Relativistic Invariance

Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
(Date: May 03, 2016)

The Dirac equation is Lorentz invariant. It means that the Dirac equation should remain from invariant under the Lorentz transformation. Here we discuss the form of transformation S such that

$$\psi' = S\psi$$

We also discuss the form of S under the **rotation, parity**.

1. Relativistic co-variance

The Dirac equation is given by

$$i\gamma^{\mu}\frac{\partial}{\partial x^{\mu}}\psi(x) - \frac{mc}{\hbar}\psi(x) = 0 .$$

The form of this equation is invariant under the Lorentz transformation

$$i\gamma^{\mu}\frac{\partial}{\partial x'^{\mu}}\psi'(x') - \frac{mc}{\hbar}\psi'(x') = 0$$

where

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} ,$$

$$x^{\mu} = (\Lambda^{-1})^{\mu}{}_{\nu}x'^{\nu} = \Lambda_{\nu}{}^{\mu}x'^{\nu}$$

Suppose that

$$\psi' = S\psi .$$

where S is a 4 x 4 matrix. We note that

$$\partial'_{\mu} = \frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}}\frac{\partial}{\partial x^{\nu}} = \Lambda_{\mu}{}^{\nu}\frac{\partial}{\partial x^{\nu}} = \Lambda_{\mu}{}^{\nu}\partial_{\nu} .$$

Thus we get

$$i\gamma^{\mu}\Lambda_{\mu}{}^{\nu}\partial_{\nu}S\psi(x) - \frac{mc}{\hbar}S\psi(x) = 0$$

Multiplying S^{-1} from the left, we have

$$iS^{-1}\gamma^{\mu}\Lambda_{\mu}{}^{\nu}\partial_{\nu}S\psi(x)-\frac{mc}{\hbar}\psi(x)=0,$$

or

$$iS^{-1}\gamma^{\mu}\Lambda_{\mu}{}^{\nu}\partial_{\nu}S\psi(x)-\frac{mc}{\hbar}\psi(x)=0.$$

Suppose that

$$S^{-1}\gamma^{\mu}S\Lambda_{\mu}{}^{\nu}=\gamma^{\nu}, \qquad \text{or} \qquad S^{-1}\gamma^{\lambda}S\Lambda_{\lambda}{}^{\nu}=\gamma^{\nu}$$

Multiplying $\Lambda^{\mu}{}_{\nu}$ from the left,

$$\begin{aligned}\Lambda^{\mu}{}_{\nu}\gamma^{\nu} &= S^{-1}\gamma^{\lambda}S\Lambda_{\lambda}{}^{\nu}\Lambda^{\mu}{}_{\nu} \\ &= S^{-1}\gamma^{\lambda}S\Lambda^{\mu}{}_{\nu}\Lambda_{\lambda}{}^{\nu} \\ &= S^{-1}\gamma^{\lambda}S\Lambda^{\mu}{}_{\nu}(\Lambda^{-1})^{\nu}{}_{\lambda} \\ &= (S^{-1}\gamma^{\lambda}S)\,g^{\mu}{}_{\lambda}=S^{-1}\gamma^{\mu}S\end{aligned}$$

we get

$$S^{-1}\gamma^{\mu}S=\Lambda^{\mu}{}_{\nu}\gamma^{\nu}, \qquad \textbf{(Pauli's fundamental theorem)}$$

Note that

$$\Lambda_{\lambda}{}^{\nu}\Lambda^{\mu}{}_{\nu}=(\Lambda^{-1})^{\nu}{}_{\lambda}\Lambda^{\mu}{}_{\nu}=\Lambda^{\mu}{}_{\nu}(\Lambda^{-1})^{\nu}{}_{\lambda}=g^{\mu}{}_{\lambda} \qquad \text{(Kronecker-delta)}$$

2. Form of *S*

The problem of demonstrating the relativistic covariance of the Dirac equation is now reduced to that of finding the form of *S*. The Dirac equation is given by

$$i\gamma^{\mu}\frac{\partial}{\partial x^{\mu}}\psi-\frac{mc}{\hbar}\psi=0.$$

Taking the Hermitian conjugate of this equation is

$$-i\frac{\partial}{\partial x^{\mu}}\psi^{+}\left(\gamma^{\mu}\right)^{+}-\frac{mc}{\hbar}\psi^{+}=0.$$

Noting that

$$(\gamma^{\mu})^{+} = \gamma^{0}\gamma^{\mu}\gamma^{0}$$

we get

$$i\frac{\partial}{\partial x^{\mu}}\psi^{+}\gamma^{0}\gamma^{\mu}\gamma^{0} + \frac{mc}{\hbar}\psi^{+} = 0$$

Multiplication of γ^{0} from the right leads to

$$i\frac{\partial}{\partial x^{\mu}}(\psi^{+}\gamma^{0})\gamma^{\mu}\gamma^{0^{2}} + \frac{mc}{\hbar}\psi^{+}\gamma^{0} = 0,$$

or

$$i\frac{\partial}{\partial x^{\mu}}(\psi^{+}\gamma^{0})\gamma^{\mu} + \frac{mc}{\hbar}(\psi^{+}\gamma^{0}) = 0,$$

where

$$\gamma^{0^{2}} = I_{4}.$$

Here we define $\overline{\psi}$ (Dirac conjugate) as

$$\overline{\psi} = \psi^{+}\gamma^{0} \qquad \text{or} \qquad \overline{\psi}\gamma^{0} = \psi^{+}\gamma^{0^{2}} = \psi^{+}.$$

Then we have

$$i(\frac{\partial}{\partial x^{\mu}}\overline{\psi})\gamma^{\mu} + \frac{mc}{\hbar}\overline{\psi} = 0.$$

This equation should be invariant under the Lorentz transformation

$$i(\frac{\partial}{\partial x'^{\mu}}\overline{\psi}')\gamma^{\mu} + \frac{mc}{\hbar}\overline{\psi}' = 0.$$

From this we get

$$i\Lambda_{\mu}{}^{\nu}\frac{\partial}{\partial x^{\nu}}\overline{\psi}'\gamma^{\mu} + \frac{mc}{\hbar}\overline{\psi}' = 0$$

where

$$\frac{\partial}{\partial x'^{\mu}} = \partial'_{\mu} = \Lambda_{\mu}{}^{\nu}\partial_{\nu}$$

We assume that

$$\overline{\psi}' = \overline{\psi}U^{-1}$$

$$i\Lambda_{\mu}{}^{\nu}\frac{\partial}{\partial x^{\nu}}\overline{\psi}U^{-1}\gamma^{\mu} + \frac{mc}{\hbar}\overline{\psi}U^{-1} = 0$$

Multiplying *U* from the left side

$$i\Lambda_{\mu}{}^{\nu}\frac{\partial}{\partial x^{\nu}}\overline{\psi}U^{-1}\gamma^{\mu}U + \frac{mc}{\hbar}\overline{\psi} = 0$$

or

$$i\frac{\partial}{\partial x^{\nu}}\overline{\psi}(\Lambda_{\mu}{}^{\nu}U^{-1}\gamma^{\mu}U) + \frac{mc}{\hbar}\overline{\psi} = 0$$

The comparison of this equation with the original Dirac equation

$$i(\frac{\partial}{\partial x^{\mu}}\overline{\psi})\gamma^{\mu} + \frac{mc}{\hbar}\overline{\psi} = 0$$

leads to the relation

$$\gamma^{\nu} = \Lambda_{\mu}{}^{\nu}U^{-1}\gamma^{\mu}U = U^{-1}\gamma^{\mu}U\Lambda_{\mu}{}^{\nu}$$

or

$$U^{-1}\gamma^{\mu}U = \Lambda^{\mu}{}_{\nu}\gamma^{\nu}$$

where

$$\begin{aligned}\Lambda^{\mu}{}_{\nu}\gamma^{\nu} &= \Lambda^{\mu}{}_{\nu}U^{-1}\gamma^{\lambda}U\Lambda_{\lambda}{}^{\nu} \\ &= (U^{-1}\gamma^{\lambda}U)\Lambda^{\mu}{}_{\nu}\Lambda_{\lambda}{}^{\nu} \\ &= (U^{-1}\gamma^{\lambda}U)\Lambda^{\mu}{}_{\nu}(\Lambda^{-1})^{\nu}{}_{\lambda} \\ &= (U^{-1}\gamma^{\lambda}U)g^{\mu}{}_{\lambda} \\ &= U^{-1}\gamma^{\mu}U\end{aligned}$$

Here we can choose

$$U = S,$$

since

$$S^{-1}\gamma^{\mu}S = \Lambda^{\mu}{}_{\nu}\gamma^{\nu}. \qquad \textbf{(Pauli's fundamental theorem)}$$

Then we have

$$\overline{\psi} = \psi^{+}\gamma^{0}, \qquad \overline{\psi}\gamma^{0} = \psi^{+}$$

$$\psi' = S\psi, \qquad \overline{\psi}' = \overline{\psi}S^{-1}$$

since

$$\overline{\psi}' = \psi'^{+}\gamma^{0} = \overline{\psi}\gamma^{0}S^{+}\gamma^{0} = \overline{\psi}S^{-1}$$

or

$$\gamma^{0}S^{+}\gamma^{0} = S^{-1}$$

or

$$S^{+}\gamma^{0} = \gamma^{0}S^{-1}$$

since

$$\psi'^{+}\gamma^{0} = (S\psi)^{+}\gamma^{0} = \psi^{+}S^{+}\gamma^{0} = \overline{\psi}\gamma^{0}S^{+}\gamma^{0}$$

In summary we have the following relations,

$$S^{-1}\gamma^{\mu}S = \Lambda^{\mu}{}_{\nu}\gamma^{\nu}$$

$$S^{+}\gamma^{0} = \gamma^{0}S^{-1}$$

$$\psi' = S\psi$$

$$\overline{\psi}' = \overline{\psi}S^{-1}$$

3. Infinitesimal Lorentz transformation

We consider the infinitesimal Lorentz transformation

$$\Lambda^{\mu}{}_{\nu} = \omega^{\mu}{}_{\nu} + g^{\mu}{}_{\nu}$$

with the addition

$$\omega^{\mu}{}_{\nu} = -\omega^{\nu}{}_{\mu}$$

((**Note**))

$$\Lambda^{\mu}{}_{\lambda}\Lambda_{\nu}{}^{\lambda} = \Lambda^{\mu}{}_{\lambda}(\Lambda^{-1})^{\lambda}{}_{\nu} = (g^{\mu}{}_{\lambda} + \omega^{\mu}{}_{\lambda})(g_{\nu}{}^{\lambda} + \omega_{\nu}{}^{\lambda}) = g^{\mu}{}_{\nu}$$

or

$$g^{\mu}{}_{\lambda}g_{\nu}{}^{\lambda} + (\omega^{\mu}{}_{\lambda}g_{\nu}{}^{\lambda} + g^{\mu}{}_{\lambda}\omega_{\nu}{}^{\lambda}) + \omega^{\mu}{}_{\lambda}\omega_{\nu}{}^{\lambda} = g^{\mu}{}_{\nu}$$

Neglecting the term $\omega^{\mu}{}_{\lambda}\omega_{\nu}{}^{\lambda}$, we get

$$g^{\mu}{}_{\lambda}g_{\nu}{}^{\lambda} + (\omega^{\mu}{}_{\lambda}g_{\nu}{}^{\lambda} + g^{\mu}{}_{\lambda}\omega_{\nu}{}^{\lambda}) = g^{\mu}{}_{\nu}$$

or

$$g^{\mu}{}_{\nu} + (\omega^{\mu}{}_{\lambda}g_{\nu}{}^{\lambda} + g^{\mu}{}_{\lambda}\omega_{\nu}{}^{\lambda}) = g^{\mu}{}_{\nu}$$

or

$$\omega^{\mu}{}_{\lambda}g_{\nu}{}^{\lambda} + g^{\mu}{}_{\lambda}\omega_{\nu}{}^{\lambda} = 0$$

$$\omega^{\mu}{}_{\nu} = -\omega_{\nu}{}^{\mu}.$$

since $g^{\mu}{}_{\nu}$ and $g_{\mu}{}^{\nu}$ are the Kronecker delta ($=\delta_{\mu,\nu}$).

5. The expression of *S*

We now consider the relation

$$S^{-1}\gamma^{\mu}S = \Lambda^{\mu}{}_{\nu}\gamma^{\nu}$$

We assume that

$$S = 1 + T + O(T^2)$$

Then we get

$$(1-T)\gamma^{\mu}(1+T) = \Lambda^{\mu}{}_{\nu}\gamma^{\nu} = (g^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu})\gamma^{\nu}$$

or

$$(\gamma^{\mu} - T\gamma^{\mu})(1+T) = \gamma^{\mu} + \omega^{\mu}{}_{\nu}\gamma^{\nu}$$

or

$$\gamma^{\mu} + [\gamma^{\mu}, T] = \gamma^{\mu} + \omega^{\mu}{}_{\nu}\gamma^{\nu}$$

or

$$[\gamma^{\mu}, T] = \omega^{\mu}{}_{\nu}\gamma^{\nu}$$

The solution of this commutation relation is seen to be

$$T = \frac{1}{4}\omega_{\mu\nu}\gamma^{\mu}\gamma^{\nu}$$

Then we have

$$\begin{aligned} S &= 1 + \frac{1}{4}\omega_{\mu\nu}\gamma^{\mu}\gamma^{\nu} \\ &= 1 + \frac{1}{8}\omega_{\mu\nu}[\gamma^{\mu}, \gamma^{\nu}] \\ &= 1 - \frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu} \end{aligned}$$

where

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}]$$

$$\omega^{\mu}{}_{\nu} = \varepsilon\chi\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \varepsilon\chi M^{\mu}{}_{\nu}$$

or

$$M^{\mu}{}_{\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Note that

$$M^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad M_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

((**Note-1**))

Noting that

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}$$

we get

$$\begin{aligned}
[\gamma^{\mu}, T] &= \frac{1}{4}[\gamma^{\mu}, \omega_{\nu\lambda}\gamma^{\nu}\gamma^{\lambda}] \\
&= \frac{1}{4}\omega_{\nu\lambda}[\gamma^{\mu}, \gamma^{\nu}\gamma^{\lambda}] \\
&= \frac{1}{4}\omega_{\nu\lambda}(\gamma^{\mu}\gamma^{\nu}\gamma^{\lambda} - \gamma^{\nu}\gamma^{\lambda}\gamma^{\mu}) \\
&= \frac{1}{4}\omega_{\nu\lambda}[(-\gamma^{\nu}\gamma^{\mu} + 2g^{\mu\nu})\gamma^{\lambda} - \gamma^{\nu}(-\gamma^{\mu}\gamma^{\lambda} + 2g^{\mu\lambda})] \\
&= \frac{1}{2}(\omega_{\nu\lambda}g^{\mu\nu}\gamma^{\lambda} - \omega_{\nu\lambda}\gamma^{\nu}g^{\mu\lambda}) \\
&= \frac{1}{2}(\omega^{\mu}{}_{\lambda}\gamma^{\lambda} - g^{\mu\lambda}\omega_{\nu\lambda}\gamma^{\nu}) \\
&= \frac{1}{2}(\omega^{\mu}{}_{\nu}\gamma^{\nu} - \omega_{\nu}{}^{\mu}\gamma^{\nu}) \\
&= \omega^{\mu}{}_{\nu}\gamma^{\nu}
\end{aligned}$$

where

$$\omega_{\nu\lambda}g^{\mu\nu} = g^{\mu\nu}\omega_{\nu\lambda} = \omega^{\mu}{}_{\lambda},$$

((**Note-2**))

$$\omega_{\mu\nu}\gamma^{\mu}\gamma^{\nu} = \frac{1}{2}(\omega_{\mu\nu}\gamma^{\mu}\gamma^{\nu} + \omega_{\nu\mu}\gamma^{\nu}\gamma^{\mu})$$
$$= \frac{1}{2}\omega_{\mu\nu}(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu})$$
$$= \frac{1}{2}\omega_{\mu\nu}[\gamma^{\mu}, \gamma^{\nu}]$$

5. The relation $S^{+}\gamma^{0} = \gamma^{0}S^{-1}$

We show that *S* satisfies the relation $S^{+}\gamma^{0} = \gamma^{0}S^{-1}$.

$$S = 1 + \frac{1}{4}\omega_{\mu\nu}\gamma^{\mu}\gamma^{\nu}, \qquad S^{-1} = 1 - \frac{1}{4}\omega_{\mu\nu}\gamma^{\mu}\gamma^{\nu}$$

$$S^{+} = 1 + \frac{1}{4}\omega_{\mu\nu}^{*}\gamma^{\nu+}\gamma^{\mu+}$$
$$= 1 + \frac{1}{4}\omega_{\mu\nu}^{*}\gamma^{0}\gamma^{\nu}\gamma^{02}\gamma^{\mu}\gamma^{0}$$
$$= 1 + \frac{1}{4}\omega_{\mu\nu}^{*}\gamma^{0}\gamma^{\nu}\gamma^{\mu}\gamma^{0}$$

where

$$\gamma^{\mu^{+}} = \gamma^{0}\gamma^{\mu}\gamma^{0}.$$

Then we get

$$S^{+}\gamma^{0} = (1 + \frac{1}{4}\omega_{\mu\nu}^{*}\gamma^{0}\gamma^{\nu}\gamma^{\mu}\gamma^{0})\gamma^{0} = \gamma^{0}(1 + \frac{1}{4}\omega_{\mu\nu}^{*}\gamma^{\nu}\gamma^{\mu})$$

$$\gamma^{0}S^{-1} = \gamma^{0}(1 - \frac{1}{4}\omega_{\mu\nu}\gamma^{\mu}\gamma^{\nu})$$

So we need to show that

$$\omega_{\mu\nu}^{*}\gamma^{\nu}\gamma^{\mu} + \omega_{\mu\nu}\gamma^{\mu}\gamma^{\nu} = 0$$

Since

$$\omega_{\mu\nu}^{*} = \omega_{\mu\nu} = -\omega_{\nu\mu} \qquad \text{(real)}$$

the left-hand side can be given as

$$\omega_{\mu\nu}\gamma^\nu\gamma^\mu + \omega_{\mu\nu}\gamma^\mu\gamma^\nu = -\omega_{\nu\mu}\gamma^\nu\gamma^\mu + \omega_{\mu\nu}\gamma^\mu\gamma^\nu = 0$$

6. The expression of *S* for the finite Lorentz transformation

We start with the definition of γ^5

$$\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\varepsilon_{\alpha\beta\gamma\delta}\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}$$

$$\begin{aligned} S^{-1}\gamma_5 S &= \frac{i}{4!}\varepsilon_{\alpha\beta\gamma\delta}S^{-1}\gamma^\alpha SS^{-1}\gamma^\beta SS^{-1}\gamma^\gamma SS^{-1}\gamma^\delta S \\ &= \frac{i}{4!}\varepsilon_{\alpha\beta\gamma\delta}\Lambda^\alpha{}_{\alpha'}\gamma^{\alpha'}\Lambda^\beta{}_{\beta'}\gamma^{\beta'}\Lambda^\gamma{}_{\gamma'}\gamma^{\gamma'}\Lambda^\delta{}_{\delta'}\gamma^{\delta'} \\ &= \frac{i}{4!}\varepsilon_{\alpha\beta\gamma\delta}\Lambda^\alpha{}_{\alpha'}\Lambda^\beta{}_{\beta'}\Lambda^\gamma{}_{\gamma'}\Lambda^\delta{}_{\delta'}\gamma^{\alpha'}\gamma^{\beta'}\gamma^{\gamma'}\gamma^{\delta'} \\ &= \frac{i}{4!}(\det\Lambda)\varepsilon_{\alpha'\beta'\gamma'\delta'}\gamma^{\alpha'}\gamma^{\beta'}\gamma^{\gamma'}\gamma^{\delta'} \end{aligned}$$

or

$$S^{-1}\gamma_5 S = (\det\Lambda)\gamma_5$$

The infinitesimal Lorentz transformation Λ is give as follows.

$$\Lambda^\mu{}_\nu = g^\mu{}_\nu + \omega^\mu{}_\nu$$

$$g_{\alpha\beta} = \Lambda^\mu{}_\alpha\Lambda_{\mu\beta}$$

where

$$\begin{aligned} \Lambda^\mu{}_\alpha\Lambda_{\mu\beta} &= \Lambda^\mu{}_\alpha g_{\lambda\beta}\Lambda_\mu{}^\lambda \\ &= g_{\lambda\beta}\Lambda^\mu{}_\alpha\Lambda_\mu{}^\lambda \\ &= g_{\lambda\beta}\Lambda^\mu{}_\alpha(\Lambda^{-1})^\lambda{}_\mu \\ &= g_{\lambda\beta}(\Lambda^{-1})^\lambda{}_\mu\Lambda^\mu{}_\alpha \\ &= g_{\lambda\beta}g^\lambda{}_\alpha \\ &= g_{\alpha\beta} \end{aligned}$$

To the first order in ω,

$$g_{\alpha\beta} = (g^{\mu}{}_{\alpha} + \omega^{\mu}{}_{\alpha})(g_{\mu\beta} + \omega_{\mu\beta}) = g_{\alpha\beta} + (\omega_{\alpha\beta} + \omega_{\beta\alpha})$$

and thus

$$\omega_{\alpha\beta} = -\omega_{\alpha\beta}$$

Now we expand $S(\Lambda)$ to the first order in ω to write

$$S = 1 - \frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}, \qquad S^{-1} = 1 + \frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}$$

Since

$$SS^{-1} = S^{-1}S = 1$$

$$(1 - \frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu})(1 + \frac{i}{2}\omega_{\nu\mu}\sigma^{\nu\mu}) = 1 - \frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu} + \frac{i}{2}\omega_{\nu\mu}\sigma^{\nu\mu} = 1$$

or

$$\omega_{\nu\mu}\sigma^{\nu\mu} = \omega_{\mu\nu}\sigma^{\mu\nu}$$

Since $\omega_{\mu\nu} = -\omega_{\nu\mu}$, we get the antisymmetric relation

$$\sigma^{\mu\nu} = -\sigma^{\nu\mu}$$

Here we assume that $A_{\mu\nu}$ is an antisymmetric tensor. $T^{\mu\nu}$ is an arbitrary tensor. Then we get

$$
\begin{aligned}
A_{\mu\nu}T^{\mu\nu} &= \frac{1}{2}(A_{\mu\nu}T^{\mu\nu} + A_{\mu\nu}T^{\mu\nu}) \\
&= \frac{1}{2}(A_{\mu\nu}T^{\mu\nu} - A_{\nu\mu}T^{\mu\nu}) \\
&= \frac{1}{2}(A_{\mu\nu}T^{\mu\nu} - A_{\mu\nu}T^{\nu\mu}) \\
&= \frac{1}{2}A_{\mu\nu}(T^{\mu\nu} - T^{\nu\mu}) \\
&= A_{\mu\nu}T^{[\mu\nu]}
\end{aligned}
$$

where

$$T^{[\mu\nu]} = \frac{T^{\mu\nu} - T^{\nu\mu}}{2}$$

((**Note**))
We start with

$$\Lambda^{\alpha}{}_{\mu}\gamma^{\mu} = S^{-1}\gamma^{\alpha}S$$

To the first order we have

$$(g^{\alpha}{}_{\mu} + \omega^{\alpha}{}_{\mu})\gamma^{\mu} = (1 + \frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu})\gamma^{\alpha}(1 - \frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu})$$

or

$$\gamma^{\alpha} + \omega^{\alpha}{}_{\mu}\gamma^{\mu} = \gamma^{\alpha} - \frac{i}{2}\omega_{\mu\nu}\gamma^{\alpha}\sigma^{\mu\nu} + \frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\gamma^{\alpha}$$

or

$$
\begin{aligned}
\omega^{\alpha}{}_{\mu}\gamma^{\mu} &= -\frac{i}{2}\omega_{\mu\nu}\gamma^{\alpha}\sigma^{\mu\nu} + \frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\gamma^{\alpha} \\
&= -\frac{i}{2}\omega_{\mu\nu}[\gamma^{\alpha}, \sigma^{\mu\nu}]
\end{aligned}
$$

Note that

$$
\begin{aligned}
\omega^{\alpha}{}_{\mu}\gamma^{\mu} &= g^{\alpha\beta}\omega_{\beta\mu}\gamma^{\mu} \\
&= \frac{1}{2}\omega_{\beta\mu}g^{\alpha\beta}\gamma^{\mu} + \frac{1}{2}\omega_{\rho\lambda}g^{\alpha\rho}\gamma^{\lambda} \\
&= \frac{1}{2}\omega_{\beta\mu}g^{\alpha\beta}\gamma^{\mu} - \frac{1}{2}\omega_{\lambda\rho}g^{\alpha\rho}\gamma^{\lambda} \\
&= \frac{1}{2}\omega_{\mu\nu}g^{\alpha\mu}\gamma^{\nu} - \frac{1}{2}\omega_{\mu\nu}g^{\alpha\nu}\gamma^{\mu} \\
&= \frac{1}{2}\omega_{\mu\nu}(g^{\alpha\mu}\gamma^{\nu} - g^{\alpha\nu}\gamma^{\mu})
\end{aligned}
$$

Thus we get

$$
i(g^{\alpha\mu}\gamma^{\nu} - g^{\alpha\nu}\gamma^{\mu}) = [\gamma^{\alpha}, \sigma^{\mu\nu}]
$$

From this equation, it is found that $\sigma^{\mu\nu}$ is antisymmetric. In fact, $\sigma^{\mu\nu}$ can be expressed as

$$
\sigma^{\mu\nu} = \frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}].
$$

Note that

$$
\begin{aligned}
\sigma^{\mu\nu +} &= \left(\frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}]\right)^{+} \\
&= -\frac{i}{2}[\gamma^{\nu +}, \gamma^{\mu +}] \\
&= -\frac{i}{2}[\gamma^{0}\gamma^{\nu}\gamma^{0}, \gamma^{0}\gamma^{\mu}\gamma^{0}] \\
&= \frac{i}{2}\gamma^{0}[\gamma^{\mu}, \gamma^{\nu}]\gamma^{0} \\
&= \gamma^{0}\sigma^{\mu\nu}\gamma^{0}
\end{aligned}
$$

Thus we obtain the infinitesimal Lorentz transformation as

$$
S = 1 - \frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu} = 1 + \frac{1}{8}\omega_{\mu\nu}[\gamma^{\mu}, \gamma^{\nu}]
$$

Next we discuss the finite Lorentz transformation with finite

Suppose that

$$\omega_{\mu\nu} = \varepsilon\chi M_{\mu\nu}$$

with

$$\varepsilon = \frac{1}{N}$$

The repeating N times of the infinitesimal Lorentz transformations leads to the

$$\begin{aligned} S(\Lambda) &= \lim_{N\to\infty}[1 + (-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu})]^N \\ &= \lim_{N\to\infty}[1 + (-\frac{i}{2}\varepsilon\chi M_{\mu\nu}\sigma^{\mu\nu})]^N \\ &= \lim_{N\to\infty}[1 + (-\frac{i}{2}\chi M_{\mu\nu}\sigma^{\mu\nu})\frac{1}{N}]^N \\ &= \lim_{N'\to\infty}(1 + \frac{1}{N'})^{N'\varsigma} \\ &= e^{\varsigma} \\ &= \exp[-\frac{i}{2}\chi M_{\mu\nu}\sigma^{\mu\nu}] \end{aligned}$$

where

$$N'\varsigma = N$$

and

$$\varsigma = -\frac{i}{2}\chi M_{\mu\nu}\sigma^{\mu\nu}$$

7. Expression of *S* for the Lorentz transformatiom (I)

$$\begin{aligned}
\omega_{\mu\nu} &= g_{\mu\alpha}\omega^{\alpha}{}_{\nu} \\
&= \varepsilon\chi(g_{\mu\alpha}M^{\alpha}{}_{\nu}) \\
&= \varepsilon\chi\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \varepsilon\chi\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \varepsilon\chi M_{\mu\nu}
\end{aligned}$$

with

$$M^{\mu}{}_{\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$M_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus we have

$$\begin{aligned}
S(\Lambda) &= \exp[-\frac{i}{2}\chi M_{\mu\nu}\sigma^{\mu\nu}] \\
&= \exp[-\frac{i}{2}\chi M_{01}\sigma^{01} - \frac{i}{2}\chi M_{10}\sigma^{10}] \\
&= \exp(\frac{i}{2}\chi\sigma^{01} - \frac{i}{2}\chi\sigma^{10}) \\
&= \exp(i\chi\sigma^{01})
\end{aligned}$$

since

$$M_{01} = -1, \qquad M_{10} = 1, \qquad \sigma^{10} = -\sigma^{01}$$

Note that

$$\sigma^{01} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}$$

$$S(\Lambda) = \exp(i\chi\sigma^{01})$$
$$= \begin{pmatrix} \cosh(\chi) & 0 & 0 & -\sinh(\chi) \\ 0 & \cosh(\chi) & -\sinh(\chi) & 0 \\ 0 & -\sinh(\chi) & \cosh(\chi) & 0 \\ -\sinh(\chi) & 0 & 0 & \cosh(\chi) \end{pmatrix}$$

((**Mathematica**))

```
k1 = σu[0, 1]; k1 // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

```
h1 = MatrixExp[i χ k1] // FullSimplify; h1 // MatrixForm
```

```
Cosh[χ]      0         0       -Sinh[χ]
   0      Cosh[χ]  -Sinh[χ]       0
   0     -Sinh[χ]   Cosh[χ]       0
-Sinh[χ]     0         0        Cosh[χ]
```

8. Derivation of the finite Lorentz transformation from the infinitesimal Lorentz transformation

The infinitesimal Lorentz transformation is given by

$$\delta\Lambda^{\mu}{}_{\nu} = (1 + \varepsilon\chi M)^{\mu}{}_{\nu}$$

The finite Lorentz transformation:

$$
\begin{aligned}
\Lambda^{\mu}{}_{\nu} &= [\lim_{N\to\infty}(1+\varepsilon\chi M)^{N}]^{\mu}{}_{\nu} \\
&= [\lim_{N\to\infty}(1+\frac{1}{N}\chi M)^{N}]^{\mu}{}_{\nu} \\
&= [\exp(\chi M)]^{\mu}{}_{\nu}
\end{aligned}
$$

where $\varepsilon = \frac{1}{N}$.

When

$$
M^{\mu}{}_{\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

we recover the result of the original Lorentz transformation given by

$$
\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \cosh(\chi) & -\sinh(\chi) & 0 & 0 \\ -\sinh(\chi) & \cosh(\chi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

using Mathematica.

((A.Z. Capri, Relativistic Quantum Mechanics and Introduction to Quantum Field Theory))

9. Expression of S for the Lorentz transformatiom (II): general case

$$\omega_{\mu\nu} = g_{\mu\alpha}\omega^{\alpha}{}_{\nu}$$

$$= \varepsilon\chi \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -\frac{\chi_x}{\chi} & -\frac{\chi_x}{\chi} & -\frac{\chi_x}{\chi} \\ -\frac{\chi_x}{\chi} & 0 & 0 & 0 \\ -\frac{\chi_y}{\chi} & 0 & 0 & 0 \\ -\frac{\chi_z}{\chi} & 0 & 0 & 0 \end{pmatrix}$$

$$= \varepsilon\chi \begin{pmatrix} 0 & -\frac{\chi_x}{\chi} & -\frac{\chi_x}{\chi} & -\frac{\chi_x}{\chi} \\ \frac{\chi_x}{\chi} & 0 & 0 & 0 \\ \frac{\chi_y}{\chi} & 0 & 0 & 0 \\ \frac{\chi_z}{\chi} & 0 & 0 & 0 \end{pmatrix}$$

$$= \varepsilon\chi M_{\mu\nu}$$

where

$$M_{\mu\nu} = \begin{pmatrix} 0 & -\frac{\chi_x}{\chi} & -\frac{\chi_x}{\chi} & -\frac{\chi_x}{\chi} \\ \frac{\chi_x}{\chi} & 0 & 0 & 0 \\ \frac{\chi_y}{\chi} & 0 & 0 & 0 \\ \frac{\chi_z}{\chi} & 0 & 0 & 0 \end{pmatrix}$$

$$\chi = \sqrt{\chi_x{}^2 + \chi_y{}^2 + \chi_z{}^2}$$

$$S(\Lambda) = \exp[-\frac{i}{2}\chi M_{\mu\nu}\sigma^{\mu\nu}]$$

$$= \exp[-\frac{i}{2}\chi_x M_{01}\sigma^{01} - \frac{i}{2}\chi_x M_{10}\sigma^{10} - \frac{i}{2}\chi_y M_{02}\sigma^{02} - \frac{i}{2}\chi_y M_{10}\sigma^{20}$$

$$-\frac{i}{2}\chi_z M_{03}\sigma^{03} - \frac{i}{2}\chi_z M_{30}\sigma^{30}]$$

$$= \exp(i\chi_x\sigma^{01} + i\chi_y\sigma^{02} + i\chi_z\sigma^{03})$$

Thus we obtain

$$S(\Lambda)=\begin{pmatrix} \cosh(\chi) & 0 & -\chi_z\frac{\sinh(\chi)}{\chi} & -(\chi_x-i\chi_y)\frac{\sinh(\chi)}{\chi} \\ 0 & \cosh(\chi) & -(\chi_x+i\chi_y)\frac{\sinh(\chi)}{\chi} & \chi_z\frac{\sinh(\chi)}{\chi} \\ -\chi_z\frac{\sinh(\chi)}{\chi} & -(\chi_x-i\chi_y)\frac{\sinh(\chi)}{\chi} & \cosh(\chi) & 0 \\ -(\chi_x+i\chi_y)\frac{\sinh(\chi)}{\chi} & \chi_z\frac{\sinh(\chi)}{\chi} & 0 & \cosh(\chi) \end{pmatrix}$$

((**Mathematica**)) **Determination of *S***

```
Clear["Global`*"]; σx = PauliMatrix[1];
σy = PauliMatrix[2]; σz = PauliMatrix[3];
I2 = IdentityMatrix[2]; αx = KroneckerProduct[σx, σx];
αy = KroneckerProduct[σx, σy];
αz = KroneckerProduct[σx, σz];
γ0 = KroneckerProduct[σz, I2]; γx = γ0.αx // Simplify;
γy = γ0.αy // Simplify; γz = γ0.αz // Simplify;
γ[1] := γx; γ[2] := γy; γ[3] := γz; γ[0] = γ0;
σ[μ_, ν_] := ⅈ ((γ[μ].γ[ν] - γ[ν].γ[μ])/2);

p1 = MatrixExp[ⅈ χ σ[0, 1]] // ExpToTrig;
p1 // MatrixForm
```

$$\begin{pmatrix} \text{Cosh}[\chi] & 0 & 0 & -\text{Sinh}[\chi] \\ 0 & \text{Cosh}[\chi] & -\text{Sinh}[\chi] & 0 \\ 0 & -\text{Sinh}[\chi] & \text{Cosh}[\chi] & 0 \\ -\text{Sinh}[\chi] & 0 & 0 & \text{Cosh}[\chi] \end{pmatrix}$$

```
p2 = MatrixExp[ⅈ χx σ[0, 1] + ⅈ χy σ[0, 2] + ⅈ χz σ[0, 3]] //
  FullSimplify;
```

$$\text{rule1} = \left\{\sqrt{\chi x^2 + \chi y^2 + \chi z^2} \to \chi,\ \frac{1}{\sqrt{\chi x^2 + \chi y^2 + \chi z^2}} \to \frac{1}{\chi}\right\};$$

```
p21 = p2 /. rule1 // FullSimplify;
p21 // MatrixForm
```

$$\begin{pmatrix} \text{Cosh}[\chi] & 0 & -\frac{\chi z\,\text{Sinh}[\chi]}{\chi} & -\frac{(\chi x - i\,\chi y)\,\text{Sinh}[\chi]}{\chi} \\ 0 & \text{Cosh}[\chi] & -\frac{(\chi x + i\,\chi y)\,\text{Sinh}[\chi]}{\chi} & \frac{\chi z\,\text{Sinh}[\chi]}{\chi} \\ -\frac{\chi z\,\text{Sinh}[\chi]}{\chi} & -\frac{(\chi x - i\,\chi y)\,\text{Sinh}[\chi]}{\chi} & \text{Cosh}[\chi] & 0 \\ -\frac{(\chi x + i\,\chi y)\,\text{Sinh}[\chi]}{\chi} & \frac{\chi z\,\text{Sinh}[\chi]}{\chi} & 0 & \text{Cosh}[\chi] \end{pmatrix}$$

10. Rotation matrices

The Infinitesimal rotation is expressed by

$$\delta\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\varepsilon\theta) & -\sin(\varepsilon\theta) & 0 \\ 0 & \sin(\varepsilon\theta) & \cos(\varepsilon\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\varepsilon\theta & 0 \\ 0 & \varepsilon\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \varepsilon\theta \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= 1_4 + \varepsilon\theta M^{\mu}{}_{\nu}$$

where

$$M^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We note that

$$
\begin{aligned}
M_{\mu\nu} &= g_{\mu\alpha} M^{\alpha}{}_{\nu} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

$$
M_{12} = 1, \qquad M_{21} = -1, \qquad \varepsilon = \frac{1}{N}
$$

Then we have

$$
\begin{aligned}
\Lambda_{\mu\nu} &= [\lim_{N\to\infty} (1 + \varepsilon\theta M)^N]_{\mu\nu} \\
&= [\lim_{N\to\infty} (1 + \frac{1}{N}\theta M)^N]_{\mu\nu} \\
&= [\exp(\theta M)]_{\mu\nu}
\end{aligned}
$$

leading to the expression

$$
\Lambda_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Similarly

$$
\begin{aligned}
\Lambda^{\mu}{}_{\nu} &= [\lim_{N\to\infty} (1 + \varepsilon\theta M)^N]^{\mu}{}_{\nu} \\
&= [\lim_{N\to\infty} (1 + \frac{1}{N}\theta M)^N]^{\mu}{}_{\nu} \\
&= [\exp(\theta M)]^{\mu}{}_{\nu}
\end{aligned}
$$

leading to

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The rotation transformation

$$\begin{aligned} S_{rot}(\varepsilon\theta) &= 1 + \frac{1}{4}\varepsilon\theta M_{\mu\nu}\gamma^{\mu}\gamma^{\nu} \\ &= 1 + \frac{1}{4}\varepsilon\theta(M_{12}\gamma^{1}\gamma^{2} + M_{21}\gamma^{2}\gamma^{1}) \\ &= 1 + \frac{1}{4}\varepsilon\theta[\gamma^{1}, \gamma^{2}] \\ &= 1 - \frac{1}{2}i\varepsilon\theta\sigma^{12} \\ &= 1 - \frac{1}{2}i\varepsilon\theta\Sigma^{3} \end{aligned}$$

where

$$M_{12} = 1, \qquad M_{21} = -1$$

$$\sigma^{12} = \Sigma^{3} = i\gamma^{1}\gamma^{2} = \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}$$

or

$$\gamma^{1}\gamma^{2} = -i\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}$$

The expression of S for the finite rotation:

$$\begin{aligned} S_{rot}(\theta) &= \lim_{N\to\infty}[S_{rot}(\varepsilon\theta)]^{N} \\ &= \lim_{N\to\infty}[1 + \frac{1}{N}(-\frac{1}{2}i\theta\Sigma_3)]^{N} \\ &= \exp(-i\frac{1}{2}\theta\Sigma_3) \\ &= I_4\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\Sigma_3 \end{aligned}$$

or

$$
\begin{aligned}
S_{rot}(\theta) &= \exp(-i\frac{1}{2}\theta\Sigma_3) \\
&= I_4\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\Sigma_3 \\
&= I_4\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\gamma^1\gamma^2
\end{aligned}
$$

where

$$
\varepsilon = \frac{1}{N}.
$$

This expression is similar to the expression

$$
\begin{aligned}
\hat{R}_z(\theta) &= \exp(-\frac{i}{2}\theta\hat{\sigma}_z) \\
&= \exp(-\frac{i}{2}\theta\hat{\sigma}_z)[|+z\rangle\langle+z| + |-z\rangle\langle-z|] \\
&= \begin{pmatrix} e^{-\frac{i\theta}{2}} & 0 \\ 0 & e^{\frac{i\theta}{2}} \end{pmatrix} \\
&= \hat{1}_2\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\hat{\sigma}_z
\end{aligned}
$$

Note that

$$
\begin{aligned}
S_{rot}(\theta) &= \begin{pmatrix} e^{-\frac{i\theta}{2}} & 0 & 0 & 0 \\ 0 & e^{-\frac{i\theta}{2}} & 0 & 0 \\ 0 & 0 & e^{-\frac{i\theta}{2}} & 0 \\ 0 & 0 & 0 & e^{\frac{i\omega}{2}} \end{pmatrix} \\
&= I_4\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\gamma^1\gamma^2
\end{aligned}
$$

$$S_{rot}{}^{-1}(\theta) = \begin{pmatrix} e^{\frac{i\theta}{2}} & 0 & 0 & 0 \\ 0 & e^{-\frac{i\theta}{2}} & 0 & 0 \\ 0 & 0 & e^{\frac{i\theta}{2}} & 0 \\ 0 & 0 & 0 & e^{-\frac{i\theta}{2}} \end{pmatrix}$$

$$= I_4 \cos\frac{\theta}{2} - \sin\frac{\theta}{2}\gamma^1\gamma^2$$

S_{rot} satisfies the following conditions

$$S_{rot}{}^{-1}(\theta)\gamma^1 S_{rot}(\theta) = \Lambda^1{}_1\gamma^1 + \Lambda^1{}_2\gamma^2$$
$$= \gamma^1\cos\theta - \gamma^2\sin\theta$$

$$S_{rot}{}^{-1}(\theta)\gamma^2 S_{rot}(\theta) = \Lambda^2{}_1\gamma^1 + \Lambda^2{}_2\gamma^2$$
$$= \gamma^1\sin\theta + \gamma^2\cos\theta$$

11. Lorentz transformation (revisited)

The Lorentz transformation is expressed by

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh(\varepsilon\chi) & -\sinh(\varepsilon\chi) & 0 & 0 \\ -\sinh(\varepsilon\chi) & \cosh(\varepsilon\chi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\varepsilon\chi & 0 & 0 \\ -\varepsilon\chi & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \varepsilon\chi \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= g^\mu{}_\nu + \omega^\mu{}_\nu$$

$$= g^\mu{}_\nu + \varepsilon\chi M^\mu{}_\nu$$

where

$$\omega^\mu{}_\nu = \varepsilon\chi M^\mu{}_\nu$$

$$M_{01} = 1, \qquad M_{10} = -1$$

Note that

$$\begin{aligned}
\omega_{\mu\nu} &= g_{\mu\alpha}\omega^{\alpha}{}_{\nu} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

The Lorentz transformation: $\Lambda^{\mu}{}_{\nu}$

$$\begin{aligned}
\Lambda^{\mu}{}_{\nu} &= \lim_{N\to\infty} (1 + \varepsilon\chi M)^{\mu}{}_{\nu} \\
&= \lim_{N\to\infty} (1 + \frac{1}{N}\chi M)^{\mu}{}_{\nu} \\
&= [\exp(\chi M)]^{\mu}{}_{\nu} \\
&= \begin{pmatrix} \cosh\chi & -\sinh\chi & 0 & 0 \\ -\sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & \\ 0 & 0 & & 1 \end{pmatrix}
\end{aligned}$$

The pure Lorentz transformation is nothing more than a rotation in the 0-1 plane by imaginary angle;

$$\sinh\chi = -i\sin(i\chi), \qquad \cos(i\chi) = \cosh\chi$$

$$
\begin{aligned}
S_L &= 1 + \frac{1}{4}(\varepsilon\chi M_{\mu\nu})\gamma^{\mu}\gamma^{\nu} \\
&= 1 + \frac{1}{4}\varepsilon\chi(M_{01}\gamma^0\gamma^1 + M_{10}\gamma^1\gamma^0) \\
&= 1 - \frac{1}{4}\varepsilon\chi[\gamma^0, \gamma^1] \\
&= 1 - \frac{1}{2}i\varepsilon\chi\sigma^{01}
\end{aligned}
$$

where

$$
\omega_{01} = -1, \qquad \omega_{10} = 1
$$

$$
\sigma^{01} = i\gamma^0\gamma^1 = i\begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix} = i\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
$$

Finite rotation

$$
\begin{aligned}
S_L(\Lambda) &= \lim_{N\to\infty}(1 - \frac{1}{2}i\varepsilon\chi\sigma^{01})^N \\
&= \lim_{N\to\infty}[1 + (-\frac{1}{2}i\frac{\chi}{N}\sigma^{01}]^N \\
&= \exp(-i\frac{1}{2}\chi\sigma^{01})
\end{aligned}
$$

where

$$
\varepsilon = \frac{1}{N}
$$

$$
\begin{aligned}
S_L(\Lambda) &= 1_4 \cosh\frac{\chi}{2} + i\sinh\frac{\chi}{2}\sigma^{01} \\
&= I_4 \cosh\frac{\chi}{2} + i\sinh\frac{\chi}{2}(i\gamma^0\gamma^1) \\
&= I_4 \cosh\frac{\chi}{2} - \sinh\frac{\chi}{2}\gamma^0\gamma^1
\end{aligned}
$$

with

$$
\sigma^{01} = i\gamma^0\gamma^1
$$

$$\gamma^0\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Then we have

$$S_L(\Lambda) = \begin{pmatrix} \cosh\frac{\chi}{2} & 0 & 0 & -\sinh\frac{\chi}{2} \\ 0 & \cosh\frac{\chi}{2} & -\sinh\frac{\chi}{2} & 0 \\ 0 & -\sinh\frac{\chi}{2} & \cosh\frac{\chi}{2} & 0 \\ -\sinh\frac{\chi}{2} & 0 & 0 & \cosh\frac{\chi}{2} \end{pmatrix}$$

The Hermitian conjugate of $S_L(\Lambda)$ is obtained by

$$\begin{aligned} S_L{}^{+}(\Lambda) &= I_4 \cosh\frac{\chi}{2} - \gamma^{1+}\gamma^{0+} \sinh\frac{\chi}{2} \\ &= I_4 \cosh\frac{\chi}{2} + \gamma^1\gamma^0 \sinh\frac{\chi}{2} \\ &= I_4 \cosh\frac{\chi}{2} - \gamma^0\gamma^1 \sinh\frac{\chi}{2} \\ &= S_L(\Lambda) \end{aligned}$$

and

$$\begin{aligned} S_L(\Lambda)^{-1} &= \begin{pmatrix} \cosh\frac{\chi}{2} & 0 & 0 & \sinh\frac{\chi}{2} \\ 0 & \cosh\frac{\chi}{2} & \sinh\frac{\chi}{2} & 0 \\ 0 & \sinh\frac{\chi}{2} & \cosh\frac{\chi}{2} & 0 \\ \sinh\frac{\chi}{2} & 0 & 0 & \cosh\frac{\chi}{2} \end{pmatrix} \\ &= \cosh\frac{\chi}{2} + \gamma^0\gamma^1 \sinh\frac{\chi}{2} \end{aligned}$$

where

$$\gamma^{0+} = \gamma^0 , \qquad \gamma^{1+} = -\gamma^1 ,$$

$$\gamma^1\gamma^0 = -\gamma^0\gamma^1$$

Thus $S_L(\Lambda)$ is not unitary since

$$S_L(\Lambda)^{-1} \neq S_L{}^+(\Lambda)$$

It is important to note that for both pure rotation and pure Lorentz transformation, we have

$$S = \gamma^0 S^+ \gamma^0 .$$

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