Hermitian operator

Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
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((**Definition**))

Hermite conjugate (definition): or Hermitian adjoint

$$\langle\psi|\hat{A}|\varphi\rangle^* = \langle\varphi|\hat{A}^+|\psi\rangle$$

1. Complex number

What is the Hermitian adjoint of the complex number?

$$\langle\psi|c|\varphi\rangle^* = \langle\varphi|c^+|\psi\rangle$$

or

$$\langle\psi|c|\varphi\rangle^* = (c\langle\psi|\varphi\rangle)^* = c^*\langle\psi|\varphi\rangle^*$$
$$= c^*\langle\varphi|\psi\rangle = \langle\varphi|c^*|\psi\rangle$$

where c is the complex number. Then we have

$$\langle\varphi|c^+|\psi\rangle = \langle\varphi|c^*|\psi\rangle$$

or

$$c^+ = c^*$$

When $c = i$ (pure imaginary),

$$i^+ = i^* = -i$$

In the quantum mechanics, the expectation value is real, i.e.,

$$\langle\psi|\hat{A}|\psi\rangle^* = \langle\psi|\hat{A}|\psi\rangle \qquad (1)$$

In this case $\hat{A}$ is called a Hermitian operator. From the definition, we have

$$\langle\psi|\hat{A}|\psi\rangle^* = \langle\psi|\hat{A}^+|\psi\rangle . \qquad (2)$$

Hermitian operator satisfies the following condition:

$$\hat{A}^{+} = \hat{A}$$

2. Dirac notation (bra vector)

$$|\alpha\rangle = \hat{A}|\varphi\rangle \rightarrow \langle\alpha| = \langle\varphi|\hat{A}^{+}$$

((Proof))

$$\langle\psi|\hat{A}|\varphi\rangle^{*} = \langle\varphi|\hat{A}^{+}|\psi\rangle$$

Then we have

$$\langle\psi|\hat{A}|\varphi\rangle^{*} = \langle\psi|\alpha\rangle^{*} = \langle\alpha|\psi\rangle = \langle\varphi|\hat{A}^{+}|\psi\rangle$$

Then

$$\langle\alpha|\psi\rangle = \langle\varphi|\hat{A}^{+}|\psi\rangle$$

or

$$\langle\alpha| = \langle\varphi|\hat{A}^{+}$$

3. The momentum operator is a Hermitian operator

$$\hat{p}^{+} = \hat{p}$$

((Proof))

$$\langle\psi|\hat{p}|\varphi\rangle^{*} = \langle\varphi|\hat{p}^{+}|\psi\rangle$$

$$\begin{aligned}\langle\psi|\hat{p}|\varphi\rangle &= \int\langle\psi|x\rangle\langle x|\hat{p}|\varphi\rangle dx = \int\langle\psi|x\rangle\frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|\varphi\rangle dx \\ &= \int\psi^{*}(x)\frac{\hbar}{i}\frac{\partial}{\partial x}\varphi(x)dx = -\int\frac{\hbar}{i}\frac{\partial}{\partial x}\psi^{*}(x)\varphi(x)dx\end{aligned}$$

Here we use the formula without proof

$$\langle x|\hat{p}|\varphi\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|\varphi\rangle$$

or

$$\langle\psi|\hat{p}|\varphi\rangle^* = \int \frac{\hbar}{i}\frac{\partial}{\partial x}\psi(x)\varphi^*(x)dx = \int \varphi^*(x)\frac{\hbar}{i}\frac{\partial}{\partial x}\psi(x)dx$$
$$= \int \langle\varphi|x\rangle\langle x|\hat{p}|\psi\rangle dx = \langle\varphi|\hat{p}|\psi\rangle$$

Thus we have

$$\langle\varphi|\hat{p}^+|\psi\rangle = \langle\varphi|\hat{p}|\psi\rangle$$

or

$$\hat{p}^+ = \hat{p}$$

4. Formula

$$\left(\hat{A}\hat{B}\right)^+ = \hat{B}^+\hat{A}^+$$

((Proof))

$$\langle\psi|(\hat{A}\hat{B})^+|\varphi\rangle = \langle\varphi|(\hat{A}\hat{B})|\psi\rangle^* = \langle\alpha|\beta\rangle^* = \langle\beta|\alpha\rangle = \langle\psi|\hat{B}^+\hat{A}^+|\varphi\rangle$$

where

$$\langle\alpha| = \langle\varphi|(\hat{A} \quad \text{and} \quad |\beta\rangle = \hat{B}|\psi\rangle$$

or

$$|\alpha\rangle = \hat{A}^+|\varphi\rangle \quad \text{and} \quad \langle\beta| = \langle\psi|\hat{B}^+$$

((**Comments**))

1. $(\hat{A}^+)^+ = \hat{A}$

2. $(c\hat{A})^+ = c^*\hat{A}^+$

3. $\left(\hat{A}+\hat{B}\right)^+ = \hat{A}^+ + \hat{B}^+$

4. If $\hat{A}^+ = \hat{A}$ and $\hat{B}^+ = \hat{B}$,

$$\left(\hat{A}\hat{B}\right)^+ = \hat{B}^+\hat{A}^+ = \hat{B}\hat{A}$$

So $\hat{A}\hat{B}$ is not Hermitian.

However,

5.

$$\left(\hat{A}\hat{B}+\hat{B}\hat{A}\right)^{+}=\hat{B}\hat{A}+\hat{A}\hat{B}$$

is Hermitian.

6.

$$\left(\hat{A}^{2}\right)^{+}=\left(\hat{A}\hat{A}\right)^{+}=\hat{A}^{+}\hat{A}^{+}=\hat{A}\hat{A}=\hat{A}^{2}$$

is Hermitian.

7. $\hat{p}$ is Hermitian.

$$\hat{p}=-i\hbar\hat{D} \qquad \text{with } \hat{D}^{+}=-\hat{D}$$

$$\hat{p}^{+}=(-i\hbar\hat{D})^{+}=-i^{+}\hbar\hat{D}^{+}=-i\hbar\hat{D}=\hat{p}$$

$$|\alpha\rangle\Rightarrow-\hat{A}|\alpha\rangle$$

Hermitian conjugate

$$\langle\alpha|\Rightarrow\langle\alpha|\hat{A}^{+}$$

Outer product of $|\beta\rangle$ and $\langle\alpha|$ is an operator

$$\hat{A}=|\beta\rangle\langle\alpha|$$

$$\hat{A}|\chi\rangle=(|\beta\rangle\langle\alpha|)|\chi\rangle=\langle\alpha|\chi\rangle|\beta\rangle$$

$$\hat{A}^{+}=|\alpha\rangle\langle\beta|$$

((Proof))

$$\langle\gamma|\hat{A}^{+}|\delta\rangle = \langle\delta|\hat{A}|\gamma\rangle^{*}$$
$$= (\langle\delta|\beta\rangle\langle\alpha|\gamma\rangle)^{*}$$
$$= \langle\delta|\beta\rangle^{*}\langle\alpha|\gamma\rangle^{*}$$
$$= \langle\beta|\delta\rangle\langle\gamma|\alpha\rangle$$
$$= \langle\gamma|\alpha\rangle\langle\beta|\delta\rangle$$

Thus we have

$$\hat{A}^{+} = |\alpha\rangle\langle\beta|$$

5. Properties of Hermitian

Eigenvalue problem:

$$\hat{A}|\varphi_n\rangle = a_n|\varphi_n\rangle$$

where

$\hat{A}$ is the Hermitian: $\hat{A}^{+} = \hat{A}$
$|\varphi_n\rangle$ is the eigenket
a_n is the eigenvalue

Since $\hat{A}$ is the Hermitian,

$$\langle\varphi_m|\hat{A}|\varphi_n\rangle^{*} = \langle\varphi_n|\hat{A}^{+}|\varphi_m\rangle = \langle\varphi_n|\hat{A}|\varphi_m\rangle \qquad (1)$$

The matrix element

$$A_{nm} = \langle\varphi_n|\hat{A}|\varphi_m\rangle$$

$$A_{mn}{}^{*} = A_{nm}$$

The matrix element of $\hat{A}^{+}$ is the complex conjugate of the matrix element for the transpose of the matrix $\hat{A}$.

n x *n* matrix elements

$$\begin{matrix} A_{11} & A_{12} & A_{13} & . & . & . & . & . & . & A_{1n} \\ A_{12}^* & A_{22} & A_{23} & . & . & . & . & . & . & A_{2n} \\ A_{13}^* & A_{23}^* & A_{33} & . & . & . & . & . & . & A_{3n} \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ A_{1n}^* & A_{2n}^* & A_{n3}^* & . & . & . & . & . & . & A_{nn} \end{matrix}$$

The diagonal elements are real.

From Eq.(1), we have

$$\langle \varphi_m | a_n | \varphi_n \rangle^* = a_n^* \langle \varphi_n | \varphi_m \rangle = \langle \varphi_n | \hat{A} | \varphi_m \rangle = a_m \langle \varphi_n | \varphi_m \rangle \qquad (2)$$

(1) *Eigen value is real*

$m = n$ in Eq.(2),

$$(a_n^* - a_n)\langle \varphi_n | \varphi_n \rangle = 0$$

$$a_n^* = a_n$$

Thus a_n is real.

(2) *The eigenkets are orthogonal*

In Eq.(2),

$$(a_n - a_m)\langle \varphi_n | \varphi_m \rangle = 0$$

If $a_n \neq a_m$ (non-degenerate case),

$\langle \varphi_n | \varphi_m \rangle = 0$ (orthogonal)

(3) The eigenkets of $\hat{A}$ form a complete set.

For any $|\psi\rangle$,

$$|\psi\rangle = \sum_{n=1}^{\infty} c_n |\varphi_n\rangle$$

with

$$c_n = \langle\varphi_n|\psi\rangle$$

6. Closure relation:

$$|\psi\rangle = \sum_{n=1}^{\infty} c_n |\varphi_n\rangle = \sum_{n=1}^{\infty} |\varphi_n\rangle\langle\varphi_n|\psi\rangle$$

Because $|\psi\rangle$ is an arbitrary ket, we must have

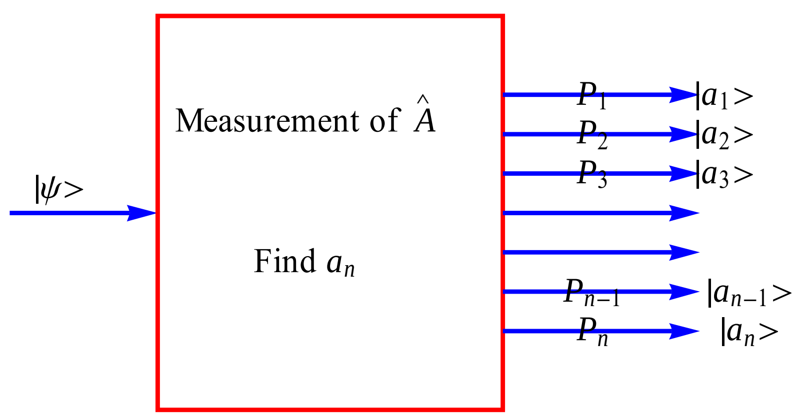
$$\sum_{n=1}^{\infty} |\varphi_n\rangle\langle\varphi_n| = \hat{1}$$

For $\hat{A}$ with $\hat{A}|\varphi_n\rangle = a_n|\varphi_n\rangle$

$$\hat{A} = \hat{A}\sum_{n=1}^{\infty} |\varphi_n\rangle\langle\varphi_n| = \sum_{n=1}^{\infty} \hat{A}|\varphi_n\rangle\langle\varphi_n| = \sum_{n=1}^{\infty} a_n|\varphi_n\rangle\langle\varphi_n|$$

7. Principle of measurement

For any $|\psi\rangle$,



$$|\psi\rangle = (\sum_{n=1}^{\infty} |\varphi_n\rangle\langle\varphi_n|)|\psi\rangle = \sum_{n=1}^{\infty} |\varphi_n\rangle\langle\varphi_n|\psi\rangle$$

Probability of finding $|\varphi_n\rangle$

$$P(a_n) = |\langle \varphi_n | \psi \rangle|^2$$

8. Commuting observables

The product of Hermitian operators $\hat{A}$ and $\hat{B}$

$$\left(\hat{A}\hat{B}\right)^+ = \hat{B}^+\hat{A}^+ = \hat{B}\hat{A}$$

If

$$[\hat{A}, \hat{B}] = 0 \text{ (commutable)}$$

$$\left(\hat{A}\hat{B}\right)^+ = \hat{B}\hat{A} = \hat{A}\hat{B}$$

Thus $\hat{A}\hat{B}$ is Hermitian.

9. Simultaneous eigenkets

We may use $|a,b\rangle$ to characterize the simultaneous eigenket.

$$\hat{A}|a,b\rangle = a|a,b\rangle$$

$$\hat{B}|a,b\rangle = b|a,b\rangle$$

Then

$$\hat{A}\hat{B}|a,b\rangle = b\hat{A}|a,b\rangle = ab|a,b\rangle$$

$$\hat{B}\hat{A}|a,b\rangle = a\hat{B}|a,b\rangle = ab|a,b\rangle$$

or

$$[\hat{A}, \hat{B}]|a,b\rangle = 0$$