_b(R)} \int_0^R u_s u_b dr &= \frac{-\kappa_b \cot(\kappa_b R) + \kappa_s \cot(\kappa_s R)}{\kappa_b^2 - \kappa_s^2} \\ &= \frac{-\kappa_b \cot(\kappa_b R) + \kappa_s \cot(\kappa_s R)}{-(\alpha^2 + k^2)} \end{aligned}$$

or

$$\frac{(\alpha^2 + k^2)}{u_s(R)u_b(R)} \int_0^R u_s u_b dr = \kappa_b \cot(\kappa_b R) - \kappa_s \cot(\kappa_s R)$$

Then we get

$$\begin{aligned} k \cot(kR + \delta_0) + \alpha &= \kappa_s \cot(\kappa_s R) - \kappa_b \cot(\kappa_b R) \\ &= \alpha + \kappa_s \cot(\kappa_s R) \end{aligned}$$

We expand  $\kappa_s \cot(\kappa_s R)$  around  $\kappa_s = \kappa_b$  by using the Taylor expansion.

$$\begin{aligned} \kappa_s \cot(\kappa_s R) &= \kappa_b \cot(\kappa_b R) + [\cot(\kappa_b R) - \kappa_b R \csc^2(\kappa_b R)](\kappa_b - \kappa_s) + \dots \\ &\approx -\alpha - (\kappa_s - \kappa_b) \left[ \frac{\alpha}{\kappa_b} + \kappa_b R (1 + \cot^2(\kappa_b R)) \right] \\ &= -\alpha - (\kappa_s - \kappa_b) \left[ \frac{\alpha}{\kappa_b} + \kappa_b R \left( 1 + \frac{\alpha^2}{\kappa_b^2} \right) \right] \\ &\approx -\alpha - (\kappa_s - \kappa_b) \kappa_b R \end{aligned}$$

We also have

$$\kappa_s - \kappa_b = \frac{\kappa_s^2 - \kappa_b^2}{\kappa_b + \kappa_s} \approx \frac{\kappa_s^2 - \kappa_b^2}{2\kappa_b} = \frac{\kappa_s^2 - \kappa_b^2}{2\kappa_b} = \frac{\alpha^2 + k^2}{2\kappa_b},$$

Finally we have

$$k \cot(kR + \delta_0) = -\alpha - \frac{1}{2}(\alpha^2 + k^2)R.$$

Suppose that  $kR \ll 1$ , then we have

$$\cot(\delta_0) = -\frac{\alpha}{k}$$

Then the total cross section is given by

$$\begin{aligned}
\sigma &= \frac{4\pi}{k^2} \sin^2 \delta_0 \\
&= \frac{4\pi}{k^2} \frac{1}{1 + \cot^2 \delta_0} \\
&= \frac{4\pi}{k^2 + \alpha^2} \\
&= \frac{2\pi\hbar^2}{m} \frac{1}{E + \frac{\hbar^2\alpha^2}{2m}}
\end{aligned}$$

with  $m = M_n/2$ .

$$\frac{2\pi\hbar^2}{m} = 5.20782 \text{ barns MeV}$$


---