Eigenvalue problem Collection
Masatsugu Sei Suzuki
Department of Physics
(Date: September 13, 2015)

---

1. D.H. McIntyre p.66 2-23 (simultaneous eigenkets, degenerate case)

2.23 Consider a three-dimensional ket space. In the basis defined by three orthogonal kets $|1\rangle$, $|2\rangle$, and $|3\rangle$, the operators A and B are represented by

$$A \doteq \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \qquad B \doteq \begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix},$$

where all the quantities are real.

a) Do the operators A and B commute?

b) Find the eigenvalues and normalized eigenvectors of both operators.

((**Solution**))

$$\hat{A} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix}$$

$$\hat{A}\hat{B} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}\begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix} = \begin{pmatrix} a_1 b_1 & 0 & 0 \\ 0 & 0 & a_2 b_2 \\ 0 & a_3 b_2 & 0 \end{pmatrix}$$

$$\hat{B}\hat{A} = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_2 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = \begin{pmatrix} a_1 b_1 & 0 & 0 \\ 0 & 0 & a_3 b_2 \\ 0 & a_2 b_2 & 0 \end{pmatrix}$$

When $[\hat{A}, \hat{B}] = 0$, we need to assume that

$$a_2 = a_3$$

Thus the eigenkets and eigenvalues for $\hat{A}$ are

$$|\phi_1\rangle = |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$ with the eigenvalue a_1

$$|\phi_2\rangle = |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$ with the eigenvalue a_2

$$|\phi_3\rangle = |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$ with the eigenvalue a_2

The eigenkets and eigen values for $\hat{B}$ are

$$|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |\phi_1\rangle$$ with the eigenvalue b_1

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}[|\phi_2\rangle + |\phi_3\rangle]$$ with the eigenvalue b_2

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}[|\phi_2\rangle - |\phi_3\rangle]$$ with the eigenvalue $-b_2$

We note that

$$\hat{A}|\psi_1\rangle = \hat{A}|\phi_1\rangle = a_1|\psi_1\rangle$$

$$\hat{A}|\psi_2\rangle = \frac{1}{\sqrt{2}}[\hat{A}|\phi_2\rangle + \hat{A}|\phi_3\rangle] = a_2|\psi_2\rangle$$

$$\hat{A}|\psi_3\rangle = \frac{1}{\sqrt{2}}[\hat{A}|\phi_2\rangle - \hat{A}|\phi_3\rangle] = a_2|\psi_3\rangle$$

$$\hat{B}|\psi_1\rangle = b_1|\psi_1\rangle$$

$$\hat{B}|\psi_2\rangle = b_2|\psi_2\rangle$$

$$\hat{B}|\psi_3\rangle = -b_2|\psi_3\rangle$$

Thus $|\psi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$ are the simultaneous eigenkets of both $\hat{A}$ and $\hat{B}$.

---

2. Schaum p.72 4-30 (degenerate case)

4.30. Consider a physical system with a three-dimensional state space. An orthonormal basis of the state space is chosen; in this basis the Hamiltonian is represented by the matrix

$$H = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad (4.30.1)$$

(*a*) What are the possible results when the energy of the system is measured? (*b*) A particle is in the state

((**Solution**))

Under the basis $\{|1\rangle, |2\rangle, |3\rangle\}$,

$$\hat{H} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

where

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Note that

$$\hat{H}|3\rangle = 3|3\rangle$$

So we have

$$|\phi_3\rangle = |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \text{with the eigenvalue 3.}$$

We also have

$$\hat{H}|1\rangle = 2|1\rangle + |2\rangle, \qquad \hat{H}|2\rangle = |1\rangle + 2|2\rangle$$

The matrix of $\hat{H}$ under the basis $\{|1\rangle, |2\rangle\}$ can be rewritten as

$$\hat{H}_{sub} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 2\hat{1} + \hat{\sigma}_x,$$

using the Pauli spin operators (2x2). Then we have

$$\hat{H}_{sub}|+x\rangle = (2\hat{1} + \hat{\sigma}_x)|+x\rangle = 3|+x\rangle,$$

$$\hat{H}_{sub}|-x\rangle = (2\hat{1} + \hat{\sigma}_x)|-x\rangle = |+x\rangle.$$

$$|+x\rangle \to |\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \text{with the eigenvalue 3.}$$

$$|-x\rangle \to |\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad \text{with the eigenvalue 1.}$$

---

3. Schaum p.73 4-31 (degenerate case, 3 x 3)

4.31. Refer to Problem 4.30. Suppose that the energy of the system was measured and a value of $E = 1$ was found. Subsequently we perform a measurement of a variable A described in the same basis by

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & i \\ 0 & -i & 2 \end{pmatrix} \tag{4.31.1}$$

(*a*) Find the possible results of A. (*b*) What are the probabilities of obtaining each of the results found in part (*a*)?

((**Solution**))

Under the basis $\{|1\rangle$, $|2\rangle$, and $|3\rangle\}$,

$$\hat{A} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & i \\ 0 & -i & 2 \end{pmatrix}$$

where

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Note that

$$\hat{A}|1\rangle = 5|1\rangle$$

So we have

$$|\phi_1\rangle = |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{with the eigenvalue 5.}$$

We also have

$$\hat{A}|2\rangle = 2|2\rangle - i|3\rangle, \qquad \hat{A}|3\rangle = i|2\rangle + 2|3\rangle$$

The submatrix of $\hat{A}$ under the basis $\{|2\rangle, |3\rangle\}$ can be rewritten as

$$\hat{A}_{sub} = \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix} = 2\hat{1} - \hat{\sigma}_y$$

using the Pauli spin operators (2x2). Then we have

$$\hat{A}_{sub}|+y\rangle = (2\hat{1} - \hat{\sigma}_y)|+y\rangle = |+y\rangle$$

$$\hat{A}_{sub}|-y\rangle = (2\hat{1} - \hat{\sigma}_y)|-y\rangle = 3|-y\rangle$$

$$|+y\rangle \to |\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$$ with the eigenvalue 1.

$$|-y\rangle \to |\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}$$ with the eigenvalue 3.

---

4. Goswami p.325 Problem 12 (non-degenerate case, 3 x 3 matrix)

12. Consider the 3×3 matrix

$$M = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

Is the matrix hermitian? Find the eigenvalues and eigenvectors (which are now column matrices with three rows) of M and normalize the eigenvectors. Find the matrix U that diagonalizes M. Is U unitary?

$$\hat{M} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

((**Solution**))

The eigenkets and eigenvalues for $\hat{M}$ are

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ i \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$ with the eigenvalue (5)

$$|\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$ with the eigenvalue (3)

$$|\phi_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -i \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$ with the eigenvalue (1)

$$\hat{U} = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{pmatrix}$$

$$\hat{U}^{+}\hat{A}\hat{U} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

---

5. Steeb p.69 problem-1 (non-degenerate 2 x 2)

Problem 1. (i) Find the eigenvalues and normalized eigenvectors of the rotational matrix

$$A = \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}.$$

(ii) Are the eigenvectors orthogonal to each other?

$$\hat{A} = \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}$$

((**Solution**))

$$\hat{A} = \begin{pmatrix} \sin\theta & 0 \\ 0 & \sin\theta \end{pmatrix} + i\cos\theta \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sin\theta\hat{1} + i\cos\theta\hat{\sigma}_y$$

$$\hat{A}|+y\rangle = (\sin\theta\hat{1} + i\cos\theta\hat{\sigma}_y)|+y\rangle = (\sin\theta + i\cos\theta)|+y\rangle = ie^{-i\theta}|+y\rangle$$

$$\hat{A}|-y\rangle = (\sin\theta\hat{1} + i\cos\theta\hat{\sigma}_y)|-y\rangle = (\sin\theta - i\cos\theta)|-y\rangle = -ie^{i\theta}|-y\rangle$$

$|+y\rangle$ is the eigenket of $\hat{A}$ with the eigenvalue $ie^{-i\theta}$

$|-y\rangle$ is the eigenket of $\hat{A}$ with the eigenvalue $-ie^{i\theta}$

---

6. Steeb p.94 Problem 41 (degenerate case, 4 x 4 matrix)

Problem 41. Calculate the eigenvalues of the 4×4 matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

by calculating the eigenvalues of A^2.

$$\hat{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

where

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |4\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

((**Solution**))

Under the basis $\{|1\rangle, |4\rangle\}$, the submatrix of $\hat{A}$ can be written as

$$\widehat{A}_{sub}(1,4) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \hat{\sigma}_z + \hat{\sigma}_x = \sqrt{2}(\cos\theta\hat{\sigma}_z + \sin\hat{\sigma}_z),$$

with $\theta = \dfrac{\pi}{4}$

$$|\phi_1\rangle = \sqrt{2}\begin{pmatrix} 0 \\ \cos\dfrac{\theta}{2} \\ 0 \\ \sin\dfrac{\theta}{2} \end{pmatrix} \qquad \text{eigenvalue } (\sqrt{2})$$

$$|\phi_4\rangle = \sqrt{2}\begin{pmatrix} 0 \\ \sin\dfrac{\theta}{2} \\ 0 \\ -\cos\dfrac{\theta}{2} \end{pmatrix} \qquad \text{eigenvalue } (-\sqrt{2})$$

Under the basis $\{|2\rangle, |3\rangle\}$, the submatrix of $\hat{A}$ can be written as

$$\widehat{A}_{sub}(2,3) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \hat{\sigma}_z + \hat{\sigma}_x = \sqrt{2}(\cos\theta\hat{\sigma}_z + \sin\hat{\sigma}_z)$$

with $\theta = \dfrac{\pi}{4}$

$$|\phi_2\rangle = \sqrt{2}\begin{pmatrix} 0 \\ \cos\frac{\theta}{2} \\ 0 \\ \sin\frac{\theta}{2} \end{pmatrix} \qquad \text{eigenvalue } (\sqrt{2})$$

$$|\phi_3\rangle = \sqrt{2}\begin{pmatrix} 0 \\ \sin\frac{\theta}{2} \\ 0 \\ -\cos\frac{\theta}{2} \end{pmatrix} \qquad \text{eigenvalue } (-\sqrt{2})$$

---

7. Shankar p.41 Exercise 1-8-2 (Non-degenerate case)

*Exercise 1.8.2.** Consider the matrix

$$\Omega = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(1) Is it Hermitian?
(2) Find its eigenvalues and eigenvectors.
(3) Verify that $U^{\dagger}\Omega U$ is diagonal, U being the matrix of eigenvectors of Ω.

((**Solution**))

$$\hat{A} = \hat{\Omega} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Under the basis $\{|1\rangle, |2\rangle, |3\rangle\}$,

$$\hat{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

where

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Note that

$$\hat{A}|1\rangle = |3\rangle, \qquad \hat{A}|2\rangle = 0, \qquad \hat{A}|3\rangle = |1\rangle$$

So we have

$$|\phi_2\rangle = |2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{with the eigenvalue 0.}$$

We also have

$$\hat{A}|1\rangle = |3\rangle, \qquad \hat{A}|3\rangle = |1\rangle$$

The matrix of $\hat{A}$ under the basis $\{|2\rangle, |3\rangle\}$ can be rewritten as

$$\hat{A}_{sub} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \hat{\sigma}_x,$$

using the Pauli spin operators (2x2). Then we have

$$\hat{A}_{sub}|+x\rangle = \hat{\sigma}_x|+x\rangle = |+x\rangle$$

$$\hat{A}_{sub}|-x\rangle = \hat{\sigma}_x|-x\rangle = -|-x\rangle$$

$$|+x\rangle \rightarrow |\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \text{with the eigenvalue 1.}$$

$$|-x\rangle \to |\phi_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$ with the eigenvalue -1.

---

8. Shankar p. 41 Exercise 1.8.3 (Non-degenerate case)

*Exercise 1.8.3.** Consider the Hermitian matrix

$$\Omega = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

(1) Show that $\omega_1 = \omega_2 = 1$; $\omega_3 = 2$.
(2) Show that $|\omega = 2\rangle$ is any vector of the form

$$\frac{1}{(2a^2)^{1/2}}\begin{bmatrix} 0 \\ a \\ -a \end{bmatrix}$$

(3) Show that the $\omega = 1$ eigenspace contains all vectors of the form

$$\frac{1}{(b^2 + 2c^2)^{1/2}}\begin{bmatrix} b \\ c \\ c \end{bmatrix}$$

either by feeding $\omega = 1$ into the equations or by requiring that the $\omega = 1$ eigenspace be orthogonal to $|\omega = 2\rangle$.

$$\hat{\Omega} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

((**Solution**))

$$\hat{\Omega} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

Note that

$$\hat{\Omega}|1\rangle = |1\rangle, \qquad \hat{\Omega}|2\rangle = \frac{3}{2}|2\rangle - \frac{1}{2}|3\rangle, \quad \hat{\Omega}|3\rangle = -|2\rangle + 3|3\rangle$$

So we have

$$|\phi_1\rangle = |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$ with the eigenvalue 0.

We also have

$$\hat{\Omega}|2\rangle = \frac{3}{2}|2\rangle - \frac{1}{2}|3\rangle, \quad \hat{\Omega}|3\rangle = -\frac{1}{2}|2\rangle + \frac{3}{2}|3\rangle$$

The matrix of $\hat{A}$ under the basis $\{|2\rangle, |3\rangle\}$ can be rewritten as

$$\hat{\Omega}_{sub} = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} = \frac{31 - \hat{\sigma}_x}{2}$$

using the Pauli spin operators (2x2). Then we have

$$\hat{\Omega}_{sub}|+x\rangle = \frac{1}{2}(31 - \hat{\sigma}_x)|+x\rangle = |+x\rangle$$

$$\hat{\Omega}_{sub}|-x\rangle = \frac{1}{2}(31 - \hat{\sigma}_x)|+x\rangle = 2|-x\rangle$$

$$|+x\rangle \to |\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$ with the eigenvalue 1.

$$|-x\rangle \to |\phi_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$ with the eigenvalue 2.

---

9. Shankar p.46 1.8.10 Simultaneous eigenstates

*Exercise 1.8.10.** By considering the commutator, show that the following Hermitian matrices may be simultaneously diagonalized. Find the eigenvectors common to both and verify that under a unitary transformation to this basis, both matrices are diagonalized.

$$\Omega = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Since Ω is degenerate and Λ is not, you must be prudent in deciding which matrix dictates the choice of basis.

((**Solution**))

$$\hat{\Omega} = \hat{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \hat{\Lambda} = \hat{B} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$\hat{A}\hat{B} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix}$$

$$\hat{B}\hat{A} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix}$$

Then we have

$$\hat{A}\hat{B} = \hat{B}\hat{A}$$

So we have simultaneous eigenkets of both $\hat{A}$ and $\hat{B}$.

The eigenkets of $\hat{B}$ (non-degenerate case) are obtained as

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \text{with the eigenvalue 3}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\-1\end{pmatrix}$$ with the eigenvalue 2

$$|\psi_3\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\-1\end{pmatrix}$$ with the eigenvalue -1

The eigenkets of $\hat{A}$ (degenerate case) are obtained as

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} = |\psi_1\rangle$$ with the eigenvalue 2

$$|\phi_2\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}$$ with the eigenvalue 0

$$|\phi_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$$ with the eigenvalue 0

Note that

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} = |\phi_1\rangle$$

$$|\psi_2\rangle = \frac{1}{\sqrt{3}}(|\phi_2\rangle + \sqrt{2}|\phi_3\rangle)$$

$$|\psi_3\rangle = \frac{1}{\sqrt{6}}(-2|\phi_2\rangle + \sqrt{2}|\phi_3\rangle)$$

Thus $|\psi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$ are the simultaneous eigenkets of both $\hat{A}$ and $\hat{B}$.

$\hat{A}|\psi_1\rangle = 2|\psi_1\rangle$, $\quad \hat{B}|\psi_1\rangle = 3|\psi_1\rangle$

$\hat{A}|\psi_2\rangle = 0$, $\quad \hat{B}|\psi_2\rangle = 2|\psi_2\rangle$

$\hat{A}|\psi_3\rangle = 0$ $\quad \hat{B}|\psi_3\rangle = -|\psi_3\rangle$

---

10. Griffiths p.457 A-26 (degenerate case, 3 x 3 matrix)

****Problem A.26** Consider the following hermitian matrix:

$$\mathbf{T} = \begin{pmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{pmatrix}.$$

(a) Calculate det(**T**) and Tr(**T**).

(b) Find the eigenvalues of **T**. Check that their sum and product are consistent with (a), in the sense of Equation A.85. Write down the diagonalized version of **T**.

(c) Find the eigenvectors of **T**. Within the degenerate sector, construct two linearly independent eigenvectors (it is this step that is always possible for a *hermitian* matrix, but not for an *arbitrary* matrix—contrast Problem A.19). Orthogonalize them, and check that both are orthogonal to the third. Normalize all three eigenvectors.

(d) Construct the unitary matrix **S** that diagonalizes **T**, and show explicitly that the similarity transformation using **S** reduces **T** to the appropriate diagonal form.

((**Solution**))

$$\hat{A} = \hat{T} = \begin{pmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{pmatrix}$$

det[$\hat{A}$]=0, Tr[$\hat{A}$]=6.

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \text{eigenvalue (3)}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2i \\ -1 \end{pmatrix}, \qquad \text{eigenvalue (3)}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ i \\ -1 \end{pmatrix} \qquad \text{eigenvalue (0)}$$

$$\hat{U} = \hat{S} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2i}{\sqrt{6}} & \frac{i}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{pmatrix}$$

$$\hat{U}^{+}\hat{A}\hat{U} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

---

11. Sakurai 1-23 (simultaneous eigenkets, degenerate cases)

1.23 Consider a three-dimensional ket space. If a certain set of orthonormal kets—say, $|1\rangle$, $|2\rangle$, and $|3\rangle$—are used as the base kets, the operators A and B are represented by

$$A \doteq \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}, \quad B \doteq \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}$$

with a and b both real.

(a) Obviously A exhibits a degenerate spectrum. Does B also exhibit a degenerate spectrum?

(b) Show that A and B commute.

(c) Find a new set of orthonormal kets that are simultaneous eigenkets of both A and B. Specify the eigenvalues of A and B for each of the three eigenkets. Does your specification of eigenvalues completely characterize each eigenket?

((**Solution**))

$$\hat{A} = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}$$

$$\hat{A}\hat{B} = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}\begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix} = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix}$$

$$\hat{B}\hat{A} = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 3 \end{pmatrix}$$

Then we have

$$\hat{A}\hat{B} = \hat{B}\hat{A}$$

So we have simultaneous eigenkets of both $\hat{A}$ and $\hat{B}$. The eigenkets of $\hat{A}$ (degenerate case) are obtained as

$$|\phi_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$ with the eigenvalue *a*

$$|\phi_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$ with the eigenvalue -*a*

$$|\phi_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$ with the eigenvalue -*a*

The eigenkets of $\hat{B}$ (degenerate case) are obtained as

$$|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$ with the eigenvalue *b*

The submatrix of $\hat{B}$ under the basis $\{|2\rangle, |3\rangle\}$ can be written as

$$\hat{B}_{sub} = \begin{pmatrix} 0 & -ib \\ ib & 0 \end{pmatrix} = b\hat{\sigma}_y$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$$ with the eigenvalue *b*

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}$$ with the eigenvalue -*b*

Then we have

$$|\psi_1\rangle = |\phi_1\rangle$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}[|\phi_2\rangle + i|\phi_3\rangle]$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}[|\phi_2\rangle - i|\phi_3\rangle]$$

Thus $|\psi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$ are the simultaneous eigenkets of both $\hat{A}$ and $\hat{B}$.

$$\hat{A}|\psi_1\rangle = a|\psi_1\rangle, \qquad \hat{B}|\psi_1\rangle = b|\psi_1\rangle$$

$$\hat{A}|\psi_2\rangle = -a|\psi_2\rangle, \qquad \hat{B}|\psi_2\rangle = b|\psi_2\rangle$$

$$\hat{A}|\psi_3\rangle = -a|\psi_3\rangle \qquad \hat{B}|\psi_3\rangle = -b|\psi_3\rangle$$

---

12. Rogalski p.134 3-3-1

Solve the eigenvalue problem for the matrix

$$\hat{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

((**Solution**))

$$\hat{A}|1\rangle = |2\rangle, \qquad \hat{A}|2\rangle = |1\rangle, \qquad \hat{A}|3\rangle = 3|3\rangle$$

$$|\phi_3\rangle = |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \text{with the eigenvalue 3}$$

The matrix of $\hat{A}$ (subsystem) under the basis $\{|1\rangle, |2\rangle\}$ can be rewritten as

$$\hat{A}_{sub} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \hat{\sigma}_x.$$

$$\hat{A}_{sub}|+x\rangle = \hat{\sigma}_x|+x\rangle = |+x\rangle, \qquad \hat{A}_{sub}|-x\rangle = \hat{\sigma}_x|-x\rangle = -|-x\rangle$$

$$|\phi_1\rangle = |+x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix},$$ with the eigenvalue (1)

$$|\phi_2\rangle = |+x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix},$$ with the eigenvalue (-1)

---

13. Rogalski p.136 (degenerate case)

Solve the eigenvalue problem for the matrix

$$\hat{A} = \begin{pmatrix}0 & 1 & 1\\1 & 0 & 1\\1 & 1 & 0\end{pmatrix}$$

((**Solution**))

$$|\phi_1\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$ with the eigenvalue 2

$$|\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$$ with the eigenvalue -1

$$|\phi_3\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix}$$ with the eigenvalue -1

The unitary operator

$$\hat{U} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$\hat{U}^{+}\hat{A}\hat{U} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

((**Mathematica**))

```
Clear["Global`*"];
exp_ * :=
 exp /. {Complex[re_, im_] :→ Complex[re, -im]};
```

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix};$$

```
eq1 = Eigensystem[A]
```

{{2, -1, -1}, {{1, 1, 1}, {-1, 0, 1}, {-1, 1, 0}}}

```
ψ1 = Normalize[eq1[[2, 1]]]
```

$$\left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}$$

```
ψ2 = Normalize[eq1[[2, 2]]]
```

$$\left\{-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}$$

```
ψ3 = -Normalize[eq1[[2, 3]]]
```

$$\left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right\}$$

```
{ψ1*.ψ2, ψ2*.ψ3, ψ3*.ψ1}
```

$$\left\{0, -\frac{1}{2}, 0\right\}$$

```
eq2 = Orthogonalize[{ψ1, ψ2, ψ3}];
ϕ1 = eq2[[1]]; ϕ2 = -eq2[[2]]; ϕ3 = eq2[[3]];
ϕ1 // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

```
ϕ2 // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

```
ϕ3 // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

```
UT = {ϕ1, ϕ2, ϕ3}
```

$$\left\{\left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}, \left\{\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{6}}, -\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{6}}\right\}\right\}$$

```
UH = UT*
```

$$\left\{\left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}, \left\{\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{6}}, -\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{6}}\right\}\right\}$$

```
U = Transpose[UT]; U // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

```
UH.U
```

$$\{\{1, 0, 0\}, \{0, 1, 0\}, \{0, 0, 1\}\}$$

```
UH.A.U // Simplify // MatrixForm
```

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

---

14. Zettili P.158 2-24 (simultaneous eigenkets, degenerate case)

Exercise 2.24
Consider two operators $\hat{A}$ and $\hat{B}$ whose matrices are

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}.$$

(a) Are $\hat{A}$ and $\hat{B}$ Hermitian?
(b) Do $\hat{A}$ and $\hat{B}$ commute?
(c) Find the eigenvalues and eigenvectors of $\hat{A}$ and $\hat{B}$.
(d) Are the eigenvectors of each operator orthonormal?
(e) Verify that $\hat{U}^\dagger \hat{B} \hat{U}$ is diagonal, $\hat{U}$ being the matrix of the normalized eigenvectors of $\hat{B}$.
(f) Verify that $\hat{U}^{-1} = \hat{U}^\dagger$.

((**Solution**))

$$\hat{A} = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$$

(a) $\hat{A}$ is not Hermitian, but $\hat{B}$ is Hermitian.

(b) $[\hat{A}, \hat{B}] \neq 0$

$$\hat{A}\hat{B} - \hat{B}\hat{A} = \begin{pmatrix} 0 & -5 & 0 \\ -1 & 0 & 2 \\ 0 & 10 & 0 \end{pmatrix}$$

(c)
The eigenvectors of $\hat{A}$ (no Hermitian)

$$|\alpha_1\rangle = \frac{1}{\sqrt{11}} \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \qquad \text{eigenvalue (2)}$$

$$|\alpha_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \text{eigenvalue (1)}$$

$$|\alpha_3\rangle = \frac{1}{\sqrt{14}}\begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} \qquad \text{eigenvalue (-1)}$$

The eigenvector of $\hat{B}$ (degenerate case)

$$|\beta_1\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \qquad \text{eigenvalue (5)}$$

$$|\beta_2\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \qquad \text{eigenvalue (0)}$$

$$|\beta_3\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \text{eigenvalue (0)}$$

The unitary operator:

$$\hat{U} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \end{pmatrix}$$

$$\hat{U}^{+}\hat{U} = \hat{1}$$

$$\hat{U}^{+}\hat{B}\hat{U} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

---

15. Das p.31 (non-degenerate case)

Solve the eigenvalue problem for the matrix

$$\hat{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

((**Solution**))

$$\hat{A}|1\rangle = |1\rangle, \qquad \hat{A}|2\rangle = |3\rangle, \qquad \hat{A}|3\rangle = -|2\rangle$$

$|1\rangle$ is the eigenket of $\hat{A}$ with the eigenvalue 1.

The matrix of $\hat{A}$ under the basis {$|2\rangle$, $|3\rangle$ } can be written as

$$\hat{A}_{sub} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -i\hat{\sigma}_y$$

$$\hat{A}_{sub}|+y\rangle = -i\hat{\sigma}_y|+y\rangle = -i|+y\rangle$$

$$\hat{A}_{sub}|-y\rangle = -i\hat{\sigma}_y|-y\rangle = i|-y\rangle$$

Then we have

$$|\phi_1\rangle = |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$ with the eigenvalue 1

$$|\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$$ with the eigenvalue (-i)

$$|\phi_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}$$ with the eigenvalue (i).

---

16. Das p.32 (degenerate case)

Solve the eigenvalue problem for the matrix

$$\hat{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

((**Solution**))

$$\hat{A}|1\rangle = |1\rangle + |3\rangle, \qquad \hat{A}|2\rangle = 2|2\rangle, \qquad \hat{A}|3\rangle = |1\rangle + |3\rangle$$

$|2\rangle$ is the eigenket of $\hat{A}$ with the eigenvalue 2.

The matrix of $\hat{A}$ under the basis $\{|1\rangle, |3\rangle\}$ can be written as

$$\hat{A}_{sub} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \hat{1} + \hat{\sigma}_x$$

$$\hat{A}_{sub}|+x\rangle = (\hat{1} + \hat{\sigma}_x)|+x\rangle = 2|+x\rangle$$

$$\hat{A}_{sub}|-x\rangle = (\hat{1} + \hat{\sigma}_x)|-x\rangle = 0|-x\rangle$$

Then we have

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \text{with the eigenvalue 2}$$

$$|\phi_2\rangle = |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \text{with the eigenvalue 2}$$

$$|\phi_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \text{with the eigenvalue 0}$$

17. Gasiorowicz p.156 Chapter 9 Problem 9 (non-degenerate)

9. Consider the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Calculate the eigenvalues and the eigenvectors of this matrix.

((**Solution**))

$$\hat{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$|\phi_1\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \text{eigenvalue (4)}$$

$$|\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \qquad \text{eigenvalue (0)}$$

$$|\phi_3\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix} \qquad \text{eigenvalue (0)}$$

$$|\phi_4\rangle = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 \\ -3 \\ 1 \\ 1 \end{pmatrix} \qquad \text{eigenvalue (0)}$$

The unitary operator:

$$\hat{U} = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2} & 0 & 0 & -\frac{3}{2\sqrt{3}} \\ \frac{1}{2} & 0 & -\frac{2}{\sqrt{6}} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} \end{pmatrix}$$

$$\hat{U}^{+}\hat{A}\hat{U} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

---

18. 4 x 4 matrix (degenerate case)

Solve the eigenvalue problem for the matrix

$$\hat{A} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

((Mathematica))

```
Clear["Global`*"];
exp_ * :=
 exp /. {Complex[re_, im_] :> Complex[re, -im]};
```

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix};$$

```
HermitianMatrixQ[A]
```

True

```
eq1 = Eigensystem[A] // Simplify
```

$$\left\{\left\{\frac{1}{2}\left(1+\sqrt{17}\right), \frac{1}{2}\left(1-\sqrt{17}\right), -1, 0\right\}, \left\{\left\{\frac{1}{4}\left(1+\sqrt{17}\right), 1, \frac{1}{4}\left(1+\sqrt{17}\right), 1\right\}, \left\{\frac{1}{4}\left(1-\sqrt{17}\right), 1, \frac{1}{4}\left(1-\sqrt{17}\right), 1\right\}, \{-1, 0, 1, 0\}, \{0, -1, 0, 1\}\right\}\right\}$$

```
ϕ1 =  -Normalize[eq1[[2, 1]]] // Simplify;
ϕ2 = - Normalize[eq1[[2, 4]]] // Simplify;
ϕ3 = -Normalize[eq1[[2, 3]]] // Simplify;
ϕ4 =  Normalize[eq1[[2, 2]]] // Simplify;
```

```
{ϕ1*.ϕ2, ϕ2*.ϕ3, ϕ3*.ϕ4, ϕ4*.ϕ1 , ϕ4*.ϕ2} //
 Simplify
```

{0, 0, 0, 0, 0}

```
UT = {ϕ1, ϕ2, ϕ3, ϕ4 }; UH = UT*; U = Transpose[UT];
U // MatrixForm
```

$$\begin{pmatrix} \frac{-1-\sqrt{17}}{2\sqrt{17+\sqrt{17}}} & 0 & \frac{1}{\sqrt{2}} & \frac{1-\sqrt{17}}{2\sqrt{17-\sqrt{17}}} \\ -\frac{2}{\sqrt{17+\sqrt{17}}} & \frac{1}{\sqrt{2}} & 0 & \frac{2}{\sqrt{17-\sqrt{17}}} \\ \frac{-1-\sqrt{17}}{2\sqrt{17+\sqrt{17}}} & 0 & -\frac{1}{\sqrt{2}} & \frac{1-\sqrt{17}}{2\sqrt{17-\sqrt{17}}} \\ -\frac{2}{\sqrt{17+\sqrt{17}}} & -\frac{1}{\sqrt{2}} & 0 & \frac{2}{\sqrt{17-\sqrt{17}}} \end{pmatrix}$$

```
UH.U // Simplify
```

{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}

```
UH.A.U // Simplify // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2}\left(1+\sqrt{17}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\left(1-\sqrt{17}\right) \end{pmatrix}$$

19. 4 x 4 matrix (degenerate case)

Solve the eigenvalue problem for the matrix

$$\hat{A} = \begin{pmatrix} 0 & -i & 0 & i \\ i & 0 & -i & 0 \\ 0 & i & 0 & -i \\ -i & 0 & i & 0 \end{pmatrix}$$

Eigenkets and eigenvalues:

$$|\phi_1\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix}.$$ eigenvalue (2)

$$|\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$ eigenvalue (0)

$$|\phi_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$ eigenvalue (0)

$$|\phi_4\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix}$$ eigenvalue (-2)

((**Mathematica**))

```
Clear["Global`*"];
exp_* :=
 exp /. {Complex[re_, im_] ↦ Complex[re, -im]};
```

$$A = \begin{pmatrix} 0 & -i & 0 & i \\ i & 0 & -i & 0 \\ 0 & i & 0 & -i \\ -i & 0 & i & 0 \end{pmatrix};$$

```
HermitianMatrixQ[A]
```

```
True
```

```
eq1 = Eigensystem[A]
```

```
{{-2, 2, 0, 0}, {{-i, -1, i, 1},
  {i, -1, -i, 1}, {0, 1, 0, 1}, {1, 0, 1, 0}}}
```

```
ϕ1 = -i Normalize[eq1[[2, 2]]]
```

$$\left\{\frac{1}{2}, \frac{i}{2}, -\frac{1}{2}, -\frac{i}{2}\right\}$$

```
ϕ2 = Normalize[eq1[[2, 3]]]
```

$$\left\{0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}$$

```
ϕ3 = Normalize[eq1[[2, 4]]]
```

$$\left\{\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right\}$$

```
ϕ4 = ⅈ Normalize[eq1[[2, 1]]]
```

$$\left\{\frac{1}{2}, -\frac{i}{2}, -\frac{1}{2}, \frac{i}{2}\right\}$$

```
{ϕ1*.ϕ2, ϕ2*.ϕ3, ϕ3*.ϕ4, ϕ4*.ϕ1 , ϕ4*.ϕ2}
```

{0, 0, 0, 0, 0}

```
UT = {ϕ1, ϕ2, ϕ3, ϕ4 };

UH = UT*;

U = Transpose[UT]; U // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{\sqrt{2}} & 0 & -\frac{i}{2} \\ -\frac{1}{2} & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ -\frac{i}{2} & \frac{1}{\sqrt{2}} & 0 & \frac{i}{2} \end{pmatrix}$$

```
UH.U
```

{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}

```
UH.A.U // MatrixForm
```

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

REFERENCES

A. Das, Lectures on Quantum Mechanics, 2nd edition (World Scientific, 2012).

S. Gasiorowicz, Quantum Physics, 3rd edition (John Wiley & Sons, 2003).

A. Goswami, Quantum Mechanics, 2nd edition (Wm. C. Brown, 1997).

D.J. Griffiths, Introduction to Quantum Mechanics (Pearson, 2005).

D.H. McIntyre, Quantum Mechanics, A Paradigms Approach (Pearson, 2012).

Y. Peleg, R. Pnini, and E.Z. ZaarurSchaum's Outlines of Theory and Problems of Quantum Mechanics (McGraw-Hill, 1998).

M.S. Rogalski and S.B. Palmer, Quantum Physics (Gordon and Breach Science Publishers, 1999).

J.J. Sakurai and J. Napolitano, Modern Quantum Mechanics, 2nd edition (Addison-Wesley, 2011).

R. Shankar, Principles of Quantum Mechanics, 2nd edition (Plenum Press, 1994).

W.-H. Steeb and Y. Hardy, Problems and Solutions in Introductory and Advanced Matrix Calculus (World Scientific, 2006).

N. Zettili, Quantum Mechanics, Concepts and Applications, 2nd edition (Wiley, 2009).