Here we discuss the bound states in a three-dimensional square well potential. The potential depends only on the radial coordinates.

1. Schrödinger equation for the finite spherical well

The Hamiltonian for the finite spherical well is given by

\[
H = \frac{p_r^2}{2\mu} + \frac{\hbar^2 l(l+1)}{2\mu r^2} + V(r),
\]

with the radial momentum

\[
p_r = \frac{i}{r} \frac{\partial}{\partial r} r.
\]

\(V(r)\) is the potential energy for the spherical well

\[
V(r) = -V_0 \quad \text{for } r < a, \quad V(r) = 0 \quad \text{for } r > a.
\]

Then the Schrödinger equation can be written as
\[-\frac{\hbar^2}{2\mu r} \frac{1}{r} \frac{\partial}{\partial r} \left[ r \psi(r) \right] + \frac{\hbar^2(l+1)}{2\mu r^2} \psi(r) + V(r) \psi(r) = E \psi(r)\]

where

\[E < 0\]

and

\[p_r^2 \psi(r) = \frac{\hbar}{i} \frac{1}{r} \frac{\partial}{\partial r} \left[ \frac{\hbar}{i} \frac{1}{r} \frac{\partial}{\partial r} \psi(r) \right] = -\frac{\hbar^2}{r} \frac{1}{r} \frac{\partial}{\partial r} \left[ r \psi(r) \right]\]

(i) For \(r < a\),

\[-\frac{\hbar^2}{2\mu r} \frac{1}{r} \frac{\partial}{\partial r} \left[ r \psi(r) \right] + \frac{\hbar^2(l+1)}{2\mu r^2} \psi(r) - V_o \psi(r) = E \psi(r)\]

or

\[-\frac{1}{r} \frac{\partial}{\partial r} \left[ r \psi(r) \right] + \frac{l(l+1)}{r^2} \psi(r) - \frac{2\mu}{\hbar^2} (E + V_o) \psi(r) = 0\]

We put

\[r \psi(r) = u(r)\]

or

\[\frac{d^2}{dr^2} u(r) - \frac{l(l+1)}{r^2} u(r) + \frac{2\mu}{\hbar^2} (E + V_o) u(r) = 0\]  \(\text{for } r < a\).

(ii) \(r > a\),

\[\frac{d^2}{dr^2} u(r) - \frac{l(l+1)}{r^2} u(r) + \frac{2\mu}{\hbar^2} E u(r) = 0\]  \(\text{for } r > a\).

2. \(l = 0\) case

We consider the case when \(l = 0\), for which there is no centrifugal barrier.

\[\frac{d^2}{dr^2} u(r) = -\frac{2\mu}{\hbar^2} (E + V_o) u(r) = -k_o^2 u(r)\]  \((r < a)\)
\[ \frac{d^2}{dr^2} u(r) = -\frac{2\mu}{\hbar^2} E u(r) = q^2 u(r) \quad (r>a) \]

where

\[ E + V_0 = \frac{\hbar^2 k_0^2}{2\mu}, \quad E = -\frac{\hbar^2 q^2}{2\mu}. \]

This leads to the condition that

\[ (k_0a)^2 + (qa)^2 = \frac{2\mu}{\hbar^2} V_0a^2 \]  \( (1) \)

The solution of \( u(r) \) is obtained as

\[ u(r) = A \sin(k_0r) \quad (r<a) \]
\[ u(r) = C \exp(-qr) \quad (r>a) \]

The continuity of \( u(r) \) and \( u'(r) \) at \( r = a \) leads to

\[ A \sin(k_0a) = Ce^{-qa} \]
\[ Ak_0 \cos(k_0a) = C(-q)e^{-qa} \]

From these two equations, we have

\[ qa = -k_0a \cot(k_0a) \]  \( (2) \)

We assume that

\[ qa = y, \quad k_0a = x \]

Then we have

\[ y = -x \cot(x), \]  \( (3) \)
\[ x^2 + y^2 = \frac{2\mu}{\hbar^2} V_0a^2 = r_0^2 \]  \( (4) \)

Figure shows a plot of Eqs.(3) and (4) in the \( x-y \) plane. There are no bound states for

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\[
\frac{2\mu V_0 a^2}{\hbar^2} < \left( \frac{\pi}{2} \right)^2 .
\]

![Graph showing the region of interest with labeled curves](image)

**Fig.** \( l = 0 \). A plot of \( y = -x \cot(x) \) and \( x^2 + y^2 = \frac{2\mu V_0 a^2}{\hbar^2} = r^2 \). \( x = k_0 a \). \( y = qa \). The intersection of two curves leads to the solution (graphically solved). \( E = -\frac{\hbar^2 q^2}{2\mu} \).

The curve \[ y = -x \cot(x) \] crosses the \( Y=0 \) line at \( X = \pi/2, 3\pi/2, 5\pi/2 \).

There is no bound state for
\[
\frac{2\mu V_0 a^2}{\hbar^2} < \left( \frac{\pi}{2} \right)^2.
\]

There is a single bound state for
\[
\left( \frac{\pi}{2} \right)^2 < \frac{2\mu V_0 a^2}{\hbar^2} < \left( \frac{3\pi}{2} \right)^2.
\]

There are two bound states for
\[
\left( \frac{3\pi}{2} \right)^2 < \frac{2\mu V_0 a^2}{\hbar^2} < \left( \frac{5\pi}{2} \right)^2.
\]

3. **Solution for \( r<a \) with finite \( l \)**

We solve the differential equation for \( r<a \)
\[
\frac{d^2}{dr^2} u(r) - \frac{l(l+1)}{r^2} u(r) + \frac{2\mu}{\hbar^2} (E + V_0) u(r) = 0 \quad \text{(for } r < a). \]

or

\[
\frac{d^2}{dr^2} u(r) + \left[ k^2 - \frac{l(l+1)}{r^2} \right] u(r) = 0
\]

where the wave number \( k \) is defined as

\[
E + V_0 = \frac{\hbar^2}{2\mu} k^2.
\]

Now we introduce a dimensionless variable \( \rho \),

\[
\rho = kr
\]

Then we have

\[
\left[ \frac{d^2}{d\rho^2} - \frac{l(l+1)}{\rho^2} + 1 \right] u_{k,i}(\rho) = 0
\]

The solution of this differential equation is obtained as

\[
u_{k,i}(\rho) = A_{ij} j_{l+\frac{1}{2}}(\rho) + A_{in} n_{l+\frac{1}{2}}(\rho)
\]

where the spherical Bessel function and spherical Neumann function are defined by

\[
j_{l}(\rho) = \sqrt{\frac{\pi}{2\rho}} J_{l+\frac{1}{2}}(\rho),
\]

\[
n_{l}(\rho) = \sqrt{\frac{\pi}{2\rho}} N_{l+\frac{1}{2}}(\rho).
\]

Note that \( n_{l}(\rho) \) becomes infinity in the limit of \( \rho \to 0 \). So we choose the first term

\[
R_{k,i}(r) = A_{ij} j_{l}(kr).
\]

where \( rR_{k,i}(r) = u_{k,i}(r) \) and \( a_i \) is constant.

(((Mathematica)))
Clear["Global`*"];

eq1 = y''[x] + \left(1 - \frac{L (L + 1)}{x^2}\right)y[x] = 0;

DSolve[eq1, y[x], x]

\[\{\{y[x] \rightarrow \sqrt{x} \text{BesselJ}\left[\frac{1}{2} (1 + 2 L), x\right] C[1] + \sqrt{x} \text{BesselY}\left[\frac{1}{2} (1 + 2 L), x\right] C[2]\}\}\]

(Note)
We note that

\[u(\rho) \propto \sqrt{\rho} J_{\frac{1}{2}}(\rho) = \rho \frac{J_{\frac{1}{2}}(\rho)}{\sqrt{\rho}} = \sqrt{\frac{2}{\pi}} \rho_1(\rho)\]

Thus we have

\[R(r) = \frac{u(r)}{r} \propto \frac{u(\rho)}{\rho} = j_1(\rho).\]

3. \textbf{Solution for }r>a \textbf{ with finite }l
We solve the differential equation for \(r>a\)

\[\frac{d^2}{dr^2} u(r) - \frac{l(l+1)}{r^2} u(r) + \frac{2\mu}{\hbar^2} Eu(r) = 0 \quad \text{(for }r>a).\]

where

\[E = -\frac{\hbar^2}{2\mu} \kappa^2\]

where \(\kappa\) is the wavenumber.

\[\frac{d^2}{dr^2} u(r) + [(i\kappa)^2 - \frac{l(l+1)}{r^2}] u(r) = 0\]

Now we introduce a dimensionless variable \(\rho\),

\[\rho = ikr\]
Then we get

\[
\frac{d^2}{d\rho^2}u_{k,l}(\rho) + [1 - \frac{l(l+1)}{\rho^2}]u_{k,l}(\rho) = 0
\]

The solution of this differential equation is obtained as

\[
R_{k,l}(r) = B_1 \cdot j_l(i\kappa r) + B_2 \cdot n_l(i\kappa r)
\]

\[
= B_1 h^{(1)}_l(i\kappa r) + B_2 h^{(2)}_l(i\kappa r)
\]

where \(h^{(1)}_l(x)\) and \(h^{(2)}_l(x)\) are the spherical Hankel function of the first and second kind.

\[
h^{(1)}_l(x) = \sqrt{\frac{\pi}{2x}} H^{(1)}_{l+\frac{1}{2}}(x) = j_l(x) + in_l(x)
\]

\[
h^{(2)}_l(x) = \sqrt{\frac{\pi}{2x}} H^{(2)}_{l+\frac{1}{2}}(x) = j_l(x) - in_l(x)
\]

The asymptotic forms of \(h^{(1)}_n(x)\) and \(h^{(2)}_n(x)\) are given by

\[
h^{(1)}_l(x) \approx -i \frac{e^{i(x-l\pi/2)}}{x},
\]

\[
h^{(2)}_l(x) \approx i \frac{e^{-i(x-l\pi/2)}}{x},
\]

in the limit of large \(x\). Then we get

\[
h^{(1)}_l(i\kappa r) \approx -i \frac{e^{i(x\kappa-l\pi/2)}}{i\kappa r} = -\frac{e^{i(x\kappa-l\pi/2)}}{\kappa r}
\]

\[
h^{(2)}_l(i\kappa r) \approx i \frac{e^{-i(x\kappa-l\pi/2)}}{i\kappa r} = \frac{e^{i(x\kappa+l\pi/2)}}{\kappa r}
\]

which means that \(h^{(2)}_l(i\kappa r)\) becomes diverging for large \(r\), while \(h^{(1)}_l(i\kappa r)\) becomes zero for large \(r\). So we choose \(h^{(1)}_l(i\kappa r)\) as the solution of \(R_{k,l}(r)\) for \(r>a\),

\[
R_{k,l}(r) = B_1 h^{(1)}_l(i\kappa r).
\]
where \(B_1\) is constant.

**4. Bound states with finite \(l\)**

Using the boundary condition at \(r = a\), we determine the energy eigenvalues. We note that the wave function and its derivative should be continuous at \(r = a\).

\[ A_j j_l(ka) = B_l h^{(1)}_l(i\kappa a), \]

\[ A_k j_l(ka) = B_k i\kappa h^{(1)}_l(i\kappa a) \]

or

\[
ka \left( \frac{1}{j_l(x)} \frac{\partial j_l(x)}{\partial x} \right)_{x=ka} = i\kappa a \left( \frac{1}{h^{(1)}_l(x)} \frac{\partial h^{(1)}_l(x)}{\partial x} \right)_{x=i\kappa a}
\]

where

\[ \xi = ka, \quad \eta = \kappa a \]

((Note))

The following equations for the condition of the continuity of the wave function and its derivative with respect to \(r\) at \(r = a\) are equivalent.

\[
(i\kappa a) \left( \frac{1}{h^{(1)}_l(x)} \frac{\partial h^{(1)}_l(x)}{\partial x} \right)_{x=i\kappa a} = \frac{x}{h^{(1)}_l(x)} \frac{\partial h^{(1)}_l(x)}{\partial x} \bigg|_{x=i\kappa a} = \frac{y}{h^{(1)}_l(iy)} \frac{\partial h^{(1)}_l(iy)}{\partial y} \bigg|_{y=i\kappa a}
\]

From the conditions of

\[ E = -\frac{\hbar^2}{2\mu} k^2, \quad \text{and} \quad E + V_0 = \frac{\hbar^2}{2\mu} k^2, \]

we have

\[ (ka)^2 + (\kappa a)^2 = \frac{2\mu V_0}{\hbar^2} a^2 = r_0^2, \]

or
We solve the problem using the graphs. These graphs can be drawn in the \((\xi, \eta)\) plane by using the Mathematica (ContourPlot), where the radius \(r_0\) is changed as a parameter.

5. **A finite spherical well with \(l = 1\)**

\[
\xi^2 + \eta^2 = \frac{2\mu V_0}{\hbar^2} a^2 = r_0^2.
\] (2)

**Fig.** \(l = 1\). Curve -1 denoted by Eq.(1), which cross the \(\eta = 0\) line at \(\xi = \pi, 2\pi,\) and \(3\pi\).
The curve-2 denoted by Eq.(2) (circle with radius \(r_0\)), where \(r_0\) is changed as a parameter.

There is no bound state for \(\frac{2\mu V_0 a^2}{\hbar^2} < \pi^2\).
There is a single bound state for \((\pi)^2 < \frac{2\mu V_0 a^2}{\hbar^2} < (2\pi)^2\).

There are two bound states for \((2\pi)^2 < \frac{2\mu V_0 a^2}{\hbar^2} < (3\pi)^2\).

There are three bound states for \((3\pi)^2 < \frac{2\mu V_0 a^2}{\hbar^2} < (4\pi)^2\).

((Schiff))

For \(l = 1\), Eq. (1) can be expressed as

\[
\cot \xi - \frac{1}{\xi^2} = \frac{1}{\eta} + \frac{1}{\eta^2}
\]

or

\[
\eta^2 \xi \cos \xi = [\xi^2 (1 + \eta) + \eta^2] \sin \xi
\]

(1’)

together with

\[
\xi^2 + \eta^2 = \frac{2\mu V_0 a^2}{\hbar^2} = r_0^2
\]

(2)

where

\[
\xi = ka , \quad \eta = \kappa a
\]

These two equations can be solved graphically by using the ContourPlot of the Mathematica. See the detail the APPENDIX.

((a part of Mathematica))
Clear["Global`*"]; L = 1;

f1[x_] := \(\sin[x]\) - \(\cos[x]\) / \(x^2\);

f2[x_] := \(i \frac{1}{x} \frac{1}{x^2}\) Exp[-x];

\[eq1 = \frac{x}{f1[x]} D[f1[x], x] /. x \to X // Simplify;\]

\[eq2 = \frac{x}{f2[x]} D[f2[x], x] /. x \to Y // Simplify;\]

\[h1 = eq1 - eq2 // Simplify;\]

\[h1\]

\[
\frac{X Y^2 \cos[X] - \left(Y^2 + X^2 (1 + Y)\right) \sin[X]}{(1 + Y) (X \cos[X] - \sin[X])}
\]

6. A finite spherical well with \(l = 2\)
Fig. \( l = 2 \). Curve -1 denoted by Eq.(1), which cross the \( \eta = 0 \) line at \( \xi = 3\pi/2, 5\pi/2, \) and \( 7\pi/2 \) The curve-2 denoted by Eq.(2) (circle with radius \( r_0 \)), where \( r_0 \) is changed as a parameter.

There is no bound state for

\[
\frac{2\mu V_0 a^2}{\hbar^2} < \left( \frac{3\pi}{2} \right)^2.
\]

There is a single bound state for

\[
\frac{2\mu V_0 a^2}{\hbar^2} < \left( \frac{3\pi}{2} \right)^2 < \left( \frac{5\pi}{2} \right)^2.
\]

There are two bound states for

\[
\left( \frac{5\pi}{2} \right)^2 < \frac{2\mu V_0 a^2}{\hbar^2} < \left( \frac{7\pi}{2} \right)^2.
\]

7. **Conclusion**

The minimum value of \( V_0 a^2 \) for the \( p \)-wave binding \((l = 1)\) is larger than that for the \( s \)-wave binding \((l = 0)\), and so on.
\[
\frac{2\mu V_0 a^2}{\hbar^2} \geq \left( \frac{\pi}{2} \right)^2 \quad (l = 0)
\]
\[
\frac{2\mu V_0 a^2}{\hbar^2} \geq \left( \pi \right)^2 \quad (l = 1)
\]
\[
\frac{2\mu V_0 a^2}{\hbar^2} \geq \left( \frac{3\pi}{2} \right)^2 \quad (l = 2).
\]

Physically, the meaning of this is as follows. In the case of \(l = 2\) and \(l = 1\), there exists a centrifugal barrier and, therefore, a particle requires stronger attraction for the binding.

REFERENCES
F.S. Levin, An Introduction to Quantum Theory (Cambridge University Press, 2002).

APPENDIX-I
(a) Spherical Hankel function of the first kind.

\[ h_0^{(1)}(x) = -ie^{ix} \frac{1}{x}, \]

\[ h_1^{(1)}(x) = -e^{ix} \left( \frac{x + i}{x^2} \right), \]

\[ h_2^{(1)}(x) = ie^{ix} \left( \frac{x^2 + 3ix - 3}{x^3} \right), \]

\[ h_3^{(1)}(x) = e^{ix} \left( \frac{x^3 + 6ix^2 - 15x - 15i}{x^4} \right). \]

(b) Spherical Bessel function

\[ j_0(x) = \frac{\sin x}{x}, \]

\[ j_1(x) = \frac{\sin x - x \cos x}{x^2}, \]
\[ j_2(x) = \frac{(3 - x^2) \sin x - 3x \cos x}{x^3}, \]

\[ j_3(x) = \frac{3(5 - 2x^2) \sin x + x(-15 + x^2) \cos x}{x^4}. \]

Rayleigh formula:

\[ j_l(x) = (-1)^l x^l \left( \frac{1}{x} \frac{d}{dx} \right)^l \frac{\sin x}{x} \]

\[ h_l^{(1)}(x) = -i(-1)^l x^l \left( \frac{1}{x} \frac{d}{dx} \right)^l e^{ix} \]

**APPENDIX II**

Finite spherical well with \( l = 1 \)

Mathematica program
Clear["Global`*"]; L = 1;
f1[x_] := SphericalBesselJ[L, x];
f2[x_] := SphericalHankelH1[L, x];

\[eq1 = \frac{x}{f1[x]} D[f1[x], x] /. x \to X // Simplify;\]
\[eq2 = \frac{x}{f2[x]} D[f2[x], x] /. x \to Y // Simplify;\]
h1 = eq1 - eq2;

g1 = ContourPlot[Evaluate[h1 == 0], {X, 0, 10}, {Y, 0, 10}, ContourStyle \to \{Red, Thick\}, PlotPoints \to 40] // Simplify;

g2 = ContourPlot[
  Evaluate[Table[X^2 + Y^2 == a^2, {a, 3, 10, 1}]],
  {X, 0, 10}, {Y, 0, 10},
  ContourStyle \to Table[{Hue[0.1 i], Thick},
  {i, 0, 10}]];
g3 =
Graphics[{
  Black, Thin,
  Line[{{0, 0}, {10, 0}}],
  Line[{{0, 0}, {0, 10}}],
  Text[Style["\(\xi\)", Black, 15], {9.5, 0.2}],
  Text[Style["\(\eta\)", Black, 15], {0.2, 9.5}],
  Text[Style["l = " <> ToString[L], Black, 15], {2.5, 7.5}]]
]
Show[g1, g2, g3]