The Foldy-Wouthuysen Transformation

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In the non-relativistic limit, a Dirac particle (one with spin 1/2) is well described by the Pauli equation with a two component wave function. In the non-relativistic limit, the particles and antiparticles are separate. In order to understand how to find a representation in which we can eliminate two components of the wavefunction it is necessary to understand why the Dirac equation requires a four-componnet wave function in the first place. The reason is in the α-matrices. They are odd matrices. Thus the term $c\boldsymbol{\alpha}\cdot\boldsymbol{p}$ in the Hamiltonian connects particle and antiparticle parts of the wavefunction. Foldy and Wouthuysen (1950) have shown that the transformation is possible in which the particles and antiparticles are separated for any value of the momentum.

1. Foldy–Wouthuysen transform (I)

The Foldy–Wouthuysen transform is widely used in high energy physics. It was historically formulated by Leslie Lawrance Foldy and Siegfried Adolf Wouthuysen in 1949 to understand the nonrelativistic limit of the Dirac equation, the equation for spin-1/2 particles. A detailed general discussion of the Foldy–Wouthuysen-type transformations in particle interpretation of relativistic wave equations is in Acharya and Sudarshan (1960).

$$\left|\psi'\right\rangle = \hat{U}\left|\psi\right\rangle, \qquad \left|\psi\right\rangle = \hat{U}^{+}\left|\psi'\right\rangle$$

The Hamiltonian:

$$\hat{H} = c\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc^2$$

$$\left\langle\psi'\right|\hat{H}'\left|\psi'\right\rangle = \left\langle\psi\right|\hat{H}\left|\psi\right\rangle$$

or

$$\left\langle\psi'\right|\hat{H}'\left|\psi'\right\rangle = \left\langle\psi'\right|\hat{U}\hat{H}\hat{U}^{+}\left|\psi'\right\rangle$$

leading to

$$\hat{H}' = \hat{U}\hat{H}\hat{U}^{+}$$

The unitary operator:

$$\hat{U} = \exp(i\hat{S})$$

with

$$i\hat{S} = \frac{1}{2mc}\beta(\alpha \cdot \boldsymbol{p})\theta$$

Thus we have

$$\begin{aligned}\hat{U} &= \exp(i\hat{S}) \\ &= \exp[\frac{1}{2mc}\beta(\boldsymbol{\alpha} \cdot \boldsymbol{p})\theta] \\ &= \cos[\frac{|\boldsymbol{p}|\theta}{2mc}]\hat{1} + \frac{\beta(\boldsymbol{\alpha} \cdot \boldsymbol{p})}{|\boldsymbol{p}|}\sin[\frac{|\boldsymbol{p}|\theta}{2mc}]\end{aligned}$$

where θ is a function of $\left(\frac{|\boldsymbol{p}|}{mc}\right)$.

$$\begin{aligned}\hat{U}^{+} &= \exp(-i\hat{S}) \\ &= \cos[\frac{|\boldsymbol{p}|\theta}{2mc}]\hat{1} + \frac{(\boldsymbol{\alpha}^{+} \cdot \boldsymbol{p})\beta^{+}}{|\boldsymbol{p}|}\sin[\frac{|\boldsymbol{p}|\theta}{2mc}] \\ &= \cos[\frac{|\boldsymbol{p}|\theta}{2mc}]\hat{1} + \frac{(\boldsymbol{\alpha} \cdot \boldsymbol{p})\beta}{|\boldsymbol{p}|}\sin[\frac{|\boldsymbol{p}|\theta}{2mc}] \\ &= \cos[\frac{|\boldsymbol{p}|\theta}{2mc}]\hat{1} - \frac{\beta(\boldsymbol{\alpha} \cdot \boldsymbol{p})}{|\boldsymbol{p}|}\sin[\frac{|\boldsymbol{p}|\theta}{2mc}]\end{aligned}$$

$$\hat{U}\hat{U}^{+} = \hat{U}^{+}\hat{U} = \hat{1}$$

((**Mathematica**))

```
Clear["Global`*"];
σx = PauliMatrix[1];
σy = PauliMatrix[2];
σz = PauliMatrix[3];
I2 = IdentityMatrix[2];
I4 = IdentityMatrix[4];
αx = KroneckerProduct[σx, σx];
αy = KroneckerProduct[σx, σy];
αz = KroneckerProduct[σx, σz];
β = KroneckerProduct[σz, I2];
rule1 = {px → p Sin[θ1] Cos[ϕ1], py → p Sin[θ1] Sin[ϕ1],
   pz → p Cos[θ1]};
h1 = 1/(2 m c) ((β.αx) px θ + (β.αy) py θ + (β.αz) pz θ) // Simplify;
h11 = h1 /. rule1 // Simplify;
h12 = MatrixExp[h11] // Simplify;
k11 =
  Cos[ρ] I4 + ((β.αx) px Sin[ρ] + (β.αy) py Sin[ρ] + (β.αz) pz Sin[ρ])/p;
k12 = k11 /. ρ → (p θ)/(2 c m) /. rule1 // Simplify;
h12 - k12 // Simplify
{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}
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The Hamiltonian $\hat{H}$

$$H = c(\boldsymbol{\alpha} \cdot \boldsymbol{p}) + \beta mc^2$$

$$\hat{U}\hat{H}\hat{U}^+ = \begin{pmatrix} f & 0 & g\cos\theta_1 & e^{-i\phi_1} g\sin\theta_1 \\ 0 & f & e^{i\phi_1} g\sin\theta_1 & -g\cos\theta_1 \\ g\cos\theta_1 & e^{-i\phi_1} g\sin\theta_1 & -f & 0 \\ e^{i\phi_1} g\sin\theta_1 & -g\cos\theta_1 & 0 & -f \end{pmatrix}$$

where

$$\boldsymbol{p} = p(\sin\theta_1 \cos\phi_1, \sin\theta_1 \sin\phi_1, \cos\theta_1)$$

$$f = c[mc\cos(\frac{p\theta}{mc}) + p\sin(\frac{p\theta}{mc})], \qquad g = c[p\cos(\frac{p\theta}{mc}) - mc\sin(\frac{p\theta}{mc})]$$

We choose θ such that

$$g = 0$$

So we get

$$\tan\alpha = \tan(\frac{p\theta}{mc}) = \frac{cp}{mc^2}$$

$$\cos\alpha = \frac{mc^2}{E_R}, \qquad \sin\alpha = \frac{cp}{E_R}$$

Using this value of $\alpha = \frac{p\theta}{mc}$, we get the value of f as

$$\begin{aligned} f &= c[mc\cos(\alpha) + p\sin(\alpha)] \\ &= mc^2 \frac{mc^2}{E_R} + cp\frac{cp}{E_R} \\ &= \frac{m^2c^4 + c^2p^2}{E_R} \\ &= E_R \end{aligned}$$

Finally we get

$$\hat{U}\hat{H}\hat{U}^{+} = \begin{pmatrix} E_R & 0 & 0 & 0 \\ 0 & E_R & 0 & 0 \\ 0 & 0 & -E_R & 0 \\ 0 & 0 & 0 & -E_R \end{pmatrix} = E_R \beta$$

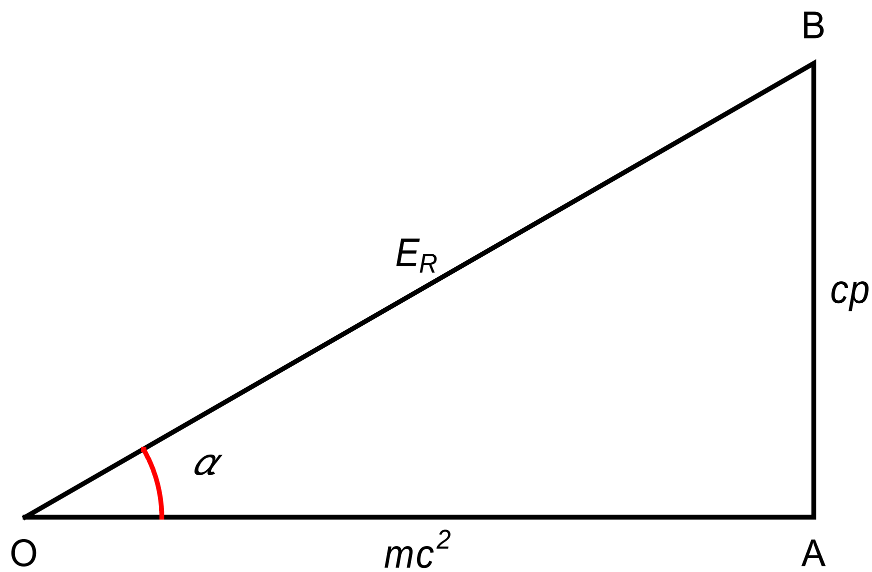


Fig. Graphic representation of the Foldy-Wouthuysen transformation. $\alpha = \dfrac{p\theta}{mc}$.

$E_R = c\sqrt{\boldsymbol{p}^2 + m^2c^2}$. $\tan\alpha = \dfrac{cp}{mc^2}$. $|\boldsymbol{p}| = p$.

((Mathematica))

```
Clear["Global`*"]; σx = PauliMatrix[1]; σy = PauliMatrix[2];
σz = PauliMatrix[3]; I2 = IdentityMatrix[2]; I4 = IdentityMatrix[4];
αx = KroneckerProduct[σx, σx]; αy = KroneckerProduct[σx, σy];
αz = KroneckerProduct[σx, σz]; β = KroneckerProduct[σz, I2];
rule1 = {px → p Sin[θ1] Cos[ϕ1], py → p Sin[θ1] Sin[ϕ1], pz → p Cos[θ1]};

h1 = 1/(2 m c) ((β.αx) px θ + (β.αy) py θ + (β.αz) pz θ) // Simplify;

h11 = h1 /. rule1 // Simplify; h12 = MatrixExp[h11] // Simplify;
k11 = Cos[ρ] I4 + ((β.αx) px Sin[ρ] + (β.αy) py Sin[ρ] + (β.αz) pz Sin[ρ])/p;
k12 = k11 /. ρ → (p θ)/(2 c m) /. rule1 // Simplify;
h12 - k12 // Simplify

{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

αxH = ConjugateTranspose[αx]; αyH = ConjugateTranspose[αy];
αzH = ConjugateTranspose[αz];
βH = ConjugateTranspose[β];

h1 = 1/(2 m c) ((β.αx) px θ + (β.αy) py θ + (β.αz) pz θ) // Simplify;

k11H = Cos[ρ] I4 + ((αxH.βH) px Sin[ρ] + (αyH.βH) py Sin[ρ] + (αzH.βH) pz Sin[ρ])/p;
k12H = k11H /. ρ → (p θ)/(2 c m) /. rule1 // Simplify;
```

```
k12.k12H // Simplify
```

{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}

```
H = c (αx px + αy py + αz pz) + β m c^2; H1 = H /. rule1 // Simplify;

s1 = k12.H1.k12H /. rule1 // FullSimplify;

s11 =
  s1 //. {(p Cos[p θ/(c m)] - c m Sin[p θ/(c m)]) → g, (-p Cos[p θ/(c m)] + c m Sin[p θ/(c m)]) → -g,
     (p Sin[p θ/(c m)] + c m Cos[p θ/(c m)]) → f} // Simplify;

s11 // MatrixForm
```

$$\begin{pmatrix} c\,f & 0 & c\,g\,\text{Cos}[\theta 1] & c\,e^{-i\,\phi 1}\,g\,\text{Sin}[\theta 1] \\ 0 & c\,f & c\,e^{i\,\phi 1}\,g\,\text{Sin}[\theta 1] & -c\,g\,\text{Cos}[\theta 1] \\ c\,g\,\text{Cos}[\theta 1] & c\,e^{-i\,\phi 1}\,g\,\text{Sin}[\theta 1] & -c\,f & 0 \\ c\,e^{i\,\phi 1}\,g\,\text{Sin}[\theta 1] & -c\,g\,\text{Cos}[\theta 1] & 0 & -c\,f \end{pmatrix}$$

2. Derivation of FW transformation without Mathematica

(a) Formula-1

$$\hat{U} = \exp[\frac{1}{2mc}\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\theta] = \cos[\frac{|\boldsymbol{p}|\theta}{2mc}]\hat{1} + \frac{\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})}{|\boldsymbol{p}|}\sin[\frac{|\boldsymbol{p}|\theta}{2mc}]$$

((**Proof**))

$$\hat{U} = \exp[\frac{1}{2mc}\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\theta]$$
$$= 1 + \frac{1}{2mc}\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\theta + \frac{1}{2!}\frac{1}{(2mc)^2}\{\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\}^2\theta^2 + \frac{1}{3!}\frac{1}{(2mc)^3}\{\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\}^3\theta^3 + ..$$

Using the formula

$$\{\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\}^2 = -|\boldsymbol{p}|^2$$

$$(\boldsymbol{\alpha}\cdot\boldsymbol{p})^2 = |\boldsymbol{p}|^2$$

$$(\boldsymbol{\alpha}\cdot\boldsymbol{p})\{\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\} = -\{\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\}(\boldsymbol{\alpha}\cdot\boldsymbol{p})$$

$$\beta\{\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\} = -\{\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\}\beta$$

with

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij} I_4, \qquad \{\alpha_i, \beta\} = 0, \qquad \beta^2 = I_4$$

where $i = x$, y, and z, and the curly bracket denotes an anti-commutator, we get

$$\begin{aligned}
\hat{U} &= \exp[\frac{1}{2mc}\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\theta] \\
&= [1 - \frac{1}{2!}\frac{|\boldsymbol{p}|^2\theta^2}{(2mc)^2} + \frac{1}{4!}\frac{|\boldsymbol{p}|^4\theta^4}{(2mc)^4} + ... \\
&\quad + \frac{\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})}{|\boldsymbol{p}|}[\frac{|\boldsymbol{p}|\theta}{2mc} - \frac{1}{3!}\frac{(|\boldsymbol{p}|\theta)^3}{(2mc)^3} + ...] \\
&= \cos(\frac{|\boldsymbol{p}|\theta}{2mc}) + \frac{\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})}{|\boldsymbol{p}|}\sin(\frac{|\boldsymbol{p}|\theta}{2mc})
\end{aligned}$$

(b) Formula-II

$$[\hat{H}, \hat{S}]_+ = \hat{H}\hat{S} + \hat{S}\hat{H} = 0$$

((Proof))

$$\hat{S} = -\frac{i}{2mc}\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\theta, \qquad \hat{H} = c(\boldsymbol{\alpha}\cdot\boldsymbol{p}) + \beta mc^2$$

$$\begin{aligned}
&[c(\boldsymbol{\alpha}\cdot\boldsymbol{p}) + \beta mc^2]\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p}) + \beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})[c(\boldsymbol{\alpha}\cdot\boldsymbol{p}) + \beta mc^2] \\
&= c(\boldsymbol{\alpha}\cdot\boldsymbol{p})\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p}) + mc^2\beta^2(\boldsymbol{\alpha}\cdot\boldsymbol{p}) + c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})^2 + mc^2\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\beta \\
&= -c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})^2 + mc^2(\boldsymbol{\alpha}\cdot\boldsymbol{p}) + c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})^2 - mc^2(\boldsymbol{\alpha}\cdot\boldsymbol{p}) \\
&= 0
\end{aligned}$$

(c) Formula III

$$\hat{H}\exp(-i\hat{S}) = \exp(i\hat{S})\hat{H}$$

((Proof))

$$\begin{aligned}\hat{H}\exp(-i\hat{S}) &= \hat{H}[1+\frac{1}{1!}(-i\hat{S})+\frac{1}{2!}(-i\hat{S})^2+\frac{1}{3!}(-i\hat{S})^3+...] \\ &= \hat{H}+\frac{1}{1!}\hat{H}(-i\hat{S})+\frac{1}{2!}\hat{H}(-i\hat{S})^2+\frac{1}{3!}\hat{H}(-i\hat{S})^3+... \\ &= \hat{H}+\frac{1}{1!}(i\hat{S})\hat{H}+\frac{1}{2!}(i\hat{S})^2\hat{H}+\frac{1}{3!}(i\hat{S})^3\hat{H}+... \\ &= \hat{H}\exp(i\hat{S})\end{aligned}$$

(d) Formula IV

$$\hat{U}=\exp(i\hat{S})$$

$$\begin{aligned}\hat{H}' &= \hat{U}\hat{H}\hat{U}^+ \\ &= \exp(i\hat{S})\hat{H}\exp(-i\hat{S}) \\ &= \exp(i\hat{S})\exp(i\hat{S})\hat{H} \\ &= \exp(2i\hat{S})\hat{H}\end{aligned}$$

(e)

$$\begin{aligned}\exp(2i\hat{S})\hat{H} &= [\cos(\frac{|\boldsymbol{p}|\theta}{mc})\hat{1}+\frac{\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})}{|\boldsymbol{p}|}\sin(\frac{|\boldsymbol{p}|\theta}{mc})][c(\boldsymbol{\alpha}\cdot\boldsymbol{p})+\beta mc^2] \\ &= c[\cos(\frac{|\boldsymbol{p}|\theta}{mc})(\boldsymbol{\alpha}\cdot\boldsymbol{p})+\frac{\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})^2}{|\boldsymbol{p}|}\sin(\frac{|\boldsymbol{p}|\theta}{mc})] \\ &\quad +mc^2[\beta\cos(\frac{|\boldsymbol{p}|\theta}{mc})\hat{1}+\frac{\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\beta}{|\boldsymbol{p}|}\sin(\frac{|\boldsymbol{p}|\theta}{mc})] \\ &= c[(\boldsymbol{\alpha}\cdot\boldsymbol{p})\cos(\frac{|\boldsymbol{p}|\theta}{mc})+\beta|\boldsymbol{p}|\sin(\frac{|\boldsymbol{p}|\theta}{mc})] \\ &\quad +mc^2[\beta\cos(\frac{|\boldsymbol{p}|\theta}{mc})\hat{1}-\frac{(\boldsymbol{\alpha}\cdot\boldsymbol{p})}{|\boldsymbol{p}|}\sin(\frac{|\boldsymbol{p}|\theta}{mc})]\end{aligned}$$

or

$$\exp(2i\hat{S})\hat{H} = \beta[mc^2\cos(\frac{|\boldsymbol{p}|\theta}{mc})\hat{1} + c|\boldsymbol{p}|\sin(\frac{|\boldsymbol{p}|\theta}{mc})] + \frac{(\boldsymbol{\alpha}\cdot\boldsymbol{p})}{|\boldsymbol{p}|}[c|\boldsymbol{p}|\cos(\frac{|\boldsymbol{p}|\theta}{mc}) - mc^2\sin(\frac{|\boldsymbol{p}|\theta}{mc})]$$

where

$$\exp(2i\hat{S}) = \exp[\frac{1}{mc}\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\theta] = \cos(\frac{|\boldsymbol{p}|\theta}{mc})\hat{1} + \frac{\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})}{|\boldsymbol{p}|}\sin(\frac{|\boldsymbol{p}|\theta}{mc})$$

We want the term containing $\frac{(\boldsymbol{\alpha}\cdot\boldsymbol{p})}{|\boldsymbol{p}|}$ to vanish. The necessary condition is

$$\tan(\frac{|\boldsymbol{p}|\theta}{mc}) = \frac{c|\boldsymbol{p}|}{mc^2},$$

$$\cos(\frac{|\boldsymbol{p}|\theta}{mc}) = \frac{mc^2}{E_R}, \qquad \sin(\frac{|\boldsymbol{p}|\theta}{mc}) = \frac{c|\boldsymbol{p}|}{E_R}$$

So that

$$\begin{aligned}\hat{H}' &= \exp(2i\hat{S})\hat{H} \\ &= \beta[mc^2\cos(\frac{|\boldsymbol{p}|\theta}{mc})\hat{1} + c|\boldsymbol{p}|\sin(\frac{|\boldsymbol{p}|\theta}{mc})] \\ &= \frac{\beta}{E_R}(m^2c^4 + c^2|\boldsymbol{p}|^2) \\ &= \beta E_R\end{aligned}$$

4. Separation between particle and antiparticle components

$$H'\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = E_R\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad H'\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = E_R\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \qquad \text{(degenerate state)}$$

$$H'\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = -E_R\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \qquad H'\begin{pmatrix}0\\0\\0\\11\end{pmatrix} = -E_R\begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \quad \text{(degenerate state)}$$

There are 4 independent states.

$$|\psi\rangle = \hat{U}^+|\psi'\rangle = \exp(-i\hat{S})|\psi'\rangle$$

and

$$\begin{aligned}\hat{U}^+ &= \exp(-i\hat{S})\\ &= \cos[\frac{|\boldsymbol{p}|\theta}{2mc}]\hat{1} + \frac{(\boldsymbol{\alpha}\cdot\boldsymbol{p})\beta}{|\boldsymbol{p}|}\sin[\frac{|\boldsymbol{p}|\theta}{2mc}]\\ &= \frac{mc^2}{E_R}\hat{1} + \frac{1}{E_R}c(\boldsymbol{\alpha}\cdot\boldsymbol{p})\beta\\ &= \frac{1}{E_R}\hat{H}\beta\end{aligned}$$

$$\begin{aligned}\hat{U} &= \exp(i\hat{S})\\ &= \cos(\frac{|\boldsymbol{p}|\theta}{2mc})\hat{1} + \frac{\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})}{|\boldsymbol{p}|}\sin(\frac{|\boldsymbol{p}|\theta}{2mc})\\ &= \frac{mc^2}{E_R}\hat{1} + \frac{c}{E_R}\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\\ &= \frac{1}{E_R}[mc^2 + c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})]\\ &= \frac{1}{E_R}[mc^2\beta^2 + c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})]\\ &= \frac{1}{E_R}[mc^2\beta^2 + c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})]\\ &= \frac{1}{E_R}\beta\hat{H}\end{aligned}$$

$$\begin{aligned}\psi &= \hat{U}^+\psi'\\ &= \frac{1}{E_R}\hat{H}\beta\psi'\end{aligned}$$

$$\hat{U}\hat{U}^{+} = \frac{1}{E_R{}^2}(\beta\hat{H})(\hat{H}\beta) = \hat{1}$$

$$\beta\hat{H} = c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p}) + mc^2, \qquad \hat{H}\beta = c(\boldsymbol{\alpha}\cdot\boldsymbol{p})\beta + mc^2$$

$$\begin{aligned}\beta\hat{H}\hat{H}\beta &= [c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p}) + mc^2][c(\boldsymbol{\alpha}\cdot\boldsymbol{p})\beta + mc^2] \\ &= m^2c^4 + c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})c(\boldsymbol{\alpha}\cdot\boldsymbol{p})\beta + mc^3(\boldsymbol{\alpha}\cdot\boldsymbol{p})\beta + mc^3\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p}) \\ &= m^2c^4 + c^2\beta^2(\boldsymbol{\alpha}\cdot\boldsymbol{p})^2 \\ &= m^2c^4 + c^2|\boldsymbol{p}|^2 \\ &= E_R{}^2\end{aligned}$$

$$\psi = \hat{U}\psi', \qquad \psi' = \hat{U}^{+}\psi$$

$$\hat{H}'\psi' = \hat{U}\hat{H}\hat{U}^{+}\psi' = E_R\beta\psi'$$

We note that $\hat{H}'$ is the diagonal matrix.

$$\psi' = \psi'_{+} + \psi'_{-},$$

$$\frac{1}{2}(1+\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\frac{1}{2}(1-\beta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\psi' = \begin{pmatrix} \psi_1' \\ \psi_2' \\ \psi_3' \\ \psi_4 \end{pmatrix}$$

$$\psi_+' = \frac{1}{2}(1+\beta)\psi' = \begin{pmatrix} \psi_1' \\ \psi_2' \\ 0 \\ 0 \end{pmatrix}, \qquad \psi_-' = \frac{1}{2}(1-\beta)\psi' = \begin{pmatrix} 0 \\ 0 \\ \psi_3' \\ \psi_4' \end{pmatrix}$$

$$H'\psi_+' = E_R\psi_+', \qquad H'\psi_-' = -E_R\psi_-'$$

Note that ψ_+' (ψ_+') is now essentially a two-component wave function, since its lower (upper) components are identically zero.

$$\psi' = \begin{pmatrix} \psi_+' \\ \psi_-' \end{pmatrix}$$

Helicity:

$$\frac{(\boldsymbol{\alpha}\cdot\boldsymbol{p})}{|p|} = \begin{pmatrix} 0 & \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{|p|} \\ \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{|p|} & 0 \end{pmatrix}$$

$$\frac{(\boldsymbol{\alpha}\cdot\boldsymbol{p})}{|p|}\psi' = \begin{pmatrix} 0 & \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{|p|} \\ \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{|p|} & 0 \end{pmatrix} \begin{pmatrix} \psi_+' \\ \psi_-' \end{pmatrix} = \begin{pmatrix} \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{|\boldsymbol{p}|}\psi_-' \\ \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{|\boldsymbol{p}|}\psi_+' \end{pmatrix}$$

$$\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2}{|\boldsymbol{p}|^2} = 1$$

$$\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{|\boldsymbol{p}|}\psi_+' = \pm\psi_+'$$

The eigenket for the helicity $h = 1$

$$\begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} \end{pmatrix}$$

The eigenket for the helicity $h = -1$

$$\begin{pmatrix} -e^{-i\phi/2}\sin\frac{\theta}{2} \\ e^{i\phi/2}\cos\frac{\theta}{2} \end{pmatrix}$$

In summary

(a) $E = E_R$, $h = 1$

$$\begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} \\ 0 \\ 0 \end{pmatrix}$$

(b) $E = E_R$, $h = -1$

$$\begin{pmatrix} -e^{-i\phi/2}\sin\frac{\theta}{2} \\ e^{i\phi/2}\cos\frac{\theta}{2} \\ 0 \\ 0 \end{pmatrix}$$

(c) $E = -E_R$, $h = 1$

$$\begin{pmatrix} 0 \\ 0 \\ e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} \end{pmatrix}$$

(d) $E = -E_R, \qquad h = -1$

$$\begin{pmatrix} 0 \\ 0 \\ -e^{-i\phi/2}\sin\frac{\theta}{2} \\ e^{i\phi/2}\cos\frac{\theta}{2} \end{pmatrix}$$

5. Foldy–Wouthuysen transform (II)

Here we define $i\hat{S}$ as

$$i\hat{S} = \frac{1}{|\boldsymbol{p}|}\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\theta = \frac{1}{p}\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\theta$$

The unitary operator:

$$\begin{aligned} \hat{U} &= \exp(i\hat{S}) \\ &= \exp[\frac{1}{p}\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\theta] \\ &= \cos\theta\, I_4 + \frac{1}{p}\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\sin\theta \end{aligned}$$

$$\hat{U}^{+} = \cos\theta\, I_4 - \frac{1}{p}\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\sin\theta$$

$$\hat{U}\hat{H}\hat{U}^{+} = \begin{pmatrix} f & 0 & g\cos\theta_1 & e^{-i\phi_1}g\sin\theta_1 \\ 0 & f & e^{i\phi_1}g\sin\theta_1 & -g\cos\theta_1 \\ g\cos\theta_1 & e^{-i\phi_1}g\sin\theta_1 & -f & 0 \\ e^{i\phi_1}g\sin\theta_1 & -g\cos\theta_1 & 0 & -f \end{pmatrix}$$

where

$$\boldsymbol{p} = p(\sin\theta_1\cos\phi_1, \sin\theta_1\sin\phi_1, \cos\theta_1)$$

$$f = c[mc\cos(2\theta) + p\sin(2\theta)], \qquad g = c[p\cos(2\theta) - mc\sin(2\theta)]$$

We choose θ such that

$$g = 0$$

So we get

$$p\cos(2\theta) = mc\sin(2\theta)$$

or

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{p}{mc}$$

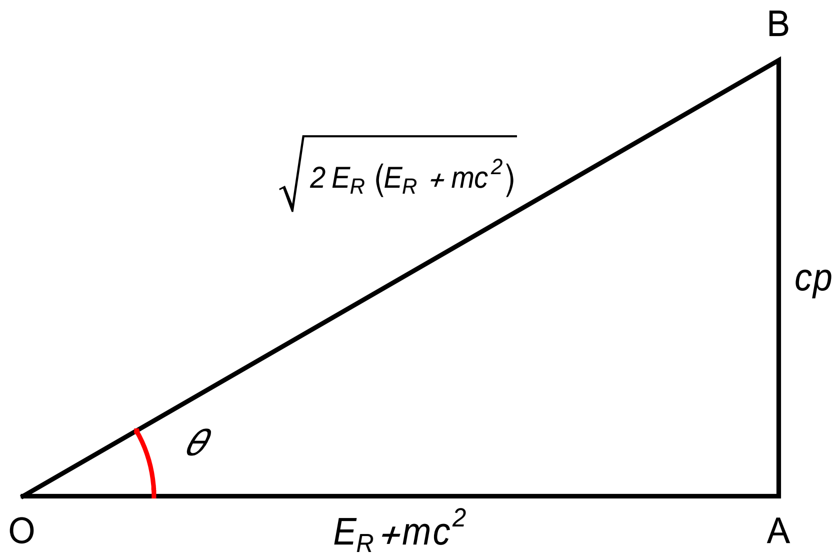
or

$$\tan^2\theta + \frac{2mc}{p}\tan\theta - 1 = 0$$

leading to

$$\tan\theta = \frac{cp}{c\sqrt{p^2+m^2c^2}+mc^2} = \frac{cp}{E_R+mc^2}(>0)$$

$$\cos\theta = \frac{E_R+mc^2}{\sqrt{2E_R(E_R+mc^2)}}, \qquad \sin\theta = \frac{cp}{\sqrt{2E_R(E_R+mc^2)}}$$



Then we have

$$\begin{aligned}\hat{U} &= \exp(i\hat{S}) \\ &= \cos\theta + \frac{1}{p}\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\sin\theta \\ &= \frac{E_R + mc^2 + c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})}{\sqrt{2E_R(E_R+mc^2)}} \quad , \\ &= \frac{E_R + \beta\hat{H}}{\sqrt{2E_R(E_R+mc^2)}}\end{aligned}$$

and

$$\hat{U}^+ = \frac{E_R + \hat{H}\beta}{\sqrt{2E_R(E_R+mc^2)}}$$

with

$$\tan\theta = \frac{cp}{E_R+mc^2}\,, \qquad\qquad E_R = \sqrt{c^2p^2+m^2c^4}$$

$$\sin\theta = \frac{cp}{\sqrt{c^2p^2 + (E_R + mc^2)^2}} = \frac{cp}{\sqrt{2E_R(E_R + mc^2)}}, \qquad \cos\theta = \frac{E_R + mc^2}{\sqrt{c^2p^2 + (E_R + mc^2)^2}} = \frac{E_R + mc^2}{\sqrt{2E_R(E_R + mc^2)}}$$

since

$$c^2p^2 + (E_R + mc^2)^2 = c^2p^2 + E_R{}^2 + m^2c^4 + 2mc^2E_R$$
$$= 2E_R{}^2 + 2mc^2E$$
$$= 2E_R(E_R + mc^2)$$

$$\hat{U}H\hat{U}^+ = \frac{E_R + mc^2 + c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})}{\sqrt{2E_R(E_R + mc^2)}}[c(\boldsymbol{\alpha}\cdot\boldsymbol{p}) + \beta mc^2]\frac{E_R + mc^2 - c\beta\cdot(\boldsymbol{\alpha}\cdot\boldsymbol{p})}{\sqrt{2E_R(E_R + mc^2)}}$$
$$= E_R\beta$$

$$f = c[mc\cos(2\theta) + p\sin(2\theta)]$$
$$= mc^2(\cos^2\theta - \sin^2\theta) + 2pc\sin\theta\cos\theta$$
$$= mc^2\frac{(E_R + mc^2)^2 - c^2p^2}{2E_R(E_R + mc^2)} + 2pc\frac{(E_R + mc^2)cp}{2E_R(E_R + mc^2)}$$
$$= E_R$$

$$\hat{U}\hat{H}\hat{U}^+ = E_R\beta$$

6. FW transformation of operators

We note that

$$\hat{U}\boldsymbol{p}\hat{U}^+ = \boldsymbol{p}$$

$$[\boldsymbol{r}, E_R] = i\hbar\frac{\partial}{\partial\boldsymbol{p}}E_R = i\hbar\frac{c^2\boldsymbol{p}}{E_R}$$

$$\hat{U}\boldsymbol{r}\hat{U}^+ = \frac{1}{\sqrt{2E_R(E_R + mc^2)}}\{E_R + mc^2 + c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\}\boldsymbol{r}\{E_R + mc^2 - c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\}\frac{1}{\sqrt{2E_R(E_R + mc^2)}}$$

Here

$$\boldsymbol{r}\{E_R + mc^2 - c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\} - \{E_R + mc^2 - c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\}\boldsymbol{r} = i\hbar\frac{\partial}{\partial\boldsymbol{p}}\{E_R + mc^2 - c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\}$$

$$= i\hbar(\frac{c^2 p}{E_R} - c\beta\boldsymbol{\alpha})$$

We note that

$$\hat{U}\boldsymbol{r}\hat{U}^+ = \frac{1}{\sqrt{2E_R(E_R + mc^2)}}\{E_R + mc^2 + c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\}\{[E_R + mc^2 - c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})]\boldsymbol{r} + i\hbar(\frac{c^2\boldsymbol{p}}{E_R} - c\beta\boldsymbol{\alpha})\}\frac{1}{\sqrt{2E_R(E_R + mc^2)}}$$

$$= \frac{1}{\sqrt{2E_R(E_R + mc^2)}}[\{(E_R + mc^2)^2 + c^2p^2\}\boldsymbol{r} + [E_R + mc^2 + c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})]i\hbar(\frac{c^2\boldsymbol{p}}{E_R} - c\beta\boldsymbol{\alpha})\}\frac{1}{\sqrt{2E_R(E_R + mc^2)}}$$

$$= \frac{1}{\sqrt{2E_R(E_R + mc^2)}}[2E_R(E_R + mc^2)\boldsymbol{r} + [E_R + mc^2 + c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})]i\hbar(\frac{c^2\boldsymbol{p}}{E_R} - c\beta\boldsymbol{\alpha})\}\frac{1}{\sqrt{2E_R(E_R + mc^2)}}$$

$$= \sqrt{2E_R(E_R + mc^2)}\boldsymbol{r}\frac{1}{\sqrt{2E_R(E_R + mc^2)}} - \frac{1}{2E_R(E_R + mc^2)}[E_R + mc^2 + c\beta\cdot(\boldsymbol{\alpha}\cdot\boldsymbol{p})]i\hbar(c\beta\boldsymbol{\alpha} - \frac{c^2\boldsymbol{p}}{E_R})$$

$$[\boldsymbol{r}, \sqrt{2E_R(E_R + mc^2)}] = \boldsymbol{r}\sqrt{2E_R(E_R + mc^2)} - \sqrt{2E_R(E_R + mc^2)}\boldsymbol{r}$$

$$= i\hbar\frac{\partial}{\partial\boldsymbol{p}}\sqrt{2E_R(E_R + mc^2)}$$

$$= i\hbar\frac{2E_R + mc^2}{\sqrt{2E_R(E_R + mc^2)}}\frac{\partial E_R}{\partial\boldsymbol{p}}$$

$$= i\hbar\frac{2E_R + mc^2}{\sqrt{2E_R(E_R + mc^2)}}\frac{c^2\boldsymbol{p}}{E_R}$$

or

$$\sqrt{2E_R(E_R + mc^2)}\boldsymbol{r} - \boldsymbol{r}\sqrt{2E_R(E_R + mc^2)} = -i\hbar\frac{2E_R + mc^2}{\sqrt{2E_R(E_R + mc^2)}}\frac{c^2\boldsymbol{p}}{E_R}$$

or

$$\sqrt{2E_R(E_R+mc^2)}\boldsymbol{r}\frac{1}{\sqrt{2E_R(E_R+mc^2)}}=\boldsymbol{r}-i\hbar\frac{2E_R+mc^2}{\sqrt{2E_R(E_R+mc^2)}}\frac{c^2\boldsymbol{p}}{E_R}\frac{1}{\sqrt{2E_R(E_R+mc^2)}}$$
$$=\boldsymbol{r}-i\hbar\frac{2E_R+mc^2}{2E_R{}^2(E_R+mc^2)}c^2\boldsymbol{p}$$

(a) $\boldsymbol{r}_{FW}$

Using this relation, we have

$$\begin{aligned}
\boldsymbol{r}_{FW} &= \hat{U}\boldsymbol{r}\hat{U}^+ \\
&= \boldsymbol{r}-i\hbar\frac{2E_R+mc^2}{2E_R{}^2(E_R+mc^2)}c^2\boldsymbol{p}-\frac{[E_R+mc^2+c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})]}{2E_R(E_R+mc^2)}i\hbar(c\beta\boldsymbol{\alpha}-\frac{c^2\boldsymbol{p}}{E_R}) \\
&= \boldsymbol{r}-\frac{i\hbar c\beta\boldsymbol{\alpha}}{2E_R}+\frac{i\hbar c\beta\boldsymbol{\alpha}}{2E_R}-i\hbar\frac{2E_R+mc^2}{2E_R{}^2(E_R+mc^2)}c^2\boldsymbol{p}-i\hbar\frac{[E_R+mc^2+c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})]}{2E_R(E_R+mc^2)}(c\beta\boldsymbol{\alpha}-\frac{c^2\boldsymbol{p}}{E_R}) \\
&= \boldsymbol{r}-\frac{i\hbar c\beta\boldsymbol{\alpha}}{2E_R}+\frac{i\hbar c\beta\boldsymbol{\alpha}}{2E_R}-i\hbar\frac{2E_R+mc^2}{2E_R{}^2(E_R+mc^2)}c^2\boldsymbol{p}-\frac{i\hbar}{2E_R(E_R+mc^2)}[E_R+mc^2+c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})](c\beta\boldsymbol{\alpha}-\frac{c^2\boldsymbol{p}}{E_R}) \\
&= \boldsymbol{r}-\frac{i\hbar c\beta\boldsymbol{\alpha}}{2E_R}+\frac{i\hbar c\beta\boldsymbol{\alpha}}{2E_R}-i\hbar\frac{2E_R+mc^2}{2E_R{}^2(E_R+mc^2)}c^2\boldsymbol{p}-\frac{i\hbar}{2E_R(E_R+mc^2)}\{(E_R+mc^2)c\beta\boldsymbol{\alpha}+c^2\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\beta\boldsymbol{\alpha} \\
&\quad +(E_R+mc^2)(-\frac{c^2\boldsymbol{p}}{E_R})-c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\frac{c^2\boldsymbol{p}}{E_R}\} \\
&= \boldsymbol{r}-\frac{i\hbar c\beta\boldsymbol{\alpha}}{2E_R}-i\hbar\frac{E_R}{2E_R{}^2(E_R+mc^2)}c^2\boldsymbol{p}+\frac{i\hbar}{2E_R{}^2(E_R+mc^2)}c^2(\boldsymbol{\alpha}\cdot\boldsymbol{p})\boldsymbol{\alpha}E_R \\
&\quad +\frac{i\hbar}{2E_R{}^2(E_R+mc^2)}c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})c^2\boldsymbol{p}\}
\end{aligned}$$

or

$$\begin{aligned}
\boldsymbol{r}_{FW} &= \hat{U}\boldsymbol{r}\hat{U}^+ \\
&= \boldsymbol{r}-\frac{i\hbar c\beta\boldsymbol{\alpha}}{2E_R}+\frac{i\hbar c^2}{2E_R{}^2(E_R+mc^2)}[c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\boldsymbol{p}+(\boldsymbol{\alpha}\cdot\boldsymbol{p})\boldsymbol{\alpha}E_R-E_R\boldsymbol{p}\} \\
&= \boldsymbol{r}-\frac{i\hbar c\beta\boldsymbol{\alpha}}{2E_R}+\frac{i\hbar c^2}{2E_R{}^2(E_R+mc^2)}\{c\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\boldsymbol{p}+E_R[(\boldsymbol{\alpha}\cdot\boldsymbol{p})\boldsymbol{\alpha}-\boldsymbol{p}]\} \\
&= \boldsymbol{r}-\frac{i\hbar c\beta\boldsymbol{\alpha}}{2E_R}+\frac{\hbar c^2}{2E_R{}^2(E_R+mc^2)}\{ic\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\boldsymbol{p}-E_R(\boldsymbol{\Sigma}\times\boldsymbol{p})]\}
\end{aligned}$$

((**Note**))

$$\boldsymbol{\Sigma}=\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix}$$

$$(\boldsymbol{\alpha}\cdot\boldsymbol{p})\boldsymbol{\alpha}-\boldsymbol{p}=i(\boldsymbol{\Sigma}\times\boldsymbol{p})$$

((**Proof**))

$$(\boldsymbol{\alpha}\cdot\boldsymbol{p})\boldsymbol{\alpha}-\boldsymbol{p}=\begin{pmatrix}0 & \boldsymbol{\sigma}\cdot\boldsymbol{p}\\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & 0\end{pmatrix}\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix}-\begin{pmatrix}\boldsymbol{p} & 0\\ 0 & \boldsymbol{p}\end{pmatrix}$$
$$=\begin{pmatrix}(\boldsymbol{\sigma}\cdot\boldsymbol{p})\boldsymbol{\sigma}-\boldsymbol{p} & 0\\ 0 & (\boldsymbol{\sigma}\cdot\boldsymbol{p})\boldsymbol{\sigma}-\boldsymbol{p}\end{pmatrix}$$

$$(\boldsymbol{\alpha}\cdot\boldsymbol{p})\boldsymbol{\alpha}_x-\boldsymbol{p}_x=\begin{pmatrix}(\boldsymbol{\sigma}\cdot\boldsymbol{p})\sigma_x-p_x & 0\\ 0 & (\boldsymbol{\sigma}\cdot\boldsymbol{p})\sigma_x-p_x\end{pmatrix}$$

$$(\boldsymbol{\sigma}\cdot\boldsymbol{p})\sigma_x-p_x=\begin{pmatrix}p_z & p_x-ip_y\\ p_x+ip_y & -p_z\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}-p_x\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$
$$=\begin{pmatrix}p_x-ip_y & p_z\\ -p_z & p_x+ip_y\end{pmatrix}-p_x\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$
$$=\begin{pmatrix}-ip_y & p_z\\ -p_z & ip_y\end{pmatrix}$$
$$=i\sigma_y p_z-i\sigma_z p_y$$
$$=i(\boldsymbol{\sigma}\times\boldsymbol{p})_x$$

$$(\boldsymbol{\sigma}\cdot\boldsymbol{p})\sigma_y-p_y=\begin{pmatrix}p_z & p_x-ip_y\\ p_x+ip_y & -p_z\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}-p_y\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$
$$=\begin{pmatrix}i(p_x-ip_y) & -ip_z\\ -ip_z & -i(p_x+ip_y)\end{pmatrix}-p_y\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$
$$=\begin{pmatrix}ip_x & -ip_z\\ -ip_z & -ip_x\end{pmatrix}$$
$$=i(\sigma_z p_x-\sigma_x p_z)$$
$$=i(\boldsymbol{\sigma}\times\boldsymbol{p})_y$$

$$(\boldsymbol{\sigma}\cdot\boldsymbol{p})\sigma_z - p_z = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - p_z\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} p_z & -(p_x - ip_y) \\ p_x + ip_y & p_z \end{pmatrix} - p_z\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -(p_x - ip_y) \\ p_x + ip_y & 0 \end{pmatrix}$$
$$= i(\sigma_x p_y - \sigma_y p_x)$$
$$= i(\boldsymbol{\sigma}\times\boldsymbol{p})_z$$

Then we have

$$(\boldsymbol{\alpha}\cdot\boldsymbol{p})\boldsymbol{\alpha} - \boldsymbol{p} = i\begin{pmatrix} (\boldsymbol{\sigma}\times\boldsymbol{p}) & 0 \\ 0 & (\boldsymbol{\sigma}\times\boldsymbol{p}) \end{pmatrix}$$
$$= i(\boldsymbol{\Sigma}\times\boldsymbol{p})$$

where

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}$$

Velocity:

$$\boldsymbol{v}_{FW} = \hat{U}\boldsymbol{v}\hat{U}^{+} = c\boldsymbol{\alpha} + \frac{c^2\beta}{E_R}\boldsymbol{p} - \frac{c^3}{E_R(E_R + mc^2)}(\boldsymbol{\alpha}\cdot\boldsymbol{p})\boldsymbol{p}$$

(b) Angular momentum:

$$\boldsymbol{L}_{FW} = \hat{U}(\boldsymbol{r}\times\boldsymbol{p})\hat{U}^{+}$$
$$= \boldsymbol{r}'\times\boldsymbol{p}$$

(c) Spin:

$$\boldsymbol{S}_{FW} = \hat{U}\frac{\hbar(\boldsymbol{\alpha}\times\boldsymbol{\alpha})}{4i}\hat{U}^{+} = \frac{\hbar}{2}[\boldsymbol{\Sigma} + \frac{ic}{E_R}\beta(\boldsymbol{\alpha}\times\boldsymbol{p}) - \frac{c^2}{E_R(E_R + mc^2)}\boldsymbol{p}\times(\boldsymbol{\Sigma}\times\boldsymbol{p})]$$

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APPENDIX-I

Formula

One further very useful property of both α-matrices and σ-matrices also arises from being able to write them in terms of the Pauli matrices. If A and B are arbitrary vectors, then

$$
\begin{aligned}
(\boldsymbol{\alpha}\cdot\boldsymbol{A})(\boldsymbol{\alpha}\cdot\boldsymbol{B}) &= \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{A} \\ \boldsymbol{\sigma}\cdot\boldsymbol{A} & 0 \end{pmatrix}\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{B} \\ \boldsymbol{\sigma}\cdot\boldsymbol{B} & 0 \end{pmatrix} \\
&= \begin{pmatrix} (\boldsymbol{\sigma}\cdot\boldsymbol{A})(\boldsymbol{\sigma}\cdot\boldsymbol{B}) & 0 \\ 0 & (\boldsymbol{\sigma}\cdot\boldsymbol{A})(\boldsymbol{\sigma}\cdot\boldsymbol{B}) \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{A}\cdot\boldsymbol{B}+i\boldsymbol{\sigma}\cdot(\boldsymbol{A}\times\boldsymbol{B}) & 0 \\ 0 & \boldsymbol{A}\cdot\boldsymbol{B}+i\boldsymbol{\sigma}\cdot(\boldsymbol{A}\times\boldsymbol{B}) \end{pmatrix} \\
&= [\boldsymbol{A}\cdot\boldsymbol{B}+i\boldsymbol{\sigma}\cdot(\boldsymbol{A}\times\boldsymbol{B})]\, I_4
\end{aligned}
$$

$$(\boldsymbol{\alpha}\times\boldsymbol{\alpha}) = 2i\boldsymbol{\Sigma}$$

$$
\begin{aligned}
\alpha_2\alpha_3-\alpha_3\alpha_2 &= \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \sigma_2\sigma_3-\sigma_3\sigma_2 & 0 \\ 0 & \sigma_2\sigma_3-\sigma_3\sigma_2 \end{pmatrix} \\
&= \begin{pmatrix} 2i\sigma_1 & 0 \\ 0 & 2i\sigma_1 \end{pmatrix} \\
&= 2i\Sigma_1
\end{aligned}
$$

(b) Odd matrix and even matrix

Odd matrix (example, matrix $\boldsymbol{\alpha}$)

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}$$

Even matrix (example $\boldsymbol{\Sigma}$)

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}$$

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

φ: large component

χ: small component

APPENDIX-II Application of FW transformation

(a) Energy eigenkets

Hamiltonian: $H = c(\boldsymbol{\alpha} \cdot \boldsymbol{p}) + \beta mc^2$

$$\hat{H}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \pm E_R \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

with the eigenvalue $\pm E_R$ since

$$\hat{H}^2 = E_R{}^2$$

Using the FW transformation, we have the new eigenket

$$\psi_{FW} = \hat{U}\psi ,$$

with

$$\hat{U} = \frac{1}{\sqrt{2E_R(E_R + mc^2)}}(E_R + \beta H),$$

$$\hat{U}^{+} = \frac{1}{\sqrt{2E_R(E_R + mc^2)}}(E_R + H\beta)$$

(a) For the energy eigenvalue $+E_R$, (particle)

$$\psi_{FW}^{(+)} = \hat{U}\psi$$

$$= \frac{1}{\sqrt{2E_R(E_R + mc^2)}}(E_R + \beta H)\psi$$

$$= \frac{E_R}{\sqrt{2E_R(E_R + mc^2)}}(1 + \beta)\psi$$

$$= \frac{E_R}{\sqrt{2E_R(E_R + mc^2)}}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

$$= \sqrt{\frac{2E_R}{E_R + mc^2}}\begin{pmatrix} u_1 \\ u_2 \\ 0 \\ 0 \end{pmatrix}$$

with the normalization condition

$$|u_1|^2 + |u_2|^2 = \frac{E_R + mc^2}{2E_R}$$

(b) For the energy eigenvalue $-E_R$ (anti-particle)

$$
\begin{aligned}
\psi_{FW}{}^{(-)} &= \hat{U}\psi \\
&= \frac{1}{\sqrt{2E_R(E_R+mc^2)}}(E_R+\beta H)\psi \\
&= \frac{E_R}{\sqrt{2E_R(E_R+mc^2)}}(1-\beta)\psi \\
&= \frac{E_R}{\sqrt{2E_R(E_R+mc^2)}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \\
&= \sqrt{\frac{2E_R}{E_R+mc^2}}\begin{pmatrix} 0 \\ 0 \\ u_3 \\ u_4 \end{pmatrix}
\end{aligned}
$$

From the normalization condition, we have

$$
|u_3|^2 + |u_4|^2 = \frac{E_R+mc^2}{2E_R}
$$

Helicity:

$$
\boldsymbol{\Sigma}\cdot\frac{\boldsymbol{p}}{|p|} = \begin{pmatrix} \boldsymbol{\sigma}\cdot\frac{\boldsymbol{p}}{|p|} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\frac{\boldsymbol{p}}{|p|} \end{pmatrix}
$$

$$
\boldsymbol{\Sigma}\cdot\frac{\boldsymbol{p}}{|p|}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\sigma}\cdot\frac{\boldsymbol{p}}{|p|} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\frac{\boldsymbol{p}}{|p|} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}
$$

with

$$
\boldsymbol{\sigma}\cdot\frac{\boldsymbol{p}}{|p|}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \pm\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad \boldsymbol{\sigma}\cdot\frac{\boldsymbol{p}}{|p|}\begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = \pm\begin{pmatrix} u_3 \\ u_4 \end{pmatrix}
$$

with the eigenvalues ± 1 since $(\boldsymbol{\sigma} \cdot \frac{\boldsymbol{p}}{|\boldsymbol{p}|})^2 = 1$. When we define $\frac{\boldsymbol{p}}{|\boldsymbol{p}|} = \boldsymbol{n}$

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} |\pm \boldsymbol{n}\rangle = \pm |\pm \boldsymbol{n}\rangle$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\phi/2} \\ \sin\frac{\theta}{2} e^{i\phi/2} \end{pmatrix} \text{ for the helicity 1}$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \sin\frac{\theta}{2} e^{-i\phi/2} \\ -\cos\frac{\theta}{2} e^{i\phi/2} \end{pmatrix} \text{ for the helicity -1}$$

$$\hat{U}^{+} = \frac{1}{\sqrt{2E_R(E_R + mc^2)}}(E_R + H\beta)$$

$$H = c(\boldsymbol{\alpha} \cdot \boldsymbol{p}) + \beta mc^2 = \begin{pmatrix} mc^2 & c(\boldsymbol{\sigma} \cdot \boldsymbol{p}) \\ c(\boldsymbol{\sigma} \cdot \boldsymbol{p}) & -mc^2 \end{pmatrix}$$

$$\begin{aligned}(E_R + H\beta) &= \begin{pmatrix} E_R & 0 \\ 0 & E_R \end{pmatrix} + \begin{pmatrix} mc^2 & c(\boldsymbol{\sigma} \cdot \boldsymbol{p}) \\ c(\boldsymbol{\sigma} \cdot \boldsymbol{p}) & -mc^2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} E_R & 0 \\ 0 & E_R \end{pmatrix} + \begin{pmatrix} mc^2 & -c(\boldsymbol{\sigma} \cdot \boldsymbol{p}) \\ c(\boldsymbol{\sigma} \cdot \boldsymbol{p}) & mc^2 \end{pmatrix} \\ &= \begin{pmatrix} E_R + mc^2 & -c(\boldsymbol{\sigma} \cdot \boldsymbol{p}) \\ c(\boldsymbol{\sigma} \cdot \boldsymbol{p}) & E_R + mc^2 \end{pmatrix}\end{aligned}$$

(a)

$$\hat{U}^{+}\begin{pmatrix} u_1 \\ u_2 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2E_R(E_R+mc^2)}}(E_R + H\beta)\begin{pmatrix} u_1 \\ u_2 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2E_R(E_R+mc^2)}}\begin{pmatrix} E_R+mc^2 & -c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) \\ c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) & E_R+mc^2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2E_R(E_R+mc^2)}}\begin{pmatrix} (E_R+mc^2)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ c(\boldsymbol{\sigma}\cdot\boldsymbol{p})\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2E_R(E_R+mc^2)}}\begin{pmatrix} (E_R+mc^2)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ \pm cp\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{pmatrix}$$

or

$$\hat{U}^{+}\begin{pmatrix} u_1 \\ u_2 \\ 0 \\ 0 \end{pmatrix} == \frac{(E_R+mc^2)}{\sqrt{2E_R(E_R+mc^2)}}\begin{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ \pm\frac{cp}{E_R+mc^2}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{pmatrix}$$

This leads to

$$\begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = \pm\frac{cp}{E_R+mc^2}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

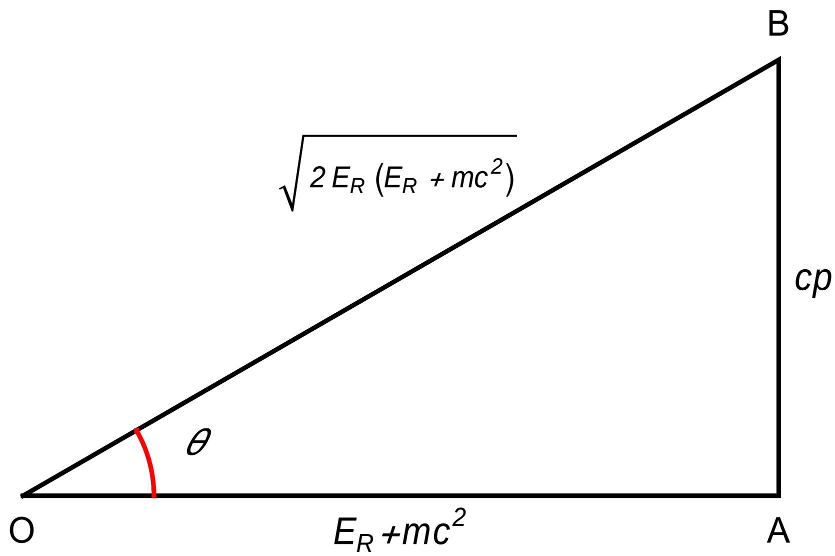
The normalization condition is satisfied,

$$\frac{(E_R+mc^2)}{2E_R(E_R+mc^2)}(|u_1|^2+|u_2|^2+|u_3|^2+|u_4|^2)=\frac{(E_R+mc^2)}{2E_R(E_R+mc^2)}(|u_1|^2+|u_2|^2)[1+\frac{c^2p^2}{(E_R+mc^2)^2}]$$

$$=\frac{(E_R+mc^2)}{2E_R(E_R+mc^2)}(|u_1|^2+|u_2|^2)\frac{(E_R+mc^2)^2+c^2p^2}{(E_R+mc^2)^2}$$

$$=\frac{E_R+mc^2}{2E_R}[\frac{2E_R(E_R+mc^2)}{(E_R+mc^2)^2}]$$

$$=1$$



(b)

$$\hat{U}^{+}\begin{pmatrix}0\\0\\u_3\\u_4\end{pmatrix}=\frac{1}{\sqrt{2E_R(E_R+mc^2)}}(E_R+H\beta)\begin{pmatrix}0\\0\\u_3\\u_4\end{pmatrix}$$

$$=\frac{1}{\sqrt{2E_R(E_R+mc^2)}}\begin{pmatrix}E_R+mc^2 & -c(\boldsymbol{\sigma}\cdot\boldsymbol{p})\\ c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) & E_R+mc^2\end{pmatrix}\begin{pmatrix}0\\0\\u_3\\u_4\end{pmatrix}$$

$$=\frac{1}{\sqrt{2E_R(E_R+mc^2)}}\begin{pmatrix}-c(\boldsymbol{\sigma}\cdot\boldsymbol{p})\begin{pmatrix}u_3\\u_4\end{pmatrix}\\ (E_R+mc^2)\begin{pmatrix}u_3\\u_4\end{pmatrix}\end{pmatrix}$$

$$=\frac{E_R+mc^2}{\sqrt{2E_R(E_R+mc^2)}}\begin{pmatrix}\mp\frac{cp}{E_R+mc^2}\begin{pmatrix}u_3\\u_4\end{pmatrix}\\ \begin{pmatrix}u_3\\u_4\end{pmatrix}\end{pmatrix}$$

This leads to

$$\begin{pmatrix}u_1\\u_2\end{pmatrix}=\mp\frac{cp}{E_R+mc^2}\begin{pmatrix}u_3\\u_4\end{pmatrix}$$

where

$$\begin{pmatrix}u_3\\u_4\end{pmatrix}=\begin{pmatrix}\cos\frac{\theta}{2}e^{-i\phi/2}\\ \sin\frac{\theta}{2}e^{i\phi/2}\end{pmatrix}\quad\text{for the helicity +1}$$

$$\begin{pmatrix}u_3\\u_4\end{pmatrix}=\begin{pmatrix}\sin\frac{\theta}{2}e^{-i\phi/2}\\ -\cos\frac{\theta}{2}e^{i\phi/2}\end{pmatrix}\quad\text{for the helicity -1}$$

The normalization condition is satisfied,

$$\frac{(E_R+mc^2)}{2E_R(E_R+mc^2)}(|u_1|^2+|u_2|^2+|u_3|^2+|u_4|^2)=\frac{(E_R+mc^2)}{2E_R(E_R+mc^2)}(|u_3|^2+|u_4|^2)[1+\frac{c^2p^2}{(E_R+mc^2)^2}]$$

$$=\frac{(E_R+mc^2)}{2E_R(E_R+mc^2)}(|u_3|^2+|u_4|^2)\frac{(E_R+mc^2)^2+c^2p^2}{(E_R+mc^2)^2}$$

$$=\frac{E_R+mc^2}{2E_R}[\frac{2E_R(E_R+mc^2)}{(E_R+mc^2)^2}]$$

$$=1$$

In summary we have the solution of Dirac equation

(a) The energy eigenvalue $+E_R$, helicity (+1)

$$\sqrt{\frac{E_R+mc^2}{2E_R}}\begin{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}e^{-i\phi/2}\\ \sin\frac{\theta}{2}e^{i\phi/2}\end{pmatrix}\\ \frac{cp}{E_R+mc^2}\begin{pmatrix}\cos\frac{\theta}{2}e^{-i\phi/2}\\ \sin\frac{\theta}{2}e^{i\phi/2}\end{pmatrix}\end{pmatrix}$$

(b) The energy eigenvalue $+E_R$, helicity (-1)

$$\sqrt{\frac{E_R+mc^2}{2E_R}}\begin{pmatrix}\begin{pmatrix}-\sin\frac{\theta}{2}e^{-i\phi/2}\\ \cos\frac{\theta}{2}e^{i\phi/2}\end{pmatrix}\\ \frac{-cp}{E_R+mc^2}\begin{pmatrix}-\sin\frac{\theta}{2}e^{-i\phi/2}\\ \cos\frac{\theta}{2}e^{i\phi/2}\end{pmatrix}\end{pmatrix}$$

(c) The energy eigenvalue $-E_R$, helicity (+1)

$$\sqrt{\frac{E_R + mc^2}{2E_R}} \begin{pmatrix} \dfrac{-cp}{E_R + mc^2} \begin{pmatrix} \cos\dfrac{\theta}{2} e^{-i\phi/2} \\ \sin\dfrac{\theta}{2} e^{i\phi/2} \end{pmatrix} \\ \begin{pmatrix} \cos\dfrac{\theta}{2} e^{-i\phi/2} \\ \sin\dfrac{\theta}{2} e^{i\phi/2} \end{pmatrix} \end{pmatrix}$$

(d) The energy eigenvalue $-E_R$, helicity (-1)

$$\sqrt{\frac{E_R + mc^2}{2E_R}} \begin{pmatrix} \dfrac{cp}{E_R + mc^2} \begin{pmatrix} -\sin\dfrac{\theta}{2} e^{-i\phi/2} \\ \cos\dfrac{\theta}{2} e^{i\phi/2} \end{pmatrix} \\ \begin{pmatrix} -\sin\dfrac{\theta}{2} e^{-i\phi/2} \\ \cos\dfrac{\theta}{2} e^{i\phi/2} \end{pmatrix} \end{pmatrix}$$