Isospin

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Isospin was introduced by Werner Heisenberg in 1932 to explain symmetries of the then newly discovered neutron. The mass of the neutron and the proton are almost identical: they are nearly degenerate, and both are thus often called nucleons. Although the proton has a positive charge, and the neutron is neutral, they are almost identical in all other respects. The name *isospin* however, was introduced by Eugene Wigner in 1937.

If two or more nucleons are treated as identical particles, the state vector describing then in a compound nucleus must refer not only to space and spin variables but also to isospins. The total state vector must be antisymmetric with respect to exchange.

1. Introduction of the concept of isospin ((Tomonaga))

In his book on the Story of Spin, Tomonaga explained how Heisenberg got the idea of isospin.

(a) Heisenberg's explanation for the attractive interaction between proton and neutron

It is an experimental fact that if the atomic number Z is not very large and therefore the charge Ze is not very large, then Z is approximately $A/2$ for many nuclei, where A is the mass number. This shows that nuclei with approximately equal numbers of neutrons and protons are most stable.

From this fact, Heisenberg concluded that the attraction between neutrons and protons plays the biggest role in the nucleus. If the attraction between neutrons were stronger, then nuclei composed only of neutrons would be more stable, and therefore more such nuclei should exist. But this contradicts the facts. The same thing can be said if the attraction between protons is stronger- namely nuclei with only protons must be abundant.

(b) Nuclear force is an exchange interaction

Next Heisenberg noticed the experimental fact that the binding energies of nuclei are approximately proportional to the mass number A (the number of particles in nucleus). From this he was led to the idea that the nuclear force is not the usual attractive force but is an exchange force. He reasoned as follows. If the force acting between a neutron and a proton is the usual two-body force, then if the potential between the K-th neutron and the L-th proton is written as $V_{K,L}$, and if the number of neutrons is written as N and the number of protons is written as P, then total potential is

$$\sum_{K=1}^{L}\sum_{L=1}^{P} V_{K,L} \,,$$

and the total binding energy must approximately equal the number of combinations pf pairs (K, L), which is

$$NP \approx \frac{A^2}{4} .$$

In reality, it is proportional only to A.

(c) Possibility of boson electron with spin zero

In the case of a neutron and a proton, both are fermions, and statistics is constant. But if we can consider the neutron to composed of a proton and a particle with spin zero which may be called boson electron, then it is not impossible to imagine that this boson electron is going back and forth between the neutron and the proton. Heisenberg did refer to this idea. Nevertheless, he concluded that it was better to ignore the existence of the boson electron, perhaps because he was not sure whether he could use quantum mechanics for the shuttling of this particle even if this idea were adopted. He did not adopt this idea.

(d) Introduction of the isospin

Heisenberg introduced isospin. Instead of considering the neutron and proton as different elementary particles, he considered them as two different states of the same elementary particle; the proton state and the neutron state. Both the neutron and proton are the fermion with spin 1/2.

(e) Exchange force

There is no force if the nucleons are both neutrons of both protons. By using the wave packet

$$\left|\chi_s\right\rangle \exp[-\frac{i}{\hbar}tJ_s(r)] \pm \left|\chi_a\right\rangle \exp[-\frac{i}{\hbar}tJ_a(r)]$$

we find that the transformation from a neutron into a proton or a proton into a neutron occurs with the angular frequency,

$$\omega = \frac{1}{\hbar}[J_a(r) - J_s(r)] = \frac{2J(\boldsymbol{r})}{\hbar}$$

2. Isospin: proton state $\left|p\right\rangle$ and neutron states $\left|n\right\rangle$

Because of the symmetry between the proton and the neutron, it is convenient to regard proton and neutron as two distinct states of the same particle, the nucleon. The state vector of a nucleon can be expressed by

The proton state:

$$\left|p\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

The neutron state

$$\left|n\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

These states correspond to spin-up and spin-down .In analogy to the ordinary spin we introduce three matrices that connect neutron and proton states (the isotopic spin matrices)

$$\hat{\tau}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\tau}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{\tau}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with

$$\hat{\boldsymbol{\tau}} \times \hat{\boldsymbol{\tau}} = 2i\hat{\boldsymbol{\tau}} .$$

Heisenberg chose the state for which the eigenvalue of $\hat{\tau}_3$ is +1 to be the neutron state and the state for which the eigenvalue is -1 to be the proton state.

$$|p\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |n\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\hat{\tau}_3 |p\rangle = |p\rangle , \qquad \hat{\tau}_3 |n\rangle = -|n\rangle .$$

The charge operator:

$$\hat{q} = \frac{1}{2}(\hat{1} + \hat{\tau}_3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\hat{q}|p\rangle = |p\rangle, \qquad \hat{q}|n\rangle = 0 .$$

The isospin is defined by

$$\hat{\boldsymbol{t}} = \frac{1}{2}\hat{\boldsymbol{\tau}} ,$$

with the commutation relation

$$\hat{\boldsymbol{t}} \times \hat{\boldsymbol{t}} = i\hat{\boldsymbol{t}} ,$$

where

$$\hat{t}_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{t}_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{t}_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\hat{t}_+ = \hat{t}_1 + i\hat{t}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \hat{t}_- = \hat{t}_1 - i\hat{t}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

satisfying

$$\hat{t}_+|p\rangle = 0 , \qquad \hat{t}_+|n\rangle = |p\rangle , \qquad \hat{t}_-|p\rangle = |n\rangle , \qquad \hat{t}_-|n\rangle = 0 .$$

We introduce the charge-symmetry operator $\hat{S}$, which interchanges neutron and proton. Among many possible choices of $\hat{S}$, we take

$$\hat{S} = i\hat{\tau}_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Here

$$\hat{S}|p\rangle = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -|n\rangle, \qquad \hat{S}|n\rangle = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = |p\rangle.$$

3. The system composed of two nucleons

For a system of two interacting nucleons such as the combinations of (neutron-neutron, proton-neutron), the total isospin operator is given by

$$\hat{\boldsymbol{T}} = \hat{\boldsymbol{t}}_1 + \hat{\boldsymbol{t}}_2.$$

If we neglect the electromagnetic interaction and the mass difference of proton and neutron, the interaction Hamiltonian conserves isospin and so commutes with all the components of isospin

$$[\hat{H}, \hat{\boldsymbol{T}}] = 0.$$

Then since $\hat{H}$ is invariant under rotation in isospin space, it can only depend on the isospin through $\hat{\boldsymbol{T}}^2$, where

$$\hat{\boldsymbol{T}}^2 = (\hat{\boldsymbol{t}}_1 + \hat{\boldsymbol{t}}_2)^2 = \hat{\boldsymbol{t}}_1^{\,2} + \hat{\boldsymbol{t}}_2^{\,2} + 2\hat{\boldsymbol{t}}_1 \cdot \hat{\boldsymbol{t}}_2 = \frac{3}{2}\hat{1} + \frac{1}{2}\hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2.$$

So $\hat{H}$ can be a function of the operator $\hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2$. The commutation relation $[\hat{H}, \hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2] = 0$ implies that there are simultaneous states of $\hat{H}$ and $\hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2$.

4. Eigenvalue problem for $\hat{\boldsymbol{T}}^2$

We consider the eigenvalue problem of $\hat{\boldsymbol{T}}^2$

$$\begin{aligned}
\hat{\boldsymbol{T}}^2 &= \frac{3}{2}\hat{1} + \frac{1}{2}\hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2 \\
&= \frac{3}{2}\hat{1}_2 \otimes \hat{1}_2 + \frac{1}{2}(\hat{\tau}_{1x} \otimes \hat{\tau}_{2x} + \hat{\tau}_{1y} \otimes \hat{\tau}_{2y} + \hat{\tau}_{1z} \otimes \hat{\tau}_{2z}) \\
&= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}
\end{aligned}$$

We note that

$$|p\rangle_1 |p\rangle_2 = |p\rangle_1 \otimes |p\rangle_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad |p\rangle_1 |n\rangle_2 = |p\rangle_1 \otimes |n\rangle_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|n\rangle_1 |p\rangle_2 = |p\rangle_1 \otimes |p\rangle_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |n\rangle_1 |n\rangle_2 = |n\rangle_1 \otimes |n\rangle_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The state in which nucleon 1 is a neutron and nucleon 2 is a proton can be expressed as the product,

$$|n\rangle_1 \otimes |p\rangle_2 .$$

The state in which nucleon 1 is a proton and nucleon 2 is a neutron can be expressed as the product,

$$|p\rangle_1 \otimes |n\rangle_2 .$$

Using the Mathematica we obtain the eigenstates and eigenvalues.

$$D_{1/2} \times D_{1/2} = D_1 + D_0 .$$

leading to $T = 1$ (triplet) and $T = 0$ (singlet).

(i) The eigenvalue 2 [$=T(T+1)$]; $T = +1$.

$$|\psi_1\rangle = |p\rangle_1 |p\rangle_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}[|p\rangle_1 |n\rangle_2 + |n\rangle_1 |p\rangle_2] = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$|\psi_3\rangle = |p\rangle_1 |p\rangle_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

where

$$\hat{\boldsymbol{T}}^2|\psi_1\rangle = 2|\psi_1\rangle, \qquad \hat{\boldsymbol{T}}^2|\psi_2\rangle = 2|\psi_2\rangle, \qquad \hat{\boldsymbol{T}}^2|\psi_3\rangle = 2|\psi_3\rangle$$

or

$$\hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2|\psi_1\rangle = |\psi_1\rangle, \qquad \hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2|\psi_2\rangle = |\psi_2\rangle, \qquad \hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2|\psi_3\rangle = |\psi_3\rangle$$

(ii) The eigenvalue 0 [$=T$(T+1)]; $T = 0$.

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}[|p\rangle_1|n\rangle_2 - |n\rangle_1|p\rangle_2] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

where

$$\hat{\boldsymbol{T}}^2|\psi_4\rangle = 0.$$

or

$$\hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2|\psi_4\rangle = -3|\psi_4\rangle,$$

Using the Dirac exchange operator $\hat{P}_{12} = \frac{1}{2}(\hat{1} + \hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2)$, We note that $\hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2$ can be expressed as

$$\hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2 = 2\hat{P}_{12} - \hat{1}.$$

5. Eigenstate of $\hat{T}_z$

$$\hat{T}_z = \hat{t}_{1z} + \hat{t}_{2z} = \hat{t}_{1z} \otimes \hat{I}_2 + \hat{I}_2 \otimes \hat{t}_{2z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We note that $[\hat{T}_z, \hat{\boldsymbol{T}}^2] = 0$, which mean that there are simultaneous eigenkets of $\hat{\boldsymbol{T}}^2$ and $\hat{T}_z$.

$$\hat{T}_z|p\rangle_1|p\rangle_2 = |p\rangle_1|p\rangle_2,$$

So $|p\rangle_1|p\rangle_2$ is the eigenket of $\hat{T}_z$ with the eigenvalue 1.

$$\hat{T}_z|n\rangle_1|n\rangle_2 = -|n\rangle_1|n\rangle_2,$$

So $|n\rangle_1 |n\rangle_2$ is the eigenket of $\hat{T}_z$ with the eigenvalue -1.

$$\hat{T}_z |p\rangle_1 |n\rangle_2 = 0, \qquad \hat{T}_z |n\rangle_1 |p\rangle_2 = 0$$

$|p\rangle_1 |n\rangle_2$ and $|n\rangle_1 |p\rangle_2$ are the degenerate states with the same eigenvalue. These states are not the eigenstates of $\hat{\boldsymbol{T}}^2$. However, from the two equations, we get the super-positions such that

$$\hat{T}_z \left(\frac{|p\rangle_1 |n\rangle_2 + |n\rangle_1 |p\rangle_2}{\sqrt{2}}\right) = 0,$$

$$\hat{T}_z \left(\frac{|p\rangle_1 |n\rangle_2 - |n\rangle_1 |p\rangle_2}{\sqrt{2}}\right) = 0,$$

which means that $\frac{|p\rangle_1 |n\rangle_2 + |n\rangle_1 |p\rangle_2}{\sqrt{2}}$ and $\frac{|p\rangle_1 |n\rangle_2 - |n\rangle_1 |p\rangle_2}{\sqrt{2}}$ are the simultaneous eigenkets of $\hat{\boldsymbol{T}}^2$ and $\hat{T}_z$.

6. Simultaneous eigenstate of $\hat{\boldsymbol{T}}^2$ and $\hat{T}_z$

From the above discussion, we have the simultaneous eigenstate of $\hat{\boldsymbol{T}}^2$ and $\hat{T}_z$ which can be expressed by the kets $|T, T_z\rangle$.

Eigenvalue: Eigenkets

Triplet

$T = 1 \quad T_z = 1,$ $\quad \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 = 1 \quad |1,1\rangle = |p\rangle_1 |p\rangle_2$ (symmetric)

$T = 1 \quad T_z = 0$ $\quad \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 = 1 \quad |1,0\rangle = \frac{1}{\sqrt{2}}[|p\rangle_1 |n\rangle_2 + |n\rangle_1 |p\rangle_2],$ (symmetric)

$T = 1 \quad T_z = -1$ $\quad \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 = 1 \quad |1,-1\rangle = |n\rangle_1 |n\rangle_2,$ (symmetric)

Singlet:

$T = 0 \quad T_z = 0$ $\quad \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 = -3 \quad |0,0\rangle = \frac{1}{\sqrt{2}}[|p\rangle_1 |n\rangle_2 - |n\rangle_1 |p\rangle_2].$ (antisymmetric)

((**Note**)) Clebsch-Gordan coefficients

```
Clear["Global`*"];
CCGG[{j1_, m1_}, {j2_, m2_}, {j_, m_}] :=
 Module[{s1},
  s1 = If[Abs[m1] ≤ j1 && Abs[m2] ≤ j2 && Abs[m] ≤ j,
    ClebschGordan[{j1, m1}, {j2, m2}, {j, m}],
    0]]; CG[{j_, m_}, j1_, j2_] :=
 Sum[CCGG[{j1, m1}, {j2, m - m1}, {j, m}]
   a[j1, m1] b[j2, m - m1], {m1, -j1, j1}]
```

j1=1/2 and j2=1/2

```
j1 = 1 / 2; j2 = 1 / 2;
```

```
CG[{1, 1}, j1, j2]
```

$$a\left[\frac{1}{2}, \frac{1}{2}\right] b\left[\frac{1}{2}, \frac{1}{2}\right]$$

```
CG[{1, 0}, j1, j2]
```

$$\frac{a\left[\frac{1}{2}, \frac{1}{2}\right] b\left[\frac{1}{2}, -\frac{1}{2}\right]}{\sqrt{2}} + \frac{a\left[\frac{1}{2}, -\frac{1}{2}\right] b\left[\frac{1}{2}, \frac{1}{2}\right]}{\sqrt{2}}$$

```
CG[{1, -1}, j1, j2]
```

$$a\left[\frac{1}{2}, -\frac{1}{2}\right] b\left[\frac{1}{2}, -\frac{1}{2}\right]$$

```
CG[{0, 0}, j1, j2]
```

$$\frac{a\left[\frac{1}{2}, \frac{1}{2}\right] b\left[\frac{1}{2}, -\frac{1}{2}\right]}{\sqrt{2}} - \frac{a\left[\frac{1}{2}, -\frac{1}{2}\right] b\left[\frac{1}{2}, \frac{1}{2}\right]}{\sqrt{2}}$$

7. Eigenstates of charge operator

The charge operator is defined by

$$\hat{Q} = \hat{q}_1 + \hat{q}_2 = \hat{1} + \frac{1}{2}(\hat{\tau}_{31} + \hat{\tau}_{32}) = \hat{1} + \hat{t}_{31} + \hat{t}_{32}$$

or

$$\hat{Q} = \hat{I}_2 \otimes \hat{I}_2 + t_z \otimes \hat{I}_2 + \hat{I}_2 \otimes t_z = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We note that $[\hat{Q}, \hat{\boldsymbol{T}}^2] = 0$, which mean that there are simultaneous eigenkets of $\hat{\boldsymbol{T}}^2$ and $\hat{Q}$.

$$\hat{Q}|p\rangle_1|p\rangle_2 = 2|p\rangle_1|p\rangle_2,$$

So $|1,1\rangle = |p\rangle_1|p\rangle_2$ is the eigenket of $\hat{Q}$ with the eigenvalue 2.

$$\hat{Q}|n\rangle_1|n\rangle_2 = 0|n\rangle_1|n\rangle_2,$$

So $|1,-1\rangle = |n\rangle_1|n\rangle_2$ is the eigenket of $\hat{Q}$ with the eigenvalue 0.

$$\hat{Q}|p\rangle_1|n\rangle_2 = |p\rangle_1|n\rangle_2, \qquad \hat{Q}|n\rangle_1|p\rangle_2 = |n\rangle_1|p\rangle_2$$

$|p\rangle_1|n\rangle_2$ and $|n\rangle_1|p\rangle_2$ are the degenerate states with the same eigenvalue. These states are not the eigenstates of $\hat{\boldsymbol{T}}^2$. However, from the two equations, we get the super-positions such that

$$\hat{Q}(\frac{|p\rangle_1|n\rangle_2 + |n\rangle_1|p\rangle_2}{\sqrt{2}}) = \frac{|p\rangle_1|n\rangle_2 + |n\rangle_1|p\rangle_2}{\sqrt{2}},$$

$$\hat{Q}(\frac{|p\rangle_1|n\rangle_2 - |n\rangle_1|p\rangle_2}{\sqrt{2}}) = \frac{|p\rangle_1|n\rangle_2 - |n\rangle_1|p\rangle_2}{\sqrt{2}},$$

which means that $|1,0\rangle = \frac{|p\rangle_1|n\rangle_2 + |n\rangle_1|p\rangle_2}{\sqrt{2}}$ and $|0,0\rangle = \frac{|p\rangle_1|n\rangle_2 - |n\rangle_1|p\rangle_2}{\sqrt{2}}$ are the simultaneous eigenkets of $\hat{\boldsymbol{T}}^2$ and $\hat{Q}$.

In summary, we have

$\hat{Q}|1,1\rangle = 2|1,1\rangle$, charge $2e$ (e>0)

$\hat{Q}|1,0\rangle = 1|1,1\rangle$, charge e

$\hat{Q}|1,-1\rangle = 0$, charge 0

$\hat{Q}|0,0\rangle = 1|0,0\rangle$, charge e

8. The π-mesons, π^+, π^0, π^-

The pi mesons π^+, π^0, π^- have zero spin and nearly equal masses

$$m(\pi^{\pm}) = 139.6 \text{ MeV}/c^2, \qquad m(\pi^0) = 135 \text{ MeV}/c^2.$$

The relatively small mass difference is thought to be due to electromagnetic interaction, in the absence of which the pions would form a perfect isospin triplet with $T = 1$.

The π-mesons, π^+, π^0, π^- can be considered as constituting three states of a particle having an isotopic spin equal to unity, with base vectors

$$|\pi^+\rangle = |1,1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |\pi^0\rangle = |1,0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |\pi^-\rangle = |1,-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

In this case the operations of the three components ($\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$ of the isotopic spin operator are given by the matrices for isospin $T = 1$.

Isospin and scattering amplitude

$$D_1 \times D_{1/2} = D_{3/2} + D_{1/2}$$

Clebsch-Gordan coefficients

$$\left|\pi, N; \frac{3}{2}, \frac{3}{2}\right\rangle = |\pi^+\rangle |p\rangle$$

$$\left|\pi, N; \frac{3}{2}, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}} |\pi^+\rangle |n\rangle + \sqrt{\frac{2}{3}} |\pi^0\rangle |p\rangle$$

$$\left|\pi, N; \frac{3}{2}, -\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}} |\pi^0\rangle |n\rangle + \frac{1}{\sqrt{3}} |\pi^-\rangle |p\rangle$$

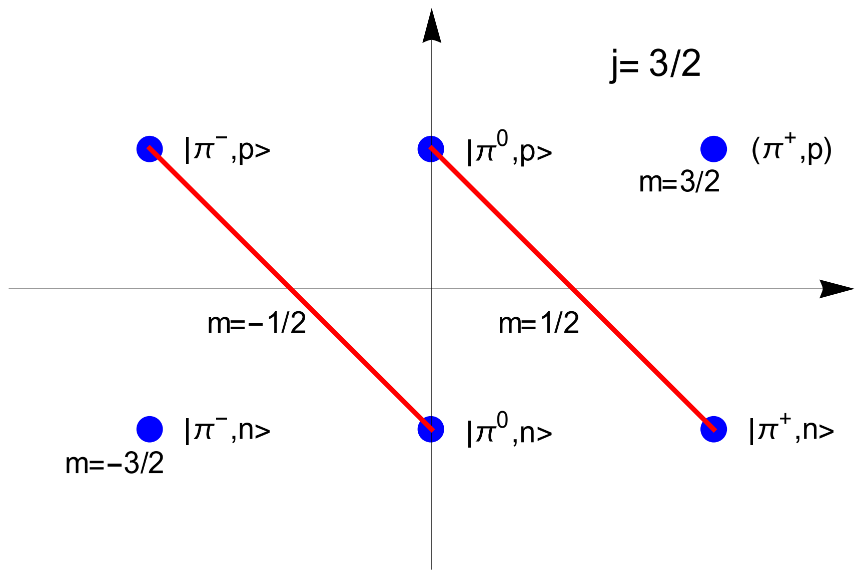
$$\left|\pi, N; \frac{3}{2}, -\frac{3}{2}\right\rangle = |\pi^-\rangle |n\rangle$$

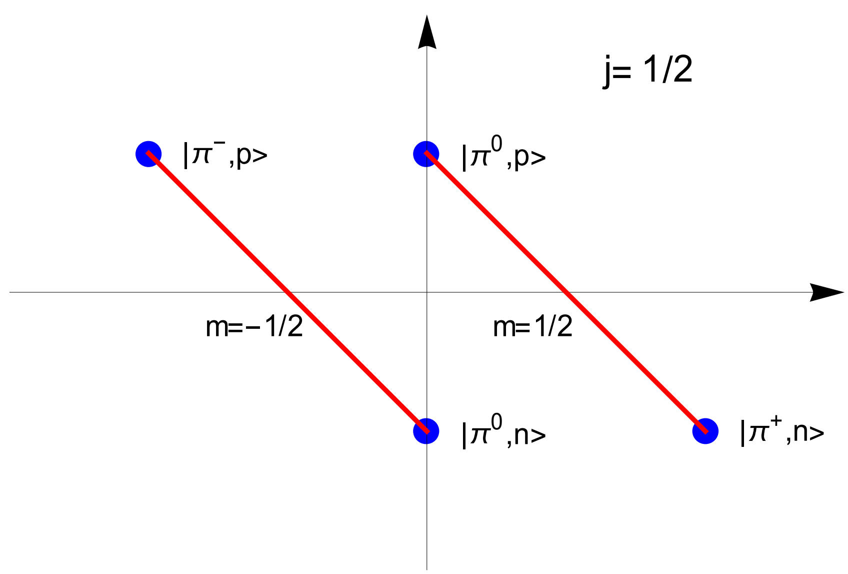
and

$$\left|\pi, N; \frac{1}{2}, \frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}} |\pi^+\rangle |n\rangle - \frac{1}{\sqrt{3}} |\pi^0\rangle |p\rangle ,$$

$$\left|\pi, N; \frac{1}{2}, -\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|\pi^{0}\right\rangle\left|n\right\rangle - \sqrt{\frac{2}{3}}\left|\pi^{-1}\right\rangle\left|p\right\rangle$$

where *N*:nucleon and π: meson.





From these equations, we also get

$$\left|\pi^{+}\right\rangle\left|p\right\rangle = \left|\pi, N; \frac{3}{2}, \frac{3}{2}\right\rangle$$

$$\left|\pi^{+}\right\rangle\left|n\right\rangle = \frac{1}{\sqrt{3}}\left|\pi, N; \frac{3}{2}, \frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|\pi, N; \frac{1}{2}, \frac{1}{2}\right\rangle$$

$$\left|\pi^0\right\rangle\left|p\right\rangle = \sqrt{\frac{2}{3}}\left|\pi, N; \frac{3}{2}, \frac{1}{2}\right\rangle - \frac{1}{\sqrt{3}}\left|\pi, N; \frac{1}{2}, \frac{1}{2}\right\rangle$$

$$\left|\pi^0\right\rangle\left|n\right\rangle = \sqrt{\frac{2}{3}}\left|\pi, N; \frac{3}{2}, -\frac{1}{2}\right\rangle + \frac{1}{\sqrt{3}}\left|\pi, N; \frac{1}{2}, -\frac{1}{2}\right\rangle$$

$$\left|\pi^-\right\rangle\left|p\right\rangle = \frac{1}{\sqrt{3}}\left|\pi, N; \frac{3}{2}, -\frac{1}{2}\right\rangle - \sqrt{\frac{2}{3}}\left|\pi, N; \frac{1}{2}, -\frac{1}{2}\right\rangle$$

$$\left|\pi^-\right\rangle\left|n\right\rangle = \left|\pi, N; \frac{3}{2}, -\frac{3}{2}\right\rangle$$

((**Mathematica**)) Clebsch-Gordan co-efficient : $T_1 = 1$ and $t_2 = 1/2$.

```
Clear["Global`*"];

CCGG[{j1_, m1_}, {j2_, m2_}, {j_, m_}] :=
 Module[{s1},
  s1 = If[Abs[m1] ≤ j1 && Abs[m2] ≤ j2 && Abs[m] ≤ j,
    ClebschGordan[{j1, m1}, {j2, m2}, {j, m}], 0]]

CG[{j_, m_}, j1_, j2_] :=
 Sum[CCGG[{j1, m1}, {j2, m - m1}, {j, m}] a[j1, m1]
   b[j2, m - m1], {m1, -j1, j1}]
```

j1 = 1 and j2 = 1/2;

```
j1 = 1; j2 = 1/2;

CG[{3/2, 3/2}, j1, j2]
```

$$a[1,1]\, b\left[\frac{1}{2}, \frac{1}{2}\right]$$

```
CG[{3/2, 1/2}, j1, j2]
```

$$\frac{a[1,1]\, b\left[\frac{1}{2}, -\frac{1}{2}\right]}{\sqrt{3}} + \sqrt{\frac{2}{3}}\, a[1,0]\, b\left[\frac{1}{2}, \frac{1}{2}\right]$$

```
CG[{3/2, -1/2}, j1, j2]
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$$\sqrt{\frac{2}{3}}\, a[1,0]\, b\left[\frac{1}{2}, -\frac{1}{2}\right] + \frac{a[1,-1]\, b\left[\frac{1}{2}, \frac{1}{2}\right]}{\sqrt{3}}$$

```
CG[{3/2, -3/2}, j1, j2]
```

$$a[1,-1]\, b\left[\frac{1}{2}, -\frac{1}{2}\right]$$

```
CG[{1/2, 1/2}, j1, j2]
```

$$\sqrt{\frac{2}{3}}\, a[1,1]\, b\left[\frac{1}{2}, -\frac{1}{2}\right] - \frac{a[1,0]\, b\left[\frac{1}{2}, \frac{1}{2}\right]}{\sqrt{3}}$$

```
CG[{1/2, -1/2}, j1, j2]
```

We consider the reactions

Elastic process:

(a) $\pi^+ + p \to \pi^+ + p$

(b) $\pi^0 + p \to \pi^0 + p$

(c) $\pi^- + p \to \pi^- + p$

(d) $\pi^+ + n \to \pi^+ + n$

(e) $\pi^0 + n \to \pi^0 + n$

(f) $\pi^- + n \to \pi^- + n$

Charge exchange processes:

(g) $\pi^+ + n \to \pi^0 + p$

(h) $\pi^0 + p \to \pi^+ + n$

(i) $\pi^0 + n \to \pi^- + p$

(j) $\pi^- + p \to \pi^0 + n$

Then the reactions (a) and (f) are pure 3/2.

(a) $\pi^+ + p \to \pi^+ + p$,

(f) $\pi^- + n \to \pi^- + n$

$$A_a = A_f = A_{3/2}$$

Other reactions are mixture (coefficients given by the Clebsch-Gordans),

(c) $\pi^- + p \to \pi^- + p$

(j) $\pi^- + p \to \pi^0 + n$

$$A_c = \frac{1}{3}A_{3/2} + \frac{2}{3}A_{1/2}, \qquad A_j = \frac{\sqrt{2}}{3}A_{3/2} - \frac{\sqrt{2}}{3}A_{1/2}$$

With a proportionality constant K equal for all we obtain

$$\sigma(\pi^+ + p \to \pi^+ + p) = K\left|A_{3/2}\right|^2$$

$$\sigma(\pi^- + n \to \pi^- + n) = K\left|A_{3/2}\right|^2$$

$$\sigma(\pi^- + p \to \pi^- + p) = K\left|\frac{1}{3}A_{3/2} + \frac{2}{3}A_{1/2}\right|^2$$

$$\sigma(\pi^- + p \to \pi^0 + n) = K\left|\frac{\sqrt{2}}{3}A_{3/2} - \frac{\sqrt{2}}{3}A_{1/2}\right|^2$$

If $A_{12} = 0$, then we have

$$\begin{aligned}
&\sigma(\pi^+ + p \to \pi^+ + p) : \sigma(\pi^- + p \to \pi^- + p) : \sigma(\pi^- + p \to \pi^0 + n) \\
&\quad = K|A_{3/2}|^2 : \frac{K}{9}|A_{3/2}|^2 : \frac{2K}{9}|A_{3/2}|^2 \\
&\quad = 9 : 1 : 2
\end{aligned}$$

We define

$$\sigma_{total}(\pi^+ p) = \sigma(\pi^+ + p \to \pi^+ + p)$$

$$\sigma_{total}(\pi^- p) = \sigma(\pi^- + p \to \pi^- + p) + \sigma(\pi^- + p \to \pi^0 + n)$$

Then the isospin symmetry predicts

$$\frac{\sigma_{total}(\pi^+ p)}{\sigma_{total}(\pi^- p)} = \frac{9}{3} = 3$$

Figure shows the two total cross-sections at low energies. There are clear peaks with Breit-Wigner forms at a mass of 1232 MeV corresponding to the production of the hadronic resonance D(1232) (I = 3/2) and the ratio of the peaks is in good agreement with the prediction (=3).

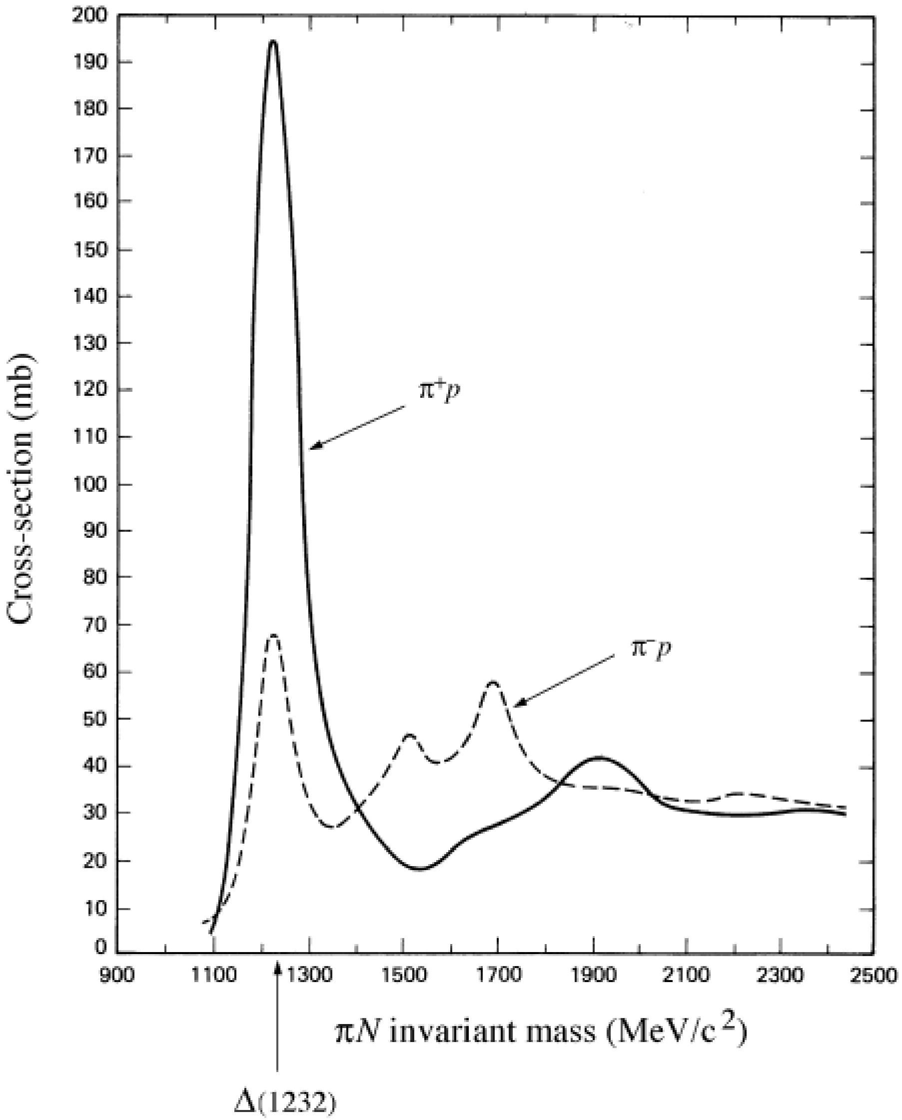

Fig. Total cross-sections for $\pi^- p$ and $\pi^+ p$ scattering. (B.R. Martin, Nuclear and Particle Physics).

REFERENCES

S. Tomonaga, *The Story of Spin* (University of Chicago Press, 1997).
E.D. Commins, *Quantum Mechanics: An Experimentalist's Approach* (Cambridge, 2014).
M. Thomson, *Modern Particle Physics* (Cambridge, 2013).
A. Das and T. Ferbel, Introduction to Nuclear and Particle Physics, 2nd edition (World Scientific, 2005).
A. Bettini; Introduction to Elementary Particle Physics (Cambridge, 2008).
B.R. Martin: Nuclear and Particle Physics: An Introduction (John Wiley & Sons, 2006)

APPENDIX

Hadron	*Mass* (MeV/c^2)	I	I_3
p	938.3	1/2	1/2
n	939.6	1/2	−1/2
π^+	139.6	1	1
π^0	135.0	1	0
π^-	139.6	1	−1
K^+	494.6	1/2	1/2
K^0	497.7	1/2	−1/2
$\overline{K}^0$	497.7	1/2	1/2
K^-	494.6	1/2	−1/2
η^0	548.8	0	0
Λ^0	1115.6	0	0
Σ^+	1189.4	1	1
Σ^0	1192.6	1	0
Σ^-	1197.4	1	−1
Ω^-	1672.4	0	0

Table: Isotopic spin assignments of a representative group of relatively long-lived hadrons