Dirac equation

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In particle physics, the **Dirac equation** is a relativistic wave equation derived by British physicist Paul Dirac in 1928 and later seen to be an elaboration of the work of Wolfgang Pauli. In its free form, or including electromagnetic interactions, it describes all spin-½ particles, such as electrons and quarks, and is consistent with both the principles of quantum mechanics and the theory of special relativity, and was the first theory to account fully for special relativity in the context of quantum mechanics.

It accounted for the fine details of the hydrogen spectrum in a completely rigorous way. The equation also implied the existence of a new form of matter, *antimatter*, hitherto unsuspected and unobserved, and actually predated its experimental discovery. It also provided a *theoretical* justification for the introduction of several-component wave functions in Pauli's phenomenological theory of spin; the wave functions in the Dirac theory are vectors of four complex numbers (known as bispinors), two of which resemble the Pauli wavefunction in the non-relativistic limit, in contrast to the Schrödinger equation which described wave functions of only one complex value. Moreover, in the limit of zero mass, the Dirac equation reduces to the Weyl equation.

http://en.wikipedia.org/wiki/Dirac_equation

http://en.wikipedia.org/wiki/Paul_Dirac

((S. Brandt))
The Harvest of a Century Discoveries of Modern Physics in 100 Episodes (Oxford, 2009)

"In 1955 Dirac gave lectures in Moscow. When asked about his philosophy of physics, he wrote on the blackboard:

PHYSICAL LAWS SHOULD HAVE MATHEMATICAL BEAUTY.

Dalitz, who relates this story, adds that 'this has been preserved to this day'. It was this philosophy that Dirac used when he found the equation that now bears his name. Unlike many results in theoretical physics it was neither inspired by unexplained measurements nor by physical insight but only by considerations of mathematical 'beauty' or, in other words, simplicity. In the *Dirac equation* not only quantum mechanics and the special theory of relativity were married, but also the spin of the electron is contained in it without any ad hoc assumption. So far, so good. But the equation not just beautifully described known phenomena, it did more. It predicted the existence of electrons with negative energy. This was at first held to be a severe problem of the theory but was finally understood as great progress, because negative-energy electrons could be interpreted as hitherto unknown particles. Thus, the existence of new particles was predicted which had all properties of the electron except for the electric charge, which must be positive rather than negative (positron). These particles were indeed found four years after the equation. Dirac is often quoted to have said that his equation 'contains most of physics and all of chemistry'. This, however, is not the case, although in a paper on the (non-relativistic) *Quantum Mechanics of Many-Electron Systems*, quite unrelated to the Dirac equation, similar words appear: 'The underlying physical laws necessary for the mathematical theory of a large part of physics and the whole of chemistry are thus completely known, and the difficulty is only that the exact application of these laws leads to equations much too complicated to be soluble.' We begin this episode by mentioning briefly the work, previous to Dirac's, on the reconciliation of quantum mechanics with special relativity and with spin, respectively."

We discuss the relativistic theory of electron which was derived by Dirac. Here we use the notations

(a) The space-time position four vector (contravariant and covariant):

$x^{\mu} = (ct, \boldsymbol{r})$ (contravariant vector)

where

$x^0 = ct \qquad x^1 = x, \qquad x^2 = y, \qquad x^3 = z,$

$x_{\mu} = g_{\mu\nu} x^{\nu} = (ct, -x, -y, -z)$ (covariant vector)

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \text{(metric tensor)}$$

$$x^\mu = g^{\mu\nu} x_\nu$$

Any time the same index appears in an upper and a lower position summation over the index is assumed without explicitly noting.

(b)
The one-particle differential operator that represents energy-momentum is given by

$$p^\mu = (\frac{E}{c}, \boldsymbol{p}), \qquad p_\mu = g_{\mu\nu} p^\nu = (\frac{E}{c}, -\boldsymbol{p})$$

where

$$p^0 = \frac{E}{c} \qquad p^1 = p_x, \qquad p^2 = p_y, \qquad p^3 = p_z,$$

(c)

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\partial_0, \partial_i) = (\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = g^{\mu\nu}\partial_\nu = (\partial^0, \partial^i) = (\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z})$$

(d)

$$p_\mu = i\hbar\partial_\mu = i\hbar\frac{\partial}{\partial x^\mu} = i\hbar(\frac{\partial}{c\partial t}, \nabla) = (\frac{\varepsilon}{c}, -\boldsymbol{p}),$$

Note that

$$\boldsymbol{p} = \frac{\hbar}{i}\nabla, \qquad \varepsilon = i\hbar\frac{\partial}{\partial t}$$

1 Hamiltonian

In this section, for convenience, we do not use the notation of the contravariant and covariant vectors. We use the notations

$$p_1 = p_x, \quad p_2 = p_y, \quad p_3 = p_z$$

$$\boldsymbol{\alpha} = (\alpha_x, \alpha_y, \alpha_z).$$

The time dependent Schrödinger equation for the particle is given by

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi$$

We find that the time t and the space position (x, y, z) are treated very non symmetrically. We need to search for relativistic equation for the particle of first order in t, x, y, and z, where the equation should be symmetrical in space and time coordinates. Thus H is required to be linear in the momentum operator.

$$i\hbar\frac{\partial}{\partial t}\psi = H\psi$$

The form of H is introduced by Dirac as

$$H = c\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc^2$$

By squaring H, we should get the relation from the relativity (the Einstein relation),

$$\begin{aligned}(H)^2 &= (c\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc^2)^2 \\ &= c^2[p_i p_j \frac{1}{2}\{\alpha_i, \alpha_j\} + mcp_i\{\alpha_i, \beta\} + \beta^2 m^2 c^2] \\ &= m^2c^4 + c^2\boldsymbol{p}^2 \\ &= E^2\end{aligned}$$

which leads to the relations

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}I_4, \qquad \{\alpha_i, \beta\} = 0, \qquad \beta^2 = I_4$$

where $i = x$, y, and z, the curly bracket denotes an anti-commutator,

$$\{\alpha_i, \alpha_j\} = \alpha_i\alpha_j + \alpha_j\alpha_i$$

Here we do not show how to derive the form of matrices α and β (4 x 4). The matrices are Hermitian matrices (4x4). Thus the Hamiltonian is also Hermitian. We have the expressions for the matrices as

$$\alpha_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}$$

$$\alpha_y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}$$

$$\alpha_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} == \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} == \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$$

Note that we do not use the contravariant and variant form of Dirac matrices.

The Hamiltonian H is described by

$$\begin{aligned} H &= c\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc^2 \\ &= c\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & 0 \end{pmatrix} + mc^2\begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \\ &= \begin{pmatrix} mc^2 & c\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ c\boldsymbol{\sigma}\cdot\boldsymbol{p} & -mc^2 \end{pmatrix} \end{aligned}$$

or

$$H = \begin{pmatrix} mc^2 & 0 & cp_z & c(p_x - ip_y) \\ 0 & mc^2 & c(p_x + ip_y) & -cp_z \\ cp_z & c(p_x - ip_y) & -mc^2 & 0 \\ c(p_x + ip_y) & -cp_z & 0 & -mc^2 \end{pmatrix}$$

$(H)^2$ can be evaluated as

$$(H)^2 = E^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = E^2 I_4$$

where

$$E^2 = m^2 c^4 + c^2 \boldsymbol{p}^2$$

((**Mathematica**))

```
Clear["Global`*"]; σx = PauliMatrix[1];
σy = PauliMatrix[2] ;; σz = (1  0
                              0 -1);
I2 = IdentityMatrix[2];
exp_* :=
 exp /. {Complex[re_, im_] :> Complex[re, -im]};

αx = KroneckerProduct[σx, σx];

αy = KroneckerProduct[σx, σy];

αz = KroneckerProduct[σx, σz];

β = KroneckerProduct[σz, I2];

f1 = c px αx + c py αy + c pz αz + β m^2 c^2 // Simplify;

g1 = f1.f1 // FullSimplify;
```

```
Clear["Global`*"]; σx = (0 1
                         1 0); σy = (0 -𝕚
                                     𝕚  0); σz = (1  0
                                                  0 -1);
I2 = IdentityMatrix[2];
```

```
αx = KroneckerProduct[σx, σx]; αx // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

```
αy = KroneckerProduct[σx, σy]; αy // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

```
αz = KroneckerProduct[σx, σz]; αz // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

```
β = KroneckerProduct[σz, I2]; β // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

```
f1 // MatrixForm
```

$$\begin{pmatrix} c^2 m^2 & 0 & c\,pz & c\,(px - i\,py) \\ 0 & c^2 m^2 & c\,(px + i\,py) & -c\,pz \\ c\,pz & c\,(px - i\,py) & -c^2 m^2 & 0 \\ c\,(px + i\,py) & -c\,pz & 0 & -c^2 m^2 \end{pmatrix}$$

```
g1 /. {c^2 (c^2 m^4 + px^2 + py^2 + pz^2) → E1^2} // MatrixForm
```

$$\begin{pmatrix} E1^2 & 0 & 0 & 0 \\ 0 & E1^2 & 0 & 0 \\ 0 & 0 & E1^2 & 0 \\ 0 & 0 & 0 & E1^2 \end{pmatrix}$$

3. The matrices α and β

The matrices α and β can be expressed in terms of the Pauli spin matrices,

$$\sigma^1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and identity matrix

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using the Kronecker product, the matrices α^1, α^2, α^3, and β are given by

$$\alpha^1 = \alpha_x = \sigma^1 \otimes \sigma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}$$

$$\alpha^2 = \alpha_y = \sigma^1 \otimes \sigma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}$$

$$\alpha^3 = \alpha_z = \sigma^1 \otimes \sigma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}$$

$$\gamma^0 = \beta = \sigma^3 \otimes I_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$$

$$\gamma^1 = \beta\alpha^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix},$$

$$\gamma^2 = \beta\alpha^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}$$

$$\gamma^3 = \beta\alpha^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}$$

$$\alpha^k = \gamma^0\gamma^k = \beta\gamma^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}$$

$$\gamma^0\gamma^k = -\gamma^k\gamma^0$$

((**Note**)) Hermitian conjugate of γ^μ

$$\gamma^{0+} = \gamma^0 = \gamma^0\gamma^0\gamma^0$$

$$\gamma^{k+} = (\beta\alpha^k)^+ = \alpha^{k+}\beta^+ = \alpha^k\beta = \beta^2\alpha^k\beta = \beta\gamma^k\beta = \gamma^0\gamma^k\gamma^0$$

where k = 1, 2, 3.

Hence we have the very useful quantity

$$\left(\gamma^{\mu}\right)^{+} = \gamma^{0}\gamma^{\mu}\gamma^{0}$$

where $\mu = 0, 1, 2, 3, 4$.

4. Dirac equation

We now have the Dirac equation given by

$$i\hbar\frac{\partial}{\partial t}\psi = H\psi$$

with

$$H = c\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc^{2}$$

Then we get

$$i\hbar\frac{\partial}{\partial(ct)}\psi = (\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc)\psi = (\gamma^{0}\gamma^{k}\frac{\hbar}{i}\partial_{k} + \beta mc)\psi$$

or

$$i\hbar\frac{\partial}{\partial(ct)}\psi = (\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc)\psi = (-i\hbar\gamma^{0}\gamma^{k}\partial_{k} + \gamma^{0}mc)\psi \qquad (1)$$

where we use the notations

$$x^{\mu} = (ct, x, y, z)$$

$$p_{\mu} = (\frac{\varepsilon}{c}, -\boldsymbol{p}) = i\hbar\partial_{\mu} = i\hbar\frac{\partial}{\partial x^{\mu}} = i\hbar(\frac{\partial}{\partial(ct)}, \nabla)$$

$$p_{k} = i\hbar\partial_{k} = -\boldsymbol{p} \qquad (k = 1, 2, 3)$$

$$\boldsymbol{\alpha} = (\alpha_{x}, \alpha_{y}, \alpha_{z}) = (\alpha^{1}, \alpha^{2}, \alpha^{3}) = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}$$

$$\alpha^{k} = \gamma^{0}\gamma^{k}, \qquad \beta = \gamma^{0}$$

$$\left(\gamma^{0}\right)^{2} = 1$$

The multiplication of Eq.(1) by γ^0 from the right leads to

$$i\hbar\gamma^0\partial_0\psi = (\gamma^0)^2(-i\hbar\gamma^k\partial_k + mc)\psi = (-i\hbar\gamma^k\partial_k + mc)\psi$$

or

$$[i\hbar(\gamma^k\partial_k + \gamma^0\partial_0) - mc]\psi = 0\,,$$

where k = 1, 2, 3. Thus we have the Dirac equation

$$(i\gamma^\mu \frac{\partial}{\partial x^\mu} - \frac{mc}{\hbar})\psi = 0$$

or

$$(i\gamma^\mu\partial_\mu - \frac{mc}{\hbar})\psi = 0 \qquad \text{(Dirac equation)} \qquad (2)$$

where

$$p_\mu = i\hbar\partial_\mu$$

$$\gamma^\mu \frac{\partial}{\partial x^\mu} = \gamma^\mu\partial_\mu \qquad \text{(Feynman dagger, Feynman slash notation)}$$

5. Probability current density operator

Now we take the adjoint of the Dirac equation

$$(i\partial_\mu\psi^+\gamma^{\mu+} + \frac{mc}{\hbar}\psi^+) = 0$$

or

$$(i\partial_\mu\psi^+\gamma^0\gamma^\mu\gamma^0 + \frac{mc}{\hbar}\psi^+) = 0 \qquad (3)$$

since

$$(\gamma^\mu)^+ = \gamma^0\gamma^\mu\gamma^0\,.$$

Multiplying Eq.(3) from the right by γ^0,

$$(i\partial_\mu \psi^+ \gamma^0 \gamma^\mu \left(\gamma^0\right)^2 + \frac{mc}{\hbar}\psi^+ \gamma_0 = 0 .$$

We define

$$\psi^+ \gamma^0 = \overline{\psi} \qquad \text{(Dirac conjugate)}$$

Noting that $\left(\gamma^0\right)^2 = 1$, we get

$$i\partial_\mu \overline{\psi}\gamma^\mu + \frac{mc}{\hbar}\overline{\psi} = 0 \qquad (4)$$

In order to get a probability current, we multiply Eq.(2) from the left by $\overline{\psi}$ and Eq.(4) from the right by ψ and add to obtain

$$\overline{\psi}(i\gamma^\mu \partial_\mu - \frac{mc}{\hbar})\psi + (i\partial_\mu \overline{\psi}\gamma^\mu + \frac{mc}{\hbar}\overline{\psi})\psi = 0$$

or

$$\overline{\psi}\gamma^\mu(\partial_\mu \psi) + (\partial_\mu \overline{\psi})\gamma^\mu \psi = 0$$

or simply

$$\partial_\mu(\overline{\psi}\gamma^\mu \psi) = 0$$

The probability four current is given by

$$j^\mu = c\overline{\psi}\gamma^\mu \psi = (c\rho, \boldsymbol{J})$$

This satisfies the equation of continuity

$$\partial_\mu j^\mu = 0$$

6. Alternative method (Sakurai)

$$\frac{E^2}{c^2} - \boldsymbol{p}^2 = m^2 c^2$$

$$(\frac{E^{(op)}}{c} - \boldsymbol{\sigma} \cdot \boldsymbol{p})(\frac{E^{(op)}}{c} + \boldsymbol{\sigma} \cdot \boldsymbol{p}) = m^2 c^2$$

where

$$E^{(op)} = i\hbar\frac{\partial}{\partial t} = i\hbar c\frac{\partial}{\partial x_0}$$

with $x_0 = ct$

This enables us to write a second order equation

$$(i\hbar\frac{\partial}{\partial x_0} + i\hbar\boldsymbol{\sigma}\cdot\nabla)(i\hbar\frac{\partial}{\partial x_0} - i\hbar\boldsymbol{\sigma}\cdot\nabla)\phi = m^2c^2\phi$$

for a free electron. ϕ is now a two component wave function

$$\phi^{(R)} = \frac{1}{mc}(i\hbar\frac{\partial}{\partial x_0} - i\hbar\boldsymbol{\sigma}\cdot\nabla)\phi$$

and

$$\phi^{(L)} = \phi$$

$$(i\hbar\frac{\partial}{\partial x_0} + i\hbar\boldsymbol{\sigma}\cdot\nabla)\phi^{(R)} = mc\phi^{(L)}$$

Then we have

$$(-i\hbar\frac{\partial}{\partial x_0} - i\hbar\boldsymbol{\sigma}\cdot\nabla)\phi^{(R)} = -mc\phi^{(L)} \tag{1a}$$

and

$$(-i\hbar\frac{\partial}{\partial x_0} + i\hbar\boldsymbol{\sigma}\cdot\nabla)\phi^{(L)} = -mc\phi^{(R)} \tag{1b}$$

((Derivation of Dirac equation))

Taking the sum and the difference of Eqs.(1a) and (1b)

$$-i\hbar\boldsymbol{\sigma}\cdot\nabla(\phi^{(R)} - \phi^{(L)}) - i\hbar\frac{\partial}{\partial x_0}(\phi^{(R)} + \phi^{(L)}) = -mc(\phi^{(R)} + \phi^{(L)})$$

$$i\hbar\boldsymbol{\sigma}\cdot\nabla(\phi^{(R)}+\phi^{(L)})+i\hbar\frac{\partial}{\partial x_0}(\phi^{(R)}-\phi^{(L)})=-mc(\phi^{(R)}-\phi^{(L)})$$

We define

$$\psi_A=\phi^{(R)}+\phi^{(L)}$$

and

$$\psi_B=\phi^{(R)}-\phi^{(L)}$$

Thus we have

$$-i\hbar\boldsymbol{\sigma}\cdot\nabla\psi_B-i\hbar\frac{\partial}{\partial x_0}\psi_A=-mc\psi_A$$

$$i\hbar\boldsymbol{\sigma}\cdot\nabla\psi_A+i\hbar\frac{\partial}{\partial x_0}\psi_B=-mc\psi_B$$

$$\begin{pmatrix}-i\hbar\frac{\partial}{\partial x_0} & -i\hbar\boldsymbol{\sigma}\cdot\nabla\\ i\hbar\boldsymbol{\sigma}\cdot\nabla & i\hbar\frac{\partial}{\partial x_0}\end{pmatrix}\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix}=-mc\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix}$$

$\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix}$ is the 4 x 1 column matrix.

Here note that

$$\begin{pmatrix}-i\hbar\frac{\partial}{\partial x_0} & -i\hbar\boldsymbol{\sigma}\cdot\nabla\\ i\hbar\boldsymbol{\sigma}\cdot\nabla & i\hbar\frac{\partial}{\partial x_0}\end{pmatrix}=-i\hbar\frac{\partial}{\partial x^0}\begin{pmatrix}I_2 & 0\\ 0 & -I_2\end{pmatrix}-i\hbar\begin{pmatrix}0 & \boldsymbol{\sigma}\\ -\boldsymbol{\sigma} & 0\end{pmatrix}\cdot\nabla$$

$$=-i\hbar\gamma^0\frac{\partial}{\partial x_0}-i\hbar\boldsymbol{\gamma}\cdot\nabla$$

$$=-i\hbar\gamma^\mu\partial_\mu$$

Note that

$$x^0=x_0\,,\qquad \partial_\mu=\frac{\partial}{\partial x^\mu}=(\frac{\partial}{\partial x_0},\nabla)$$

γ^{μ} : gamma matrices (or Dirac matrices)

$$\gamma^0 = \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$\gamma^1 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^3 = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

with

$$\sigma^1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Here we define the Dirac spinor (4 x 1 matrix)

$$\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

Using these definitions, we have Dirac equation

$$(i\gamma^{\mu}\partial_{\mu} - \frac{mc}{\hbar})\psi = 0$$ (Dirac equation)

((**Property of γ matrices**))

$$\gamma^{0+} = \gamma^0, \qquad \gamma^{\mu+} = \gamma^0\gamma^{\mu}\gamma^0$$

$$\{\gamma^{\mu}, \gamma^{\nu}\} = \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\delta^{\mu\nu} I_4$$ (Clifford algebra)

((Note))

$$\begin{aligned}\{\gamma^1, \gamma^2\} &= \gamma^1\gamma^2 + \gamma^2\gamma^1 \\ &= \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\{\sigma^1, \sigma^2\} & 0 \\ 0 & \{\sigma^1, \sigma^2\} \end{pmatrix} \\ &= 0\end{aligned}$$

where

$$\{\sigma^1, \sigma^2\} = \sigma^1\sigma^2 + \sigma^2\sigma^1 = 0$$

7. Dirac equation of particle in the presence of electromagnetic field

Dirac equation for free particle is given by

$$(i\gamma^{\mu}\partial_{\mu} - \frac{mc}{\hbar})\psi = 0$$

where

$$p^{\mu} = i\hbar\partial^{\mu} = i\hbar\frac{\partial}{\partial x_{\mu}} = (\frac{\varepsilon}{c}, \boldsymbol{p})$$

$$p_{\mu} = i\hbar\partial_{\mu} = i\hbar\frac{\partial}{\partial x^{\mu}} = (\frac{\varepsilon}{c}, -\boldsymbol{p})$$

$$\varepsilon = i\hbar\frac{\partial}{\partial t}, \qquad \boldsymbol{p} = -i\hbar\nabla$$

In the presence of the electromagnetic field $A^{\mu} = (A^0, \boldsymbol{A})$

$$p^{\mu} \rightarrow p^{\mu} - \frac{e}{c} A^{\mu}, \qquad p_{\mu} \rightarrow p_{\mu} - \frac{e}{c} A_{\mu}$$

where A^0 is a scalar potential and $\boldsymbol{A}$ is a vector potential,

$$A^{\mu} = (A^0, \boldsymbol{A}), \qquad A_{\mu} = (A^0, -\boldsymbol{A})$$

The magnetic field $\boldsymbol{B}$ and the electric field $\boldsymbol{E}$ are expressed as

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \qquad \text{(magnetic field)}$$

$$\boldsymbol{E} = -\nabla A^0 - \frac{1}{c}\frac{\partial A}{\partial t} \qquad \text{(electric field)}$$

Noting that

$$\partial^{\mu} \rightarrow \partial^{\mu} + \frac{ie}{c\hbar} A^{\mu}, \qquad \partial_{\mu} \rightarrow \partial_{\mu} + \frac{ie}{c\hbar} A_{\mu}$$

Dirac equation of particle in the presence of the electromagnetic field can be expressed by

$$[i\gamma^{\mu}(\partial_{\mu} + \frac{ie}{c\hbar} A_{\mu}) - \frac{mc}{\hbar}]\psi = 0.$$

where e is the charge of the particle (e<0 for electron)

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8. Two component wave functions ψ_A and ψ_B under the parity operation

Since

$$\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

we get

$$\psi'(\mathbf{r},t) = \begin{pmatrix} \psi_A'(\mathbf{r},t) \\ \psi_B'(\mathbf{r},t) \end{pmatrix}$$
$$= \gamma^0 \psi(-\mathbf{r},t)$$
$$= \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \begin{pmatrix} \psi_A(-\mathbf{r},t) \\ \psi_B(-\mathbf{r},t) \end{pmatrix}$$
$$= \begin{pmatrix} \psi_A(-\mathbf{r},t) \\ -\psi_B(-\mathbf{r},t) \end{pmatrix}$$

under the spatial reflection [$\psi'(\mathbf{r},t) = \gamma^0 \psi(-\mathbf{r},t)$, see the definition in the symmetry of the Dirac equation]. Then we have

$$\psi_A'(\boldsymbol{r},t) = \psi_A(-\boldsymbol{r},t)$$ (the same as the non-relativistic case)

$$\psi_B'(\boldsymbol{r},t) = -\psi_B(-\boldsymbol{r},t)$$

The upper and lower components of the wave functions have different behaviors under a parity transformation. (see the discussion of the parity for the non-relativistic case below).

This is expected from the following discussion of Dirac equation.

$$H\psi = E\psi$$

$$\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}, \qquad H = c\boldsymbol{\alpha} \cdot (\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}) + \beta mc^2$$

where

$$\boldsymbol{p} \to \boldsymbol{p} - \frac{e}{c}\boldsymbol{A}.$$

Eigenvalue problem:

$$H\psi = \begin{pmatrix} mc^2 & c\boldsymbol{\sigma} \cdot (\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}) \\ c\boldsymbol{\sigma} \cdot (\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}) & -mc^2 \end{pmatrix} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = E \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

or

$$c\boldsymbol{\sigma}\cdot(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A})\psi_A - mc^2\psi_B = E\psi_B$$

Then we have

$$\psi_B(\boldsymbol{r},t) = \frac{1}{E+mc^2}c\boldsymbol{\sigma}\cdot(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A})\psi_A(\boldsymbol{r},t)$$

Under the parity,

$$\boldsymbol{p}\to-\boldsymbol{p}, \qquad \boldsymbol{A}\to-\boldsymbol{A},$$

$$\psi_A(\boldsymbol{r},t)\to\psi_A'(\boldsymbol{r},t)=\psi_A(-\boldsymbol{r},t),$$

then we get

$$\begin{aligned}\psi_B'(\boldsymbol{r},t) &= -\frac{1}{E+mc^2}c\boldsymbol{\sigma}\cdot(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A})\psi_A'(\boldsymbol{r},t)\\ &= -\frac{1}{E+mc^2}c\boldsymbol{\sigma}\cdot(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A})\psi_A(-\boldsymbol{r},t)\\ &= -\psi_B(-\boldsymbol{r},t)\end{aligned}$$

where

$$\psi_A'(-\boldsymbol{r},t)=\psi_A(\boldsymbol{r},t)$$

8. Even or odd parity in ψ_A and ψ_B

Suppose that $|\psi_A\rangle$ and $|\psi_B\rangle$ has different parities (even or odd).

$$\hat{\pi}|\psi_A\rangle=\pm|\psi_A\rangle, \qquad \hat{\pi}|\psi_B\rangle=\mp|\psi_B\rangle$$

or

$$\langle\boldsymbol{r}|\hat{\pi}|\psi_A\rangle=\langle-\boldsymbol{r}|\psi_A\rangle=\pm\langle\boldsymbol{r}|\psi_A\rangle$$

and

$$\langle\boldsymbol{r}|\hat{\pi}|\psi_B\rangle=\langle-\boldsymbol{r}|\psi_B\rangle=\mp\langle\boldsymbol{r}|\psi_B\rangle$$

Then we get

$$\gamma^0\psi(-\boldsymbol{r},t)=\begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}\begin{pmatrix} \psi_A(-\boldsymbol{r},t) \\ \psi_B(-\boldsymbol{r},t) \end{pmatrix}=\begin{pmatrix} \psi_A(-\boldsymbol{r},t) \\ -\psi_B(-\boldsymbol{r},t) \end{pmatrix}=\pm\begin{pmatrix} \psi_A(\boldsymbol{r},t) \\ \psi_B(\boldsymbol{r},t) \end{pmatrix}=\pm\psi(\boldsymbol{r},t)$$

or, formally

$$\gamma^0\psi(-\boldsymbol{r},t)=\pm\psi(\boldsymbol{r},t).$$

since

$$\psi_A(-\boldsymbol{r},t)=\pm\psi_A(\boldsymbol{r},t), \qquad \psi_B(-\boldsymbol{r},t)=\mp\psi_B(\boldsymbol{r},t).$$

We assume that ψ_A and ψ_B are the eigenstates of the orbital angular momentum.

$$\psi_A(-\boldsymbol{r},t)=(-1)^{l_A}\psi_A(\boldsymbol{r},t)=\pm\psi_A(\boldsymbol{r},t)$$

$$\psi_B(-\boldsymbol{r},t)=(-1)^{l_B}\psi_B(\boldsymbol{r},t)=\mp\psi_B(\boldsymbol{r},t)$$

where l_A and l_B are the orbital angular momenta of the two-component wave function $\psi_A(\boldsymbol{r},t)$ and $\psi_B(\boldsymbol{r},t)$, respectively. Thus we have

$$(-1)^{l_A}=(-1)^{l_B+1}$$

This implies that if $\psi_A(\boldsymbol{r},t)$ is a two-component wave function with an even (odd) orbital angular momentum, then $\psi_B(\boldsymbol{r},t)$ is a two-component wave function with an odd (even) orbital angular momentum.

((**Example**))

We consider the case of a central force.

$$\boldsymbol{A}=0, \qquad A_0=\phi, \qquad eA_0=V(r)$$

$$\psi=\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}, \qquad H=c\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta mc^2+V(r)$$

$$H\psi=\begin{pmatrix} mc^2+V(r) & c\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ c\boldsymbol{\sigma}\cdot\boldsymbol{p} & -mc^2+V(r) \end{pmatrix}\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}=E\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

Then we get

$$\psi_B(\boldsymbol{r},t)=\frac{c}{E-V(r)+mc^2}(\boldsymbol{\sigma}\cdot\boldsymbol{p})\psi_A(\boldsymbol{r},t)$$

Let us suppose that $\psi_A(\boldsymbol{r},t)$ is an $^2S_{1/2}$ state wave function with spin up (l = 0, s = 1/2)

$$\psi_A(\boldsymbol{r},t) = R(r)\begin{pmatrix}1\\0\end{pmatrix}e^{-\frac{iEt}{\hbar}}$$

Then

$$\psi_B(\boldsymbol{r},t) = \frac{-i\hbar c}{E-V(r)+mc^2}\begin{pmatrix}\frac{\partial}{\partial z} & \frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\\ \frac{\partial}{\partial x}+i\frac{\partial}{\partial y} & -\frac{\partial}{\partial z}\end{pmatrix}\begin{pmatrix}R(r)\\0\end{pmatrix}e^{-\frac{iEt}{\hbar}}$$

Note that

$$\frac{\partial}{\partial z}R(r) = \frac{\partial r}{\partial z}\frac{dR}{dr} = \frac{z}{r}\frac{dR}{dr},$$

$$\frac{\partial}{\partial x}R(r) == \frac{x}{r}\frac{dR}{dr}, \qquad \frac{\partial}{\partial y}R(r) = \frac{y}{r}\frac{dR}{dr}$$

Then we get

$$\begin{aligned}\psi_B(\boldsymbol{r},t) &= \frac{-i\hbar c}{E-V(r)+mc^2}\frac{1}{r}\frac{dR}{dr}\begin{pmatrix}z & x-iy\\ x+iy & -z\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}e^{-\frac{iEt}{\hbar}}\\ &= \frac{-i\hbar c}{E-V(r)+mc^2}\frac{1}{r}\frac{dR}{dr}\begin{pmatrix}z\\ x+iy\end{pmatrix}e^{-\frac{iEt}{\hbar}}\\ &= \frac{-i\hbar c}{E-V(r)+mc^2}\frac{1}{r}\frac{dR}{dr}[z\begin{pmatrix}1\\0\end{pmatrix}+(x+iy)\begin{pmatrix}0\\1\end{pmatrix}]e^{-\frac{iEt}{\hbar}}\end{aligned}$$

Here we note that

$$Y_1^1 = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{i\phi} = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\frac{x+iy}{r}$$

$$Y_1^0 = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\frac{z}{r}$$

$$Y_1^{-1} = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{-i\phi} = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\frac{x-iy}{r}$$

where

$$x = r\sin\theta\cos\phi\,, \qquad y = r\sin\theta\sin\phi\,, \qquad z = r\cos\theta$$

Thus we have

$$\psi_B(\boldsymbol{r},t) = \frac{-i\hbar c}{E - V(r) + mc^2}\frac{dR}{dr}[\sqrt{\frac{4\pi}{3}}Y_1^0\begin{pmatrix}1\\0\end{pmatrix} + \sqrt{\frac{8\pi}{3}}Y_1^1\begin{pmatrix}0\\1\end{pmatrix}]e^{-\frac{iEt}{\hbar}}$$

The first term of the parenthesis is $l = 1$ and $s = 1/2$
$j = 1/2$ $(^2P_{1/2})$

The second term of the parenthesis is $l = 1$ and $s = 1/2$
$j = 3/2$ $(^2P_{3/2})$
$j = 1/2$ $(^2P_{1/2})$

((Note)) Parity operator π in non-relativistic quantum mechanics

$$|\psi'\rangle = \hat{\pi}|\psi\rangle \quad \text{with} \quad \hat{\pi}^+\hat{\pi} = \hat{1} \quad \text{and} \quad \hat{\pi}^+ = \hat{\pi}$$

$$\langle\psi'|\hat{x}|\psi'\rangle = -\langle\psi|\hat{x}|\psi\rangle\,, \qquad \hat{\pi}^+\hat{x}\hat{\pi} = -\hat{x}$$

$$\hat{\pi}|x\rangle = |-x\rangle$$

$$\langle x|\psi'\rangle = \langle x|\hat{\pi}|\psi\rangle\,, \quad \text{or} \quad \psi'(x) = \psi(-x)$$

Even parity: $\langle x|\psi'\rangle = \langle x|\hat{\pi}|\psi\rangle = \langle x|\psi\rangle$, $\psi'(x) = \psi(-x) = \psi(x)$

Odd parity $\langle x|\hat{\pi}|\psi\rangle = -\langle x|\psi\rangle$, $\psi'(x) = \psi(-x) = -\psi(x)$

---

9. Eigenvalue problem (degenerate case)

We solve the eigenvalue problem using the Mathematica.

$$H - EI_4 = \begin{pmatrix} mc^2 - E & 0 & cp_z & c(p_x - ip_y) \\ 0 & mc^2 - E & c(p_x + ip_y) & -cp_z \\ cp_z & c(p_x - ip_y) & -mc^2 - E & 0 \\ c(p_x + ip_y) & -cp_z & 0 & -mc^2 - E \end{pmatrix}$$

From the condition that det($H-\varepsilon I_4$)=0, we get

$$E=\pm R=\pm\sqrt{m^2c^4+c^2\boldsymbol{p}^2}$$

where

$$R=\sqrt{m^2c^4+c^2\boldsymbol{p}^2} \quad (>0)$$

Thus we see that there are four eigenvalues which are degenerate in pairs, i.e.

E = +R, +R, -R, and -R

For simplicity we assume that

$$p_x=p_y=0$$

For E = +R

$$\begin{pmatrix} mc^2-R & 0 & cp_z & 0 \\ 0 & mc^2-R & 0 & -cp_z \\ cp_z & 0 & -mc^2-R & 0 \\ 0 & -cp_z & 0 & -mc^2-R \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

or

$$(mc^2-R)u_1+cp_zu_3=0$$
$$(mc^2-R)u_2-cp_zu_4=0$$
$$cp_zu_1-(R+mc^2)u_3=0$$
$$-cp_zu_2-(mc^2+R)u_4=0$$

It is clear from the above equations that at the zero momentum limit ($p_z\rightarrow 0$) the first two equations do not give us any information on the unknowns. Thus we need to solve the second two equations. The two independent solutions, corresponding to the eigenvalue +R,

$$u_1=1,\ u_2=0,\ \ u_3=\frac{cp_z}{R+mc^2},\ u_4=0.$$

$$u_1=0,\ u_2=1,\ \ u_3=0,\ u_4=-\frac{cp_z}{R+mc^2}.$$

For $E = -R$

$$\begin{pmatrix} mc^2+R & 0 & cp_z & 0 \\ 0 & mc^2+R & 0 & -cp_z \\ cp_z & 0 & -mc^2+R & 0 \\ 0 & -cp_z & 0 & -mc^2+R \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

or

$$(mc^2+R)u_1 + cp_z u_3 = 0$$
$$(mc^2+R)u_2 - cp_z u_4 = 0$$
$$cp_z u_1 - (R - mc^2)u_3 = 0$$
$$-cp_z u_2 - (mc^2 - R)u_4 = 0$$

It is clear from the above equations that at the zero momentum limit ($p_z \to 0$) the second two equations do not give us any information on the unknowns. Thus we need to solve the first two equations. The two independent solutions, corresponding to the eigenvalue -R,

$$u_1 = -\frac{cp_z}{R+mc^2}, \; u_2 = 0, \; u_3 = 1, \; , \; u_4 = 0.$$

$$u_1 = 0, \;\; u_2 = \frac{cp_z}{R+mc^2}, \;\; u_3 = 0, \;\; u_4 = 1.$$

((**Summary**))

For $E = R$ (positive energy)

$$\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{R+mc^2} \\ 0 \end{pmatrix} \qquad \sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{cp_z}{R+mc^2} \end{pmatrix},$$

For $E = -R$ (negative energy)

$$\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} \frac{-cp_z}{R+mc^2} \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} 0 \\ \frac{cp_z}{R+mc^2} \\ 0 \\ 1 \end{pmatrix}$$

If $p_z = 0$, we have

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

The non-relativistic spin states. These are degenerate and have energy eigenvalue +*R*.

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The nonrelativistic spin states. These are degenerate and have energy eigenvlaue -*R*.

10. The use of the Mathematica to derive the eigenkets of *H*

The eigenvalue problem can be solved Using the Eigensystem[*H*] of the Mathematica.

((**Mathematica**))

```
Clear["Global`*"]; exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]};
σx = (0 1
      1 0); σy = (0 -ⅈ
                  ⅈ  0); σz = (1  0
                               0 -1);
I2 = IdentityMatrix[2]; I4 = IdentityMatrix[4];
αx = KroneckerProduct[σx, σx]; αy = KroneckerProduct[σx, σy];
αz = KroneckerProduct[σx, σz]; β = KroneckerProduct[σz, I2];
H = m c^2 β + c (αx px + αy py + αz pz);
rule1 = {Sqrt[c^2 m^2 + (px^2 + py^2 + pz^2)] -> R/c};
rule2 = {(px^2 + py^2 + pz^2) -> R^2/c^2 - c^2 m^2};
rule3 = {R -> c Sqrt[c^2 m^2 + px^2 + py^2 + pz^2]};
```

Eigenvalue problem of the Hamiltonian of the Dirac free particle

```
eq1 = Eigensystem[H] /. rule1 // FullSimplify
```

$$\left\{\{-R, -R, R, R\}, \left\{\left\{-\frac{c\,(px - i\,py)}{c^2 m + R}, \frac{c\,pz}{c^2 m + R}, 0, 1\right\}, \left\{-\frac{c\,pz}{c^2 m + R}, -\frac{c\,(px + i\,py)}{c^2 m + R}, 1, 0\right\}, \left\{\frac{c\,(px - i\,py)}{-c^2 m + R}, \frac{c\,pz}{c^2 m - R}, 0, 1\right\}, \left\{\frac{c\,pz}{-c^2 m + R}, \frac{c\,(px + i\,py)}{-c^2 m + R}, 1, 0\right\}\right\}\right\}$$

Orthogonality

```
eq1[[2, 1]]*.eq1[[2, 2]] /. rule3 // Simplify
0

eq1[[2, 1]]*.eq1[[2, 3]] /. rule3 // Simplify
0

eq1[[2, 1]]*.eq1[[2, 4]] /. rule3 // Simplify
0

eq1[[2, 2]]*.eq1[[2, 3]] /. rule3 // Simplify
0

eq1[[2, 2]]*.eq1[[2, 4]] /. rule3 // Simplify
0
```

```
eq1[[2, 3]]*.eq1[[2, 4]] /. rule3 // Simplify
0
```

Normalization constant

```
A1 = eq1[[2, 1]]*.eq1[[2, 1]] // Simplify; A11 = A1 /. rule2 // Simplify
```

$$\frac{2\,R}{c^2\,m + R}$$

```
A2 = eq1[[2, 2]]*.eq1[[2, 2]] // Simplify; A21 = A2 /. rule2 // Simplify
```

$$\frac{2\,R}{c^2\,m + R}$$

```
A3 = eq1[[2, 3]]*.eq1[[2, 3]] // Simplify; A31 = A3 /. rule2 // Simplify
```

$$-\frac{2\,R}{c^2\,m - R}$$

```
A4 = eq1[[2, 4]]*.eq1[[2, 4]] // Simplify; A41 = A4 /. rule2 // Simplify
```

$$-\frac{2\,R}{c^2\,m - R}$$

From the Mathematica, the results are obtained as follows. The eigenket of H with $E = R$ is not a real eigen ket, while the eigenket of H with $E = -R$ is a real eigen ket. The reason for this is that the real eigenkets are the simultaneous eigen ket of both H and helicity $(\frac{1}{p}\boldsymbol{\Sigma}\cdot\boldsymbol{p})$.

(a) The eigenstate with positive energy

For $E = R$ (positive energy), the energy eigenkets are obtained from the mathematica shown above.

$$\psi_a = \begin{pmatrix} \dfrac{cp_z}{R - mc^2} \\ \dfrac{c(p_x + ip_y)}{R - mc^2} \\ 1 \\ 0 \end{pmatrix} \qquad \psi_b = \begin{pmatrix} \dfrac{c(p_x - ip_y)}{R - mc^2} \\ \dfrac{-cp_z}{R - mc^2} \\ 0 \\ 1 \end{pmatrix},$$

which are not appropriate solutions, since these states are not the eigenket of the helicity. The real eigenkets are the simultaneous eigenkets of H and the helicity. These are expressed by the superposition of these two states.

$$\psi_{ab} = a\psi_a + b\psi_b = \begin{pmatrix} \dfrac{c[ap_z + b(p_x - ip_y)]}{R - mc^2} \\ \dfrac{c[-bp_z + a(p_x + ip_y)]}{R - mc^2} \\ a \\ b \end{pmatrix}$$

with constant *a* and *b* to be determined. When we choose

$$a = \frac{cp_z}{R + mc^2}, \qquad b = \frac{c(p_x + ip_y)}{R + mc^2}$$

we get the real eigenstate

$$\begin{pmatrix} 1 \\ 0 \\ \dfrac{cp_z}{R + mc^2} \\ \dfrac{c(p_x + ip_y)}{R + mc^2} \end{pmatrix}$$

When we choose *a* and *b* as

$$a = \frac{c(p_x - ip_y)}{R + mc^2}, \qquad b = \frac{-cp_z}{R + mc^2}$$

we get

$$\begin{pmatrix} 0 \\ 1 \\ \dfrac{c(p_x - ip_y)}{R + mc^2} \\ \dfrac{-cp_z}{R + mc^2} \end{pmatrix}$$

((**Summary**))
The normalized simultaneous eigenstates are given by

$$\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{R+mc^2} \\ \frac{c(p_x+ip_y)}{R+mc^2} \end{pmatrix} \qquad \sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{R+mc^2} \\ \frac{-cp_z}{R+mc^2} \end{pmatrix},$$

For $E = R$ (positive energy)

(b) The eigenstates with negative energy

For $E = -R$ (negative energy)

$$\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} \frac{-cp_z}{R+mc^2} \\ \frac{-c(p_x+ip_y)}{R+mc^2} \\ 1 \\ 0 \end{pmatrix}$$

and

$$\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} \frac{-c(p_x-ip_y)}{R+mc^2} \\ \frac{cp_z}{R+mc^2} \\ 0 \\ 1 \end{pmatrix}$$

which are right solutions.

11. Solution of Dirac equation

(a) The case of $p_x = p_y = p_z = 0$

The Hamiltonian H and the helicity operator (Σ_3) are given by

$$H = mc^2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \Sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$H|1\rangle = mc^2|1\rangle, \qquad H|2\rangle = mc^2|2\rangle$$

$$H|3\rangle = -mc^2|3\rangle, \qquad H|4\rangle = -mc^2|4\rangle$$

$$\Sigma_3|1\rangle = |1\rangle, \qquad \Sigma_3|2\rangle = -|2\rangle$$

$$\Sigma_3|3\rangle = |3\rangle, \qquad \Sigma_3|4\rangle = -|4\rangle$$

where

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |4\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

We note that

$$[H, \Sigma_3] = 0.$$

Note that the helicity operator is defined by

$$\Sigma \cdot \frac{p_z}{p} = \Sigma_3$$

Thus $|1\rangle$, $|2\rangle$, $|3\rangle$, and $|4\rangle$ are the simultaneous eigenkets of H and Σ_3.

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad E = mc^2, \qquad h = 1$$

$$|2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad E = mc^2, \qquad h = -1$$

$$|3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad E = -mc^2, \qquad h = 1$$

$$|4\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad E = mc^2, \qquad h = -1$$

(b) Free motion of a Dirac particle

$$H = c\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta mc^2$$

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\boldsymbol{\alpha} \cdot \boldsymbol{p} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \cdot \boldsymbol{p} \\ \boldsymbol{\sigma} \cdot \boldsymbol{p} & 0 \end{pmatrix}$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{p} = \sigma_x p_x + \sigma_y p_y + \sigma_z p_z = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

$$H = \begin{pmatrix} mc^2 & c\boldsymbol{\sigma} \cdot \boldsymbol{p} \\ c\boldsymbol{\sigma} \cdot \boldsymbol{p} & -mc^2 \end{pmatrix}$$

$$= \begin{pmatrix} mc^2 & 0 & cp_z & c(p_x - ip_y) \\ 0 & mc^2 & c(p_x + ip_y) & -cp_z \\ cp_z & c(p_x - ip_y) & -mc^2 & 0 \\ c(p_x + ip_y) & -cp_z & 0 & -mc^2 \end{pmatrix}$$

Eigenvalue problem

$$Hu = Eu$$

$$\det\begin{pmatrix} mc^2 - E & c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) \\ c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) & -mc^2 - E \end{pmatrix} = 0$$

The energy eigenvalues:

$$(E - mc^2)(E + mc^2) - c^2(\boldsymbol{\sigma}\cdot\boldsymbol{p})(\boldsymbol{\sigma}\cdot\boldsymbol{p}) = 0$$

or

$$E^2 = m^2c^4 + c^2\boldsymbol{p}^2$$

where we use the formula

$$(\boldsymbol{\sigma}\cdot\boldsymbol{p})(\boldsymbol{\sigma}\cdot\boldsymbol{p}) = \boldsymbol{p}^2$$

Then we get

$$E = \pm R = \pm\sqrt{m^2c^4 + c^2\boldsymbol{p}^2} \quad \text{with} \qquad R>0$$

For $E = R$ (positive energy solution))

The state vector is given by $u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$ (4 x 1 matrix)

$$\begin{pmatrix} mc^2 - R & c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) \\ c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) & -mc^2 - R \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$c(\boldsymbol{\sigma}\cdot\boldsymbol{p})u_A = (mc^2 + R)u_B, \qquad c(\boldsymbol{\sigma}\cdot\boldsymbol{p})u_B = (mc^2 - R)u_A$$

Suppose that

(i) $u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$u_B = \frac{c(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{R+mc^2}u_A = \frac{c}{R+mc^2}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{c}{R+mc^2}\begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix}$$

(ii) $u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$u_B = \frac{c(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{R+mc^2}u_A = \frac{c}{R+mc^2}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{c}{R+mc^2}\begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix}$$

$$\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} 1 \\ 0 \\ \dfrac{cp_z}{R+mc^2} \\ \dfrac{c(p_x+ip_y)}{R+mc^2} \end{pmatrix}, \qquad (E = R)$$

$$\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} 0 \\ 1 \\ \dfrac{c(p_x-ip_y)}{R+mc^2} \\ \dfrac{-cp_z}{R+mc^2} \end{pmatrix} \qquad (E = R)$$

The normalization factor ($1/A$) of the state vector can be determined as follows.

$$
\begin{aligned}
A^2 &= 1 + \frac{c^2({p_x}^2 + {p_y}^2 + {p_z}^2)}{(R + mc^2)^2} \\
&= 1 + \frac{c^2 \boldsymbol{p}^2}{(R + mc^2)^2} \\
&= \frac{(R + mc^2)^2 + (R + mc^2)(R - mc^2)}{(R + mc^2)^2} \\
&= \frac{R + mc^2 + R - mc^2}{R + mc^2} \\
&= \frac{2R}{R + mc^2}
\end{aligned}
$$

or

$$\frac{1}{A} = \sqrt{\frac{R + mc^2}{2R}}$$

For $E = -R$ (negative energy solution)

$$\begin{pmatrix} mc^2 + R & c(\boldsymbol{\sigma} \cdot \boldsymbol{p}) \\ c(\boldsymbol{\sigma} \cdot \boldsymbol{p}) & -mc^2 + R \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$c(\boldsymbol{\sigma} \cdot \boldsymbol{p}) u_B = -(mc^2 + R) u_A$$

Suppose that

(i) $u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$u_A = -\frac{c(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{R+mc^2}u_B$$

$$= -\frac{c}{R+mc^2}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= -\frac{c}{R+mc^2}\begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix}$$

(ii) $u_B = \begin{pmatrix}0\\1\end{pmatrix}$

$$u_A = -\frac{c(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{R+mc^2}u_B$$

$$= -\frac{c}{R+mc^2}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= -\frac{c}{R+mc^2}\begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix}$$

Then we have

$$\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} \frac{-cp_z}{R+mc^2} \\ -\frac{c(p_x+ip_y)}{R+mc^2} \\ 1 \\ 0 \end{pmatrix}, \qquad (E=-R)$$

$$\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} -\frac{c(p_x-ip_y)}{R+mc^2} \\ \frac{cp_z}{R+mc^2} \\ 0 \\ 1 \end{pmatrix} \qquad (E=-R)$$

12. Helicity operator

The helicity is defined as the relation between a particle's spin $\boldsymbol{s}$ and the direction of motion ($\boldsymbol{p}$), i.e., $\boldsymbol{s}\cdot\boldsymbol{p}$ is the component of angular momentum along the spin.

$$\Lambda_s = \frac{\hbar}{2}\boldsymbol{\Sigma}\cdot\frac{\boldsymbol{p}}{|\boldsymbol{p}|} = \boldsymbol{S}\cdot\frac{\boldsymbol{p}}{|\boldsymbol{p}|}$$

This is the component of spin in the direction of the momentum $\boldsymbol{p}$. The 4D generalization of the spin vector operator

$$\boldsymbol{S} = \frac{\hbar}{2}\boldsymbol{\Sigma}$$

where

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}$$

Suppose that the particle propagates in the z direction.

$$\Lambda_s = \frac{\hbar}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

When $p_x = 0$ and $p_y = 0$.

$$\psi_1 = \sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{R+mc^2} \\ 0 \end{pmatrix}, \qquad (E = R)$$

$\Lambda_s\psi_1 = +\psi_1$ (helicity: +1, energy: +R)

$$\psi_2 = \sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{cp_z}{R+mc^2} \end{pmatrix} \qquad (E = R)$$

$\Lambda_s\psi_2 = -\psi_2$ (helicity: -1, energy +R)

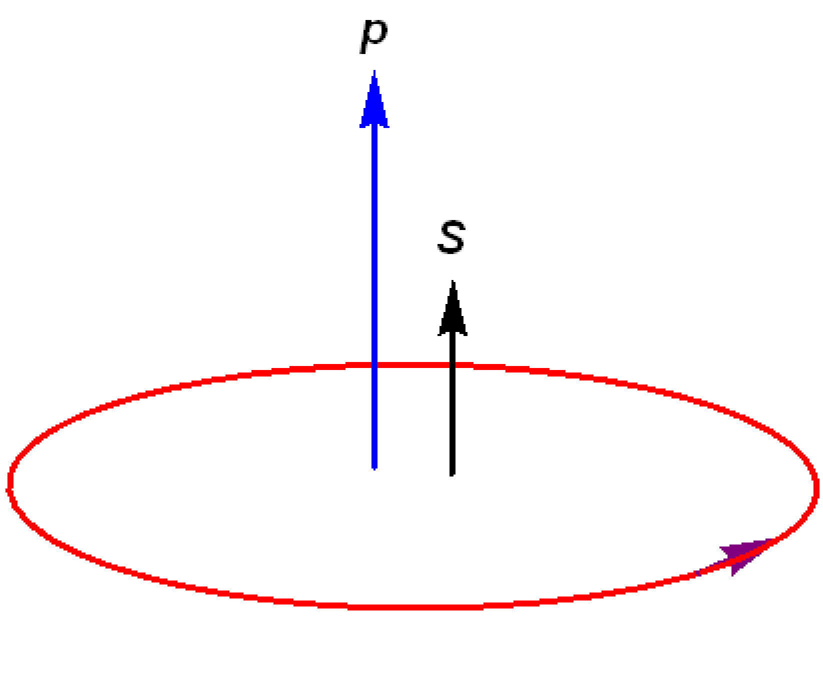
$$\psi_3 = \sqrt{\frac{R+mc^2}{2R}} \begin{pmatrix} \frac{-cp_z}{R+mc^2} \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad (E = -R)$$

$\Lambda_s \psi_3 = \psi_3$ (helicity: +1, energy -R)

$$\psi_4 = \sqrt{\frac{R+mc^2}{2R}} \begin{pmatrix} 0 \\ \frac{-cp_z}{R+mc^2} \\ 0 \\ 1 \end{pmatrix} \qquad (E = -R)$$

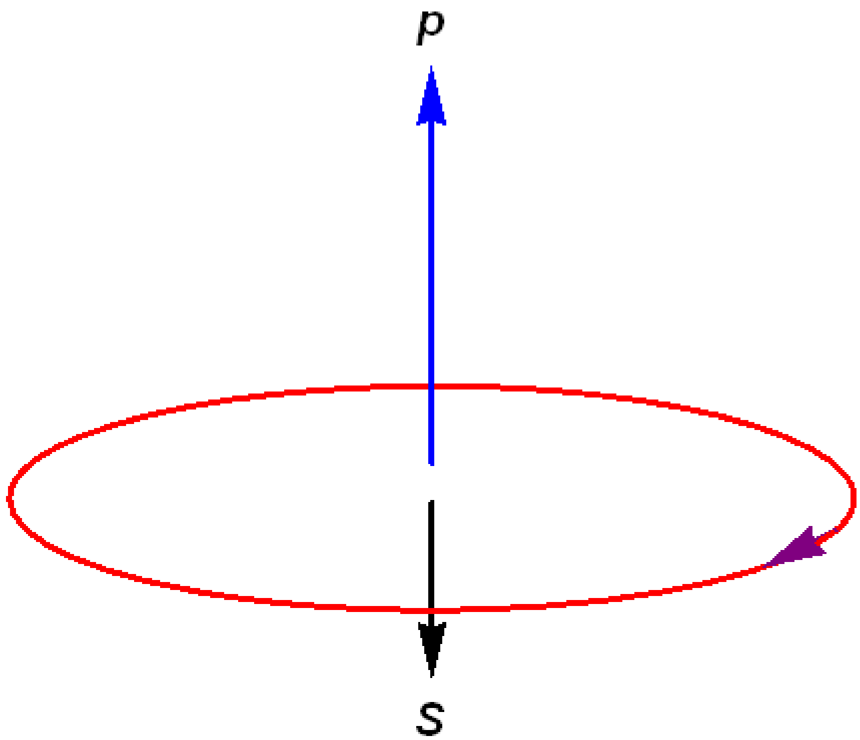
$\Lambda_s \psi_4 = -\psi_4$ (helicity: -1, energy -R)

(i) Right helicity



Positive helicity (+1): right handed (right helicity). The diection of the spin ***S*** is parallel to that of ***p***.

(ii) Left helicity



Negative helicity (-1): left handed (left helicity). The diection of the spin ***S*** is antiparallel to that of ***p***.

13. Energy eigenvalue and helicity (Mathematica)

```
Clear["Global`*"];
```

$$H1 = \begin{pmatrix} m\,c^2 & 0 & c\,pz & c\,(px - i\,py) \\ 0 & m\,c^2 & c\,(px + i\,py) & -c\,pz \\ c\,pz & c\,(px - i\,py) & -m\,c^2 & 0 \\ c\,(px + i\,py) & -c\,pz & 0 & -m\,c^2 \end{pmatrix};$$

$$\Sigma 1 = \frac{1}{p}\begin{pmatrix} pz & px - i\,py & 0 & 0 \\ px + i\,py & -pz & 0 & 0 \\ 0 & 0 & pz & px - i\,py \\ 0 & 0 & px + i\,py & -pz \end{pmatrix}; \quad N1 = \sqrt{\frac{R + m\,c^2}{2\,R}};$$

$$\psi 1 = N1\begin{pmatrix} 1 \\ 0 \\ \frac{c\,pz}{R + m\,c^2} \\ \frac{c\,(px + i\,py)}{R + m\,c^2} \end{pmatrix}; \quad \psi 2 = N1\begin{pmatrix} 0 \\ 1 \\ \frac{c\,(px - i\,py)}{R + m\,c^2} \\ \frac{-c\,pz}{R + m\,c^2} \end{pmatrix}; \quad \psi 3 = N1\begin{pmatrix} \frac{-c\,pz}{R + m\,c^2} \\ \frac{-c\,(px + i\,py)}{R + m\,c^2} \\ 1 \\ 0 \end{pmatrix};$$

$$\psi 4 = N1\begin{pmatrix} \frac{-c\,(px - i\,py)}{R + m\,c^2} \\ \frac{c\,pz}{R + m\,c^2} \\ 0 \\ 1 \end{pmatrix};$$

```
H1.Σ1 - Σ1.H1 // FullSimplify
```

{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

```
h1 = H1.ψ1 - R ψ1 // Simplify;
```

$$h1\ /.\ \left\{\left(px^2 + py^2 + pz^2\right) \to \left(\frac{R^2}{c^2} - m^2 c^2\right)\right\}\ //\ \text{FullSimplify}$$

{{0}, {0}, {0}, {0}}

```
h2 = H1.ψ2 - R ψ2 // Simplify;
```

$$h2\ /.\ \left\{\left(px^2 + py^2 + pz^2\right) \to \left(\frac{R^2}{c^2} - m^2 c^2\right)\right\}\ //\ \text{FullSimplify}$$

{{0}, {0}, {0}, {0}}

```
h3 = H1.ψ3 + R ψ3 // Simplify;
```

$$h3\ /.\ \left\{\left(px^2 + py^2 + pz^2\right) \to \left(\frac{R^2}{c^2} - m^2 c^2\right)\right\}\ //\ \text{FullSimplify}$$

{{0}, {0}, {0}, {0}}

```
h4 = H1.ψ4 + R ψ4 // Simplify;
```

$$h4\ /.\ \left\{\left(px^2 + py^2 + pz^2\right) \to \left(\frac{R^2}{c^2} - m^2 c^2\right)\right\}\ //\ \text{FullSimplify}$$

{{0}, {0}, {0}, {0}}

```
s1 = Σ1.ψ1 - ψ1 // Simplify;
s1 /. {px → 0, py → 0, pz → p} // FullSimplify
```

```
{{0}, {0}, {0}, {0}}
```

```
s2 = Σ1.ψ2 + ψ2 // Simplify;
s2 /. {px → 0, py → 0, pz → p} // FullSimplify
```

```
{{0}, {0}, {0}, {0}}
```

```
s3 = Σ1.ψ3 - ψ3 // Simplify;
s3 /. {px → 0, py → 0, pz → p} // FullSimplify
```

```
{{0}, {0}, {0}, {0}}
```

```
s4 = Σ1.ψ4 + ψ4 // Simplify;
s4 /. {px → 0, py → 0, pz → p} // FullSimplify
```

```
{{0}, {0}, {0}, {0}}
```

```
s1 /. {px → 0, py → 0, pz → p} // FullSimplify
```

```
{{0}, {0}, {0}, {0}}
```

14 Chirality for the massless particle

(K. Huang)

We consider the Hamiltonian of the massless particle

$$H = c\boldsymbol{\alpha}\cdot\boldsymbol{p}$$

Eigenvalue problem:

$$c(\boldsymbol{\alpha}\cdot\boldsymbol{p})u = \pm c|\boldsymbol{p}|u, \qquad \text{or} \qquad (\boldsymbol{\alpha}\cdot\boldsymbol{p})u = \pm|\boldsymbol{p}|u$$

$$\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & 0 \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \pm|\boldsymbol{p}|\begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

or

$$\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & 0 \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \pm|\boldsymbol{p}|\begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$(\boldsymbol{\sigma}\cdot\boldsymbol{p})u_B = \pm|\boldsymbol{p}|u_A, \qquad (\boldsymbol{\sigma}\cdot\boldsymbol{p})u_A = \pm|\boldsymbol{p}|u_B$$

We consider the matrix

$$\gamma^5 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Note that

$$[\gamma^5, \alpha^k] = 0.$$

Then we get

$$\gamma^5(\boldsymbol{\alpha}\cdot\boldsymbol{p})u = (\boldsymbol{\alpha}\cdot\boldsymbol{p})\gamma^5 u = \pm|\boldsymbol{p}|\gamma^5 u$$

we can diagonalize γ^5 , whose eigenvalue ±1 is called "chirality". The solution with chirality +1 is called "right-handed", denoted $u(\boldsymbol{p}, R)$; one with chirality -1 is called "left-handed", denoted $u(\boldsymbol{p}, L)$:

(i) $u(\boldsymbol{p}, R)$

$\gamma^5 u(\boldsymbol{p}, R) = u(\boldsymbol{p}, R)$, chirality (+1)

$(\boldsymbol{\alpha}\cdot\boldsymbol{p})u(\boldsymbol{p}, R) = +|\boldsymbol{p}|u(\boldsymbol{p}, R)$; energy $(+c|\boldsymbol{p}|)$,

(ii) $u(\boldsymbol{p}, L)$

$\gamma^5 u(\boldsymbol{p}, L) = -u(\boldsymbol{p}, L)$ chirality (-1)

$(\boldsymbol{\alpha}\cdot\boldsymbol{p})u(\boldsymbol{p}, L) = -|\boldsymbol{p}|u(\boldsymbol{p}, R)$: energy $(-c|\boldsymbol{p}|)$

Projection operator:

$$P_R = \frac{1}{2}(I_4 + \gamma^5) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad P_L = \frac{1}{2}(I_4 - \gamma^5) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$P_R u(\boldsymbol{p},R) = \frac{1}{2}(I_4 + \gamma^5)u(\boldsymbol{p},R) = u(\boldsymbol{p},R)$$

$$P_R u(\boldsymbol{p},L) = \frac{1}{2}(I_4 + \gamma^5)u(\boldsymbol{p},L) = 0$$

$$P_L u(\boldsymbol{p},L) = \frac{1}{2}(I_4 - \gamma^5)u(\boldsymbol{p},R) = 0$$

$$P_L u(\boldsymbol{p},L) = \frac{1}{2}(I_4 - \gamma^5)u(\boldsymbol{p},L) = u(\boldsymbol{p},L)$$

15. Conserved current

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

$$\psi^+ = \begin{pmatrix} \psi^*{}_0 & \psi_1^* & \psi_2^* & \psi_3^* \end{pmatrix}$$

$$\overline{\psi} = \psi^+ \gamma^0$$

$$= \begin{pmatrix} \psi^*{}_0 & \psi_1^* & \psi_2^* & \psi_3^* \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \psi^*{}_0 & \psi_1^* & -\psi_2^* & -\psi_3^* \end{pmatrix}$$

In order to obtain the wave equation for $\overline{\psi}$, we start from the Dirac equation

$$(i\gamma^\mu \frac{\partial}{\partial x^\mu} - \frac{mc}{\hbar})\psi = 0 \qquad (\mu = 0, 1, 2, 3).$$

We take the Hermitian conjugate of the Dirac equation,

$$-i\frac{\partial}{\partial x^\mu}\psi^+ \left(\gamma^\mu\right)^+ - \frac{mc}{\hbar}\psi^+ = 0$$

or

$$-i\frac{\partial}{\partial x^{\mu}}\psi^{+}\gamma^{0}\gamma^{\mu}\gamma^{0}-\frac{mc}{\hbar}\psi^{+}=0$$

Multiplying Eq.(1) by γ^{0} from the right, we get

$$-i\frac{\partial}{\partial x^{\mu}}\psi^{+}\gamma^{0}\gamma^{\mu}\left(\gamma^{0}\right)^{2}-\frac{mc}{\hbar}\psi^{+}\gamma^{0}=0\,.$$

or

$$i\frac{\partial}{\partial x^{\mu}}\left(\psi^{+}\gamma^{0}\right)\gamma^{\mu}+\frac{mc}{\hbar}\left(\psi^{+}\gamma^{0}\right)=0$$

or

$$i\frac{\partial}{\partial x^{\mu}}\overline{\psi}\gamma^{\mu}+\frac{mc}{\hbar}\overline{\psi}=0$$

Now we have

$$i\gamma^{\mu}\frac{\partial}{\partial x^{\mu}}\psi-\frac{mc}{\hbar}\psi=0 \qquad (1)$$

and

$$i\frac{\partial}{\partial x^{\mu}}\overline{\psi}\gamma^{\mu}+\frac{mc}{\hbar}\overline{\psi}=0 \qquad (2)$$

Multiplication of Eq.(1) by $\overline{\psi}$ from the left leads to

$$i\overline{\psi}\gamma^{\mu}\frac{\partial}{\partial x^{\mu}}\psi-\frac{mc}{\hbar}\overline{\psi}\psi=0 \qquad (1')$$

Multiplication of Eq.(2) by ψ from the right leads to

$$i\frac{\partial}{\partial x^{\mu}}\overline{\psi}\gamma^{\mu}\psi+\frac{mc}{\hbar}\overline{\psi}\psi=0 \qquad (2')$$

The addition of Eq.(1') from Eq.(2') yields

$$\overline{\psi}\gamma^{\mu}\frac{\partial}{\partial x^{\mu}}\psi+(\frac{\partial}{\partial x^{\mu}}\overline{\psi})\gamma^{\mu}\psi=0\,.$$

or

$$\frac{\partial}{\partial x^{\mu}}(\overline{\psi}\gamma^{\mu}\psi) = 0$$

Thus we see that

$$\begin{aligned} S^{\varpi} &= c\overline{\psi}\gamma^{\mu}\psi \\ &= (c\overline{\psi}\gamma^{0}\psi, c\overline{\psi}\gamma^{k}\psi) \\ &= (c\psi^{+}(\gamma^{0})^{2}\psi, c\overline{\psi}\gamma^{k}\psi) \\ &= (c\psi^{+}\psi, c\overline{\psi}\gamma^{k}\psi) \end{aligned}$$

$$\frac{\partial}{\partial x^{\mu}}S^{\mu} = 0$$

or

$$\nabla\cdot\boldsymbol{S} + \frac{\partial}{\partial(ct)}c\psi^{+}\psi = 0$$

or

$$\frac{\partial}{\partial t}\psi^{+}\psi + \nabla\cdot\boldsymbol{S} = 0$$

The flux density $\boldsymbol{S}$ is defined by

$$S^{k} = c\overline{\psi}\gamma^{k}\psi = c\psi^{+}\gamma^{0}\gamma^{k}\psi = c\psi^{+}\alpha^{k}\psi$$

with

$$\alpha^{k} = \gamma^{0}\gamma^{k} = \beta\gamma^{k} = \begin{pmatrix} 0 & \sigma^{k} \\ \sigma^{k} & 0 \end{pmatrix}$$

The probability density is defined by

$$\rho = \psi^{+}\psi$$

((**Note**))

$$c\overline{\psi}\gamma^{k}\psi = c\psi^{+}\gamma^{0}\gamma^{k}\psi = c\psi^{+}\alpha^{k}\psi = S^{k}$$

$$c\bar{\psi}\gamma^0\psi = c\psi^+\left(\gamma^0\right)^2\psi = c\psi^+\psi = c\rho$$

where

$$\alpha_k = \gamma^0\gamma^k$$

16. Simple solutions: nonrelativistic approximation

In the presence of electromagnetic fields,

$$p_\mu = i\hbar\partial_\mu = i\hbar\frac{\partial}{\partial x^\mu} \to p_\mu - \frac{e}{c}A_\mu$$

or

$$\partial_\mu \to -\frac{i}{\hbar}(p_\mu - \frac{e}{c}A_\mu)$$

where e is the charge ($e<0$ for electron).
Dirac equation

$$(i\gamma^\mu\frac{\partial}{\partial x_\mu} - \frac{mc}{\hbar})\psi = 0 \to [i\gamma^\mu(-\frac{i}{\hbar})(p_\mu - \frac{e}{c}A_\mu) - \frac{mc}{\hbar}]\psi = 0$$

or

$$[\gamma^\mu(p_\mu - \frac{e}{c}A_\mu) - mc]\psi = 0$$

or

$$\left\{\begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}\cdot(-\boldsymbol{p} + \frac{e}{c}\boldsymbol{A}) + \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}(p_0 - \frac{e}{c}A_0) - mc\right\}\psi = 0$$

or

$$\begin{pmatrix} 0 & -\boldsymbol{\sigma}\cdot(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}) \\ \boldsymbol{\sigma}\cdot(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}) & 0 \end{pmatrix}\psi + \begin{pmatrix} p_0 - \frac{e}{c}A_0 & 0 \\ 0 & -(p_0 - \frac{e}{c}A_0) \end{pmatrix}\psi = mc\psi$$

Noting that and $p_0 = i\hbar\frac{\partial}{\partial x^0} = \frac{i\hbar}{c}\frac{\partial}{\partial t}$

$$\begin{pmatrix} 0 & -\boldsymbol{\sigma}\cdot(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A}) \\ \boldsymbol{\sigma}\cdot(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A}) & 0 \end{pmatrix}\begin{pmatrix}\psi_A \\ \psi_B\end{pmatrix} + \begin{pmatrix} \frac{i\hbar}{c}\frac{\partial}{\partial t} - \frac{e}{c}A_0 & 0 \\ 0 & -\frac{i\hbar}{c}\frac{\partial}{\partial t} + \frac{e}{c}A_0) \end{pmatrix}\begin{pmatrix}\psi_A \\ \psi_B\end{pmatrix} = mc\begin{pmatrix}\psi_A \\ \psi_B\end{pmatrix}$$

or

$$-\boldsymbol{\sigma}\cdot(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A})\psi_B + (\frac{i\hbar}{c}\frac{\partial}{\partial t} - \frac{e}{c}A_0)\psi_A = mc\psi_A$$

$$\boldsymbol{\sigma}\cdot(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A})\psi_A + (-\frac{i\hbar}{c}\frac{\partial}{\partial t} + \frac{e}{c}A_0)\psi_B = mc\psi_B$$

or

$$\boldsymbol{\sigma}\cdot(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A})\psi_B = (\frac{i\hbar}{c}\frac{\partial}{\partial t} - \frac{e}{c}A_0 - mc)\psi_A$$

$$-\boldsymbol{\sigma}\cdot(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A})\psi_A = (-\frac{i\hbar}{c}\frac{\partial}{\partial t} + \frac{e}{c}A_0 - mc)\psi_B$$

Assuming that

$$\psi = \psi_0 e^{-iEt/\hbar}$$

$$i\hbar\frac{\partial}{\partial t}\psi_A = E\psi_A \qquad \text{or} \qquad \frac{\partial}{\partial t}\psi_A = -\frac{iE}{\hbar}\psi_A$$

$$i\hbar\frac{\partial}{\partial t}\psi_B = E\psi_B \qquad \text{or} \qquad \frac{\partial}{\partial t}\psi_B = -\frac{iE}{\hbar}\psi_B$$

Then we have

$$\boldsymbol{\sigma}\cdot(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A})\psi_B = \frac{1}{c}(E - eA_0 - mc^2)\psi_A$$

$$-\boldsymbol{\sigma}\cdot(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A})\psi_A = -\frac{1}{c}(E - eA_0 + mc^2)\psi_B$$

((**Note**)) We assume that A_μ is time-independent.

$$\psi_B = \frac{c}{E - eA_0 + mc^2}[\sigma \cdot (\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})]\psi_A$$

Substitution of this eq. into the first equation

$$[\sigma \cdot (\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})]\frac{c}{E - eA_0 + mc^2}[\sigma \cdot (\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})]\psi_A = \frac{1}{c}(E - eA_0 - mc^2)\psi_A$$

We now assume that $E \approx mc^2$ and $|eA_0| << mc^2$.

Defining the energy measured from mc^2, we have

$$E^{(NR)} = E - mc^2$$

$$\begin{aligned}
\frac{c}{E - eA_0 + mc^2} &= \frac{1}{2m}\left(\frac{2mc^2}{2mc^2 + E^{(NR)} - eA_0}\right) \\
&= \frac{1}{2m}\frac{1}{1 + \frac{E^{(NR)} - eA_0}{2mc^2}} \\
&= \frac{1}{2m}[1 - (\frac{E^{(NR)} - eA_0}{2mc^2}) + ...] \\
&= \frac{1}{2m}
\end{aligned}$$

Then we get

$$\frac{1}{2m}[\boldsymbol{\sigma} \cdot (\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})][\boldsymbol{\sigma} \cdot (\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})]\psi_A = (E^{(NR)} - eA_0)\psi_A$$

of

$$[\frac{1}{2m}(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})^2 - \frac{e\hbar}{2mc}\boldsymbol{\sigma} \cdot \boldsymbol{B} + eA_0]\psi_A = E^{(NR)}\psi_A$$

((**Note**))

$$\begin{aligned}
[\boldsymbol{\sigma} \cdot (\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})][\boldsymbol{\sigma} \cdot (\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})] &= (\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})^2 + i\boldsymbol{\sigma} \cdot [(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}) \times (\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})] \\
&= (\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})^2 - \frac{e\hbar}{c}\boldsymbol{\sigma} \cdot B
\end{aligned}$$

since

$$(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A})\times(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A})=-\frac{e}{c}(\boldsymbol{p}\times\boldsymbol{A}+\boldsymbol{A}\times\boldsymbol{P})$$
$$=\frac{ie\hbar}{c}\nabla\times\boldsymbol{A}$$
$$=\frac{ie\hbar}{c}\boldsymbol{B}$$

((**Comment**))
To zeroth order in $(v/c)^2$, ψ_A is nothing more than the Schrödinger-Pauli two component wave function in nonrelativistic quantum mechanics, multiplied by $\exp(-imc^2t/\hbar)$.

ψ_B is "smaller" than ψ_A by a factor of roughly $|\boldsymbol{p}-e\boldsymbol{A}/c|/2mc\approx v/(2c)$ if $E\approx mc^2$ and $|eA_0|<<mc^2$.

For this reason with mc^2, ψ_A and ψ_B are known as the large and small components of the Dirac wave function ψ.

Since

$$-\boldsymbol{\sigma}\cdot(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A})\psi_A=-\frac{1}{c}(E-eA_0+mc^2)\psi_B=-2mc\psi_B$$

$$\psi_B=\frac{1}{2mc}\boldsymbol{\sigma}\cdot(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A})\psi_A$$

17. Approximate Hamiltonian for an electrostatic problem

$$[\boldsymbol{\sigma}\cdot(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A})]\frac{c^2}{E-eA_0+mc^2}[\boldsymbol{\sigma}\cdot(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A})]\psi_A=(E-eA_0-mc^2)\psi_A$$

For simplicity $\boldsymbol{A}=0$.

$$\frac{1}{2m}(\boldsymbol{\sigma}\cdot\boldsymbol{p})[1-\frac{E^{(NR)}-eA_0}{2mc^2}](\boldsymbol{\sigma}\cdot\boldsymbol{p})\psi_A=(E^{(NR)}-eA_0)\psi_A$$

or

$$H_A^{(NR)}\psi_A=E^{(NR)}\psi_A$$

with

$$H_A^{(NR)} = \frac{1}{2m}(\boldsymbol{\sigma}\cdot\boldsymbol{p})[1-(\frac{E^{(NR)}-eA_0}{2mc^2})](\boldsymbol{\sigma}\cdot\boldsymbol{p}) + eA_0 \qquad (1)$$

It might appear that Eq.(1) is the time-independent Schrödinger equation.

However, there are three difficulties with this interpretation.

(1) Normalization

$$\int (\psi_A{}^{+}\psi_A + \psi_B{}^{+}\psi_B) d^3x = 1$$

(2) $H_A^{(NR)}$ contains a non-Hermitian term ($i\hbar\boldsymbol{E}\cdot\boldsymbol{p}$)

(3) Since $H_A^{(NR)}$ contains $E^{(NR)}$ itself, Eq.(1) is not an eigenvalue equation.

Since

$$\psi_B = \frac{1}{2mc}(\boldsymbol{\sigma}\cdot\boldsymbol{p})\psi_A$$

$$\psi_B{}^{+} = \psi_A{}^{+}\frac{1}{2mc}(\boldsymbol{\sigma}\cdot\boldsymbol{p})$$

Normalization:

$$\int (\psi_A{}^{+}\psi_A + \frac{1}{4m^2c^2}\psi_B{}^{+}(\boldsymbol{\sigma}\cdot\boldsymbol{p})\psi_A) d^3x = 1$$

to order $(v/c)^2$.

This suggests that we should work with a new-two component wave function Ψ defined by

$$\Psi = \Omega\psi_A$$

or

$$\psi_A = \Omega^{-1}\Psi$$

where

$$\Omega = 1 + \frac{\boldsymbol{p}^2}{8m^2c^2}$$

With this choice

$$\int \Psi^+ \Psi d^3x \approx \int (\psi_A{}^+ (1 + \frac{\boldsymbol{p}^2}{8m^2c^2})^2 \psi_A d^3x$$
$$= \int (\psi_A{}^+ (1 + \frac{\boldsymbol{p}^2}{4m^2c^2}) \psi_A d^3x$$
$$= 1$$

$$H_A^{(NR)} \psi_A = E^{(NR)} \psi_A$$

$$H_A^{(NR)} \Omega^{-1} \Psi = E^{(NR)} \Omega^{-1} \Psi$$

or

$$\Omega^{-1} H_A^{(NR)} \Omega^{-1} \Psi = E^{(NR)} \Omega^{-2} \Psi$$

$$\Omega^{-1} H_A^{(NR)} \Omega^{-1} = (1 - \frac{\boldsymbol{p}^2}{8m^2c^2}) H_A^{(NR)} (1 - \frac{\boldsymbol{p}^2}{8m^2c^2})$$
$$= (H_A^{(NR)} - \frac{\boldsymbol{p}^2}{8m^2c^2} H_A^{(NR)})(1 - \frac{\boldsymbol{p}^2}{8m^2c^2})$$
$$= H_A^{(NR)} - \{\frac{\boldsymbol{p}^2}{8m^2c^2}, H_A^{(NR)}\}$$
$$= H_A^{(NR)} - \{\frac{\boldsymbol{p}^2}{8m^2c^2}, \frac{1}{2m}\boldsymbol{p}^2 + eA_0\}$$

where

$$H_A^{(NR)} = \frac{1}{2m}\boldsymbol{p}^2 + eA_0 - \frac{1}{2m}(\boldsymbol{\sigma} \cdot \boldsymbol{p})(\frac{E^{(NR)} - eA_0}{2mc^2})](\boldsymbol{\sigma} \cdot \boldsymbol{p})$$

$$E^{(NR)} \Omega^{-2} \Psi = E^{(NR)} (1 - \frac{\boldsymbol{p}^2}{4m^2c^2}) \Psi$$

Thus we have

$$[\frac{1}{2m}\boldsymbol{p}^2 + eA_0 - \{\frac{\boldsymbol{p}^2}{8m^2c^2}, \frac{1}{2m}\boldsymbol{p}^2 + eA_0\}$$
$$- \frac{1}{2m}(\boldsymbol{\sigma} \cdot \boldsymbol{p})(\frac{E^{(NR)} - eA_0}{2mc^2})](\boldsymbol{\sigma} \cdot \boldsymbol{p})] \Psi = E^{(NR)} (1 - \frac{\boldsymbol{p}^2}{4m^2c^2}) \Psi$$

Note

$$-\{\frac{\boldsymbol{p}^2}{8m^2c^2},\frac{1}{2m}\boldsymbol{p}^2+eA_0\}+E^{(NR)}\frac{\boldsymbol{p}^2}{4m^2c^2}=-\frac{\boldsymbol{p}^4}{8m^2c^2}+\frac{1}{8m^2c^2}\{\boldsymbol{p}^2,E^{(NR)}-eA_0\}$$

Then we have

$$\begin{aligned}
&[\frac{1}{2m}\boldsymbol{p}^2+eA_0-\frac{\boldsymbol{p}^4}{8m^2c^2}\\
&+\frac{1}{8m^2c^2}[\{\boldsymbol{p}^2,E^{(NR)}-eA_0\}-2(\boldsymbol{\sigma}\cdot\boldsymbol{p})(E^{(NR)}-eA_0)](\boldsymbol{\sigma}\cdot\boldsymbol{p})]\Psi\\
&=E^{(NR)}\Psi
\end{aligned}$$

Here we use the formula

$$\{A^2,B\}=2ABA+[A,[A,B]]$$

When

$$A=\boldsymbol{\sigma}\cdot\boldsymbol{p}, \qquad B=E^{(NR)}-eA_0$$

$$A^2=(\boldsymbol{\sigma}\cdot\boldsymbol{p})(\boldsymbol{\sigma}\cdot\boldsymbol{p})=\boldsymbol{p}^2$$

Thus we have

$$\begin{aligned}
\{\boldsymbol{p}^2,E^{(NR)}-eA_0\}&=2(\boldsymbol{\sigma}\cdot\boldsymbol{p})(E^{(NR)}-eA_0)(\boldsymbol{\sigma}\cdot\boldsymbol{p})\\
&\quad+[(\boldsymbol{\sigma}\cdot\boldsymbol{p}),[(\boldsymbol{\sigma}\cdot\boldsymbol{p}),E^{(NR)}-eA_0]]
\end{aligned}$$

or

$$\{\boldsymbol{p}^2,E^{(NR)}-eA_0\}-2(\boldsymbol{\sigma}\cdot\boldsymbol{p})(E^{(NR)}-eA_0)(\boldsymbol{\sigma}\cdot\boldsymbol{p})=[(\boldsymbol{\sigma}\cdot\boldsymbol{p}),[(\boldsymbol{\sigma}\cdot\boldsymbol{p}),E^{(NR)}-eA_0]]$$

Here

$$\begin{aligned}
[(\boldsymbol{\sigma}\cdot\boldsymbol{p}),E^{(NR)}-eA_0]&=[(\boldsymbol{\sigma}\cdot\boldsymbol{p}),-eA_0]\\
&=-e\{(\boldsymbol{\sigma}\cdot\boldsymbol{p})A_0-A_0(\boldsymbol{\sigma}\cdot\boldsymbol{p})\}\\
&=-e\boldsymbol{\sigma}\cdot[\boldsymbol{p},A_0]\\
&=-e\boldsymbol{\sigma}\cdot\frac{\hbar}{i}\nabla A_0\\
&=-ie\hbar\boldsymbol{\sigma}\cdot\boldsymbol{E}
\end{aligned}$$

Note that $A_0\sigma=\sigma A_0$

$$[(\boldsymbol{\sigma}\cdot\boldsymbol{p}),[(\boldsymbol{\sigma}\cdot\boldsymbol{p}),E^{(NR)}-eA_0]]=[(\boldsymbol{\sigma}\cdot\boldsymbol{p}),-ie\hbar(\boldsymbol{\sigma}\cdot\boldsymbol{E})]$$
$$=-ie\hbar[(\boldsymbol{\sigma}\cdot\boldsymbol{p}),(\boldsymbol{\sigma}\cdot\boldsymbol{E})]$$
$$=-ie\hbar[(\sigma\cdot\boldsymbol{p})(\sigma\cdot\boldsymbol{E})-(\sigma\cdot\boldsymbol{E})(\sigma\cdot\boldsymbol{p})]$$

Note that

$$(\boldsymbol{\sigma}\cdot\boldsymbol{p})(\boldsymbol{\sigma}\cdot\boldsymbol{E})=\boldsymbol{p}\cdot\boldsymbol{E}+i\boldsymbol{\sigma}\cdot(\boldsymbol{p}\times\boldsymbol{E})$$

and

$$(\boldsymbol{\sigma}\cdot\boldsymbol{E})(\boldsymbol{\sigma}\cdot\boldsymbol{p})=\boldsymbol{E}\cdot\boldsymbol{p}+i\boldsymbol{\sigma}\cdot(\boldsymbol{E}\times\boldsymbol{p})$$

Then we have

$$(\boldsymbol{\sigma}\cdot\boldsymbol{p})(\boldsymbol{\sigma}\cdot\boldsymbol{E})-(\boldsymbol{\sigma}\cdot\boldsymbol{E})(\boldsymbol{\sigma}\cdot\boldsymbol{p})=\frac{\hbar}{i}\nabla\cdot\boldsymbol{E}-2i\boldsymbol{\sigma}\cdot(\boldsymbol{E}\times\boldsymbol{p})$$

Finally we obtain

$$\{\frac{1}{2m}\boldsymbol{p}^2+eA_0-\frac{\boldsymbol{p}^4}{8m^2c^2}+\frac{1}{8m^2c^2}(-ie\hbar)[\frac{\hbar}{i}\nabla\cdot\boldsymbol{E}-2i\boldsymbol{\sigma}\cdot(\boldsymbol{E}\times\boldsymbol{p})]\}\Psi=E^{(NR)}\Psi$$

or

$$[\frac{1}{2m}\boldsymbol{p}^2+eA_0-\frac{\boldsymbol{p}^4}{8m^2c^2}-\frac{e\hbar}{4m^2c^2}\boldsymbol{\sigma}\cdot(\boldsymbol{E}\times\boldsymbol{p})-\frac{e\hbar^2}{8m^2c^2}\nabla\cdot\boldsymbol{E}]\Psi=E^{(NR)}\Psi$$

((Physical meaning))

Third term: relativistic correction

$$\sqrt{m^2c^4+\boldsymbol{p}^2c^2}-mc^2=mc^2\sqrt{1+\frac{\boldsymbol{p}^2}{m^2c^2}}-mc^2$$
$$=mc^2[1+\frac{\boldsymbol{p}^2}{2m^2c^2}-\frac{1}{8}\frac{\boldsymbol{p}^4}{m^4c^4}+...]-mc^2$$
$$=\frac{\boldsymbol{p}^2}{2m}-\frac{1}{8}\frac{\boldsymbol{p}^4}{m^3c^2}+...$$

The fourth term (Thomas correction)

$$\text{Thomas term}=-\frac{e\hbar}{4m^2c^2}\boldsymbol{\sigma}\cdot(\boldsymbol{E}\times\boldsymbol{p})$$

For a central potential

$$eA_0 = V(r)$$

$$\boldsymbol{E} = -\nabla A_0 = -\frac{1}{r}\frac{dV}{dr}\boldsymbol{r}$$

$$\boldsymbol{E}\times\boldsymbol{p} = -\frac{1}{r}\frac{dV}{dr}(\boldsymbol{r}\times\boldsymbol{p}) = -\frac{1}{r}\frac{dV}{dr}\boldsymbol{L}$$

where $\boldsymbol{L}$ is an orbital angular momentum. Then the Thomas term is rewritten as

$$-\frac{e\hbar}{4m^2c^2}\boldsymbol{\sigma}\cdot(\boldsymbol{E}\times\boldsymbol{p}) = -\frac{e\hbar}{4m^2c^2}(-\frac{1}{r}\frac{dV}{dr})\boldsymbol{\sigma}\cdot\boldsymbol{L}$$
$$= \frac{e}{2m^2c^2}\frac{1}{r}\frac{dV}{dr}\boldsymbol{S}\cdot\boldsymbol{L}$$

(Spin-orbit interaction)

The spin angular momentum is defined by

$$\boldsymbol{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$$

which is an automatic consequence of the Dirac theory.

The last term is called the Darwin term.

For a hydrogen atom,

$$\nabla\cdot\boldsymbol{E} = -\delta^{(3)}(\boldsymbol{r}).$$

It gives rise to an energy shift

$$\int\frac{e^2\hbar^2}{8m^2c^2}\delta^{(3)}(\mathbf{r})\left|\psi(\mathbf{r})^{(Schrodinger)}\right|^2 d^3x = \frac{e^2\hbar^2}{8m^2c^2}\left|\psi(\mathbf{r})^{(Schrodinger)}\right|^2_{\mathbf{r}=0}$$

which is non-vanishing only for the *s* state.

18. Free particle at rest

Each component of the four-component wave function satisfies the Klein-Gordon equation if the particle is free.

$$(i\gamma^{\mu}\partial_{\mu} - \frac{mc}{\hbar})\psi = 0 \qquad (1)$$

$$\gamma^{k} = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

$$\gamma^{0} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$$

Multiplying Eq.(1) from the left by $\gamma^{\mu}\partial_{\mu}$

$$i\gamma^{\nu}\partial_{\nu}(\gamma^{\mu}\partial_{\mu})\psi - \frac{mc}{\hbar}\gamma^{\nu}\partial_{\nu}\psi = 0$$

or

$$i\gamma^{\nu}\gamma^{\mu}\partial_{\mu}\partial_{\nu}\psi - \frac{mc}{\hbar}\gamma^{\nu}\partial_{\nu}\psi = 0$$

or

$$\frac{1}{2}(\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu})\partial_{\mu}\partial_{\nu}\psi - (\frac{mc}{\hbar})^2\psi = 0$$

Since $\{\gamma^{\nu}, \gamma^{\mu}\} = 2g^{\mu\nu}I_4$, we have

$$g^{\mu\nu}\partial_{\mu}\partial_{\nu}\psi\psi - (\frac{mc}{\hbar})^2\psi = 0$$

or

$$\partial^{\mu}\partial_{\mu}\psi - (\frac{mc}{\hbar})^2\psi = 0 \qquad (2)$$

Note that Eq.(2) is to be understood as four separate uncoupled equations for each component of ψ. Because of Eq.(2), the Dirac equation admits a free particle solution of the type

$$\psi \approx u(\boldsymbol{p})\exp[\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r} - Et)]$$

with

$$E = \pm\sqrt{c^2\boldsymbol{p}^2 + m^2c^4}$$

$u(\boldsymbol{p})$ is a four-component spinor independent of $\boldsymbol{r}$ and t.

((Note)) The following relations are always valid.

$$i\hbar\frac{\partial}{\partial t}\psi = E\psi\,, \qquad \frac{\hbar}{i}\nabla\psi = \boldsymbol{p}\psi$$

For a particle at rest ($\boldsymbol{p}$ = 0)

$$(i\gamma^{\mu}\partial_{\mu} - \frac{mc}{\hbar})\psi = 0$$

or

$$[i(\gamma^1\frac{\partial}{\partial x^1} + \gamma^2\frac{\partial}{\partial x^2} + \gamma^3\frac{\partial}{\partial x^3} + \gamma^0\frac{\partial}{\partial x^0}) - \frac{mc}{\hbar})\psi = 0$$

Since

$$\frac{\hbar}{i}\nabla\psi = \boldsymbol{p}\psi = 0\,, \qquad E = \pm mc^2$$

$$[i\gamma^0\frac{\partial}{\partial(ct)} - \frac{mc}{\hbar}]\psi = 0$$

or

$$[i\frac{1}{c}\gamma^0\frac{E}{i\hbar} - \frac{mc}{\hbar}]\psi = 0$$

or

$$\frac{E}{\hbar c}\gamma^0\psi = \frac{mc}{\hbar}\psi$$

(i) For $E = mc^2$,

$$u = \begin{pmatrix} u_A(\boldsymbol{p}=0) \\ u_B(\boldsymbol{p}=0) \end{pmatrix}$$

$$\frac{mc^2}{\hbar c}\begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}\begin{pmatrix} u_A(\boldsymbol{p}=0) \\ u_B(\boldsymbol{p}=0) \end{pmatrix} = \frac{mc}{\hbar}\begin{pmatrix} u_A(\boldsymbol{p}=0) \\ u_B(\boldsymbol{p}=0) \end{pmatrix}$$

or

$$\begin{pmatrix} u_A(\boldsymbol{p}=0) \\ -u_B(\boldsymbol{p}=0) \end{pmatrix} = \begin{pmatrix} u_A(\boldsymbol{p}=0) \\ u_B(\boldsymbol{p}=0) \end{pmatrix}$$

or

$$u_B(\boldsymbol{p}=0) = 0$$

(ii) For $E = -mc^2$,

$$u = \begin{pmatrix} u_A(\boldsymbol{p}=0) \\ u_B(\boldsymbol{p}=0) \end{pmatrix}$$

$$-\frac{mc^2}{\hbar c}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} u_A(\boldsymbol{p}=0) \\ u_B(\boldsymbol{p}=0) \end{pmatrix} = \frac{mc}{\hbar}\begin{pmatrix} u_A(\boldsymbol{p}=0) \\ u_B(\boldsymbol{p}=0) \end{pmatrix}$$

or

$$\begin{pmatrix} -u_A(\boldsymbol{p}=0) \\ u_B(\boldsymbol{p}=0) \end{pmatrix} = \begin{pmatrix} u_A(\boldsymbol{p}=0) \\ u_B(\boldsymbol{p}=0) \end{pmatrix}$$

$$u_A(\boldsymbol{p}=0) = 0$$

So there are four independent solutions

Positive energy solution

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \exp(-i\frac{mc^2 t}{\hbar}), \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \exp(-i\frac{mc^2 t}{\hbar}),$$

spin up   spin down

Negative energy solution

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \exp(i\frac{mc^2 t}{\hbar}), \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \exp(i\frac{mc^2 t}{\hbar}),$$

The existence of negative-energy solutions is intimately related to the fact that the Dirac theory can accommodate a positron.

((**Note**))
Nonrelativistic limit $E = mc^2$, the upper two component spinor ψ_A coincides with the Schrödinger wave function apart from the factor $e^{-imc^2 t/\hbar}$.

Let us define

$$\Sigma^3 = \frac{1}{2}[\gamma^1, \gamma^2] = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}$$

$$\Sigma^3 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The eigenstate of Σ^3 is interpreted as the spin component in the positive *z*-direction in units of $\hbar/2$.

19. Plane wave solutions ($p \neq 0$).

$$\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \begin{pmatrix} u_A(\boldsymbol{p}) \\ u_B(\boldsymbol{p}) \end{pmatrix} \exp[\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r} - Et)]$$

or

$$\frac{1}{c}(i\hbar\frac{\partial}{\partial t} - mc^2)\psi_A = (\boldsymbol{\sigma}\cdot\boldsymbol{p})\psi_B ,$$

$$\frac{1}{c}(i\hbar\frac{\partial}{\partial t} + mc^2)\psi_B = (\boldsymbol{\sigma}\cdot\boldsymbol{p})\psi_A$$

or

$$(\boldsymbol{\sigma}\cdot\boldsymbol{p})u_B = \frac{1}{c}(E - mc^2)u_A , \qquad (\boldsymbol{\sigma}\cdot\boldsymbol{p})u_A = \frac{1}{c}(E + mc^2)u_B$$

or

$$u_A(\boldsymbol{p}) = \frac{c}{E - mc^2}(\boldsymbol{\sigma}\cdot\boldsymbol{p})u_B(\boldsymbol{p})$$

$$u_B(\boldsymbol{p}) = \frac{c}{E + mc^2}(\boldsymbol{\sigma}\cdot\boldsymbol{p})u_A(\boldsymbol{p})$$

For simplicity we use

$$R = \sqrt{\boldsymbol{p}^2c^2 + m^2c^4}$$

(i) For $E = R > 0$ (positive energy state)

$$u_A^{(1)}(\boldsymbol{p}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad u_A^{(2)}(\boldsymbol{p}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$u_B^{(1)}(\boldsymbol{p}) = \frac{c}{R+mc^2}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{cp_z}{R+mc^2} \\ \dfrac{c(p_x+ip_y)}{R+mc^2} \end{pmatrix}$$

$$u_B^{(2)}(\boldsymbol{p}) = \frac{c}{R+mc^2}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{c(p_x-ip_y)}{R+mc^2} \\ -\dfrac{cp_z}{R+mc^2} \end{pmatrix}$$

Then we have

$$u^{(1)}(\boldsymbol{p}) = \sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} 1 \\ 0 \\ \dfrac{cp_z}{R+mc^2} \\ \dfrac{c(p_x+ip_y)}{R+mc^2} \end{pmatrix},$$

$$u^{(2)}(\boldsymbol{p}) = \sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} 0 \\ 1 \\ \dfrac{c(p_x-ip_y)}{R+mc^2} \\ -\dfrac{cp_z}{R+mc^2} \end{pmatrix}$$

We take into account of the normalization factor.

(ii) For $E = -R < 0$

$$u_B^{(1)}(\boldsymbol{p}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad u_B^{(2)}(\boldsymbol{p}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$u_A^{(1)}(\boldsymbol{p}) = \frac{-c}{R+mc^2}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\dfrac{cp_z}{R+mc^2} \\ -\dfrac{c(p_x+ip_y)}{R+mc^2} \end{pmatrix}$$

$$u_A^{(2)}(\boldsymbol{p}) = \frac{-c}{R+mc^2}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\dfrac{c(p_x-ip_y)}{R+mc^2} \\ \dfrac{cp_z}{R+mc^2} \end{pmatrix}$$

Then we have

$$u^{(3)}(\boldsymbol{p}) = \sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} -\dfrac{cp_z}{R+mc^2} \\ -\dfrac{c(p_x+ip_y)}{R+mc^2} \\ 1 \\ 0 \end{pmatrix},$$

$$u^{(4)}(\boldsymbol{p}) = \sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} -\dfrac{c(p_x-ip_y)}{R+mc^2} \\ \dfrac{cp_z}{R+mc^2} \\ 0 \\ 1 \end{pmatrix}$$

---

20. Formulation

Since

$$u^{(r)}(\boldsymbol{p})\exp[\frac{i}{\hbar}\{\boldsymbol{p}\cdot\boldsymbol{r} - Et)] = u^{(r)}(\boldsymbol{p})\exp[-\frac{i}{\hbar}p_\mu x^\mu]$$

satisfies the free-field Dirac equation

$$(i\gamma^{\mu}\partial_{\mu} - \frac{mc}{\hbar})\psi = 0\,,$$

$$\frac{\partial}{\partial x^{\mu}}\{u^{(r)}(\boldsymbol{p})u^{(r)}(\boldsymbol{p})\exp[-\frac{i}{\hbar}p_{\mu}x^{\mu}\} = u^{(r)}(\boldsymbol{p})\exp[\frac{i}{\hbar}\{\boldsymbol{p}\cdot\boldsymbol{r} - Et)](-\frac{i}{\hbar}p_{\mu})$$

Since

$$p_{\mu} = (\frac{E}{c}, \boldsymbol{p})$$

we get

$$(-\gamma^{\mu}p_{\mu} + mc)u^{(r)}(p) = 0$$

regardless of whether $E>0$ or $E<0$.

$$\gamma^{\mu}p_{\mu} - mcI_4 = \begin{pmatrix} -mc + p_0 & 0 & p_3 & p_1 - ip_2 \\ 0 & -mc + p_0 & p_1 + ip_2 & p_3 \\ -p3 & -(p_1 - ip_2) & -mc - p_0 & 0 \\ -(p_1 + ip_2) & -p_3 & 0 & -mc - p_0 \end{pmatrix}$$

$$= \begin{pmatrix} -mc + \frac{E}{c} & 0 & p_z & p_x - ip_y \\ 0 & -mc + \frac{E}{c} & p_x + ip_y & p_z \\ -p_z & -(p_x - ip_y) & -mc - \frac{E}{c} & 0 \\ -(p_x + ip_y) & -p_z & 0 & -mc - \frac{E}{c} \end{pmatrix}$$

where

$$p_1 = -p_x\,, \qquad p_2 = -p_y\,, \qquad p_3 = -p_z\,, \qquad p_0 = \frac{E}{c}$$

In summary we have

$$u^{(1)}(\boldsymbol{p}) = \sqrt{\frac{R+mc^2}{2R}} \begin{pmatrix} 1 \\ 0 \\ \dfrac{cp_z}{R+mc^2} \\ \dfrac{c(p_x+ip_y)}{R+mc^2} \end{pmatrix}, \qquad (E>0)$$

$$u^{(2)}(\boldsymbol{p}) = \sqrt{\frac{R+mc^2}{2R}} \begin{pmatrix} 0 \\ 1 \\ \dfrac{c(p_x-ip_y)}{R+mc^2} \\ -\dfrac{cp_z}{R+mc^2} \end{pmatrix}, \qquad (E>0)$$

$$u^{(3)}(\boldsymbol{p}) = \sqrt{\frac{R+mc^2}{2R}} \begin{pmatrix} -\dfrac{cp_z}{R+mc^2} \\ -\dfrac{c(p_x+ip_y)}{R+mc^2} \\ 1 \\ 0 \end{pmatrix}, \qquad (E<0)$$

$$u^{(4)}(\boldsymbol{p}) = \sqrt{\frac{R+mc^2}{2R}} \begin{pmatrix} -\dfrac{c(p_x-ip_y)}{R+mc^2} \\ \dfrac{cp_z}{R+mc^2} \\ 0 \\ 1 \end{pmatrix}, \qquad (E<0)$$

Suppose that $\boldsymbol{p} = 0$. $R = mc^2$.

$$u^{(1)}(\boldsymbol{p}=0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad u^{(2)}(\boldsymbol{p}=0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

$$u^{(3)}(\boldsymbol{p}=0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, u^{(4)}(\boldsymbol{p}=0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The first two solutions look like the spin state of the non-relativistic theory. They are degenerate and have energy eigenvalue $E = R$. In the same limit, the last two solutions also look like the non-relativistic spin states, but they belong to the energy eigenvalue $E = -R$

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21. Dirac's hole theory

21.1. Overview on Dirac's hole theory

Dirac made the astounding suggestion that all the negative-energy states should be already occupied. This ocean of occupied negative-energy states is now referred to as the 'Dirac sea'. Thus, according to Dirac, the negative energy states are already full up; by the Pauli principle, there is now no room for an electron to fall into such a state. But, as Dirac further reasoned, occasionally there might be a few negative-energy states that are unoccupied. Such a 'hole' in the Dirac sea of negative-energy states would appear just like a positive-energy particle (and hence a positive-mass particle), whose electric charge would be the opposite of the charge on the electron. Such an empty negative-energy state could now be occupied by an ordinary electron; so the electron might 'fall into' that state with the emission of energy (normally in the form of electromagnetic radiation, i.e. photons). This would result in the 'hole' and the electron annihilating one another in the manner that we now understand as a particle and its anti-particle undergoing mutual annihilation.

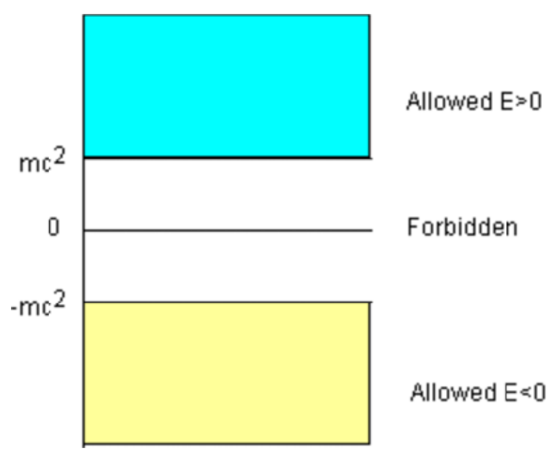


Fig. $2mc^2$ = 1.02 MeV.

Conversely, if a hole were not present initially, but a sufficient amount of energy (say in the form of photons) enters the system, then an electron can be kicked out of one of the negative-energy states to leave a hole. Dirac's 'hole' is indeed the electron's antiparticle, now referred to as the positron.

At first Dirac was cautious about making the claim that his theory actually predicted the existence of antiparticles to electrons, initially thinking (in 1929) that the 'holes' could be protons, which were the only massive particles known at the time having a positive charge. But it was not long before it became clear that the mass of each hole had to be equal to the mass of the electron, rather than the mass of a proton, which is about

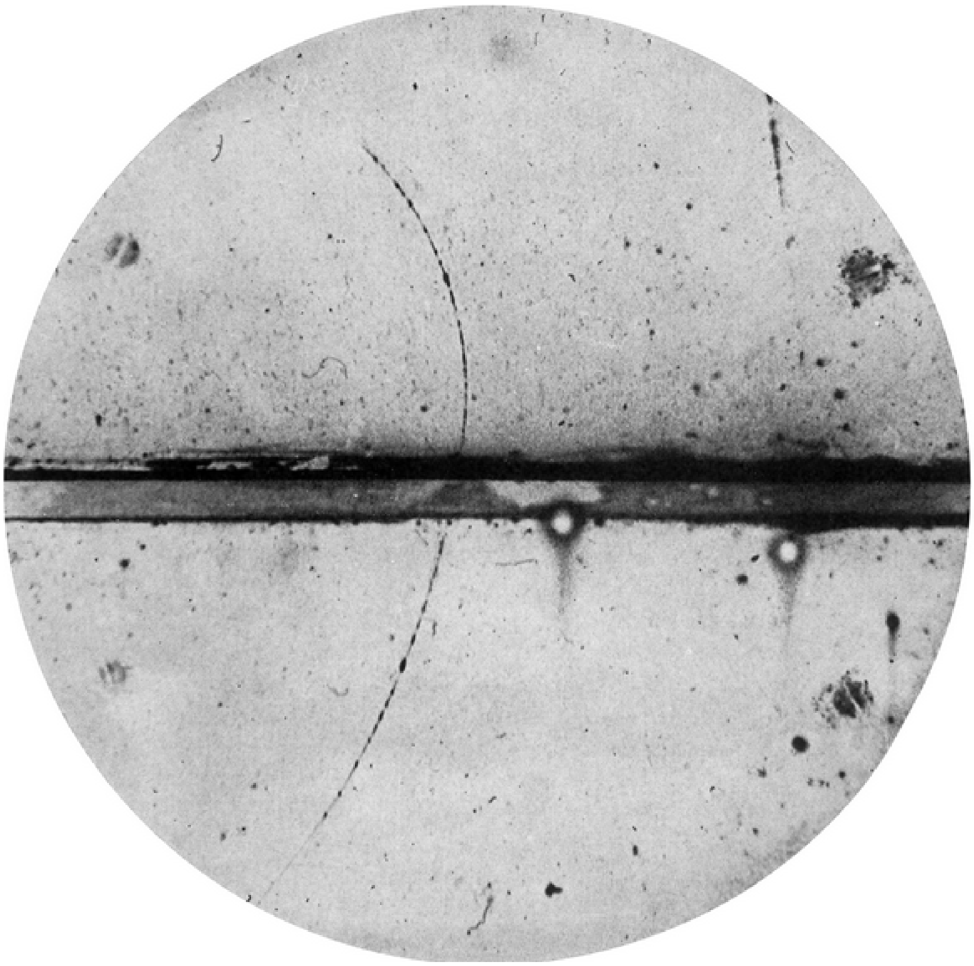
1836 times larger. In the year 1931, Dirac came to the conclusion that the holes must be 'anti-electrons'—previously unknown particles that we now call positrons.

22.2 The discovery of positron by Carl D. Anderson

In the next year after Dirac's theoretical prediction, Carl Anderson announced the discovery of a particle which indeed had the properties that Dirac had predicted: the first antiparticle had been found!

Carl David Anderson (September 3, 1905 – January 11, 1991) was an American physicist. He is best known for his discovery of the positron in 1932, an achievement for which he received the 1936 Nobel Prize in Physics, and of the muon in 1936. Anderson was born in New York City, the son of Swedish immigrants. He studied physics and engineering at Caltech (B.S., 1927; Ph.D., 1930). Under the supervision of Robert A. Millikan, he began investigations into cosmic rays during the course of which he encountered unexpected particle tracks in his (modern versions now commonly referred to as an Anderson) cloud chamber photographs that he correctly interpreted as having been created by a particle with the same mass as the electron, but with opposite electrical charge. This discovery, announced in 1932 and later confirmed by others, validated Paul Dirac's theoretical prediction of the existence of the positron. Anderson first detected the particles in cosmic rays. He then produced more conclusive proof by shooting gamma rays produced by the natural radioactive nuclide ThC" (^{208}Tl) into other materials, resulting in the creation of positron-electron pairs. For this work, Anderson shared the 1936 Nobel Prize in Physics with Victor Hess.

http://en.wikipedia.org/wiki/Carl_David_Anderson



Cloud chamber photograph of the first positron ever observed Original caption: A 63 million volt positron ($H\rho$ = 2.1×105 gauss-cm) passing through a 6 mm lead plate and emerging as a 23 million volt positron ($H\rho$ = 7.5×104 gauss-cm). The length of this latter path is at least ten times greater than the possible length of a proton path of this curvature.
https://upload.wikimedia.org/wikipedia/commons/6/69/PositronDiscovery.jpg

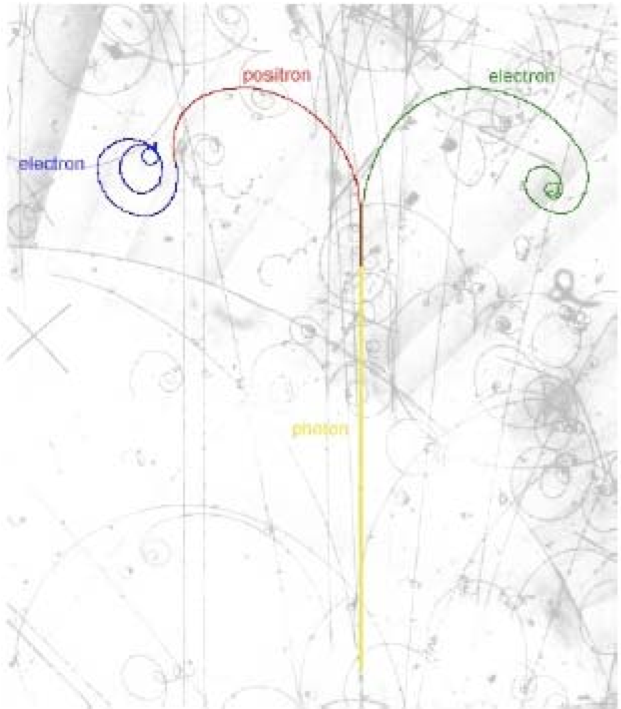


Fig. Creation of electron-positron pair due to the high energy photon. The magnetic field is applied (into the page).

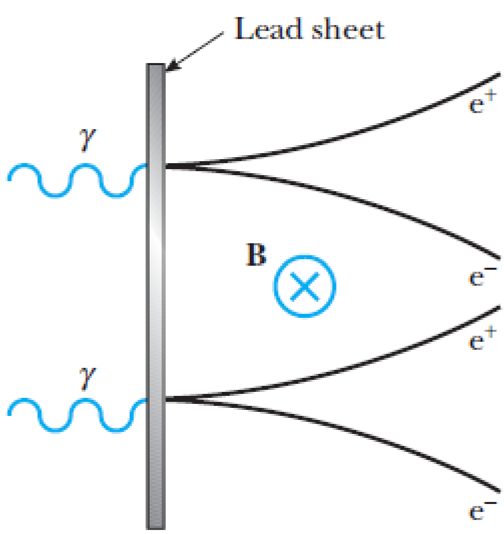


Fig. From Modern Physics, 3rd edition, Serway, Moses, and Moyer (Thomson Brooks/Cole, 2005).

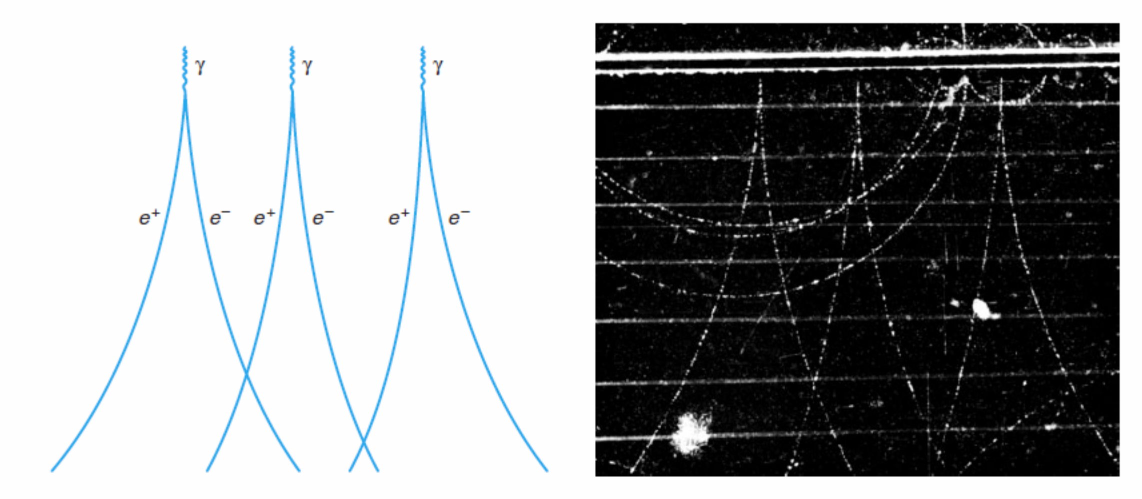


Fig. From R.A. Tipler and R.A. Llewellyn, Modern Physics 5-th edition (W.H. Freeman, 2008). The magnetic field in the chamber points out of the page.

22.3. Dirac sea

The Dirac equation for the free particle leads to a negative energy solution as well as a positive energy solution. The positive energy solutions and the negative solutions are separated by a gap shown in **Fig.1**. Classically, no transition is expected between the energy gap (= 2 mc^2). So we can restrict the energy to be positive classically. On the other hand, in quantum mechanics, it is expected that the transition can occur. Since electrons are fermions, all the negative-energy levels are filled with electrons, in accord with the Pauli exclusion principle. The vacuum state (so called Dirac sea) is one with all negative-energy levels filled and all positive-energy level empty. We note that the Dirac sea is a theoretical model of the vacuum as an infinite sea of particles with negative energy ($E<-mc^2$). The positron, the antimatter counterpart of the electron, was originally conceived of as a hole in the Dirac sea, well before its experimental discovery in 1932 (C.D. Anderson).

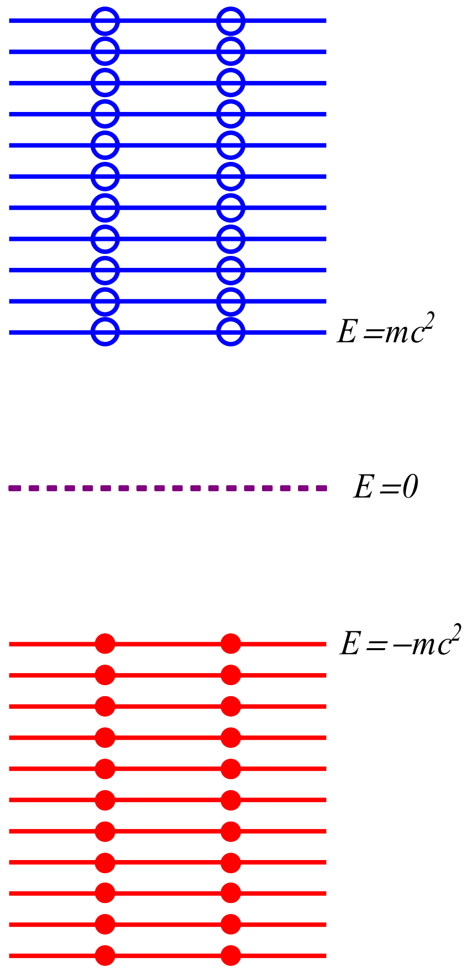


Fig.1 The Dirac sea. Positrons as holes in the Dirac sea of negative-energy electron states. Dirac proposed that almost all negative energy states of the electron are filled. Pauli principle prevents an electron from falling into such a filled state. The electron states with the energy above mc^2 (denoted by blue open circles) are empty.

Charge:

$$Q_{hole} = (Q_{vacuum} - (-|e|) - Q_{vacuum} = |e|$$

where Q_{vacuum} is infinite but we have seen such infinite renormalization before.

Momentum:

$$\boldsymbol{P}_{hole} = (\boldsymbol{P}_{vacuum} - \boldsymbol{p}) - \boldsymbol{P}_{vacuum} = -\boldsymbol{p}$$

where $\boldsymbol{P}_{vacuum} = 0$ since for each negative energy state with p there is another with (-$\boldsymbol{p}$)

Energy:

$$E_{hole} = [E_{vacuum} - (-R)] - E_{vacuum} = +R = \sqrt{m^2c^4 + c^2\boldsymbol{p}^2}$$

Spin:

$$\frac{\hbar}{2}\boldsymbol{\Sigma}_{hole} = (\frac{\hbar}{2}\boldsymbol{\Sigma}_{vacuum} - \frac{\hbar}{2}\boldsymbol{\Sigma}) - \frac{\hbar}{2}\boldsymbol{\Sigma}_{vacuum} = -\frac{\hbar}{2}\boldsymbol{\Sigma}$$

In summary

The positron is the antiparticle or the antimatter counterpart of the electron. The positron has an electric charge of +|*e*|, a spin of ½, and has the same mass as an electron.

Positive energy	$R = \sqrt{m^2c^4 + c^2\boldsymbol{p}^2}$
Positive charge	$\lvert e\rvert$ (>0)
Momentum	**-*p***
Spin	$-\frac{\hbar}{2}\boldsymbol{\Sigma}$
Helicity	$\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}}$

22.4 Pair production

If sufficient energy (more than $2mc^2$) is given to the system in the form of radiation, one of the negative energy electrons is excited into an empty state with a positive energy. Thus we observe an electron of charge -|*e*| and energy *R*, and in addition a hole in the Dirac sea. This hole (anti-particle) has the same mass as the electron but opposite charge. This hole is called the positron. The process is called the pair production. The hole registers the absence of an electron of charge -|*e*| and energy –*R*, and would be interpreted by an observer relative to the vacuum as the presence a particle of charge |e| and energy *R*.

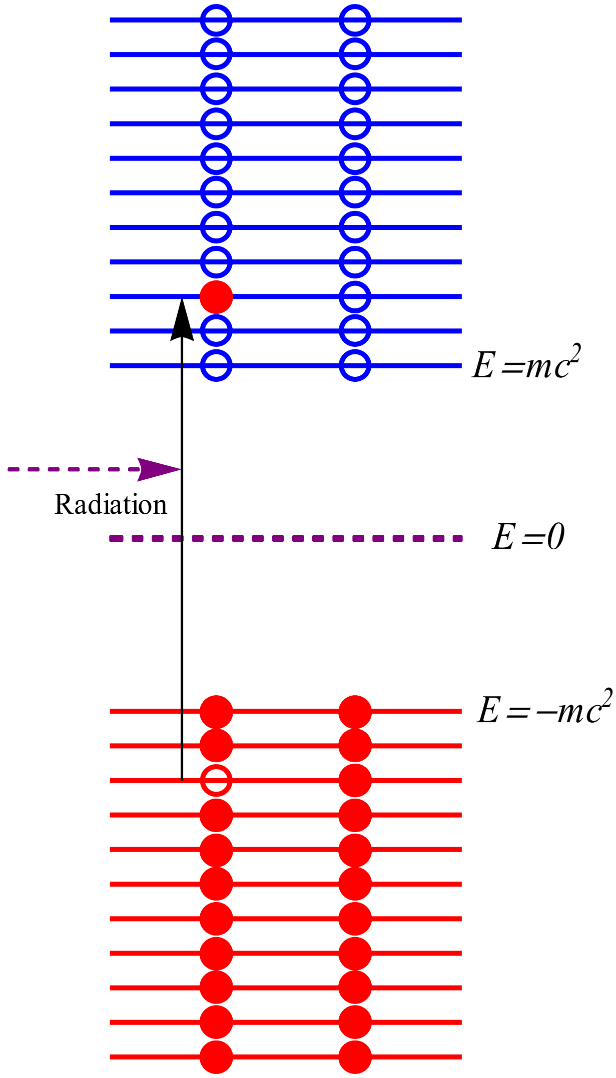


Fig.2 The pair production. The supplying of sufficient energy to the Dirac sea could produce an electron-positron pair: $\gamma \rightarrow e^{+} + e$.

22.5. Annihilation of electron and positron

When a low-energy positron collides with a low-energy electron, annihilation occurs, resulting in the production of gamma ray photons. An electron falling into a hole would be interpreted as the annihilation of the electron and the positron, with the release of energy in the form of radiation

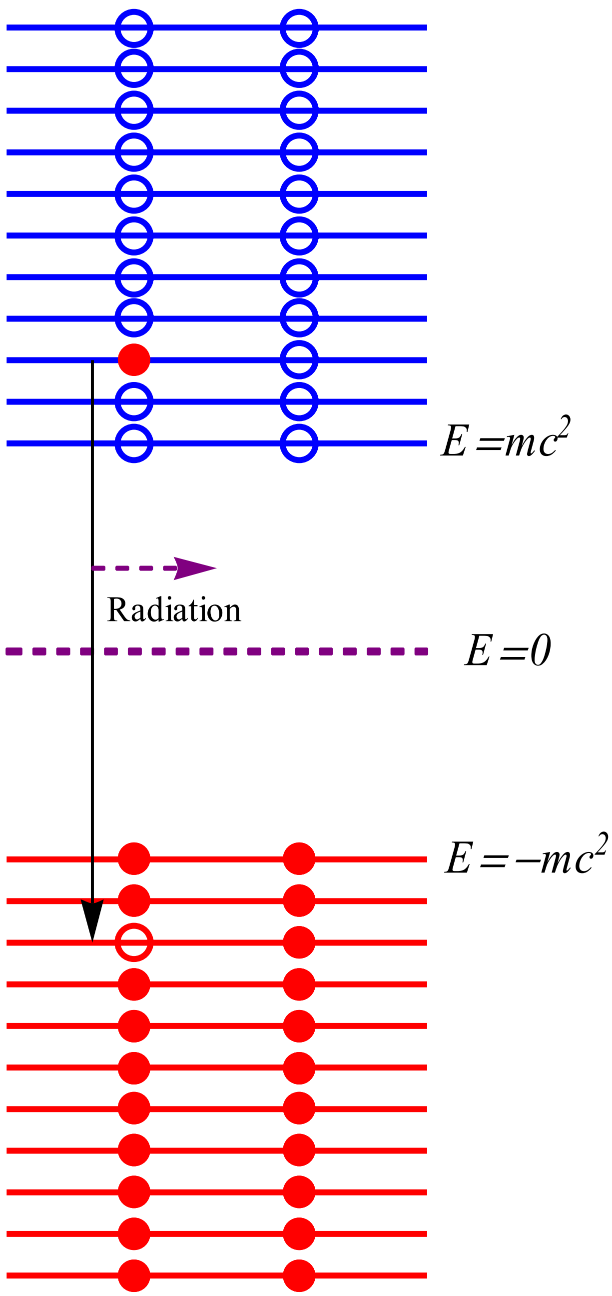


Fig.3 The annihilation of electron and positron; $e- + e+ \rightarrow \gamma + \gamma$

---

23. Orbital angular momentum *L*

The Hamiltonian of the free particle is given by

$$H = c\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta mc^2$$

The orbital angular momentum ***L*** is defined by

$$L_1 = x_2 p_3 - x_3 p_2, \qquad L_2 = x_3 p_1 - x_1 p_3, \qquad L_3 = x_1 p_2 - x_2 p_1$$

We now consider the commutation relation between these operators,

$$\begin{aligned}
[H, L_1] &= [c\alpha_k p_k + \beta mc^2, x_2 p_3 - x_3 p_2] \\
&= [c\alpha_k p_k, x_2 p_3 - x_3 p_2] \\
&= [c\alpha_2 p_2, x_2 p_3] - [c\alpha_3 p_3, x_3 p_2] \\
&= c\alpha_2 [p_2, x_2] p_3 - c\alpha_3 [p_3, x_3] p_2 \\
&= \frac{c\hbar}{i}(\alpha_2 p_3 - \alpha_3 p_2) \\
&= \frac{c\hbar}{i}(\boldsymbol{\alpha} \times \boldsymbol{p})_1
\end{aligned}$$

Then we get the Heisenberg's equation;

$$\frac{d\boldsymbol{L}}{dt} = \frac{i}{\hbar}[H, \boldsymbol{L}] = c(\boldsymbol{\alpha} \times \boldsymbol{p}).$$

24. Spin angular momentum $\frac{\hbar}{2}\boldsymbol{\Sigma}$

Here we note that

$$\Sigma^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} = i\gamma^i \gamma^j \qquad (i, j, k;\ \text{cyclic})$$

or, simply,

$$\Sigma^1 = i\gamma^2\gamma^3, \qquad \Sigma^2 = i\gamma^3\gamma^1, \qquad \Sigma^3 = i\gamma^1\gamma^2$$

$$\alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix} = \gamma^0\gamma^k = \Sigma^k\gamma^5 = \gamma^5\Sigma^k$$

or, simply,

$$\boldsymbol{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$$

$$\alpha^1 = \alpha_x = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}, \qquad \alpha^2 = \alpha_y = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix},$$

$$\alpha^3 = \alpha_z = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}$$

with

$$[\gamma^5, \Sigma^k] = 0, \quad [\beta, \Sigma^k] = 0, \quad [\gamma^5, \alpha^k] = 0, \quad [\beta, \gamma^5] = 0$$

$$[\Sigma^i, \Sigma^j] = 2i\Sigma^k, \qquad \Sigma^i\Sigma^j = -\Sigma^j\Sigma^i = i\Sigma^k \qquad (i, j, \text{ and } k; \text{ cyclic})$$

$$[\gamma^5\Sigma^k, \Sigma^j] = \gamma^5\Sigma^k\Sigma^j - \Sigma^j\gamma^5\Sigma^k = \gamma^5[\Sigma^k, \Sigma^j]$$

((**Note**))

$$\Sigma^1 = \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\Sigma^2 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix},$$

$$\Sigma^3 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \boldsymbol{1} \\ \boldsymbol{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Then we get

$$H = c\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc^2 = c\gamma^5\Sigma^k p_k + \beta mc^2 ,$$

For convenience, we use the notation

$$\boldsymbol{\alpha} = (\alpha^1, \alpha^2, \alpha^3), \qquad \boldsymbol{p} = (p_x\ p_y, p_z) = (p_1\ p_2, p_3)$$

We consider the commutation relation,

$$\begin{aligned}
[H,\Sigma^1] &= [c\gamma^5\Sigma^k p_k + \beta mc^2, \Sigma_1] \\
&= cp_k[\gamma^5\Sigma^k, \Sigma^1] + mc^2[\beta, \Sigma^1] \\
&= cp_2[\gamma^5\Sigma^2, \Sigma^1] - cp_3[\gamma^5\Sigma^3, \Sigma^1] \\
&= cp_2\gamma^5[\Sigma^1, \Sigma^2] - cp_3\gamma^5[\Sigma^3, \Sigma^1] \\
&= 2icp_2\gamma^5\Sigma^3 - 2icp_3\gamma^5\Sigma^2 \\
&= 2ic(\alpha^2 p_3 - \alpha^3 p_2) \\
&= 2ic(\boldsymbol{\alpha}\times\boldsymbol{p})_1
\end{aligned}$$

which leads to the Heisenberg's equation,

$$\frac{d}{dt}\boldsymbol{\Sigma} = \frac{i}{\hbar}[H, \boldsymbol{\Sigma}] = \frac{i}{\hbar}2ic(\boldsymbol{\alpha}\times\boldsymbol{p}) = -\frac{2c}{\hbar}(\boldsymbol{\alpha}\times\boldsymbol{p})$$

26. Total angular momentum *J*

The time derivative of the total angular momentum ***J*** is obtained

$$\frac{d}{dt}\boldsymbol{J} = \frac{d}{dt}(\boldsymbol{L} + \frac{\hbar}{2}\boldsymbol{\Sigma}) = c(\boldsymbol{\alpha}\times\boldsymbol{p}) - c(\boldsymbol{\alpha}\times\boldsymbol{p}) = 0$$

Although $\boldsymbol{L}$ and $\frac{\hbar}{2}\boldsymbol{\Sigma}$ are not constants of the motion, the total angular momentum $\boldsymbol{J}$ should be identified with the total angular momentum and is a constant of the motion.

$$\boldsymbol{J} = \boldsymbol{L} + \frac{\hbar}{2}\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{L} + \frac{\hbar}{2}\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{L} + \frac{\hbar}{2}\boldsymbol{\sigma} \end{pmatrix}.$$

As is well known, the constancy of $\boldsymbol{J}$ is a consequence of invariance under rotation. Hence $\boldsymbol{J}$ must be a constant of the motion even if a central (spherically symmetric) potential $V(r)$ is added to the free particle Hamiltonian.

((**Note**))

$$\langle\psi'|\hat{H}|\psi'\rangle = \langle\psi|\hat{H}|\psi\rangle$$

or

$$\hat{R}_\alpha{}^+\hat{H}\hat{R}_\alpha = \hat{H}$$

with

$$|\psi'\rangle = \hat{R}_\alpha|\psi\rangle$$

Since

$$\hat{R}_\alpha = \exp(-\frac{i}{\hbar}\hat{J}_\alpha\theta)$$

$$[\hat{H},\hat{J}_\alpha] = 0\,.$$

27. Helicity $\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}}$

We define the helicity operator as

$$\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}}$$

where $\hat{\boldsymbol{p}}$ is the unit vector ($\hat{\boldsymbol{p}} = \frac{\boldsymbol{p}}{p}$)

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}.$$

The eigenstate of helicity with eigenvalue +1 and -1 are referred to, respectively, as the right-handed state and the left-handed state.

$$\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}}=\boldsymbol{\Sigma}\cdot\frac{\boldsymbol{p}}{p}=\frac{1}{p}\begin{pmatrix}\sigma^1 p_1 & 0\\ 0 & \sigma^1 p_1\end{pmatrix}+\frac{1}{p}\begin{pmatrix}\sigma^2 p_2 & 0\\ 0 & \sigma^2 p_2\end{pmatrix}+\frac{1}{p}\begin{pmatrix}\sigma^3 p_3 & 0\\ 0 & \sigma^3 p_3\end{pmatrix}$$

or

$$\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}}=\frac{1}{p}\begin{pmatrix}p_3 & p_1-ip_2 & 0 & 0\\ p_1+ip_2 & -p_3 & 0 & 0\\ 0 & 0 & p_3 & p_1-ip_2\\ 0 & 0 & p_1+ip_2 & -p_3\end{pmatrix}$$

$$(\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}})u^{(1)}(\boldsymbol{p})=\frac{1}{p}\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix}p_3 & p_1-ip_2 & 0 & 0\\ p_1+ip_2 & -p_3 & 0 & 0\\ 0 & 0 & p_3 & p_1-ip_2\\ 0 & 0 & p_1+ip_2 & -p_3\end{pmatrix}\begin{pmatrix}1\\ 0\\ \dfrac{cp_3}{R+mc^2}\\ \dfrac{c(p_1+ip_2)}{R+mc^2}\end{pmatrix}$$

$$=\frac{1}{p}\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix}p_3\\ p_1+ip_2\\ \dfrac{c\boldsymbol{p}^2}{R+mc^2}\\ 0\end{pmatrix}$$

$$(\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}})u^{(2)}(\boldsymbol{p})=\frac{1}{p}\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix}p_3 & p_1-ip_2 & 0 & 0\\ p_1+ip_2 & -p_3 & 0 & 0\\ 0 & 0 & p_3 & p_1-ip_2\\ 0 & 0 & p_1+ip_2 & -p_3\end{pmatrix}\begin{pmatrix}0\\ 1\\ \dfrac{c(p_1-ip_2)}{R+mc^2}\\ \dfrac{-cp_3}{R+mc^2}\end{pmatrix}$$

$$=-\frac{1}{p}\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix}-(p_1-ip_2)\\ p_3\\ 0\\ \dfrac{-cp_3{}^2}{R+mc^2}\end{pmatrix}$$

$$(\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}})u^{(3)}(\boldsymbol{p})=\frac{1}{p}\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} p_3 & p_1-ip_2 & 0 & 0 \\ p_1+ip_2 & -p_3 & 0 & 0 \\ 0 & 0 & p_3 & p_1-ip_2 \\ 0 & 0 & p_1+ip_2 & -p_3 \end{pmatrix}\begin{pmatrix} \frac{-cp_3}{R+mc^2} \\ \frac{-c(p_1+ip_2)}{R+mc^2} \\ 1 \\ 0 \end{pmatrix}$$

$$=\frac{1}{p}\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} \frac{-c\boldsymbol{p}^2}{R+mc^2} \\ 0 \\ p_3 \\ p_1+ip_2 \end{pmatrix}$$

$$(\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}})u^{(4)}(\boldsymbol{p})=\frac{1}{p}\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} p_3 & p_1-ip_2 & 0 & 0 \\ p_1+ip_2 & -p_3 & 0 & 0 \\ 0 & 0 & p_3 & p_1-ip_2 \\ 0 & 0 & p_1+ip_2 & -p_3 \end{pmatrix}\begin{pmatrix} \frac{-c(p_1-ip_2)}{R+mc^2} \\ \frac{cp_3}{R+mc^2} \\ 0 \\ 1 \end{pmatrix}$$

$$=-\frac{1}{p}\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} 0 \\ \frac{c\boldsymbol{p}^2}{R+mc^2} \\ -(p_1-ip_2) \\ p_3 \end{pmatrix}$$

When $p_1=p_2=0$, and $p=p_3$

$$\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}}=\Sigma^3$$

$$\Sigma_3 u^{(1)}(\boldsymbol{p})=\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} 1 \\ 0 \\ \frac{cp_3}{R+mc^2} \\ 0 \end{pmatrix}=u^{(1)}(\boldsymbol{p})$$

where

$$R=\sqrt{\boldsymbol{p}^2c^2+m^2c^4}\ .$$

Similarly, we have

$$\Sigma_3 u^{(2)}(\boldsymbol{p}) = -\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \dfrac{-cp_3}{R+mc^2} \end{pmatrix} = -u^{(2)}(\boldsymbol{p}),$$

$$\Sigma_3 u^{(3)}(\boldsymbol{p}) = \sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} -\dfrac{cp_3}{R+mc^2} \\ 0 \\ 1 \\ 0 \end{pmatrix} = u^{(3)}(\boldsymbol{p})$$

$$\Sigma_3 u^{(4)}(\boldsymbol{p}) = -\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} 0 \\ \dfrac{cp_3}{R+mc^2} \\ 0 \\ 1 \end{pmatrix} = -u^{(4)}(\boldsymbol{p})$$

28. Plane wave solution

Hamiltonian

$$H = c\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}} + \beta mc^2$$

$$\alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix} = \gamma^0\gamma_k = -\Sigma_k\gamma_5$$

$$\beta = \gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3,$$

$$\Sigma^k = i\gamma^i\gamma^j \qquad (i, j, k;\ \text{cyclic}).$$

$$\Sigma^1 = i\gamma^2\gamma^3, \qquad \Sigma^2 = i\gamma^3\gamma^1, \qquad \Sigma^3 = i\gamma^1\gamma^2,$$

$$\begin{aligned} H^2 &= [c\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}} + \beta mc^2][c\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}} + \beta mc^2] \\ &= c^2(\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}})^2 + \beta^2 m^2 c^4 + c(\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}})(\beta mc^2) + c(\beta mc^2)(\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}}) \end{aligned}$$

$$
\begin{aligned}
(\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}})(\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}}) &= (\alpha^i\cdot\hat{p}_i)(\alpha^j\cdot\hat{p}_j) \\
&= \alpha^i\alpha^j\hat{p}_i\hat{p}_j \\
&= \frac{1}{2}(\alpha^i\alpha^j+\alpha^j\alpha^i)\hat{p}_i\hat{p}_j \\
&= \frac{1}{2}2\delta_{ij}I_4\hat{p}_i\hat{p}_j \\
&= \hat{\boldsymbol{p}}^2 I_4
\end{aligned}
$$

with

$$\alpha^i\alpha^j+\alpha^j\alpha^i = I_4\delta_{ij}$$

Note that

$$\beta^2 = I_4$$

$$(\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}})\beta+\beta(\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}}) = \alpha^i\hat{p}_i\beta+\beta\alpha^i\hat{p}_i = (\alpha^i\beta+\beta\alpha^i)\hat{p}_i = 0$$

Thus we have

$$H^2 = c^2\boldsymbol{p}^2+m^2c^4\,.$$

We now consider a plane wave given by

$$\psi = u(\boldsymbol{p})\exp[\frac{i}{\hbar}\{\boldsymbol{p}\cdot\boldsymbol{r}-Et)]$$

$$H = -ic\hbar\boldsymbol{\alpha}\cdot\nabla+\beta mc^2$$

$$H\psi = i\hbar\frac{\partial\psi}{\partial t}$$

Left-hand side

$$
\begin{aligned}
(-ic\hbar\boldsymbol{\alpha}\cdot\nabla+\beta mc^2)\psi &= (-ic\hbar\boldsymbol{\alpha}\cdot\nabla+\beta mc^2)u(\boldsymbol{p})\exp[\frac{i}{\hbar}\{\boldsymbol{p}\cdot\boldsymbol{r}-Et)] \\
&= (c\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta mc^2)\psi
\end{aligned}
$$

right-hand side

$$i\hbar\frac{\partial\psi}{\partial t} = i\hbar u(\mathbf{p})(-\frac{i}{\hbar}E)\exp[\frac{i}{\hbar}\{\mathbf{p}\cdot\mathbf{r} - Et)] = E\psi$$

or

$$H\psi = E\psi$$

or

$$Hu(\boldsymbol{p}) = Eu(\boldsymbol{p})$$

where

$$H = c\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc^2$$

$$H^2u(\boldsymbol{p}) = EHu(\boldsymbol{p}) = E^2u(\boldsymbol{p})$$

or

$$E^2 = c^2\boldsymbol{p}^2 + m^2c^4$$

or

$$E = \pm c\sqrt{\boldsymbol{p}^2 + m^2c^2}$$

We now discuss

$$Hu(\boldsymbol{p}) = Eu(\boldsymbol{p})$$

Since

$$H = c\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc^2 = c(\gamma^0\gamma^k)p_k + \gamma^0 mc^2$$

$$[c(\gamma^0\gamma^k)p_k + \gamma^0 mc^2]u(\boldsymbol{p}) = Eu(\boldsymbol{p})$$

or

$$(\gamma^0\gamma^k p_k - \frac{E}{c} - \gamma^0 mc)u(\boldsymbol{p}) = 0$$

Multiplying this equation by γ^0 form the left

$$(\gamma^k p_k - \frac{E}{c}\gamma^0 + mc)u(\boldsymbol{p}) = 0$$

Noting that

$$-\frac{E}{c}\gamma^0 = -p_0\gamma^0, \qquad p_0 = E/c,$$

we obtain

$$(\gamma p - mc)u(\boldsymbol{p}) = 0$$

where $p_\mu = (\frac{E}{c}, -\boldsymbol{p}) = (\frac{E}{c}, -p_k)$

29. Simultaneous eigenket of *H* and $\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}}$.

We show that *H* is commutable with $\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}}$.

$$\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}} = \Sigma^k \hat{p}_k$$

$$\Sigma^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} = i\gamma^i\gamma^j \qquad (i, j, k\text{; cyclic})$$

$$\alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix} = \gamma^0\gamma^k = \Sigma^k\gamma^5 = \gamma^5\Sigma^k$$

where

$$[\gamma^5, \Sigma^k] = 0, \quad [\beta, \Sigma^k] = 0, \quad [\gamma^5, \alpha^k] = 0, \quad \{\beta, \gamma^5\} = 0$$

Suppose that $u(\boldsymbol{p})$ is the eigenket of *H*,

$$Hu(\boldsymbol{p}) = Eu(\boldsymbol{p})$$

with

$$H = c\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc^2 = c\gamma^5\Sigma^k p_k + \beta mc^2$$

Here we note that

$$\begin{aligned}[H,\Sigma^j p_j] &= [c\gamma^5\Sigma^k p_k + \beta mc^2, \Sigma^j p_j] \\ &= cp_k p_j[\gamma^5\Sigma^k, \Sigma^j] + mc^2 p_j[\beta, \Sigma^j] \\ &= cp_k p_j[\gamma^5\Sigma^k, \Sigma^j]\end{aligned}$$

We note that

$$[\gamma^5\Sigma^k, \Sigma^j] = \gamma^5\Sigma^k\Sigma^j - \Sigma^j\gamma^5\Sigma^k = \gamma^5[\Sigma^k, \Sigma^j]$$

$$[\Sigma^i, \Sigma^j] = 2i\Sigma^k, \qquad \Sigma^i\Sigma^j = -\Sigma^j\Sigma^i = i\Sigma^k \quad (i, j, \text{ and } k; \text{ cyclic})$$

Then

$$[H,\Sigma^j p_j] = cp_k p_j\gamma^5[\Sigma^k, \Sigma^j] = \frac{c}{2}p_k p_j\gamma^5[\Sigma^k, \Sigma^j] + \frac{c}{2}p_j p_k\gamma^5[\Sigma^j, \Sigma^k]$$

or

$$[H,\Sigma^j p_j] = \frac{c}{2}p_k p_j\{\gamma^5[\Sigma^k, \Sigma^j] + \gamma^5[\Sigma^j, \Sigma^k]\} = 0$$

So we can demonstrate that

$$[H, \boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}}] = 0.$$

This implies that $u(\boldsymbol{p})$ is a simultaneous eigenket of H and $\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}}$,

$$Hu(\boldsymbol{p}) = Eu(\boldsymbol{p}), \qquad (\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}})u(\boldsymbol{p}) = hu(\boldsymbol{p})$$

with $\hat{\boldsymbol{p}} = \boldsymbol{p}/|\boldsymbol{p}|$. Since

$$(\boldsymbol{\Sigma}\cdot\boldsymbol{p})(\boldsymbol{\Sigma}\cdot\boldsymbol{p}) = (\Sigma^i p_i)(\Sigma^j p_j)$$
$$= p_i p_j (\Sigma^i \Sigma^j)$$
$$= \frac{1}{2} p_i p_j (\Sigma^i \Sigma^j + \Sigma^j \Sigma^i)$$
$$= \boldsymbol{p}^2$$

or

$$(\boldsymbol{\Sigma}\cdot\boldsymbol{p})^2 = 1$$

$$(\boldsymbol{\Sigma}\cdot\boldsymbol{p})^2 u(\boldsymbol{p}) = h^2 u(\boldsymbol{p}) = u(\boldsymbol{p})$$

$h^2 = 1$, or $h = \pm 1$.

---

30. Classification of the simultaneous eigenket of *H* and the helicity

In summary, we have

$$Hu(\boldsymbol{p}) = Eu(\boldsymbol{p}), \qquad (\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}})u(\boldsymbol{p}) = hu(\boldsymbol{p})$$

with $E = \pm c\sqrt{\boldsymbol{p}^2 + m^2c^2}$ and $h = \pm 1$.

$$\begin{pmatrix} E > 0 \\ h = 1 \end{pmatrix}, \qquad \begin{pmatrix} E > 0 \\ h = -1 \end{pmatrix}, \qquad \begin{pmatrix} E < 0 \\ h = 1 \end{pmatrix}, \qquad \begin{pmatrix} E < 0 \\ h = -1 \end{pmatrix}$$

Eigenket of the helicity

$$\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}} = \begin{pmatrix} \boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}} \end{pmatrix}, \qquad u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$(\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}})u = \begin{pmatrix} \boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}} \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = h\begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

or

$$\begin{pmatrix} (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})u_A \\ (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})u_B \end{pmatrix} = h\begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

or

$$(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})u_A = hu_A, \qquad (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})u_B = hu_B \qquad (1)$$

with $h = \pm 1$.

Hamiltonian:

$$\begin{aligned} H &= c\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc^2 \\ &= c\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & 0 \end{pmatrix} + mc^2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} mc^2 & c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) \\ c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) & -mc^2 \end{pmatrix} \end{aligned}$$

Eigenvalue problem:

$$Hu(\boldsymbol{p}) = Eu(\boldsymbol{p})$$

or

$$\begin{pmatrix} mc^2 & c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) \\ c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) & -mc^2 \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = E\begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$mc^2u_A + c(\boldsymbol{\sigma}\cdot\boldsymbol{p})u_B = Eu_A$$

$$c(\boldsymbol{\sigma}\cdot\boldsymbol{p})u_A - mc^2u_B = Eu_B$$

or

$$u_A = \frac{c(\boldsymbol{\sigma}\cdot\boldsymbol{p})u_B}{E - mc^2} = \frac{cp}{E - mc^2}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})u_B \qquad (2)$$

$$u_B = \frac{c(\boldsymbol{\sigma}\cdot\boldsymbol{p})u_A}{E + mc^2} = \frac{cp}{E + mc^2}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})u_A$$

(i) $h = 1$ and $E>0$

$$(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})u_A = u_A, \qquad (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})u_B = u_B$$

$$u_A = \frac{cp}{R - mc^2}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})u_B = \frac{cp}{R - mc^2}u_B$$

$$u_B = \frac{cp}{R + mc^2}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})u_A = \frac{cp}{R + mc^2}u_A$$

We choose

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad u_B = \begin{pmatrix} \frac{cp}{R + mc^2} \\ 0 \end{pmatrix}$$

or

$$u^{(1)} = \begin{pmatrix} 1 \\ 0 \\ \frac{cp}{R + mc^2} \\ 0 \end{pmatrix}$$

(ii) $h = -1$ and $E>0$

$$(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})u_A = -u_A, \qquad (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})u_B = -u_B$$

$$u_A = \frac{cp}{E - mc^2}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})u_B = -\frac{cp}{R - mc^2}u_B$$

$$u_B = \frac{cp}{E + mc^2}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})u_A = -\frac{cp}{R + mc^2}u_A$$

We choose

$$u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad u_B = \begin{pmatrix} 0 \\ -\frac{cp}{R + mc^2} \end{pmatrix}$$

or

$$u^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -\dfrac{cp}{R+mc^2} \\ 0 \end{pmatrix}$$

(iii) $h = 1$ and $E<0$

$$(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})u_B = u_B\,, \qquad (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})u_A = u_A$$

$$u_A = -\frac{cp}{R+mc^2}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})u_B = -\frac{cp}{R+mc^2}u_B$$

$$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad u_A = \begin{pmatrix} -\dfrac{cp}{R+mc^2} \\ 0 \end{pmatrix}$$

or

$$u^{(3)} = \begin{pmatrix} -\dfrac{cp}{R+mc^2} \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

(iv) $h = -1$ and $E<0$

$$(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})u_B = -u_B$$

$$u_A = -\frac{cp}{R+mc^2}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})u_B = \frac{cp}{R+mc^2}u_B$$

$$u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad u_A = \begin{pmatrix} 0 \\ \dfrac{cp}{R+mc^2} \end{pmatrix}$$

or

$$u^{(4)} = \begin{pmatrix} 0 \\ \dfrac{cp}{R + mc^2} \\ 0 \\ 1 \end{pmatrix}$$

31. Foldy-Wouthuysen (FW) transformation

The Foldy–Wouthuysen transformation (FWT) is one of the corner stones of relativistic quantum mechanics. It presents a straightforward and convenient way to obtain an adequate physical interpretation for relativistic wave equations. The very existence of an exact FWT for a considered relativistic problem justifies its quantum mechanical treatment, inasmuch as in this case transitions between positive and negative energy states are forbidden.

We start with an eigenvalue problem. Suppose that $|a_n\rangle$ is the eigenket of H with the eigenvalue E_n

$$H|a_n\rangle = E_n|a_n\rangle$$

with

$$|\psi\rangle = |a_n\rangle$$

We consider the unitary transformation

$$U|b_n\rangle = |a_n\rangle \qquad \text{(U is the unitary operator)}$$

where

$$|\psi'\rangle = |b_n\rangle$$

Then we have

$$HU|b_n\rangle = H|a_n\rangle = E_n|a_n\rangle = E_nU|b_n\rangle$$

or

$$U^+HU|b_n\rangle = E_n|b_n\rangle$$

When we define

$$H' = U^{+} H U$$

then we get

$$H'|b_n\rangle = E_n |b_n\rangle$$

Then $|\psi'\rangle = |b_n\rangle$ is the eigenket of *H'* with the same eigenvalue E_n.

Suppose that

$$U = e^{-iS}$$

with *S* is the Hermitian operator. Then we have

$$H' = e^{iS} H e^{iS}$$

We note that

$$\langle\psi|H|\psi\rangle = \langle\psi'|H'|\psi'\rangle \qquad \text{(from the definition)}$$

since

$$U|\psi'\rangle = |\psi\rangle, \qquad \langle\psi| = \langle\psi'|U^{+}$$

Then we have

$$\langle\psi|H|\psi\rangle = \langle\psi'|U^{+}HU|\psi'\rangle = \langle\psi'|H'|\psi'\rangle$$

or

$$H' = U^{+} H U$$

Suppose that

$$U = e^{-iS}$$

$$|\psi'\rangle = U^{+}|\psi\rangle = e^{iS}|\psi\rangle \qquad |\psi\rangle = U|\psi'\rangle = e^{-iS}|\psi'\rangle$$

$$H' = U^{+}HU = e^{iS} H e^{-iS}$$

We choose *S* of the form

$$S = -\frac{i}{p}\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\theta\,, \qquad e^{iS} = \exp[\frac{1}{p}\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\theta]$$

θ is a real function to be determined. S is a Hermitian operator.

$$S^{+} = \frac{i}{p}(\boldsymbol{\alpha}^{+}\cdot\boldsymbol{p})\beta^{+}\theta(\boldsymbol{p}) = \frac{i}{p}(\boldsymbol{\alpha}\cdot\boldsymbol{p})\beta\theta(\boldsymbol{p}) = -\frac{i}{p}\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\theta(\boldsymbol{p})$$

since $\{\alpha^{i},\beta\} = 0$.

$$\begin{aligned}
H' &= e^{iS}He^{-iS} \\
&= e^{iS}(c\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc^{2})e^{-iS} \\
&= e^{iS}[\beta(c\beta\boldsymbol{\alpha}\cdot\boldsymbol{p}) + \beta mc^{2})e^{-iS} \\
&= e^{iS}\beta(c\beta\boldsymbol{\alpha}\cdot\boldsymbol{p} + mc^{2})e^{-iS} \\
&= e^{iS}\beta e^{-iS}e^{iS}(c\beta\boldsymbol{\alpha}\cdot\boldsymbol{p} + mc^{2})e^{-iS} \\
&= e^{iS}\beta e^{-iS}(c\beta\boldsymbol{\alpha}\cdot\boldsymbol{p} + mc^{2})
\end{aligned}$$

since $[S,\beta\boldsymbol{\alpha}\cdot\boldsymbol{p}] = 0$. Furthermore

$$\beta e^{-iS} = e^{iS}\beta$$

So that

$$H' = e^{2iS}\beta(c\beta\boldsymbol{\alpha}\cdot\boldsymbol{p} + mc^{2}) = e^{2iS}(c\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc^{2})$$

where $\beta^{2} = 1$

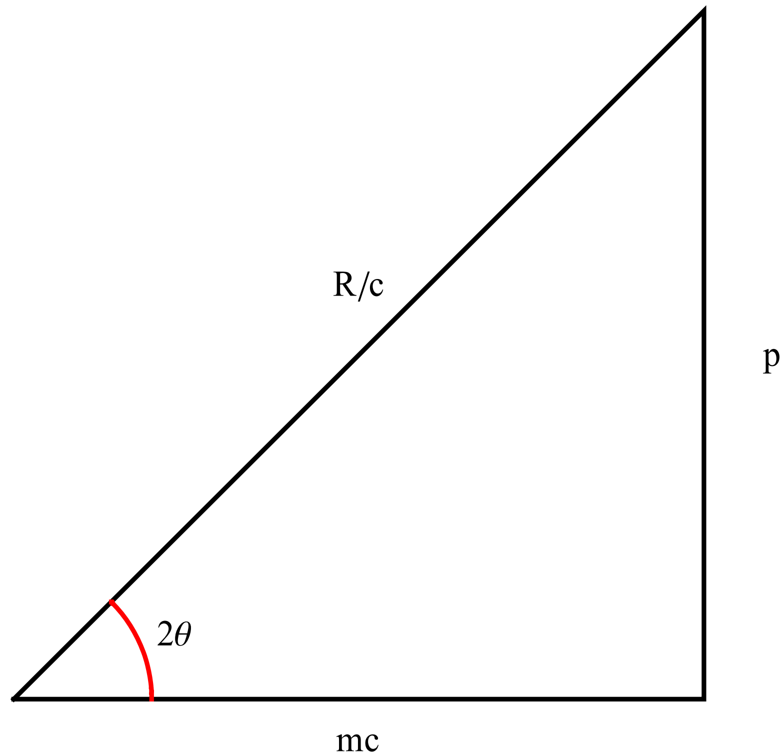
$$\begin{aligned}
H' &= (\cos 2\theta + \frac{\beta\boldsymbol{\alpha}\cdot\boldsymbol{p}}{p}\sin 2\theta)(c\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc^{2}) \\
&= \beta(mc^{2}\cos 2\theta + cp\sin 2\theta) + \frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}}{p}(pc\cos 2\theta - mc^{2}\sin 2\theta)
\end{aligned}$$

where

$$(\boldsymbol{\alpha}\cdot\boldsymbol{p})(\boldsymbol{\alpha}\cdot\boldsymbol{p}) = p^{2}$$

and

$$e^{2iS} = \cos 2\theta + \frac{\beta \boldsymbol{\alpha} \cdot \boldsymbol{p}}{p} \sin 2\theta \qquad \text{(see the Mathematica)}$$



If we choose (so that odd terms disappear)

$$\tan 2\theta = \frac{p}{mc}.$$

$$\sin 2\theta = \frac{p}{\sqrt{p^2 + m^2c^2}} = \frac{cp}{R}, \qquad \cos 2\theta = \frac{mc}{\sqrt{p^2 + m^2c^2}} = \frac{mc^2}{R}$$

Then we have

$$\begin{aligned} H' &= \beta(mc^2 \cos 2\theta + cp \sin 2\theta) \\ &= \beta(mc^2 \frac{mc}{\sqrt{p^2 + m^2c^2}} + cp \frac{p}{\sqrt{p^2 + m^2c^2}}) \\ &= \beta c(\frac{m^2c^2}{\sqrt{p^2 + m^2c^2}} + \frac{p^2}{\sqrt{p^2 + m^2c^2}}) \\ &= \beta c\sqrt{p^2 + m^2c^2} = \beta R \end{aligned}$$

where

$$R = c\sqrt{p^2 + m^2c^2}$$

So that, H' is now diagonalized. The eigenstate of H' is the same as that of β.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$|\psi\rangle = U|\psi'\rangle = e^{-iS}|\psi'\rangle$$

where

$$e^{-iS} = \begin{pmatrix} \cos\theta & 0 & -\frac{p_z}{p}\sin\theta & -\frac{(p_x - ip_y)}{p}\sin\theta \\ 0 & \cos\theta & -\frac{(p_x + ip_y)}{p}\sin\theta & \frac{p_z}{p}\sin\theta \\ \frac{p_z}{p}\sin\theta & \frac{(p_x - ip_y)}{p}\sin\theta & \cos\theta & 0 \\ \frac{(p_x + ip_y)}{p}\sin\theta & -\frac{p_z}{p}\sin\theta & 0 & \cos\theta \end{pmatrix}$$

Note that

$$\cos\theta = \sqrt{\frac{1}{2}(1 + \cos 2\theta)} = \sqrt{\frac{R + mc^2}{2R}}$$

$$\sin\theta = \sqrt{\frac{1}{2}(1 - \cos 2\theta)} = \sqrt{\frac{R - mc^2}{2R}}$$

Then the eigenstate of the original Hamiltonian H is given by

$$\begin{pmatrix} \cos\theta \\ 0 \\ \frac{p_z}{p}\sin\theta \\ \frac{(p_x + ip_y)}{p}\sin\theta \end{pmatrix} = \sqrt{\frac{R + mc^2}{2R}} \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{R + mc^2} \\ \frac{c(p_x + ip_y)}{R + mc^2} \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ \cos\theta \\ \dfrac{(p_x - ip_y)}{p}\sin\theta \\ -\dfrac{p_z}{p}\sin\theta \end{pmatrix} = \sqrt{\frac{R+mc^2}{2R}} \begin{pmatrix} 0 \\ 1 \\ \dfrac{c(p_x - ip_y)}{R+mc^2} \\ -\dfrac{cp_z}{R+mc^2} \end{pmatrix}$$

$$\begin{pmatrix} -\dfrac{p_z}{p}\sin\theta \\ -\dfrac{(p_x + ip_y)}{p}\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} = \sqrt{\frac{R+mc^2}{2R}} \begin{pmatrix} -\dfrac{cp_z}{R+mc^2} \\ \dfrac{c(p_x + ip_y)}{R+mc^2} \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\dfrac{(p_x - ip_y)}{p}\sin\theta \\ \dfrac{p_z}{p}\sin\theta \\ 0 \\ \cos\theta \end{pmatrix} = \sqrt{\frac{R+mc^2}{2R}} \begin{pmatrix} -\dfrac{c(p_x - ip_y)}{R+mc^2} \\ \dfrac{cp_z}{R+mc^2} \\ 0 \\ 1 \end{pmatrix}$$

((**Mathematica**))

```
Clear["Global`*"]; σx = {{0, 1}, {1, 0}}; σy = {{0, -ⅈ}, {ⅈ, 0}}; σz = {{1, 0}, {0, -1}};
I2 = IdentityMatrix[2]; I4 = IdentityMatrix[4];
αx = KroneckerProduct[σx, σx]; αy = KroneckerProduct[σx, σy];
αz = KroneckerProduct[σx, σz];
β = KroneckerProduct[σz, I2];

H1 = c px αx + c py αy + c pz αz + β m^2 c^2 // Simplify;
S = -ⅈ (1/p) β.(αx px + αy py + αz pz) θ;

K1 =
  MatrixExp[2 ⅈ S] //. {Sqrt[-px^2 - py^2 - pz^2] → ⅈ p,
      1/(px^2 + py^2 + pz^2) → 1/p^2} // ExpToTrig // Simplify;

K11 = K1 /. {px^2 + py^2 + pz^2 → p^2} // Simplify;
K2 = (Cos[2 θ] I4 + (1/p) β.(αx px + αy py + αz pz) Sin[2 θ]) // Simplify;
K2 // MatrixForm
```

$$\begin{pmatrix} \text{Cos}[2\,\theta] & 0 & \frac{\text{pz Sin}[2\,\theta]}{\text{p}} & \frac{(\text{px}-\mathrm{i}\,\text{py})\,\text{Sin}[2\,\theta]}{\text{p}} \\ 0 & \text{Cos}[2\,\theta] & \frac{(\text{px}+\mathrm{i}\,\text{py})\,\text{Sin}[2\,\theta]}{\text{p}} & -\frac{\text{pz Sin}[2\,\theta]}{\text{p}} \\ -\frac{\text{pz Sin}[2\,\theta]}{\text{p}} & -\frac{(\text{px}-\mathrm{i}\,\text{py})\,\text{Sin}[2\,\theta]}{\text{p}} & \text{Cos}[2\,\theta] & 0 \\ -\frac{(\text{px}+\mathrm{i}\,\text{py})\,\text{Sin}[2\,\theta]}{\text{p}} & \frac{\text{pz Sin}[2\,\theta]}{\text{p}} & 0 & \text{Cos}[2\,\theta] \end{pmatrix}$$

```
K11 - K2 // Simplify

{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

Eigensystem[β]

{{-1, -1, 1, 1}, {{0, 0, 0, 1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {1, 0, 0, 0}}}

MatrixExp[ⅈ S].β.MatrixExp[ⅈ S] - β // Simplify

{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}
```

```
K3 =
  MatrixExp[-ⅈ S] //. {√(-px² - py² - pz²) → ⅈ p,
      1/(px² + py² + pz²) → 1/p²} // ExpToTrig // Simplify;

K31 = K3 /. {px² + py² + pz² → p²} // Simplify;

K31 // MatrixForm
```

$$\begin{pmatrix} \text{Cos}[\theta] & 0 & -\frac{\text{pz Sin}[\theta]}{\text{p}} & -\frac{(\text{px}-i\,\text{py})\,\text{Sin}[\theta]}{\text{p}} \\ 0 & \text{Cos}[\theta] & -\frac{(\text{px}+i\,\text{py})\,\text{Sin}[\theta]}{\text{p}} & \frac{\text{pz Sin}[\theta]}{\text{p}} \\ \frac{\text{pz Sin}[\theta]}{\text{p}} & \frac{(\text{px}-i\,\text{py})\,\text{Sin}[\theta]}{\text{p}} & \text{Cos}[\theta] & 0 \\ \frac{(\text{px}+i\,\text{py})\,\text{Sin}[\theta]}{\text{p}} & -\frac{\text{pz Sin}[\theta]}{\text{p}} & 0 & \text{Cos}[\theta] \end{pmatrix}$$

32. Charge conjugate operator

We start with the Dirac equation for the free particle.
Dirac equation:

$$(i\gamma^{\mu}\partial_{\mu} - \frac{mc}{\hbar})\psi = 0$$

The replacement of

$$\partial_{\mu} \rightarrow \partial_{\mu} + \frac{ie}{c\hbar}A_{\mu}$$

leads to the Dirac equation in the presence of the four-potential $A_{\mu} = (A_0, -\boldsymbol{A})$

$$i\gamma^{\mu}(\partial_{\mu} + \frac{ie}{\hbar c}A_{\mu})\psi - \frac{mc}{\hbar}\psi = 0 \qquad (1)$$

Hermitian conjugate of Eq.(1):

$$-i(\partial_{\mu} - \frac{ie}{\hbar c}A_{\mu})\psi^{+}\left(\gamma^{\mu}\right)^{+} - \frac{mc}{\hbar}\psi^{+} = 0$$

Multiplying Eq.(1) by γ^0 from the right, we get

$$-i(\partial_{\mu} - \frac{ie}{\hbar c}A_{\mu})\psi^{+}\left(\gamma^{\mu}\right)^{+}\gamma^0 - \frac{mc}{\hbar}\psi^{+}\gamma^0 = 0$$

Noting that

$$\left(\gamma^{\mu}\right)^{+} = \gamma^{0}\gamma^{\mu}\gamma^{0}, \qquad \left(\gamma^{0}\right)^{2} = I_4$$

we get

$$-i(\partial_{\mu} - \frac{ie}{\hbar c}A_{\mu})\psi^{+}\gamma^{0}\gamma^{\mu} - \frac{mc}{\hbar}\psi^{+}\gamma^{0} = 0$$

Here we note that

$$\psi^{+}\gamma^{0} = \overline{\psi},$$

Then we get

$$i(\partial_{\mu} - \frac{ie}{\hbar c}A_{\mu})\overline{\psi}\gamma^{\mu} + \frac{mc}{\hbar}\overline{\psi} = 0. \tag{2}$$

We consider the Dirac equation for the charge conjugate wave function ψ^{C},

$$i\gamma^{\mu}(\partial_{\mu} - \frac{ie}{\hbar c}A_{\mu})\psi^{C} - \frac{mc}{\hbar}\psi^{C} = 0. \tag{3}$$

We assume that

$$\psi^{C} = C\overline{\psi}^{T}$$

Then we get

$$i(\partial_{\mu} - \frac{ie}{\hbar c}A_{\mu})\gamma^{\mu}C\overline{\psi}^{T} - \frac{mc}{\hbar}C\overline{\psi}^{T} = 0 \tag{4}$$

Taking the transpose of Eq.(4), we get

$$i(\partial_{\mu} - \frac{ie}{\hbar c}A_{\mu})\overline{\psi}C^{T}\gamma_{\mu}{}^{T} - \frac{mc}{\hbar}\overline{\psi}C^{T} = 0 \tag{5}$$

Multiplying Eq.(5) by $(C^{T})^{-1}$ from the right,

$$i(\partial_{\mu} - \frac{ie}{\hbar c}A_{\mu})\overline{\psi}C^{T}\gamma_{\mu}{}^{T}(C^{T})^{-1} - \frac{mc}{\hbar}\overline{\psi} = 0 \tag{6}$$

$$i(\partial_{\mu} - \frac{ie}{\hbar c}A_{\mu})(-\overline{\psi}\gamma^{\mu}) - \frac{mc}{\hbar}\overline{\psi} = 0 \tag{7}$$

Comparing Eq.(6) with Eq.(2), we have

$$C^T \gamma^{\mu^T} (C^T)^{-1} = -\gamma^\mu$$

or

$$C^{-1}\gamma_\mu C = -\gamma_\mu{}^T$$

Note that

$$\gamma^{1^T} = -\gamma^1, \qquad \gamma^{2^T} = \gamma^2, \qquad \gamma^{3^T} = -\gamma^3, \qquad \gamma^{0^T} = \gamma^0$$

Then we have

$$[\gamma^1, C] = 0, \qquad [\gamma^3, C] = 0$$

$$\{\gamma^2, C\} = 0, \qquad \{\gamma^4, C\} = 0$$

The above relations can be satisfied when

$$C = i\gamma^2\gamma^0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

$$C^T = i(\gamma^2\gamma^0)^T = i\gamma^{0^T}\gamma^{2^T} = i\gamma^0\gamma^2 = i\gamma^2\gamma^0 = -C$$

(see the Mathematica program below)

Then we have

$$\begin{aligned}\psi^C &= C\overline{\psi}^T \\ &= i\gamma^2\gamma^0(\psi^+\gamma^0)^T \\ &= i\gamma^2\gamma^0\gamma^{0^T}(\psi^+)^T \\ &= i\gamma^2\gamma^0\gamma^0\psi^* \\ &= i\gamma^2\psi^*\end{aligned}$$

where

$$i\gamma^2 = i\beta\alpha^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

((**Example**))

$$i\gamma^2 u_1^* = \sqrt{\frac{R+mc^2}{2R}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{R+mc^2} \\ \frac{c(p_x+ip_y)}{R+mc^2} \end{pmatrix} = \sqrt{\frac{R+mc^2}{2R}} \begin{pmatrix} \frac{c(p_x+ip_y)}{R+mc^2} \\ \frac{-cp_z}{R+mc^2} \\ 0 \\ 1 \end{pmatrix}$$

$$i\gamma_2 u_2^* = \sqrt{\frac{R+mc^2}{2R}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{R+mc^2} \\ -\frac{cp_z}{R+mc^2} \end{pmatrix} = -\sqrt{\frac{R+mc^2}{2R}} \begin{pmatrix} \frac{cp_z}{R+mc^2} \\ \frac{c(p_x+ip_y)}{R+mc^2} \\ 1 \\ 0 \end{pmatrix}$$

$$i\gamma_2 u_3^* = \sqrt{\frac{R+mc^2}{2R}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-cp_z}{R+mc^2} \\ \frac{-c(p_x-ip_y)}{R+mc^2} \\ 1 \\ 0 \end{pmatrix} = -\sqrt{\frac{R+mc^2}{2R}} \begin{pmatrix} 0 \\ 1 \\ \frac{-c(p_x-ip_y)}{R+mc^2} \\ \frac{cp_z}{R+mc^2} \end{pmatrix}$$

$$i\gamma_2 u_4^* = \sqrt{\frac{R+mc^2}{2R}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-c(p_x+ip_y)}{R+mc^2} \\ \frac{cp_z}{R+mc^2} \\ 0 \\ 1 \end{pmatrix} = \sqrt{\frac{R+mc^2}{2R}} \begin{pmatrix} 1 \\ 0 \\ \frac{-cp_z}{R+mc^2} \\ \frac{-c(p_x+ip_y)}{R+mc^2} \end{pmatrix}$$

We note that u_1, u_2, u_3, and u_4 are obtained from the FW transformation

$$u_1 = \sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{R+mc^2} \\ \frac{c(p_x+ip_y)}{R+mc^2} \end{pmatrix}$$

$$u_2 = \sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{R+mc^2} \\ -\frac{cp_z}{R+mc^2} \end{pmatrix}$$

$$u_3 = \sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} -\frac{cp_z}{R+mc^2} \\ \frac{c(p_x+ip_y)}{R+mc^2} \\ 1 \\ 0 \end{pmatrix}$$

$$u_4 = \sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} -\frac{c(p_x-ip_y)}{R+mc^2} \\ \frac{cp_z}{R+mc^2} \\ 0 \\ 1 \end{pmatrix}$$

When $p_1 = p_2 = p_3 = 0$

$$i\gamma_2 u_1^{\ *} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad i\gamma_2 u_2^{\ *} = -\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$i\gamma_2 u_3^{\ *} = -\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad i\gamma_2 u_4^{\ *} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

((**Mathematica**))

Charge conjugate C

```
C1 = ⅈ γu[2].γu[0] // Simplify
```

{{0, 0, 0, -1}, {0, 0, 1, 0},
 {0, -1, 0, 0}, {1, 0, 0, 0}}

```
C1 // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

```
γu[1].C1 - C1.γu[1] // Simplify
```

{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

```
γu[2].C1 + C1.γu[2] // Simplify
```

{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

```
γu[3].C1 - C1.γu[3] // Simplify
```

{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

```
γu[0].C1 + C1.γu[0] // Simplify
```

{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

```
Transpose[C1*] + C1
```

{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

```
Transpose[C1] + C1
```

{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

```
Inverse[C1] + C1
```

{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

32. Heisenberg's equation of motion (Heisenberg’s picture)

All operators are given by those in the Heisenberg picture. Here we omit the superscript (H).

$$\alpha^{k(H)} = \alpha^k , \qquad \beta^{(H)} = \beta$$

$\alpha^{k(H)}$ and $\beta^{(H)}$ must be regarded as dynamic variable.

The Hamiltonian

$$H = c\boldsymbol{\alpha}\cdot\boldsymbol{\pi} + eA_0 + \beta mc^2 = c\alpha^k \pi_k + eA_0 + \beta mc^2$$

with

$$\boldsymbol{\alpha} = \alpha^k = (\alpha^1, \alpha^2, \alpha^3) , \qquad \boldsymbol{\Sigma} = \Sigma^k = (\Sigma^1, \Sigma^2, \Sigma^3)$$

$$\boldsymbol{\pi} = \boldsymbol{p} - \frac{e}{c}\boldsymbol{A} ,$$

$$p_k = \boldsymbol{p} = (p_x, p_y, p_z) , \qquad A_k = \boldsymbol{A} = (A_x, A_y, A_z) ,$$

$$\pi_k = \boldsymbol{\pi} = (\pi_x, \pi_y, \pi_z) ,$$

$$r_k = \boldsymbol{r} = (x, y, z)$$

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} , \qquad E = -\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t} - \nabla A_0$$

Heisenberg equation for the operator in the Heisenberg picture

$$\frac{d}{dt}O = \frac{i}{\hbar}[H, O] + \frac{\partial}{\partial t}O$$

(a)

$$
\begin{aligned}
\frac{d}{dt}r_k &= \frac{i}{\hbar}[H, r_k] \\
&= \frac{i}{\hbar}c\alpha^j[\pi_j, r_k] + \frac{ie}{\hbar}[A_0, r_k] + \frac{i}{\hbar}[\beta mc^2, r_k] \\
&= \frac{i}{\hbar}c\alpha^j[p_j - \frac{e}{c}A_j, r_k] \\
&= \frac{i}{\hbar}c\alpha_j[p_{j\,j}, r_k] \\
&= \frac{i}{\hbar}c\alpha^j\,\frac{\hbar}{i}\frac{\partial}{\partial r_j}r_k \\
&= c\alpha^j\delta_{j,k} \\
&= c\alpha^k
\end{aligned}
$$

or

$$
\frac{d}{dt}\boldsymbol{r} = c\boldsymbol{\alpha} = \boldsymbol{v}
$$

(b)

$$
\begin{aligned}
\frac{d}{dt}p_k &= \frac{i}{\hbar}[H, p_k] \\
&= \frac{i}{\hbar}c\alpha^j[\pi_j, p_k] + \frac{ie}{\hbar}[A_0, p_k] + \frac{i}{\hbar}[\beta mc^2, p_k] \\
&= \frac{i}{\hbar}c\alpha^j[p_j - \frac{e}{c}A_j, p_k] + \frac{ie}{\hbar}[A_0, p_k] \\
&= \frac{i}{\hbar}c\alpha_j\,\frac{e}{c}[p_k, A_j] - \frac{ie}{\hbar}[p_k, A_0] \\
&= \frac{i}{\hbar}c\alpha^j\,\frac{e}{c}\frac{\hbar}{i}\frac{\partial A_j}{\partial x_k} - \frac{ie}{\hbar}\frac{\hbar}{i}\frac{\partial A_0}{\partial x_k} \\
&= e\alpha_j\,\frac{\partial A_j}{\partial x_k} - \frac{ie}{\hbar}\frac{\partial A_0}{\partial x_k} \\
&= e\frac{\partial(\alpha^j A_j)}{\partial x_k} - e\frac{\partial A_0}{\partial x_k}
\end{aligned}
$$

or

$$
\frac{d}{dt}\boldsymbol{p} = e\nabla(\boldsymbol{\alpha}\cdot\boldsymbol{A}) - e\nabla A_0
$$

(c)

$$\begin{aligned}
\frac{d}{dt}A_k &= \frac{i}{\hbar}[H, A_k] + \frac{\partial}{\partial t}A_k \\
&= \frac{i}{\hbar}c\alpha^j[\pi_j, A_k] + \frac{ie}{\hbar}[A_0, A_k] + \frac{i}{\hbar}[\beta mc^2, A_k] + \frac{\partial}{\partial t}A_k \\
&= \frac{i}{\hbar}c\alpha^j[p_j - \frac{e}{c}A_j, A_k] + \frac{ie}{\hbar}[A_0, A_k] + \frac{\partial}{\partial t}A_k \\
&= \frac{i}{\hbar}c\alpha^j[p_j, A_k] + \frac{\partial}{\partial t}A_k \\
&= \frac{i}{\hbar}c\alpha^j \frac{\hbar}{i}\frac{\partial A_k}{\partial x_j} + \frac{\partial}{\partial t}A_k \\
&= c\alpha^j \frac{\partial A_k}{\partial x_j} + \frac{\partial}{\partial t}A_k
\end{aligned}$$

or

$$\frac{d\boldsymbol{A}}{dt} = c(\boldsymbol{\alpha}\cdot\nabla)\boldsymbol{A} + \frac{\partial}{\partial t}\boldsymbol{A}.$$

(d)

$$\begin{aligned}
\frac{d\boldsymbol{\pi}}{dt} &= \frac{d}{dt}(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}) \\
&= e\nabla(\boldsymbol{\alpha}\cdot\boldsymbol{A}) - e\nabla A_0 - \frac{e}{c}[c(\boldsymbol{\alpha}\cdot\nabla)\boldsymbol{A} + \frac{\partial}{\partial t}\boldsymbol{A}] \\
&= e(-\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A} - \nabla A_0) + e\boldsymbol{\alpha}\times(\nabla\times\boldsymbol{A}) \\
&= e(\boldsymbol{E} + \boldsymbol{\alpha}\times\boldsymbol{B}) \\
&= e(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B})
\end{aligned}$$

where

$$\nabla(\boldsymbol{\alpha}\cdot\boldsymbol{A}) - (\boldsymbol{\alpha}\cdot\nabla)\boldsymbol{A} = \boldsymbol{\alpha}\times(\nabla\times\boldsymbol{A})$$

(e)

We note that

$$\boldsymbol{\alpha}(H - eA_0) + (H - eA_0)\boldsymbol{\alpha} = 2c(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}) = 2c\boldsymbol{\pi}$$

Using this we obtain the a quantum mechanical analogue of the Lorentz equation

$$\frac{d}{dt}\frac{1}{2}[\boldsymbol{v}(\frac{H-eA_0}{c^2})+(\frac{H-eA_0}{c^2})\boldsymbol{v}]=\dot{\boldsymbol{\pi}}=e(\boldsymbol{E}+\boldsymbol{\alpha}\times\boldsymbol{B})$$

where

$$\frac{d\boldsymbol{r}}{dt}=c\boldsymbol{\alpha}=\boldsymbol{v}\,,$$

$$\frac{d}{dt}\boldsymbol{\pi}=\frac{d}{dt}(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A})=e(\boldsymbol{E}+\boldsymbol{\alpha}\times\boldsymbol{B})\,.$$

If $H-eA_0\approx\pm mc^2$, depending on whether the state is made of positive or negative energy solutions of the Dirac equation, we get the equation of motion under the Lorentz forc,

$$m\frac{d}{dt}\boldsymbol{v}=\pm e(\boldsymbol{E}+\frac{1}{c}\boldsymbol{v}\times\boldsymbol{B})$$

((**Note**))

$$\{\boldsymbol{\alpha},H-eA_0\}=2c(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A})$$

or

$$\begin{aligned}\{\alpha^i,H-eA_0\}&=\{\alpha^i,c\alpha^j(p_j-\frac{e}{c}A_j)+\beta mc^2\}\\&=c\{\alpha^i,\alpha^j\}(p_j-\frac{e}{c}A_j)\\&=2c\delta_{ij}(p_j-\frac{e}{c}A_j)\\&=2c(p_i-\frac{e}{c}A_i)\end{aligned}$$

where

$$\{\alpha^i,\beta\}=0\,,\qquad\{\alpha^i,\alpha^j\}=2\delta_{ij}\,.$$

(f)

$$\frac{d}{dt}(\boldsymbol{\Sigma}\cdot\boldsymbol{\pi}) = \frac{i}{\hbar}[H, \boldsymbol{\Sigma}\cdot\boldsymbol{\pi}] = e\boldsymbol{\Sigma}\cdot\boldsymbol{E}$$

where

$$H = c\boldsymbol{\alpha}\cdot\boldsymbol{\pi} + eA_0 + \beta mc^2 = c\gamma^5\boldsymbol{\Sigma}\cdot\boldsymbol{\pi} + eA_0 + \beta mc^2$$

with

$$\alpha^k = \gamma^0\gamma^k = \gamma^5\Sigma^k = \Sigma^k\gamma^5, \qquad \text{and} \qquad [\gamma^5, \Sigma^k] = 0.$$

Then we get

$$\begin{aligned}\frac{d}{dt}(\boldsymbol{\Sigma}\cdot\boldsymbol{\pi}) &= \frac{i}{\hbar}[H, \boldsymbol{\Sigma}\cdot\boldsymbol{\pi}] \\ &= \frac{i}{\hbar}[c\gamma^5\boldsymbol{\Sigma}\cdot\boldsymbol{\pi} + eA_0 + \beta mc^2, \boldsymbol{\Sigma}\cdot\boldsymbol{\pi}] \\ &= \frac{ie}{\hbar}[A_0, \boldsymbol{\Sigma}\cdot\boldsymbol{\pi}] \\ &= -e\Sigma^k\frac{\partial A_0}{\partial x_k} \\ &= e\boldsymbol{\Sigma}\cdot\boldsymbol{E}\end{aligned}$$

where

$$\begin{aligned}[\beta, \boldsymbol{\Sigma}\cdot\boldsymbol{\pi}] &= [\beta, \Sigma 1\pi_1 + \Sigma 2\pi_2 + \Sigma 3\pi_3] \\ &= [\beta, \Sigma^1]\pi_1 + [\beta, \Sigma^2]\pi_2 + [\beta, \Sigma^3]\pi_3 \\ &= 0\end{aligned}$$

$$\begin{aligned}[\gamma^5\boldsymbol{\Sigma}\cdot\boldsymbol{\pi}, \boldsymbol{\Sigma}\cdot\boldsymbol{\pi}] &= \gamma^5(\boldsymbol{\Sigma}\cdot\boldsymbol{\pi})(\boldsymbol{\Sigma}\cdot\boldsymbol{\pi}) - (\boldsymbol{\Sigma}\cdot\boldsymbol{\pi})\gamma^5(\boldsymbol{\Sigma}\cdot\boldsymbol{\pi}) \\ &= 0\end{aligned}$$

(g) Zitterbewegung ((Free particle))

Suppose that

$$A_\mu = 0.$$

Then we have

$$\frac{d\boldsymbol{p}}{dt} = 0 \qquad\qquad \boldsymbol{p} = \text{constant}$$

The Hamiltonian of free particle is given by

$$H = c\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc^2$$

Heisenberg's equation for the operator $\boldsymbol{\alpha}$,

$$\begin{aligned}\frac{d\boldsymbol{\alpha}}{dt} &= \frac{1}{c}\frac{d\boldsymbol{v}}{dt} \\ &= \frac{i}{\hbar}[H,\boldsymbol{\alpha}] \\ &= \frac{i}{\hbar}(H\boldsymbol{\alpha} - \boldsymbol{\alpha}H) \\ &= -\frac{i}{\hbar}(H\boldsymbol{\alpha} + \boldsymbol{\alpha}H - 2H\boldsymbol{\alpha}) \\ &= -\frac{2i}{\hbar}(c\boldsymbol{p} - H\boldsymbol{\alpha})\end{aligned}$$

where

$$H\boldsymbol{\alpha} + \boldsymbol{\alpha}H = 2c\boldsymbol{p}$$

((Note))

$$\{H,\alpha^k\} = cp_i\{\alpha^i,\alpha^k\} + mc^2\{\beta,\alpha^i\} = 2cp_i$$

Since H = const., this equation has a simple solution:

$$\frac{d\boldsymbol{\alpha}}{dt} = \frac{2i}{\hbar}H(\boldsymbol{\alpha} - H^{-1}c\boldsymbol{p})$$

or

$$\boldsymbol{\alpha} = \frac{\boldsymbol{v}(t)}{c} = H^{-1}c\boldsymbol{p} + \exp(\frac{2i}{\hbar}Ht)[\boldsymbol{\alpha}(0) - H^{-1}c\boldsymbol{p}]$$

or

$$\boldsymbol{v}(t) = H^{-1}c^2\boldsymbol{p} + ce^{2iHt/\hbar}[\boldsymbol{\alpha}(0) - H^{-1}c\boldsymbol{p}]$$

This equation can be integrated:

$$\boldsymbol{r}(t) = \boldsymbol{r}(0) + H^{-1}c^2\boldsymbol{p}t + \frac{\hbar c}{2iH}(e^{2iHt/\hbar} - 1)[\boldsymbol{\alpha}(0) - H^{-1}c\boldsymbol{p}]$$

The first two terms on the right-hand side describe simply the uniform motion of a free particle. The last term is a feature of relativistic quantum mechanics and connotes a high-frequency vibration (Zitterbewegung) of the particle with frequency $mc^2/\hbar$ and amplitude $\hbar/(mc)$, the Compton wavelength of the particle.

(h) Free particles (continued)

For a free particle Hamiltonian,

$$\{\beta, H\} = \{\beta, c\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc^2\} = 2mc^2\beta^2 + cp_k\{\beta, \alpha^k\} = 2mc^2$$

and

$$\begin{aligned}\{\gamma^5, H\} &= \{\gamma^5, c\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc^2\} \\ &= cp_k\{\gamma^5, \alpha^k\} + mc^2\{\gamma^5, \beta\} \\ &= cp_k\{\gamma^5, \gamma^5\Sigma^k\} \\ &= 2cp_k\Sigma_k\end{aligned}$$

Note that

$$\{\gamma^5, \gamma^5\Sigma^k\} = \gamma^5\gamma^5\Sigma^k + \gamma^5\Sigma^k\gamma^5 = 2\gamma^5\gamma^5\Sigma^k = 2\Sigma^k$$

and

$$\{\gamma^5, \beta\} = 0$$

Hence, in a state of energy *E*, the operator β has the expectation value

$$<\beta> = \frac{mc^2}{E} = +\sqrt{1 - \left(\frac{cp}{E}\right)^2}$$

Similarly,

$$<\gamma_5> = -\frac{cp}{E} < \boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}} >$$

where $\hat{\boldsymbol{p}}$ is the unit vector of $\boldsymbol{p}$. The operator γ^5 is called the chirality.

(i) Lorentz force

$A_0 = 0$, $\qquad \boldsymbol{A} \neq 0$ (vector potential)

The Hamiltonian H is given by

$$H = c\boldsymbol{\alpha}\cdot(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}) + \beta mc^2 .$$

$$\frac{d}{dt}\boldsymbol{\Sigma} = \frac{i}{\hbar}[H, \boldsymbol{\Sigma}]$$

$$\alpha^k = \Sigma^k \gamma^5 = \gamma^5 \Sigma^k , \qquad [\beta, \Sigma^k] = 0$$

$$\Sigma^k = i\gamma^i \gamma^j \qquad (i, j, k\text{: cyclic})$$

$$[\gamma^5, \Sigma^k] = 0 , \quad , \qquad [\gamma^5, \alpha^k] = 0 , \quad \{\beta, \gamma^5\} = 0$$

$$[\Sigma^i, \Sigma^j] = 2i\Sigma^k , \qquad \Sigma^i \Sigma^j = -\Sigma^j \Sigma^i = i\Sigma^k \quad (i, j, \text{ and } k\text{; cyclic})$$

$$[\gamma^5 \Sigma^j, \Sigma^k] = \gamma^5 \Sigma^k \Sigma^j - \Sigma^j \gamma^5 \Sigma^k = \gamma^5 [\Sigma^k, \Sigma^j] .$$

Using the Mathematica, we get

$$\begin{aligned}
[H, \Sigma^1] &= [c\alpha^k (p_k - \frac{e}{c}A_k) + \beta mc^2, \Sigma^1] \\
&= [-c\gamma_5 \Sigma_k (p_k - \frac{e}{c}A_k), \Sigma_i] \\
&= 2ic[\alpha_2 (p_3 - \frac{e}{c}A_3) - \alpha_3 (p_2 - \frac{e}{c}A_2)] \\
&= 2ic[\boldsymbol{\alpha} \times (\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})]_1
\end{aligned}$$

or

$$[H, \boldsymbol{\Sigma}] = 2ic[\boldsymbol{\alpha} \times (\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})]$$

leading to the Heisenberg's equation,

$$\frac{d\boldsymbol{\Sigma}}{dt} = \frac{i}{\hbar}[H, \boldsymbol{\Sigma}] = -\frac{2c}{\hbar}[\boldsymbol{\alpha} \times (\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})]$$

So we have

$$H\frac{d\boldsymbol{\Sigma}}{dt}+\frac{d\boldsymbol{\Sigma}}{dt}H=-\frac{2c^2}{\hbar}[\boldsymbol{\alpha}\cdot(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A})][\boldsymbol{\alpha}\times(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A})]$$
$$+\boldsymbol{\alpha}\times(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A})][\boldsymbol{\alpha}\cdot(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A})]$$
$$=2ec\boldsymbol{\Sigma}\times\boldsymbol{B}$$

In the relativistic approximation, $H\approx mc^2$. Then we have

$$\frac{d\boldsymbol{\Sigma}}{dt}=\frac{2ec}{2mc^2}\boldsymbol{\Sigma}\times\boldsymbol{B}=\frac{e}{mc}\boldsymbol{\Sigma}\times\boldsymbol{B}$$

Since

$$\boldsymbol{\Sigma}=\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix}$$

$$\frac{d\boldsymbol{\sigma}}{dt}=\frac{e}{mc}\boldsymbol{\sigma}\times\boldsymbol{B}$$

For the one electron spin operator

$$\boldsymbol{S}=\frac{\hbar}{2}\boldsymbol{\sigma}$$

$$\frac{d\boldsymbol{S}}{dt}=\frac{e}{mc}\boldsymbol{S}\times\boldsymbol{B}=\boldsymbol{\mu}\times\boldsymbol{B}$$

or

$$\boldsymbol{\mu}=\frac{e}{mc}\boldsymbol{S}=\frac{e\hbar}{2mc}\boldsymbol{\sigma}$$

REFERENCES

E. Fermi, Notes on Quantum Mechanics (The University of Chicago Press, 1961).
J.D. Bjorken and S.D. Drell, Relativistic Quantum Mechanics (McGraw-Hill, New York, 1964)
J.J. Sakurai, Advanced Quantum Mechanics (Addison-Wesley, New York, 1967).
M.E. Rose, Relativistic Electron Theory (John Wiley & Sons, New York, 1961).
A. Das, Lectures on Quantum mechanics, second edition (World Scientific, 2012).
F.J, Dyson, Advanced Quantum Mechanics (World Scientific, 2007).

S.S. Schweber, An Introduction to Relativistic Quantum Field Theory (Row, Peterson, 1961).
F. Schwabl, Advanced *Quantum Mechanics* (Springer Verlag, Berlin, 2005).
F. Gross, Relativistic Quantum Mechanics and Field Theory (Wiley-VCH, 1993).
Eugen Merzbacher, *Quantum Mechanics*, third edition (John Wiley & Sons, New York, 1998).
Ramamurti Shankar, *Principles of Quantum Mechanics*, second edition (Springer, New York, 1994).
Leonard Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc, New York, 1955).
B.R. Holstein, Topics in ADVANCED Quantum Mechanics (Addison-Wesley, 1992).
A.Z. Capri, Relativistic Quantum Mechanics and Introduction to Quantum Field Theory (World Scientific, 2002).
R. Parthasarathy, Relativistic Quantum Mechanics (Alpha Science International, 2010).
D. Atkinson and P.W. Johnson, Quantum Field Theory: A Self-Contained Course, volume 2 (Pinton Press, 2002).
H.M. Pikuhn, Relativistic Quantum Mechanics, 2nd edition (Springer, 2005).
S. Tomonaga, The Story of Spin (University of Chicago, 1997).

APPENDIX-I

Klein-Gordon equation

((Problem))

The relativistic wave equation for bosons of rest mass *m* may be obtained by the relation

$$E^2 = \boldsymbol{p}^2 c^2 + m^2 c^4$$

through the identifications

$$E \to i\hbar \frac{\partial}{\partial t}, \qquad \boldsymbol{p} \to \frac{\hbar}{i}\nabla$$

(a) Obtain the wave equation relevant to bosons of rest mass *m*. This equation is called the Klein-Gordon equation.

(b) What form does this equation assume for photons?

(c) Suppose that the wavefunction is independent of time *t*. It depends only on *r*. Using the spherical co-ordinates; $\{r, \theta, \phi\}$, find the differential equation for the wavefunction $\psi(r)$. Show that $\psi(r)$ has the form of $\psi(r) = A\dfrac{e^{-r/a}}{r}$, where *A* and *a* are constants. We assume that $l = 0$.

(d) Find the expression for the characteristic length *a*.

(e) Use this equation to show that there is a local conservation law of the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} = 0$$

with

$$\boldsymbol{j} = \frac{\hbar}{2mi}(\psi^* \nabla \psi - \psi \nabla \psi^*) .$$

Determine the form of $\rho(\boldsymbol{r}, t)$. From this form for ρ, give an argument for why the Klein-Gordon equation is not a good candidate for a one-particle relativistic wave equation in plane of the Schrodinger equation, for which $\rho = \psi^* \psi$

((**Solution**))
(a)
We start with

$$E^2 \psi = (\boldsymbol{p}^2 c^2 + m^2 c^4)\psi ,$$

with

$$E \to i\hbar \frac{\partial}{\partial t} , \qquad \boldsymbol{p} \to \frac{\hbar}{i}\nabla$$

Then we have

$$-\hbar^2 \frac{\partial^2}{\partial t^2}\psi = -\hbar^2 c^2 \nabla^2 \psi + m^2 c^4 \psi$$

or

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = \frac{m^2 c^2}{\hbar^2}\psi \qquad \text{(Klein-Gordon equation)}$$

(b) For photon, the mass *m* is equal to zero. Then we have the wave equation as

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = 0$$

(c) Suppose that the wavefunction is independent of time *t*. It depends only on *r*.

$$\frac{1}{r}\frac{\partial}{\partial r}(\frac{\partial}{\partial r} r)\psi(r) = \frac{m^2 c^2}{\hbar^2}\psi$$

We assume that $\psi = \frac{u}{r}$.

$$\frac{d^2}{dr^2}u(r) = \frac{m^2c^2}{\hbar^2}u(r) = \frac{1}{a^2}u(r).$$

Then we have the

$$u = Ae^{-r/a}$$

or

$$\psi = A\frac{e^{-r/a}}{r}$$

(d)

a is the characteristic length and is defined by

$$a = \frac{\hbar}{mc}.$$

(e) The current density is given by

$$\boldsymbol{j} = \frac{\hbar}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*)$$

$$\begin{aligned}\nabla\cdot\boldsymbol{j} &= \frac{\hbar}{2mi}[\nabla\cdot(\psi^*\nabla\psi) - \nabla\cdot(\psi\nabla\psi^*)\\ &= \frac{\hbar}{2mi}(\nabla\psi^*\cdot\nabla\psi + \psi^*\nabla^2\psi - \nabla\psi\nabla\psi^* - \psi\nabla^2\psi^*)\\ &= \frac{\hbar}{2mi}(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*)\end{aligned}$$

Using the equation of continuity, we have

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot\boldsymbol{j} = -\frac{\hbar}{2mi}(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*)$$

We use the Klein-Gordon equation

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi + \frac{m^2c^2}{\hbar^2}\psi, \qquad \nabla^2\psi^* = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi^* + \frac{m^2c^2}{\hbar^2}\psi^*$$

Then we get

$$\frac{\partial \rho}{\partial t} = -\frac{\hbar}{2mi}[\psi^*(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi + \frac{m^2c^2}{\hbar^2}\psi) - \psi(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi^* + \frac{m^2c^2}{\hbar^2}\psi^*)]$$
$$= -\frac{\hbar}{2mc^2 i}(\psi^*\frac{\partial^2}{\partial t^2}\psi - \psi\frac{\partial^2}{\partial t^2}\psi^*)$$
$$= -\frac{\hbar}{2mc^2 i}\frac{\partial}{\partial t}(\psi^*\frac{\partial}{\partial t}\psi - \psi\frac{\partial}{\partial t}\psi^*)$$

Thus we have

$$\rho = \frac{i\hbar}{2mc^2}(\psi^*\frac{\partial}{\partial t}\psi - \psi\frac{\partial}{\partial t}\psi^*)$$

Suppose that

$$\psi^*\frac{\partial}{\partial t}\psi = \alpha + i\beta$$

where α and β are real. Then we have

$$\psi\frac{\partial}{\partial t}\psi^* = \alpha - i\beta$$

Then we have

$$\rho = \frac{i\hbar}{2mc^2}[\alpha + i\beta - (\alpha - i\beta)] = \frac{i\hbar}{2mc^2}2i\beta = -\frac{\beta\hbar}{mc^2}$$

When $\beta > 0$, the probability density could be negative, which is inconsistent with the requirement that ρ should be positive. In this sense, the Klein-Gordon equation is not a good candidate for a one-particle relativistic wave equation in plane of the Schrodinger equation, for which $\rho = \psi^*\psi$

APPENDIX-II
Another way of constructing solutions of the free Dirac equation
Construction by Lorentz transformation

Greiner: Relativistic Quantum Mechanics 3rd edition (Springer 2000)

We show that

$$S(-\boldsymbol{v}) = \exp[-\frac{\omega}{2}\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}}]$$

$$= \begin{pmatrix} \cosh\frac{\omega}{2} & 0 & -\hat{p}_z\sinh\frac{\omega}{2} & -(\hat{p}_x - i\hat{p}_y)\sinh\frac{\omega}{2} \\ 0 & \cosh\frac{\omega}{2} & -(\hat{p}_x + i\hat{p}_y)\sinh\frac{\omega}{2} & \hat{p}_z\sinh\frac{\omega}{2} \\ -\hat{p}_z\sinh\frac{\omega}{2} & -(\hat{p}_x - i\hat{p}_y)\sinh\frac{\omega}{2} & \cosh\frac{\omega}{2} & 0 \\ -(\hat{p}_x + i\hat{p}_y)\sinh\frac{\omega}{2} & \hat{p}_z\sinh\frac{\omega}{2} & 0 & \cosh\frac{\omega}{2} \end{pmatrix}$$

by using the Mathematica, where $\hat{\boldsymbol{p}}$ is the unit vector of $\boldsymbol{p}$. Then we get

$$\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}} = \begin{pmatrix} 0 & 0 & \hat{p}_z & \hat{p}_x - i\hat{p}_y \\ 0 & 0 & \hat{p}_x + i\hat{p}_y & -\hat{p}_z \\ \hat{p}_z & \hat{p}_x - i\hat{p}_y & 0 & 0 \\ \hat{p}_x + i\hat{p}_y & -\hat{p}_z & 0 & 0 \end{pmatrix}$$

((**Mathematica**))

```
Clear["Global`*"];
exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]};
S1 = (
    0          0          pz         px - i py
    0          0          px + i py  -pz
    pz         px - i py  0          0
    px + i py  -pz        0          0
   );
A1 = MatrixExp[-ω/2 S1] //. {pz^2 -> 1 - (px^2 + py^2)} // FullSimplify;
A1 // MatrixForm
```

$$\begin{pmatrix} \text{Cosh}\left[\frac{\omega}{2}\right] & 0 & -\text{pz Sinh}\left[\frac{\omega}{2}\right] & -(\text{px} - i\,\text{py})\,\text{Sinh}\left[\frac{\omega}{2}\right] \\ 0 & \text{Cosh}\left[\frac{\omega}{2}\right] & -(\text{px} + i\,\text{py})\,\text{Sinh}\left[\frac{\omega}{2}\right] & \text{pz Sinh}\left[\frac{\omega}{2}\right] \\ -\text{pz Sinh}\left[\frac{\omega}{2}\right] & -(\text{px} - i\,\text{py})\,\text{Sinh}\left[\frac{\omega}{2}\right] & \text{Cosh}\left[\frac{\omega}{2}\right] & 0 \\ -(\text{px} + i\,\text{py})\,\text{Sinh}\left[\frac{\omega}{2}\right] & \text{pz Sinh}\left[\frac{\omega}{2}\right] & 0 & \text{Cosh}\left[\frac{\omega}{2}\right] \end{pmatrix}$$

We assume that

$$\cosh\frac{\omega}{2} = \sqrt{\frac{R+mc^2}{2mc^2}}, \qquad \sinh\frac{\omega}{2} = -\sqrt{\frac{R+mc^2}{2mc^2}}\frac{c|\boldsymbol{p}|}{R+mc^2}$$

or

$$\tanh\frac{\omega}{2} = -\frac{c|\boldsymbol{p}|}{R+mc^2}$$

Then we have

$$S(-\boldsymbol{v}) = \sqrt{\frac{R+mc^2}{2mc^2}}\begin{pmatrix} 1 & 0 & \frac{p_3 c}{R+mc^2} & \frac{(p_1-ip_2)c}{R+mc^2} \\ 0 & 1 & \frac{(p_1+ip_2)c}{R+mc^2} & -\frac{p_3 c}{R+mc^2} \\ \frac{p_3 c}{R+mc^2} & \frac{(p_1-ip_2)c}{R+mc^2} & 1 & 0 \\ \frac{(p_1+ip_2)c}{R+mc^2} & \frac{-p_3 c}{R+mc^2} & 0 & 1 \end{pmatrix}$$

$$u_1 = S(-\boldsymbol{v})\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \sqrt{\frac{R+mc^2}{2mc^2}}\begin{pmatrix} 1 \\ 0 \\ \frac{p_3 c}{R+mc^2} \\ \frac{(p_1+ip_2)c}{R+mc^2} \end{pmatrix},$$

$$u_2 = S(-\boldsymbol{v})\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \sqrt{\frac{R+mc^2}{2mc^2}}\begin{pmatrix} 0 \\ 1 \\ \frac{(p_1-ip_2)c}{R+mc^2} \\ -\frac{p_3 c}{R+mc^2} \end{pmatrix}$$

$$u_3 = S(-\boldsymbol{v})\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \sqrt{\frac{R+mc^2}{2mc^2}}\begin{pmatrix} \frac{p_3 c}{R+mc^2} \\ \frac{(p_1+ip_2)c}{R+mc^2} \\ 1 \\ 0 \end{pmatrix}$$

$$u_4 = S(-\boldsymbol{v})\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \sqrt{\frac{R+mc^2}{2mc^2}}\begin{pmatrix}\frac{(p_1-ip_2)c}{R+mc^2}\\ -\frac{p_3 c}{R+mc^2}\\ 0\\ 1\end{pmatrix}$$

APPENDIX III Commutation relation of the helicity operator

$$[\boldsymbol{\Sigma}\cdot\boldsymbol{p}_1, \boldsymbol{\Sigma}\cdot\boldsymbol{p}_2] = \begin{pmatrix}\boldsymbol{\sigma}\cdot\boldsymbol{p}_1 & 0\\ 0 & \boldsymbol{\sigma}\cdot\boldsymbol{p}_1\end{pmatrix}\begin{pmatrix}\boldsymbol{\sigma}\cdot\boldsymbol{p}_2 & 0\\ 0 & \boldsymbol{\sigma}\cdot\boldsymbol{p}_2\end{pmatrix} - \begin{pmatrix}\boldsymbol{\sigma}\cdot\boldsymbol{p}_2 & 0\\ 0 & \boldsymbol{\sigma}\cdot\boldsymbol{p}_2\end{pmatrix}\begin{pmatrix}\boldsymbol{\sigma}\cdot\boldsymbol{p}_{12} & 0\\ 0 & \boldsymbol{\sigma}\cdot\boldsymbol{p}_1\end{pmatrix}$$

$$= \begin{pmatrix}(\boldsymbol{\sigma}\cdot\boldsymbol{p}_1)(\boldsymbol{\sigma}\cdot\boldsymbol{p}_2) & 0\\ 0 & (\boldsymbol{\sigma}\cdot\boldsymbol{p}_1)(\boldsymbol{\sigma}\cdot\boldsymbol{p}_2)\end{pmatrix} - \begin{pmatrix}(\boldsymbol{\sigma}\cdot\boldsymbol{p}_2)(\boldsymbol{\sigma}\cdot\boldsymbol{p}_1) & 0\\ 0 & (\boldsymbol{\sigma}\cdot\boldsymbol{p}_2)(\boldsymbol{\sigma}\cdot\boldsymbol{p}_1)\end{pmatrix}$$

$$= \begin{pmatrix}(\boldsymbol{\sigma}\cdot\boldsymbol{p}_1)(\boldsymbol{\sigma}\cdot\boldsymbol{p}_2) - (\boldsymbol{\sigma}\cdot\boldsymbol{p}_2)(\boldsymbol{\sigma}\cdot\boldsymbol{p}_1) & 0\\ 0 & (\boldsymbol{\sigma}\cdot\boldsymbol{p}_1)(\boldsymbol{\sigma}\cdot\boldsymbol{p}_2) - (\boldsymbol{\sigma}\cdot\boldsymbol{p}_2)(\boldsymbol{\sigma}\cdot\boldsymbol{p}_1)\end{pmatrix}$$

$$= \begin{pmatrix}i\boldsymbol{\sigma}\cdot(\boldsymbol{p}_1\times\boldsymbol{p}_2) & 0\\ 0 & i\boldsymbol{\sigma}\cdot(\boldsymbol{p}_1\times\boldsymbol{p}_2)\end{pmatrix}$$

We note that

$$(\boldsymbol{\sigma}\cdot\boldsymbol{p}_1)(\boldsymbol{\sigma}\cdot\boldsymbol{p}_2) = \boldsymbol{p}_1\cdot\boldsymbol{p}_2 + i\boldsymbol{\sigma}\cdot(\boldsymbol{p}_1\times\boldsymbol{p}_2)$$

$$(\boldsymbol{\sigma}\cdot\boldsymbol{p}_1)(\boldsymbol{\sigma}\cdot\boldsymbol{p}_2) - (\boldsymbol{\sigma}\cdot\boldsymbol{p}_2)(\boldsymbol{\sigma}\cdot\boldsymbol{p}_1) = i\boldsymbol{\sigma}\cdot(\boldsymbol{p}_1\times\boldsymbol{p}_2) - i\boldsymbol{\sigma}\cdot(\boldsymbol{p}_2\times\boldsymbol{p}_1)$$
$$= i\boldsymbol{\sigma}\cdot(\boldsymbol{p}_1\times\boldsymbol{p}_2)$$

So that only if $\boldsymbol{p}_1\times\boldsymbol{p}_2 = 0$, $[\boldsymbol{\Sigma}\cdot\boldsymbol{p}_1, \boldsymbol{\Sigma}\cdot\boldsymbol{p}_2] = 0$.

APPENDIX IV

$$\frac{d\boldsymbol{L}}{dt} = c(\boldsymbol{\alpha}\times\boldsymbol{p})$$

In the Heisenberg picture,

$$-i\hbar\frac{d}{dt}\boldsymbol{L} = [H, \boldsymbol{L}]$$

where

$$H = c\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc^2 = c\alpha_k p_k + \beta mc^2$$

We consider the component L_3. We write

$$L_3 = x_1 p_2 - x_2 p_1$$

$$\begin{aligned}
-i\hbar\frac{d}{dt}L_3 &= [H, L_3] \\
&= [c\alpha_k p_k + \beta mc^2, x_1 p_2 - x_2 p_1] \\
&= [c\alpha_k p_k, x_1 p_2] - [c\alpha_k p_k, x_2 p_1] \\
&= c\alpha_k [p_k, x_1 p_2] - c\alpha_k [p_k, x_2 p_1] \\
&= c\alpha_k [p_k, x_1] p_2 - c\alpha_k [p_k, x_2] p_1 \\
&= \frac{\hbar}{i}(c\alpha_k p_2 \delta_{k,1} - c\alpha_k p_1 \delta_{k,2}) \\
&= \frac{\hbar}{i}(c\alpha_1 p_2 - c\alpha_2 p_1) \\
&= \frac{\hbar}{i} c(\boldsymbol{\alpha}\times\boldsymbol{p})_3
\end{aligned}$$

This results lead to

$$\frac{d}{dt}\boldsymbol{L} = c(\boldsymbol{\alpha}\times\boldsymbol{p}) \qquad (1)$$

We now consider the operator $\boldsymbol{\Sigma}$ such that

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}$$

The Heisenberg's equation of motion:

$$
\begin{aligned}
-i\hbar\frac{d}{dt}\Sigma^3 &= [H,\Sigma^3] \\
&= [c\alpha_k p^k + \beta mc^2, \Sigma_3] \\
&= cp^k[\alpha^k,\Sigma^3] + mc^2[\beta,\Sigma^3] \\
&= cp^k[\gamma^5\Sigma^k,\Sigma^3] \\
&= cp^k\gamma^5[\Sigma^k,\Sigma^3] \\
&= -cp^1\gamma^5[\Sigma^3,\Sigma^1] + cp^2\gamma^5[\Sigma^2,\Sigma^3] \\
&= -2icp^1\gamma^5\Sigma^2 + 2icp^2\gamma^5\Sigma^1
\end{aligned}
$$

or

$$
\begin{aligned}
-i\hbar\frac{d}{dt}\Sigma^3 &= -2ic(p^1\alpha^2 - p^2\alpha^1) \\
&= 2ic(\boldsymbol{\alpha}\times\boldsymbol{p})_3
\end{aligned}
$$

where

$$
\alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix} = \gamma^0\gamma^k = \Sigma^k\gamma^5 = \gamma^5\Sigma^k
$$

$$
[\beta,\Sigma^k] = 0\,, \qquad [\gamma^5,\Sigma^k] = 0
$$

$$
[\Sigma^i,\Sigma^j] = 2i\Sigma^k
$$

Since

$$
\frac{d}{dt}\frac{\hbar\Sigma}{2} = -c(\boldsymbol{\alpha}\times\boldsymbol{p}) \tag{2}
$$

we have

$$
\frac{d}{dt}\boldsymbol{J} = \frac{d}{dt}(\boldsymbol{L} + \frac{\hbar}{2}\boldsymbol{\Sigma}) = 0
$$

$\frac{\hbar}{2}\boldsymbol{\Sigma}$ is called spin angular momentum and $\boldsymbol{J} = \boldsymbol{L} + \frac{\hbar}{2}\boldsymbol{\Sigma}$ is called total angular momentum, and is a conserved quantity. Thus we can conclude that the Dirac particles is endowed with spin $\frac{\hbar}{2}\boldsymbol{\Sigma}$.

REFERENCES

S. Rajasekar and R. Velusamy, Quantum Mechanics I: The Fundamentrals (CRC Press2015).