Symmetries of Dirac equation

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(a) The parity operator

When the time t is taken into account, we have

$$\psi'(\mathbf{r},t) = P\psi(\mathbf{r},t) = \tilde{P}\psi(-\mathbf{r},t) = \gamma^0\psi(-\mathbf{r},t)$$

with

$$P = \gamma^0\pi, \qquad \tilde{P} = \gamma^0$$

(b) The time reversal operator

$$\psi'(\mathbf{r},t) = T\psi(\mathbf{r},t) = \tilde{T}\psi^*(\mathbf{r},t) = \gamma^1\gamma^3\psi^*(\mathbf{r},t)$$

with

$$\tilde{T} = \gamma^1\gamma^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

(c) Charge conjugate operator

$$\psi'(\mathbf{r},t) = C\psi(\mathbf{r},t) = \tilde{C}\psi^*(\mathbf{r},t) = i\gamma^2\psi^*(\mathbf{r},t) = i\gamma^2\gamma^0(\overline{\psi})^T$$

with

$$\tilde{C} = i\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$i\gamma^2\gamma^0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

(d) CPT

$$CPT\psi(\mathbf{r},t) = i\gamma^0\gamma^1\gamma^2\gamma^3\psi(-\mathbf{r},t) = \gamma^{5^3}\psi(-\mathbf{r},t)$$

with

$$\gamma^5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

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1. Time reversal operator

We define the time reversal operator by

$$\hat{T} = \hat{U}_T\hat{K} .$$

where $\hat{U}_T$ is the unitary operator and $\hat{K}$ is an operator that takes the complex conjugate of any complex numbers that follow it.

Here we determine the expression of $\hat{U}_T$ in terms of the Dirac matrices. We start with the Dirac equation given by

$$i\hbar(\gamma^k\partial_k + \gamma^0\partial_0)\psi - mc\psi = 0$$

Multiplying γ^0 from the right-side of this equation, we get

$$i\hbar\gamma^0\gamma^k\partial_k + i\hbar\gamma^0\gamma^0\partial_0\psi - mc\gamma^0\psi = 0$$

or

$$i\hbar\partial_0\psi = (-i\hbar\gamma^0\gamma^k\partial_k + mc\gamma^0)\psi \tag{1}$$

where $k = 1, 2, 3,$

$$\left(\gamma^0\right)^2 = I_4$$

The left side of Eq.(1):

$$\begin{aligned}\hat{T}(i\hbar\partial_0\psi) &= \hat{T}(i\hbar\partial_0)\hat{T}^{-1}\hat{T}\psi \\ &= \hat{U}_T\hat{K}(i\hbar\partial_0)(\hat{U}_T\hat{K})^{-1}\hat{U}_T\hat{K}\psi \\ &= \hat{U}_T\hat{K}(i\hbar\partial_0)\hat{K}^{-1}\hat{U}_T{}^{-1}\hat{U}_T\psi^* \\ &= \hat{U}_T\hat{K}(i\hbar\partial_0)\hat{K}^{-1}\psi^* \\ &= -i\hbar\partial_0\hat{U}_T\psi^*\end{aligned}$$

where we use

$$\hat{K}i\hat{K}^{-1} = -i \,. \qquad (\hat{U}_T\hat{K})^{-1} = \hat{K}^{-1}(\hat{U}_T)^{-1} \,.$$

The right-side of Eq.(1):

$$\begin{aligned}\hat{T}(-i\hbar\gamma^0\gamma^k\partial_k + mc\gamma^0)\psi &= \hat{T}(-i\hbar\gamma^0\gamma^k\partial_k + mc\gamma^0)\hat{T}^{-1}\hat{T}\psi \\ &= \hat{T}(-i\hbar\gamma^0\gamma^k\partial_k + mc\gamma^0)\hat{T}^{-1}\hat{U}_T\psi^*\end{aligned}$$

Then we get the Dirac equation, given by

$$\begin{aligned}i\hbar(-\partial_0)\hat{U}_T\psi^* &= \hat{T}(-i\hbar\gamma^0\gamma^k\partial_k + mc\gamma^0)\hat{T}^{-1}(\hat{U}_T\psi^*) \\ &= [-\hbar\hat{T}(i\gamma^0\gamma^k)\hat{T}^{-1}\partial_k + mc\hat{T}\gamma^0\hat{T}^{-1}]\hat{U}_T\psi^*\end{aligned} \qquad (2)$$

In order for $\hat{U}_T\psi^*$ to satisfy the time reversed form of Eq.(1),

$$i\hbar(-\partial_0\psi) = (-i\hbar\gamma^0\gamma^k\partial_k + mc\gamma^0)\psi \qquad (1a)$$

it is required that

$$\hat{T}i\gamma^0\gamma^k\hat{T}^{-1} = i\gamma^0\gamma^k$$

$$\hat{T}\gamma^0\hat{T}^{-1} = \gamma^0$$

where in Eq.(1), $\partial_0\psi$ becomes $(-\partial_0\psi)$ after the time reversal ($t \to -t$). These conditions can be rewritten as

$$\hat{T}^{-1}\gamma^0\hat{T} = \gamma^0, \tag{3a}$$

$$\hat{T}^{-1}(i\gamma^0\gamma^k)\hat{T} = i\gamma^0\gamma^k. \tag{3b}$$

Noting the relation $\hat{T}^{-1} = (\hat{U}_T\hat{K})^{-1} = \hat{K}^{-1}\hat{U}_T{}^{-1}$, Eq.(3a) can be rewritten as

$$\hat{K}^{-1}\hat{U}_T{}^{-1}\gamma^0\hat{U}_T\hat{K} = \gamma^0,$$

or

$$\hat{U}_T{}^{-1}\gamma^0\hat{U}_T = \hat{K}\gamma^0\hat{K}^{-1} = \left(\gamma^0\right)^* = \gamma^0 \tag{4a}$$

Eq.(3b) can be rewritten as

$$\hat{K}^{-1}\hat{U}_T{}^{-1}(i\gamma^0\gamma^k)\hat{U}_T\hat{K} = i\gamma^0\gamma^k,$$

or

$$\hat{U}_T{}^{-1}(i\gamma^0\gamma^k)\hat{U}_T = \hat{K}(i\gamma^0\gamma^k)\hat{K}^{-1} = -i(\gamma^0\gamma^k)^* = -i\gamma^0(\gamma^k)^*$$

or

$$\hat{U}_T{}^{-1}\gamma^0\gamma^k\hat{U}_T = -\gamma^0(\gamma^k)^*$$

or

$$\hat{U}_T{}^{-1}\gamma^0(\hat{U}_T\hat{U}_T{}^{-1})\gamma^k\hat{U}_T = -\gamma^0(\gamma^k)^*$$

or

$$(\hat{U}_T{}^{-1}\gamma^0\hat{U}_T)\hat{U}_T{}^{-1}\gamma^k\hat{U}_T = -\gamma^0(\gamma^k)^*$$

leading to the result

$$\gamma^0 \hat{U}_T^{-1} \gamma^k \hat{U}_T = -\gamma^0 (\gamma^k)^*,$$

or

$$\hat{U}_T^{-1} \gamma^k \hat{U}_T = -(\gamma^k)^* \tag{4b}$$

since

$$\hat{U}_T^{-1} \gamma^0 \hat{U}_T = \gamma^0$$

We have the expression of $\hat{U}_T$ satisfying Eqs.(4a) and (4b) as

$$\hat{U}_T = \gamma^1 \gamma^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Note that

$$\hat{U}_T^{-1} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

((**Mathematica**))

```
Clear["Global`*"];
exp_ * :=
 exp /. {Complex[re_, im_] :> Complex[re, -im]};
g1 = ( 1  0  0  0
       0 -1  0  0
       0  0 -1  0
       0  0  0 -1 );
Gd[μ_, ν_] := g1[[μ + 1, ν + 1]];
Gu[μ_, ν_] := g1[[μ + 1, ν + 1]];
Gum = Table[Gu[μ, ν], {μ, 0, 3, 1}, {ν, 0, 3, 1}];
Gdm = Table[Gd[μ, ν], {μ, 0, 3, 1}, {ν, 0, 3, 1}];
σx = PauliMatrix[1];
σy = PauliMatrix[2];
σz = PauliMatrix[3];
I2 = IdentityMatrix[2];
I4 = IdentityMatrix[4];
αx = KroneckerProduct[σx, σx];
αy = KroneckerProduct[σx, σy];
αz = KroneckerProduct[σx, σz];
γu0 = KroneckerProduct[σz, I2];
γux = γu0.αx // Simplify;
γuy = γu0.αy // Simplify;
γuz = γu0.αz // Simplify; ; γu[0] = γu0;
γu[1] = γux; γu[2] = γuy; γu[3] = γuz ;
γu[5] = ⅈ γu0 .γux.γuy.γuz;
```

```
γu[1] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

```
γu[3] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

```
UT = γu[1].γu[3]; UT // MatrixForm
```

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

```
Inverse[UT] // MatrixForm
```

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

```
Inverse[UT].γu[1].UT + γu[1]* // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
Inverse[UT].γu[2].UT + γu[2]* // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
Inverse[UT].γu[3].UT + γu[3]* // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
Inverse[UT].γu[0].UT - γu[0]* // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2. Space Inversion (parity operator)

When $\boldsymbol{r}' = -\boldsymbol{r}$ and $t' = t$,

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

det(Λ) = -1: improper Lorentz transformation.

The form invariance of the Dirac equation under the parity gives

$$S_p^{-1}\gamma^\mu S_p = \Lambda^\mu{}_\nu \gamma^\nu , \qquad \textbf{(Pauli's fundamental theorem)}$$

or

$$S_p^{-1}\gamma^0 S_p = \Lambda^0{}_\nu \gamma^\nu = \gamma^0$$

$$S_p^{-1}\gamma^k S_p = -\gamma^k$$

We assume that

$$S_p = \eta\gamma^0$$

where η is some multiplicative constant (here we use $\eta = 1$). Here we use

$$\gamma^0 = \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Note that

$$(\gamma^0)^2 = 1, \qquad \gamma^{0^{-1}} = \gamma^0$$

Then we get the relation

$$S_p^{-1}\gamma^0 S_p = \gamma^0\gamma^0\gamma^0 = \gamma^0$$

$$S_p^{-1}\gamma^k S_p = \gamma^0\gamma^k\gamma^0 = \gamma^0\gamma^k\gamma^0 = -\gamma^k$$

3. Space Inversion (II): presence of external field

Dirac equation is given by

$$(i\gamma^\mu\partial_\mu - \frac{mc}{\hbar})\psi = 0$$

with

$$p_\mu = i\hbar\partial_\mu$$

In the presence of vector potential A^{μ},

$$p^{\mu} \to p^{\mu} - \frac{e}{c} A^{\mu} \qquad \text{or,} \qquad p_{\mu} \to p_{\mu} - \frac{e}{c} A_{\mu}$$

or

$$\partial_{\mu} \to \partial_{\mu} + \frac{ie}{c\hbar} A_{\mu}$$

Note that *e* is the charge of the particle (*e*<0 for the electron). Then we get the Dirac equation of electron in the presence of A_{μ}

$$i\gamma^{\mu}(\partial_{\mu} + \frac{ie}{c\hbar} A_{\mu})\psi - \frac{mc}{\hbar}\psi = 0$$

Under the parity operation, $\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, we have

$$\boldsymbol{E}' = -\boldsymbol{E}, \qquad \boldsymbol{B}' = -\boldsymbol{B}.$$

and

$$\boldsymbol{E} = -\nabla A_0 - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}, \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A}$$

where

$$A^{\mu} = (A^0, \boldsymbol{A}), \qquad A_{\mu} = (A^0, -\boldsymbol{A})$$

Then we get

$$A_k' \to -A_k, \qquad A_0' \to A_0.$$

under the spatial reflection.

Dirac equation is given by

$$i\gamma^{\mu}(\partial_{\mu}' + \frac{ie}{\hbar c} A_{\mu}')\psi' - \frac{mc}{\hbar}\psi' = 0$$

or equivalently

$$i[(-\partial_k - \frac{ie}{\hbar c}A_k)\gamma^k + (\partial_0 + \frac{ie}{\hbar c}A_0)\gamma^0]\psi' - \frac{mc}{\hbar}\psi' = 0.$$

since

$$\partial_k' + \frac{ie}{\hbar c}A_k' \to -\partial_k - \frac{ie}{\hbar c}A_k, \qquad \partial_0' + \frac{ie}{\hbar c}A_0' \to \partial_0 + \frac{ie}{\hbar c}A_0$$

We try as before

$$\psi' = S_p\psi$$

$$i[(-\partial_k - \frac{ie}{\hbar c}A_k)\gamma^k + (\partial_0 + \frac{ie}{\hbar c}A_0)\gamma^0]S_p\psi - \frac{mc}{\hbar}S_p\psi = 0$$

Multiplying S_p^{-1} from the left,

$$S_p^{-1}i[(-\partial_k - \frac{ie}{\hbar c}A_k)\gamma_k + (\partial_0 + \frac{ie}{\hbar c}A_0)\gamma_0]S_p\psi - \frac{mc}{\hbar}\psi = 0.$$

or

$$i[(\partial_k + \frac{ie}{\hbar c}A_k)\gamma^k + (\partial_0 + \frac{ie}{\hbar c}A_0)\gamma^0]\psi - \frac{mc}{\hbar}\psi = 0$$

since

$$S_p^{-1}\gamma^0 S_p = \Lambda^0{}_\nu\gamma^\nu = \gamma^0$$

$$S_p^{-1}\gamma^k S_p = -\gamma^k$$

This equation is the same as the original Dirac equation,

$$i(\partial_\mu + \frac{ie}{\hbar c}A_\mu)\gamma^\mu\psi - \frac{mc}{\hbar}\psi = 0,$$

or

$$i\gamma^\mu(\partial_\mu + \frac{ie}{\hbar c}A_\mu)\psi - \frac{mc}{\hbar}\psi = 0.$$

4. Definition of the parity operator *P*

From the above discussion, we have

$$\psi'(x') = \gamma^0 \psi(x) = \beta\psi(x)$$

where

$$x' = (ct, -\boldsymbol{r}), \quad x = (ct, \boldsymbol{r})$$

Then we have

$$\psi'(ct, -\boldsymbol{r}) = \beta\psi(ct, \boldsymbol{r})$$

When $\boldsymbol{r} \to -\boldsymbol{r}$, we get

$$\psi'(ct, \boldsymbol{r}) = \gamma^0 \psi(ct, -\boldsymbol{r})$$

with

$$\beta = \gamma^0$$

or

$$\psi'(x) = \beta\psi(ct, -\boldsymbol{r}) = \beta\pi\psi(ct, \boldsymbol{r}) = P\psi(x)$$

So we define the parity operator as

$$P = \beta\pi$$

where β operates on the Dirac space and π operates on the coordinate space. Note that

$$P^2 = \beta\pi\beta\pi = \beta^2\pi^2 = 1$$

The commutation relation between H (for the free particle) and P is given by

$$\begin{aligned}
[P, H] &= [\beta\pi, c\alpha_k p_k + \beta mc^2] \\
&= [\beta\pi, c\alpha_k p_k] \\
&= \beta\pi c\alpha_k p_k - c\alpha_k p_k \beta\pi \\
&= c\beta\alpha_k \pi p_k - c\alpha_k \beta p_k \pi \\
&= c(\beta\alpha_k + \alpha_k\beta)\pi p_k \\
&= c\{\beta, \alpha_k\}\pi p_k \\
&= 0
\end{aligned}$$

where $\pi p_k + p_k\pi = 0$. Then the eigenket of P is the same as that of H. What happened to the wave function?

$$|\psi'\rangle = \beta\pi|\psi\rangle;$$

$$\begin{aligned}\psi'(\mathbf{r}) &= \langle\mathbf{r}|\psi'\rangle \\ &= \langle\mathbf{r}|\beta\pi|\psi\rangle \\ &= \beta\langle\mathbf{r}|\pi|\psi\rangle \\ &= \beta\langle-\mathbf{r}|\psi\rangle \\ &= \beta\psi(-\mathbf{r})\end{aligned}$$

For the change in $\boldsymbol{r} \to -\boldsymbol{r}$, we have

$$\psi'(-\mathbf{r}) = \beta\psi(\mathbf{r}) = \gamma^0\psi(\mathbf{r})$$

When the time t is taken into account, we have

$$\psi'(\mathbf{r},t) = \gamma^0\psi(-\mathbf{r},t).$$

5. Charge conjugation

We start with the Dirac equation

$$i\gamma^\mu(\partial_\mu + \frac{ie}{c\hbar}A_\mu)\psi - \frac{mc}{\hbar}\psi = 0$$

or

$$(i\gamma^\mu\partial_\mu - \frac{e}{c\hbar}\gamma^\mu A_\mu - \frac{mc}{\hbar})\psi(\mathbf{r},t) = 0 \qquad (1)$$

where e is the charge (e<0 for electron). We take the complex conjugate of this equation.

$$[-i(\gamma^\mu)^*\partial_\mu - \frac{e}{c\hbar}(\gamma^\mu)^* A_\mu - \frac{mc}{\hbar}]\psi^*(\mathbf{r},t) = 0.$$

We insert $1 = \widetilde{C}^{-1}\widetilde{C}$ before the wave function and multiply $\widetilde{C}$ from the left-side of the equation.

$$\tilde{C}[-i(\gamma^{\mu})^{*}\partial_{\mu}-\frac{e}{c\hbar}(\gamma^{\mu})^{*}A_{\mu}-\frac{mc}{\hbar}]\tilde{C}^{-1}\tilde{C}\psi^{*}(\boldsymbol{r},t)=0$$

or

$$[-i\tilde{C}(\gamma^{\mu})^{*}\tilde{C}^{-1}\partial_{\mu}-\frac{e}{c\hbar}\tilde{C}(\gamma^{\mu})^{*}\tilde{C}^{-1}A_{\mu}-\frac{mc}{\hbar}]\tilde{C}\psi^{*}(\boldsymbol{r},t)=0 \qquad (2)$$

We choose $\tilde{C}$ such that

$$\tilde{C}(\gamma^{\mu})^{*}\tilde{C}^{-1}=-\gamma^{\mu}$$

Equation (2) reduces to

$$[i\gamma^{\mu}\partial_{\mu}+\frac{e}{c\hbar}\gamma^{\mu}A_{\mu}-\frac{mc}{\hbar}]\tilde{C}\psi^{*}(\boldsymbol{r},t)=0. \qquad (3)$$

Thus the wavefunction $\tilde{C}\psi^{*}(\boldsymbol{r},t)$ satisfies the positron equation, while $\psi(\boldsymbol{r},t)$ satisfies the electron equation. Note that $\tilde{C}$ is given by

$$\tilde{C}=i\gamma^{2}$$

This will be proved later.

$$C\psi(\boldsymbol{r},t)=\tilde{C}\psi^{*}(\boldsymbol{r},t)=i\gamma^{2}\psi^{*}(\boldsymbol{r},t)$$

where

$$i\gamma^{2}=\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}$$

It is convenient to write this wave function in terms of

$$\overline{\psi}=\psi^{+}\gamma^{0}=(\psi^{*})^{T}\gamma^{0}$$

or

$$\overline{\psi}\gamma^0 = (\psi^*)^T$$

or

$$(\overline{\psi}\gamma^0)^T = \psi^*$$

In fact

$$\begin{aligned}\tilde{C}\psi^*(\mathbf{r},t) &= i\gamma^2\psi^* \\ &= i\gamma^2(\overline{\psi}\gamma^0)^T \\ &= i\gamma^2\gamma^{0T}(\overline{\psi})^T \\ &= i\gamma^2\gamma^0(\overline{\psi})^T\end{aligned}$$

Thus we get the new definition of the operator C as

$$C\psi(\mathbf{r},t) = \tilde{C}\psi^*(\mathbf{r},t) = i\gamma^2\psi^* = i\gamma^2\gamma^0(\overline{\psi})^T$$

with

$$i\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad i\gamma^2\gamma^0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

((**Note**))

We show that $\tilde{C}^* = i\gamma^2$ satisfies the relation; $\tilde{C}(\gamma^\mu)^*\tilde{C}^{-1} = -\gamma^\mu$.

$$(\gamma^0)^* = \gamma^0, \qquad (\gamma^1)^* = \gamma^1, \qquad (\gamma^2)^* = -\gamma^2, \qquad (\gamma^3)^* = -\gamma^3$$

In fact, we have

$$i\gamma^2(\gamma^0)^*(i\gamma^2)^{-1} = i\gamma^2(\gamma^0)^* i^{-1}(\gamma^2)^{-1} = \gamma^2\gamma^0(\gamma^2)^{-1} = -\gamma^0\gamma^2(\gamma^2)^{-1} = -\gamma^0$$

$$i\gamma^2(\gamma^1)^*(i\gamma^2)^{-1} = i\gamma^2(\gamma^1)^* i^{-1}(\gamma^2)^{-1} = \gamma^2\gamma^1(\gamma^2)^{-1} = -\gamma^1\gamma^2(\gamma^2)^{-1} = -\gamma^1$$

where

$$\left(\gamma^{0}\right)^{2} = I_4, \qquad \left(\gamma^{k}\right)^{2} = -I_4$$

and

$$\frac{1}{2}\{\gamma^{\mu},\gamma^{\nu}\} = g^{\mu\nu}\, I_4 \qquad \text{(Clifford relation)}$$

6. Example: Charge conjugation

We check that the relation between the electron and positron wave functions implied by the above discussion is consistent with the free-particle wavefunctions. We consider the wave function of the free electron with the momentum with ***p***.

$$u_R^{(+)}(\boldsymbol{r},t) = \sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} 1 \\ 0 \\ \dfrac{cp_z}{R+mc^2} \\ \dfrac{c(p_x+ip_y)}{R+mc^2} \end{pmatrix} \exp[\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r} - Rt)$$

with the helicity (+1) and positive energy $E = R$ given by

$$E = R = \sqrt{m^2c^4 + c^2p^2} > 0\,.$$

$$u_L^{(+)}(\boldsymbol{r},t) = \sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} 0 \\ 1 \\ \dfrac{c(p_x-ip_y)}{R+mc^2} \\ \dfrac{-cp_z}{R+mc^2} \end{pmatrix} \exp[\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r} - Rt)$$

with the helicity (-1) and positive energy $E = R$.

$$u_R^{(-)}(\boldsymbol{r},t)=\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix}\frac{-cp_z}{R+mc^2}\\ \frac{-c(p_x+ip_y)}{R+mc^2}\\ 1\\ 0\end{pmatrix}\exp[\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}+Rt)$$

with the helicity (+1) and positive energy $E=-R$.

$$u_L^{(-)}(\boldsymbol{r},t)=\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix}\frac{-c(p_x-ip_y)}{R+mc^2}\\ \frac{cp_z}{R+mc^2}\\ 0\\ 1\end{pmatrix}\exp[\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}+Rt)$$

with the helicity (-1) and positive energy $E=-R$.

(i)

$$Cu_R^{(+)}(\boldsymbol{r},t)=i\gamma^2u_R^{(+)^*}(\boldsymbol{r},t)$$

$$=\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&-1&0&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}1\\ 0\\ \frac{cp_z}{R+mc^2}\\ \frac{c(p_x-ip_y)}{R+mc^2}\end{pmatrix}\exp[\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}-Rt)^*$$

$$=\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix}\frac{c(p_x-ip_y)}{R+mc^2}\\ \frac{-cp_z}{R+mc^2}\\ 0\\ 1\end{pmatrix}\exp[-\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}-Rt)]$$

where

$$i\gamma^2=\begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&-1&0&0\\1&0&0&0\end{pmatrix}$$

Similarly, we have

(ii)

$$Cu_L^{(+)}(\boldsymbol{r},t) = i\gamma^2 u_L^{(+)^*}(\boldsymbol{r},t)$$

$$= \sqrt{\frac{R+mc^2}{2R}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x + ip_y)}{R+mc^2} \\ \frac{-cp_z}{R+mc^2} \end{pmatrix} \exp[\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r} - Rt)]^*$$

$$= -\sqrt{\frac{R+mc^2}{2R}} \begin{pmatrix} \frac{cp_z}{R+mc^2} \\ \frac{c(p_x + ip_y)}{R+mc^2} \\ 1 \\ 0 \end{pmatrix} \exp[-\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r} - Rt)]$$

(iii)

$$Cu_R^{(-)}(\boldsymbol{r},t) = i\gamma^2 u_R^{(-)^*}(\boldsymbol{r},t)$$

$$= \sqrt{\frac{R+mc^2}{2R}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-cp_z}{R+mc^2} \\ \frac{-c(p_x - ip_y)}{R+mc^2} \\ 1 \\ 0 \end{pmatrix} \exp[\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r} + Rt)^*$$

$$= -\sqrt{\frac{R+mc^2}{2R}} \begin{pmatrix} 0 \\ 1 \\ \frac{-c(p_x - ip_y)}{R+mc^2} \\ \frac{cp_z}{R+mc^2} \end{pmatrix} \exp[-\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r} + Rt)]$$

(iv)

$$Cu_L^{(-)}(\boldsymbol{r},t) = i\gamma^2 u_L^{(-)^*}(\boldsymbol{r},t)$$

$$= \sqrt{\frac{R+mc^2}{2R}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-c(p_x+ip_y)}{R+mc^2} \\ \frac{cp_z}{R+mc^2} \\ 0 \\ 1 \end{pmatrix} \exp[\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}+Rt)^*$$

$$= \sqrt{\frac{R+mc^2}{2R}} \begin{pmatrix} 1 \\ 0 \\ \frac{-cp_z}{R+mc^2} \\ \frac{-c(p_x+ip_y)}{R+mc^2} \end{pmatrix} \exp[-\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}+Rt)]$$

In summary we have the following relation. Under the charge cognate the charge is changed. The momentum and the helicity are also reversed.

$$Cu_R^{(+)}(\boldsymbol{p})\exp[\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}-Rt)] = u_L^{(-)}(-\boldsymbol{p})\exp[-\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}-Rt)]$$

$$Cu_L^{(+)}(\boldsymbol{p})\exp[\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}-Rt)] = -u_R^{(-)}(-\boldsymbol{p})\exp[-\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}-Rt)]$$

$$Cu_R^{(-)}(\boldsymbol{p})\exp[\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}-Rt)] = -u_L^{(+)}(-\boldsymbol{p})\exp[-\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}+Rt)]$$

$$Cu_L^{(-)}(\boldsymbol{p})\exp[\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}-Rt)] = u_R^{(+)}(-\boldsymbol{p})\exp[-\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}+Rt)]$$

7. Example: Parity operator

$$Pu_R^{(+)}(\boldsymbol{r},t)=\gamma^0 u_R^{(+)}(-\boldsymbol{r},t)$$

$$=\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}1\\0\\\frac{cp_z}{R+mc^2}\\\frac{c(p_x-ip_y)}{R+mc^2}\end{pmatrix}\exp[-\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}+Rt)$$

$$=\sqrt{\frac{E+mc^2}{2R}}\begin{pmatrix}1\\0\\\frac{-cp_z}{E+mc^2}\\-\frac{c(p_x-ip_y)}{R+mc^2}\end{pmatrix}\exp[-\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}+Rt)$$

$$\gamma^0=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$$

$$Pu_L^{(+)}(\boldsymbol{r},t)=\gamma^0 u_L^{(+)}(-\boldsymbol{r},t)$$

$$=\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}0\\1\\\frac{c(p_x-ip_y)}{R+mc^2}\\\frac{-cp_z}{R+mc^2}\end{pmatrix}\exp[-\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}+Rt)$$

$$=\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix}0\\1\\\frac{-c(p_x-ip_y)}{R+mc^2}\\\frac{cp_z}{R+mc^2}\end{pmatrix}\exp[-\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}+Rt)$$

$$Pu_R^{(-)}(\boldsymbol{r},t)=\gamma^0 u_R^{(-)}(-\boldsymbol{r},t)$$

$$=\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}\frac{-cp_z}{R+mc^2}\\ \frac{-c(p_x+ip_y)}{R+mc^2}\\1\\0\end{pmatrix}\exp[-\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}-Rt)$$

$$=-\sqrt{\frac{E+mc^2}{2R}}\begin{pmatrix}\frac{cp_z}{E+mc^2}\\ \frac{c(p_x+ip_y)}{R+mc^2}\\1\\0\end{pmatrix}\exp[-\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}-Rt)$$

$$Pu_L^{(-)}(\boldsymbol{r},t)=\gamma^0 u_L^{(-)}(-\boldsymbol{r},t)$$

$$=\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}\frac{-c(p_x-ip_y)}{R+mc^2}\\ \frac{cp_z}{E+mc^2}\\0\\1\end{pmatrix}\exp[-\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}-Rt)$$

$$=-\sqrt{\frac{E+mc^2}{2R}}\begin{pmatrix}\frac{c(p_x-ip_y)}{R+mc^2}\\ \frac{-cp_z}{E+mc^2}\\0\\1\end{pmatrix}\exp[-\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}-Rt)$$

8. Example Time reversal operator

$$\widetilde{T}=\gamma^1\gamma^3=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}$$

$$\gamma^1\gamma^3 u_R^{(+)^*}(\boldsymbol{r},t)=\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}\begin{pmatrix}1\\0\\\frac{cp_z}{R+mc^2}\\\frac{c(p_x-ip_y)}{R+mc^2}\end{pmatrix}\exp[\frac{i}{\hbar}(-\boldsymbol{p}\cdot\boldsymbol{r}+Rt)$$

$$=-\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix}0\\1\\\frac{-c(p_x-ip_y)}{R+mc^2}\\\frac{cp_z}{R+mc^2}\end{pmatrix}\exp[\frac{i}{\hbar}(-\boldsymbol{p}\cdot\boldsymbol{r}+Rt)$$

$$\gamma^1\gamma^3 u_L^{(+)^*}(\boldsymbol{r},t)=\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}\begin{pmatrix}0\\1\\\frac{c(p_x+ip_y)}{R+mc^2}\\\frac{-cp_z}{R+mc^2}\end{pmatrix}\exp[\frac{i}{\hbar}(-\boldsymbol{p}\cdot\boldsymbol{r}+Rt)$$

$$=\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix}1\\0\\\frac{-cp_z}{E+mc^2}\\\frac{-c(p_x+ip_y)}{R+mc^2}\end{pmatrix}\exp[\frac{i}{\hbar}(-\boldsymbol{p}\cdot\boldsymbol{r}+Rt)$$

$$\gamma^1\gamma^3 u_R^{(-)^*}(\boldsymbol{r},t)=\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}\begin{pmatrix}\frac{-cp_z}{R+mc^2}\\\frac{-c(p_x-ip_y)}{R+mc^2}\\1\\0\end{pmatrix}\exp[-\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}+Rt)]$$

$$=-\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix}\frac{c(p_x-ip_y)}{R+mc^2}\\\frac{-cp_z}{R+mc^2}\\0\\1\end{pmatrix}\exp[-\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}+Rt)]$$

$$\gamma^1\gamma^3 u_L^{(-)*}(\boldsymbol{r},t) = \sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} \frac{-c(p_x+ip_y)}{R+mc^2} \\ \frac{cp_z}{R+mc^2} \\ 0 \\ 1 \end{pmatrix}\exp[-\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}+Rt)]$$

$$= \sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix} \frac{cp_z}{R+mc^2} \\ \frac{c(p_x+ip_y)}{R+mc^2} \\ 1 \\ 0 \end{pmatrix}\exp[-\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}+Rt)]$$

<u>In summary (time reversal operator)</u>

$$\gamma^1\gamma^3 u_R^{(+)*}(\boldsymbol{p})\exp[\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}-Rt)]^* = -u_L^{(+)}(-\boldsymbol{p})\exp[-\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}-Rt)]$$

$$\gamma^1\gamma^3 u_L^{(+)*}(\boldsymbol{p})\exp[\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}-Rt)]^* = u_R^{(+)}(-\boldsymbol{p})\exp[-\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}-Rt)]$$

$$\gamma^1\gamma^3 u_R^{(-)*}(\boldsymbol{p})\exp[\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}+Rt)]^* = -u_L^{(-)}(-\boldsymbol{p})\exp[-\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}+Rt)]$$

$$\gamma^1\gamma^3 u_L^{(-)*}(\boldsymbol{p})\exp[\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}-Rt)]^* = u_R^{(-)}(-\boldsymbol{p})\exp[-\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}+Rt)]$$

9. CPT

We consider the operator combination CPT.

$$\begin{aligned} CPT\psi(\boldsymbol{r},t) &= i\gamma^2[PT\psi(\boldsymbol{r},t)]^* \\ &= i\gamma^2[\gamma^0 T\psi(-\boldsymbol{r},t)]^* \\ &= i\gamma^2\gamma^0[T\psi(-\boldsymbol{r},t)]^* \\ &= i\gamma^2\gamma^0[\gamma^1\gamma^3\psi^*(-\boldsymbol{r},t)]^* \\ &= i\gamma^2\gamma^0\gamma^1\gamma^3\psi(-\boldsymbol{r},t) \end{aligned}$$

where

$$P\psi(\mathbf{r},t)=\gamma^0\psi(-\mathbf{r},t)\,,\; T\psi(\mathbf{r},t)=\gamma^1\gamma^3\psi^*(\mathbf{r},t)\,,$$

$$C\psi(\mathbf{r},t)=i\gamma^2\psi^*(\mathbf{r},t)$$

Noting that

$$\gamma^2\gamma^0\gamma^1\gamma^3=\gamma^0\gamma^1\gamma^2\gamma^3=-i\gamma^5$$

with

$$\gamma^5=i\gamma^0\gamma^1\gamma^2\gamma^3=\begin{pmatrix}0 & I_2\\ I_2 & 0\end{pmatrix}$$

We get the relation

$$CPT\psi(\mathbf{r},t)=\gamma^5\psi(-\mathbf{r},t)$$

((**Example**))

$$CPTu_R^{(+)}(\mathbf{r},t)=\gamma^5u_R^{(+)}(-\mathbf{r},t)$$

$$=\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}1\\0\\\frac{cp_z}{R+mc^2}\\\frac{c(p_x-ip_y)}{R+mc^2}\end{pmatrix}\exp[-\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}+Rt)$$

$$=\sqrt{\frac{E+mc^2}{2R}}\begin{pmatrix}\frac{cp_z}{R+mc^2}\\\frac{c(p_x-ip_y)}{R+mc^2}\\1\\0\end{pmatrix}\exp[-\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}+Rt)$$

where

$$u_R^{(-)}(\mathbf{r},t)=\sqrt{\frac{R+mc^2}{2R}}\begin{pmatrix}\frac{-cp_z}{R+mc^2}\\\frac{-c(p_x+ip_y)}{R+mc^2}\\1\\0\end{pmatrix}\exp[\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{r}+Rt)$$

10. CP

We consider the operator combination CP.

$$
\begin{aligned}
CP\psi(\mathbf{r},t) &= i\gamma^2[P\psi(\mathbf{r},t)]^* \\
&= i\gamma^2[\gamma^0\psi(-\mathbf{r},t)]^* \\
&= i\gamma^2\gamma^0[\psi(-\mathbf{r},t)]^* \\
&= i\gamma^2\gamma^0\psi^*(-\mathbf{r},t)
\end{aligned}
$$

with

$$
i\gamma^2\gamma^0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
$$

where

$$
i\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
$$