Orbital magnetic moment

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The Current density for the electron with a charge (-e) and a mass μ, is given by

$$\boldsymbol{J} = \frac{-e}{\mu}\mathrm{Re}[\psi_{nlm}(\boldsymbol{r})^* \frac{\hbar}{i}\nabla\psi_{nlm}(\boldsymbol{r})],$$

with

$$\nabla = \boldsymbol{e}_r\frac{\partial}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \boldsymbol{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}. \qquad \text{(the spherical co-ordinate)}$$

In an electron for the hydrogen-like atom,

$$R_{nl}(r) = \sqrt{\frac{4Z^3(n-l-1)!}{a^3n^4(n+l)!}}e^{-\frac{Zr}{na}}(\frac{2Zr}{na})^l L_{n-l-1}^{2l+1}(\frac{2Zr}{na}),$$

$$\psi_{nlm}(\boldsymbol{r}) = R_{nl}(r)Y_l^m(\theta,\phi) = R_{nl}(r)\Theta_l^m(\theta)\Phi_m(\phi),$$

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi}.$$

and the spherical harmonics,

$$Y_l^m(\theta,\phi) = \frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)}{4\pi}\frac{(l+m)!}{(l-m)!}}e^{im\phi}\frac{1}{\sin^m\theta}\frac{d^{l-m}}{d(\cos\theta)^{l-m}}(\sin\theta)^{2l},$$

Note that $R_{nl}(r)$ and $\Theta_l^m(\theta)$ are real functions.

We define $Y_l^{-m}(\theta,\phi)$ ($m \geq 0$) by

$$Y_l^{-m}(\theta,\phi) = (-1)^m[Y_l^m(\theta,\phi)]^*, \qquad \text{for } m \geq 0,$$

or

$$[Y_l^m(\theta,\phi)]^* = (-1)^m Y_l^{-m}(\theta,\phi) .$$

Then we have the current density as

$$\mathbf{J} = -\frac{e\hbar}{\mu}\mathrm{Re}[R_{nl}(r)\Theta_l^m(\theta)\Phi_m{}^*(\phi)\frac{1}{i}\nabla R_{nl}(r)\Theta_l^m(\theta)\Phi_m{}^*(\phi)] .$$

Noting that

$$\mathrm{Re}[R_{nl}(r)\Theta_l^m(\theta)\Phi_m{}^*(\phi)\frac{1}{i}\{\boldsymbol{e}_r\frac{\partial}{\partial r}+\boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\boldsymbol{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\}R_{nl}(r)\Theta_l^m(\theta)\Phi_m(\phi)]$$

$$= \mathrm{Re}\{\frac{1}{i}\boldsymbol{e}_r R_{nl}(r)\frac{\partial R_{nl}(r)}{\partial r}[\Theta_l^m(\theta)]^2+\frac{1}{ir}\boldsymbol{e}_\theta[R_{nl}(r)]^2[\Theta_l^m(\theta)\frac{\partial\Theta_l^m(\theta)}{\partial\theta}]$$

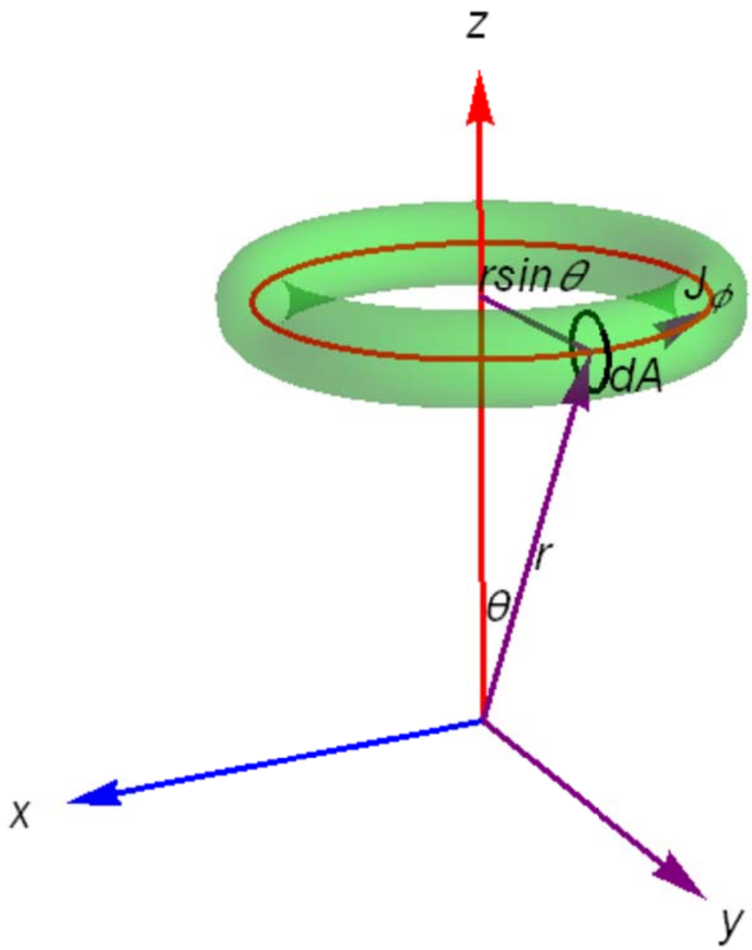
$$+\frac{1}{ir\sin\theta}\boldsymbol{e}_\phi[R_{nl}(r)]^2[\Theta_l^m(\theta)]^2\Phi_m{}^*(\phi)\frac{\partial\Phi_m(\phi)}{\partial\phi}]\}$$

$$=\frac{m}{r\sin\theta}\boldsymbol{e}_\phi[R_{nl}(r)]^2[\Theta_l^m(\theta)]^2\left|\Phi_m(\phi)\right|^2$$

$$=\frac{m}{r\sin\theta}\boldsymbol{e}_\phi\psi^*{}_{nlm}(\boldsymbol{r})\psi_{nlm}(\boldsymbol{r})$$

we obtain the expression for the current density

$$J_\phi = \frac{-e}{\mu}m\hbar\frac{\left|\psi_{nlm}(\mathbf{r})\right|^2}{r\sin\theta} .$$



The magnetic moment along the z axis is

$$
\begin{aligned}
d\mu_z &= \frac{\pi(r\sin\theta)^2}{c} J_\phi dA \\
&= \left(\frac{-e}{\mu}\right)\frac{\pi(r\sin\theta)^2}{c}[m\hbar\frac{\psi_{nlm}{}^*(\mathbf{r})\psi_{nlm}(\mathbf{r})}{r\sin\theta}]dA \\
&= \left(\frac{-e}{\mu}\right)\frac{\pi r\sin\theta}{c} m\hbar|\psi_{nlm}(\mathbf{r})|^2]dA \\
&= (\frac{-e\hbar}{2\mu c})m|\psi_{nlm}(\mathbf{r})|^2 2\pi r\sin\theta dA
\end{aligned}
$$

Since $dV = 2\pi r \sin\theta dA$

$$\mu_z = (\frac{-e\hbar}{2\mu c}) m \int \left|\psi_{nlm}(\mathbf{r})\right|^2 dV = (\frac{-e}{2\mu c}) m\hbar = (\frac{-e\hbar}{2\mu c}) \frac{m\hbar}{\hbar} .$$

or

$$\hat{\mu}_L = -\frac{\mu_B}{\hbar} \hat{\mathbf{L}} , \qquad \text{(orbital magnetic moment).}$$

where

$$\int \left|\psi_{nlm}(\mathbf{r})\right|^2 dV = 1$$

$$\mu_B = \frac{e\hbar}{2\mu c} . \qquad \text{(Bohr magneton).}$$