# Dirac matrices

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**(Date: January 21, 2016)**

Here we present the properties of Dirac matrices. The manipulation of the matrices can be expressed using the Mathematica. All the notations including the contravariant and covariant tensors, which are conventionally used for the special relativity and general relativity are presented in the other chapter. With the use of the Mathematica program which is prepared here, it is possible to calculate any kind of the combination of Dirac matrices such as the commutation relations. We use the following notations in the Mathematica program

γu[μ]→ $\gamma^{\mu}$,  γd[μ]→ $\gamma_{\mu}$,

Σu[μ]→ $\Sigma^{\mu}$,  Σd[μ]→ $\gamma_{\mu}$,

σu[μ, ν]  $\sigma^{\mu\nu}$,  σd[μ, ν]  $\sigma_{\mu\nu}$,

gu[μ, ν]  $g^{\mu\nu}$,  gd[μ, ν]  $g_{\mu\nu}$,

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The metric tensor is defined by

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \text{(metric tensor)}$$

The contravariant vector is described by

$$x^{\mu} = g^{\mu\nu} x_{\nu}$$

The matrices $\boldsymbol{\alpha}$ and $\beta$ can be expressed in terms of the Pauli spin matrices,

$$\sigma^1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the identity matrix

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using the Kronecker product, the matrices $\alpha^1$, $\alpha^2$, $\alpha^3$, and $\beta$ are given by

$$\alpha^1 = \alpha_x = \sigma^1 \otimes \sigma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}$$

$$\alpha^2 = \alpha_y = \sigma^1 \otimes \sigma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}$$

$$\alpha^3 = \alpha_z = \sigma^1 \otimes \sigma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}$$

$$\gamma^0 = \beta = \sigma^3 \otimes I_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$$

$$\gamma^1 = \beta\alpha^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix},$$

$$\gamma^2 = \beta\alpha^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}$$

$$\gamma^3 = \beta\alpha^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}$$

$$\alpha^k = \gamma^0\gamma^k = \beta\gamma^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}$$

$\gamma^0\gamma^k = -\gamma^k\gamma^0$ or $\{\gamma^0, \gamma^k\} = 0$

$\gamma^{0+} = \gamma^0$ (Hermitian)

$\gamma^{k+} = -\gamma^k$ (anti-Hermitian)

$(\gamma^0)^2 = I_4, \quad (\gamma^k)^2 = -I_4$

$\{\beta, \alpha^k\} = 0 \quad (k = 1, 2, 3)$

$\beta = \gamma^0$

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$\gamma_\mu = g_{\mu\nu}\gamma^\nu$

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma_1 = \begin{pmatrix} 0 & -\sigma_x \\ \sigma_x & 0 \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} 0 & -\sigma_y \\ \sigma_y & 0 \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} 0 & -\sigma_z \\ \sigma_z & 0 \end{pmatrix}$$

$\gamma_\mu\gamma^\mu = 4\, I_4$

$\gamma^\mu = \gamma_\mu{}^+ = \gamma_\mu{}^{-1}$

$$(\gamma^\mu)^{-1} = \gamma_\mu$$

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$\gamma^5$ is the fifth gamma matrix. It plays very important role in particle physics (related to chiral symmetry) and in mathematical physics (index theorem for Dirac operator).

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\varepsilon_{\alpha\beta\gamma\delta}\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$$

$$\gamma^{5+} = \gamma^5$$

$$(\gamma^5)^2 = I_4$$

$$\gamma^5\gamma^1 + \gamma^1\gamma^5 = 0$$

$$\gamma^5\gamma^2 + \gamma^2\gamma^5 = 0$$

$$\gamma^5\gamma^3 + \gamma^3\gamma^5 = 0$$

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$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$$

$$\begin{aligned} \gamma_5 &= (\gamma^5)^{-1} \\ &= i^{-1}(\gamma^0\gamma^1\gamma^2\gamma^3)^{-1} \\ &= -i(\gamma^3)^{-1}(\gamma^2)^{-1}(\gamma^1)^{-1}(\gamma^0)^{-1} \\ &= -i\gamma_3\gamma_2\gamma_1\gamma_0 \end{aligned}$$

or

$$\gamma_5 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$$

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Trace

$$Tr[\gamma^5] = 0$$

$Tr[\gamma^5\gamma^0\gamma^1\gamma^2\gamma^3] = -4i$

$Tr[\gamma^1\gamma^2 + \gamma^2\gamma^1] = 0$

---

Hermitian conjugate of $\gamma^\mu$

$$(\gamma^\mu)^+ = \gamma^0\gamma^\mu\gamma^0$$

$\gamma^{0^+} = \gamma^0$, $\qquad \gamma^{k^+} = -\gamma^k \qquad (k = 1, 2, 3)$

---

$$\gamma^1\gamma^0 = \begin{pmatrix} 0 & -\sigma_x \\ -\sigma_x & 0 \end{pmatrix}$$

$$\gamma^2\gamma^0 = \begin{pmatrix} 0 & -\sigma_y \\ -\sigma_y & 0 \end{pmatrix}$$

$$\gamma^3\gamma^0 = \begin{pmatrix} 0 & -\sigma_z \\ -\sigma_z & 0 \end{pmatrix}$$

$$\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & -i\,I_2 \\ i\,I_2 & 0 \end{pmatrix}$$

$$\gamma^1\gamma^2 = \begin{pmatrix} -i\sigma_z & 0 \\ 0 & -i\sigma_z \end{pmatrix}$$

$$\gamma^2\gamma^3 = \begin{pmatrix} -i\sigma_x & 0 \\ 0 & -i\sigma_x \end{pmatrix}$$

$$\gamma^3\gamma^1 = \begin{pmatrix} -i\sigma_y & 0 \\ 0 & -i\sigma_y \end{pmatrix}$$

$$\gamma^0\gamma^2\gamma^3 = \begin{pmatrix} -i\sigma_x & 0 \\ 0 & i\sigma_x \end{pmatrix}$$

$$\gamma^0\gamma^3\gamma^1 = \begin{pmatrix} -i\sigma_y & 0 \\ 0 & i\sigma_y \end{pmatrix}$$

$$\gamma^0\gamma^1\gamma^2 = \begin{pmatrix} -i\sigma_z & 0 \\ 0 & i\sigma_z \end{pmatrix}$$

$$\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu} I_4$$ (Clifford algebra)

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$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] = i\gamma^\mu \gamma^\nu$$

$$\sigma^{01} = i\begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}$$

$$\sigma^{03} = i\begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}$$

$$\sigma^{12} = \Sigma^3 = \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}$$

$$\sigma^{23} = \Sigma^1 = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}$$

$$\sigma^{31} = \Sigma^2 = \begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_y \end{pmatrix}$$

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$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$$

$$\sigma_{01} = \begin{pmatrix} 0 & i\sigma_x \\ i\sigma_x & 0 \end{pmatrix}$$

$$\sigma_{02} = \begin{pmatrix} 0 & i\sigma_y \\ i\sigma_y & 0 \end{pmatrix}$$

$$\sigma_{03} = \begin{pmatrix} 0 & i\sigma_z \\ i\sigma_z & 0 \end{pmatrix}$$

$$\sigma_{12} = \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}$$

$$\sigma_{23} = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}$$

$$\sigma_{31} = \begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_y \end{pmatrix}$$

---

$$\Sigma^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} = i\gamma^i\gamma^j \qquad (i, j, k;\ \text{cyclic})$$

$$\Sigma^3 = i\gamma^1\gamma^2, \qquad \Sigma^1 = i\gamma^2\gamma^3, \qquad \Sigma^2 = i\gamma^3\gamma^1$$

or, simply,

$$\alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix} = \gamma^0\gamma^k = \Sigma^k\gamma^5 = \gamma^5\Sigma^k$$

or, simply,

$$\boldsymbol{\alpha} = \left(\alpha^1, \alpha^2, \alpha^3\right)$$

$$\alpha^1 = \alpha_x = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}, \qquad \alpha^2 = \alpha_y = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix},$$

$$\alpha^3 = \alpha_z = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}$$

with

$[\gamma^5, \Sigma^k] = 0$,

$[\beta, \Sigma^k] = 0$,

$[\gamma^5, \alpha^k] = 0$,

$\{\beta, \gamma^5\} = 0$

$[\Sigma^i, \Sigma^j] = 2i\Sigma^k$, $\Sigma^i\Sigma^j = -\Sigma^j\Sigma^i = i\Sigma^k$ ($i$, $j$, and $k$; cyclic)

$[\gamma^5\Sigma^k, \Sigma^j] = \gamma^5\Sigma^k\Sigma^j - \Sigma^j\gamma^5\Sigma^k = \gamma^5[\Sigma^k, \Sigma^j]$

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((**Mathematica**))

We use the following Mathematica program to evaluate the properties of Dirac maticres.

```
Clear["Global`*"];
exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]};
g1 = ( 1  0  0  0
       0 -1  0  0
       0  0 -1  0
       0  0  0 -1 );
Gd[μ_, ν_] := g1[[μ + 1, ν + 1]];
Gu[μ_, ν_] := g1[[μ + 1, ν + 1]];
Gum = Table[Gu[μ, ν], {μ, 0, 3, 1}, {ν, 0, 3, 1}];
Gdm = Table[Gd[μ, ν], {μ, 0, 3, 1}, {ν, 0, 3, 1}];
σx = PauliMatrix[1];
σy = PauliMatrix[2];
σz = PauliMatrix[3];
I2 = IdentityMatrix[2];
I4 = IdentityMatrix[4];
αx = KroneckerProduct[σx, σx];
αy = KroneckerProduct[σx, σy];
αz = KroneckerProduct[σx, σz];
γu0 = KroneckerProduct[σz, I2];
γux = γu0.αx // Simplify;
γuy = γu0.αy // Simplify;
γuz = γu0.αz // Simplify; ; γu[0] = γu0; γu[1] = γux; γu[2] = γuy;
γu[3] = γuz ; γu[5] = i γu0 .γux.γuy.γuz;
γd[μ_] := Sum[Gd[μ, ν] γu[ν], {ν, 0, 3, 1}];
σu[μ_, ν_] := i/2 (γu[μ].γu[ν] - γu[ν].γu[μ]);
σd[μ_, ν_] := i/2 (γd[μ].γd[ν] - γd[ν].γd[μ]);
γd[5] = -i γd[3] .γd[2] .γd[1] .γd[0]; Σu[3] = σu[1, 2]; Σu[2] = σu[3, 1];
Σu[1] = σu[2, 3]; αu[1] = αx; αu[2] = αy; αu[3] = αz;
β = γu[0];
```

```
γu[0] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

```
γu[0].γu[0] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

```
γu[1] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

```
γu[1].γu[1] // MatrixForm
```

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

```
γu[2].γu[2] // MatrixForm
```

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

```
γu[3].γu[3] // MatrixForm
```

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

```
γu[5] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

```
ⅈ/4! Sum[Signature[{a0, a1, a2, a3}]
   γu[a0].γu[a1].γu[a2].γu[a3], {a0, 0, 3, 1},
  {a1, 0, 3, 1}, {a2, 0, 3, 1}, {a3, 0, 3, 1}] //
 MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

```
γu[5].γu[5] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

```
γu[5].γu[1] + γu[1].γu[5] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
γu[5].γu[2] + γu[2].γu[5] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
γu[5].γu[3] + γu[3].γu[5] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
γu[5].γu[0] + γu[0].γu[5] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
γd[1] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

```
γd[2] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

```
γd[3] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

```
γd[5] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

```
Transpose[γd[1]*] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

```
γu[1] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

```
Transpose[γd[2]*] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

```
γu[2] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

```
Inverse[γd[2]] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

```
Sum[γd[μ].γu[μ], {μ, 0, 3, 1}] // MatrixForm
```

$$\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

```
Tr[γu[5]]
```

0

```
Tr[γu[5].γu[0].γu[1].γu[2].γu[3]]
```

$-4\,i$

```
Tr[γu[1].γu[2] + γu[2].γu[1]] // MatrixForm
```

0

```
γu[1].γu[0] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

```
γu[0].γu[1] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

```
γu[2].γu[0] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

```
γu[3].γu[0] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

```
γu[1].γu[2].γu[3] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

```
γu[1].γu[2] // MatrixForm
```

$$\begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$$

```
γu[2].γu[3] // MatrixForm
```

$$\begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix}$$

```
γu[3].γu[1] // MatrixForm
```

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

```
γu[1].γu[2] // MatrixForm
```

$$\begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$$

```
γu[0].γu[2].γu[3] // MatrixForm
```

$$\begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}$$

```
γu[0].γu[3].γu[1] // MatrixForm
```

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

```
γu[0].γu[1].γu[2] // MatrixForm
```

$$\begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

```
γu[0] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

```
γu[3] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

```
σu[0, 1] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

```
σu[0, 2] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

```
σu[0, 3] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

```
σu[1, 2] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

```
σu[2, 3] // MatrixForm
```

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

```
σu[3, 1] // MatrixForm
```

$$\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

```
σd[0, 2] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

```
Transpose[γu[1]*] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

```
γu[0].γu[1].γu[0] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

```
Transpose[γu[2]*] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

```
γu[0].γu[2].γu[0] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

```
γd[0] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

```
σd[0, 1] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

```
σd[0, 2] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

```
σd[0, 3] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

```
σd[1, 2] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

```
σd[2, 3] // MatrixForm
```

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

```
σd[3, 1] // MatrixForm
```

$$\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$