Laser Physics

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Laser: Light Amplification by Stimulated Emission of Radiation

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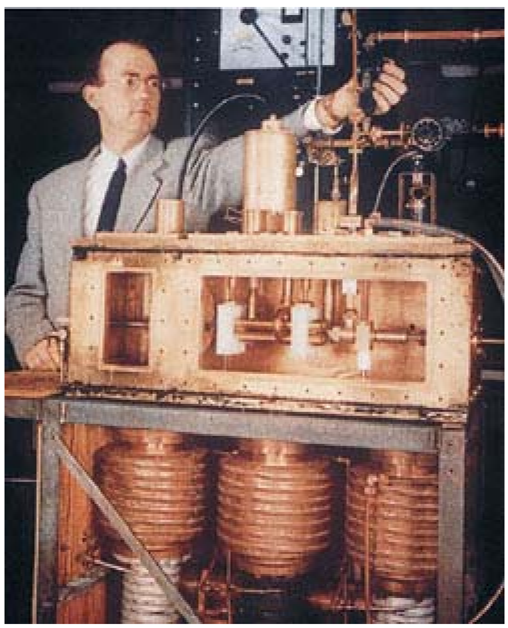
Charles Hard Townes (born July 28, 1915) is an American Nobel Prize-winning physicist and educator. Townes is known for his work on the theory and application of the maser, on which he got the fundamental patent, and other work in quantum electronics connected with both maser and laser devices. He shared the Nobel Prize in Physics in 1964 with Nikolay Basov and Alexander Prokhorov. The Japanese FM Towns computer and game console is named in his honor.
http://en.wikipedia.org/wiki/Charles_Hard_Townes



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Nikolay Gennadiyevich Basov (Russian: Никола́й Генна́диевич Ба́сов; 14 December 1922 – 1 July 2001) was a Soviet physicist and educator. For his fundamental work in the field of quantum electronics that led to the development of laser and maser, Basov shared the 1964 Nobel Prize in Physics with Alexander Prokhorov and Charles Hard Townes.

http://en.wikipedia.org/wiki/Nikolay_Basov

Alexander Mikhaylovich Prokhorov (Russian: Алекса́ндр Миха́йлович Про́хоров) (11 July 1916– 8 January 2002) was a Russian physicist known for his pioneering research on lasers and masers for which he shared the Nobel Prize in Physics in 1964 with Charles Hard Townes and Nikolay Basov.

http://en.wikipedia.org/wiki/Alexander_Prokhorov

Nobel prizes related to laser physics

1964

Charles H. Townes, Nikolai G. Basov, and *Alexandr M. Prokhorov* for developing masers (1951–1952) and lasers.

1981

Nicolaas Bloembergen and *Arthur L. Schawlow* for developing laser spectroscopy and *Kai M. Siegbahn* for developing high-resolution electron spectroscopy (1958).

1989

Norman Ramsay for various techniques in atomic physics; and *Hans Dehmelt* and *Wolfgang Paul* for the development of techniques for trapping single charge particles.

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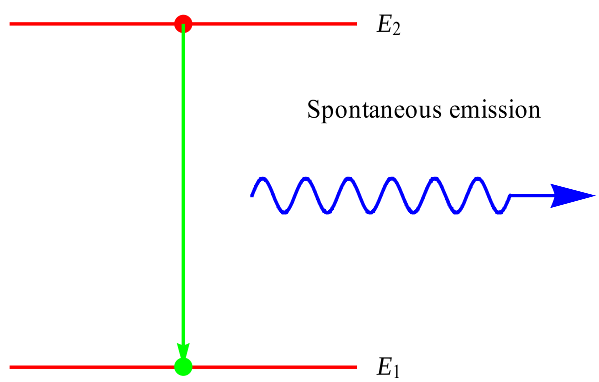
1. Type of transition in atoms due to radiation

We consider the transitions between two energy states of an atom in the presence of an electromagnetic field. There are three types of transitions, spontaneous emission, stimulated emission , and absorption.

(i) Spontaneous emission

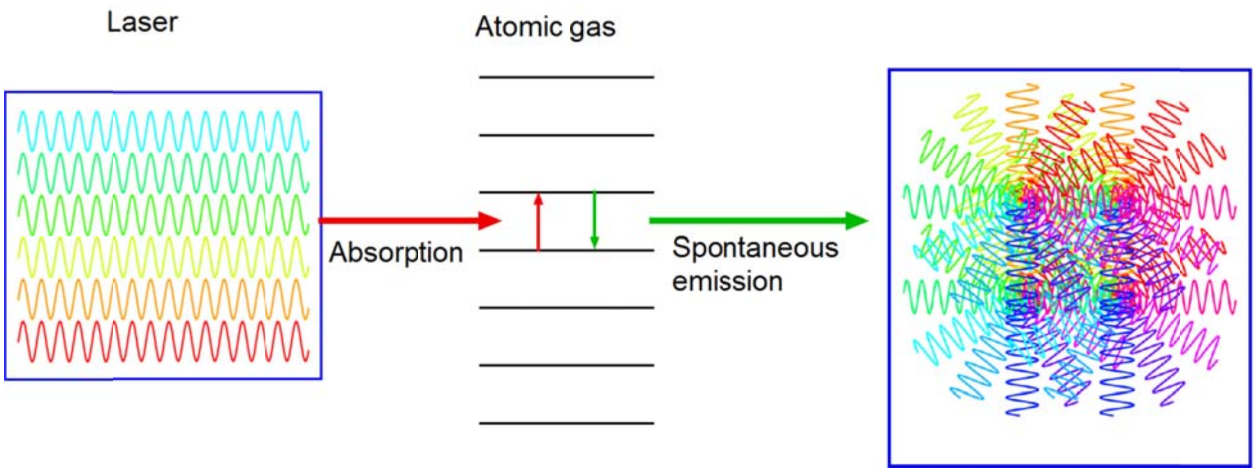
In the spontaneous emission, process, the atom is initially in the upper state of energy E_2 and decays to the lower state of energy E_1 by the emission of a photon with the energy $\hbar\omega$,

$$\hbar\omega = E_2 - E_1$$



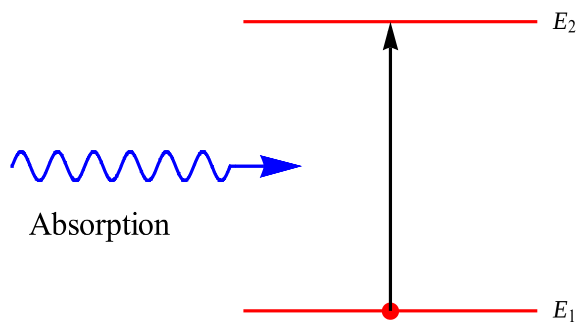
A: Spontaneous probability (completely independent of the incident beam). The light produced by spontaneous emission (random orientation)

((**Spontaneous emission**))



(ii) Absorption

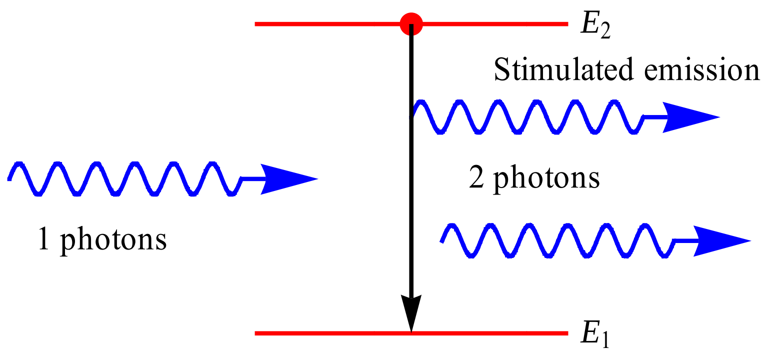
An incident photon with the energy $\hbar\omega$, from an electromagnetic field applied to the atom, stimulates the atom to make a transition of atom from the lower to the higher energy state, the photon being absorbed by the atom.



$B\overline{W}$: absorption probability

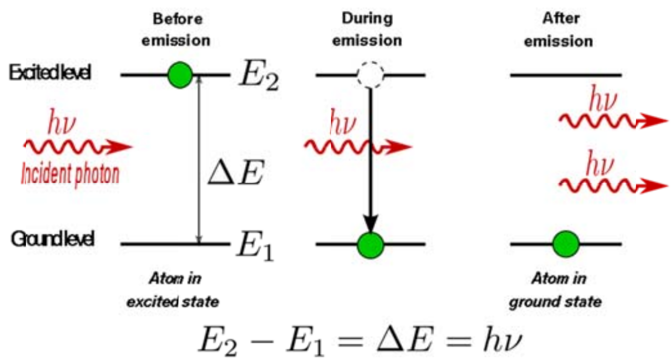
(iii) Stimulated emission

An incident photon with the energy $\hbar\omega$ stimulate the atom to make a transition from the higher to the lower energy state. The atom is left in this lower state at the emergence of two photons of the same energy, the incident one and the emitted one.

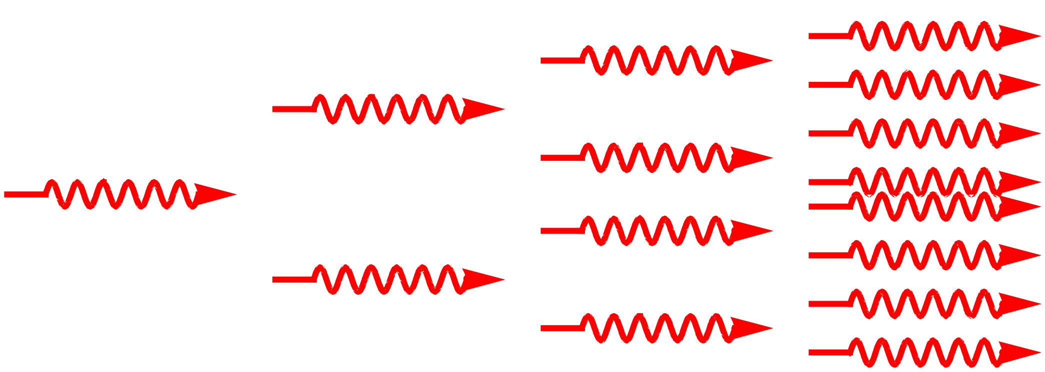


$B\overline{W}$: Stimulated emission probability
(The emitted light the same properties as the incident beam.)

((Note)) Stimulate emission process



((**Amplification**))

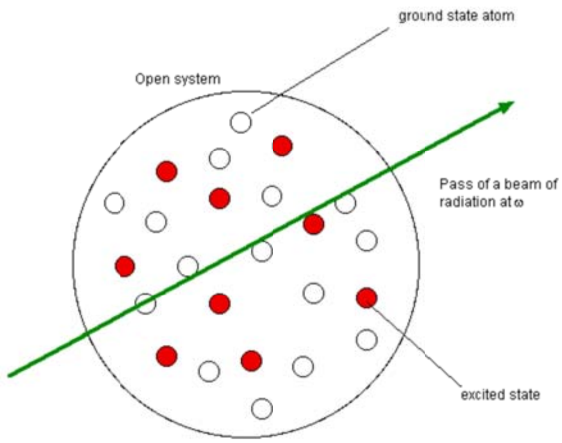


2. Einstein *A* and *B* coefficient

Suppose that a gas of *N* identical atoms is placed in the interior of the cavity:

$$\hbar\omega = E_2 - E_1 .$$

Suppose that two atomic levels are not degenerate. The level populations for the lower level and th upper level are defined by N_1 and N_2.



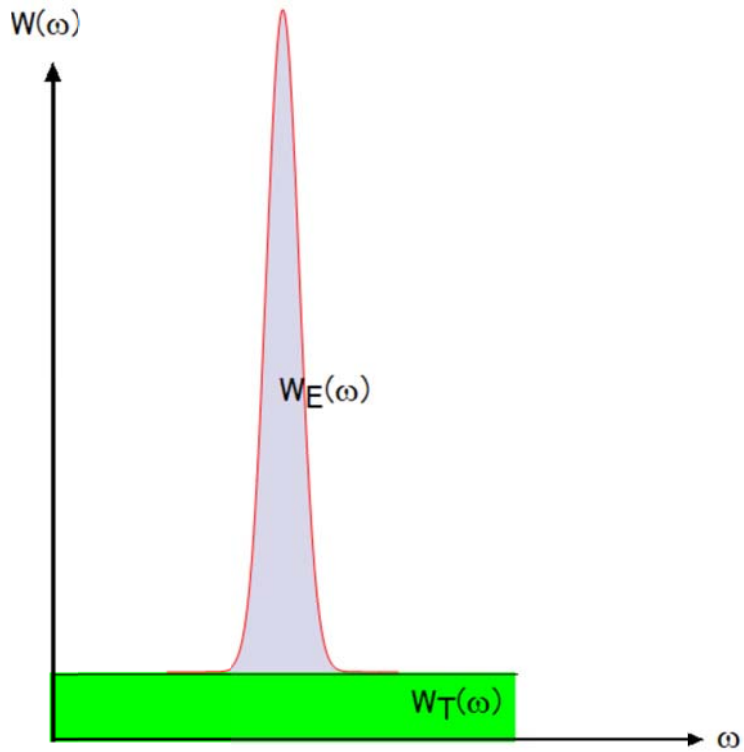
The radioactive process of interest involves the absorption and stimulated emissin associated with the external source.

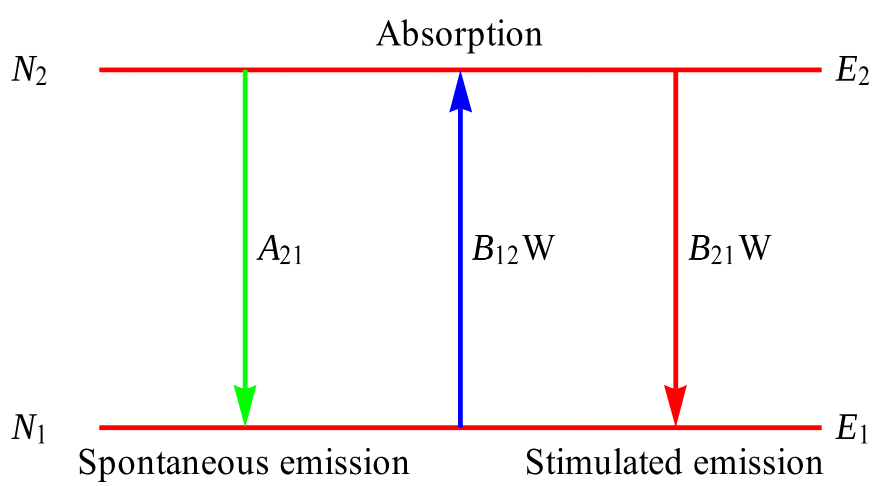
$$\overline{W}(\omega) = \overline{W_T}(\omega) + \overline{W_E}(\omega) ,$$

where

$\overline{W}(\omega)$: cycle-average energy density of radiation at ω
$\overline{W_T}(\omega)$: thermal part
$\overline{W_E}(\omega)$: contribution from some external source of electromagnetic radiation

((**Note**))
The unit of $\overline{W}(\omega)$ is [Js/m^3]





3. Planck's law

The rate of change in N_1 and N_2 is given by

$$\frac{dN_1}{dt} = A_{21}N_2 - N_1B_{12}\overline{W}(\omega) + N_2B_{21}\overline{W}(\omega)$$
$$= -N_1B_{12}\overline{W}(\omega) + N_2[A_{21} + B_{21}\overline{W}(\omega)]$$

$$\frac{dN_2}{dt} = -A_{21}N_2 + N_1B_{12}\overline{W}(\omega) - N_2B_{21}\overline{W}(\omega)$$

$$= N_1B_{12}\overline{W}(\omega) - [A_{21} + N_2B_{21}\overline{W}(\omega)]N_2$$

We consider the case of thermal equilibrium. There is no contribution from the external source. We use the following notation,
Then we have

$$\overline{W}(\omega) = \overline{W_T}(\omega).$$

Since

$$\frac{dN_1}{dt} = \frac{dN_2}{dt} = 0$$

we get

$$N_2A_{21} - N_1B_{12}\overline{W_T}(\omega) + N_2B_{21}\overline{W}_T(\omega) = 0$$

or

$$\overline{W_T}(\omega) = \frac{A_{21}}{\frac{N_1}{N_2}B_{12} - B_{21}}$$

The level populations N_1 and N_2 are related in thermal equilibrium by the Boltzmann's law

$$\frac{N_1}{N_2} = \frac{e^{-\beta E_1}}{e^{-\beta E_2}} = \exp(\beta\hbar\omega), \qquad (\beta = 1/k_B T)$$

Then we get

$$\overline{W}_T(\omega) = \frac{A_{21}}{B_{12}e^{\beta\hbar\omega} - B_{21}}$$

In the Black body problem, we show that the Planck's law for the radiative energy density is given by

$$\overline{W}_T(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{e^{\beta\hbar\omega} - 1}$$

Then we have

$$\Rightarrow \quad \begin{cases} B_{12} = B_{21} \\ \dfrac{A_{21}}{B_{12}} = \dfrac{\hbar\omega^3}{\pi^2 c^3} \end{cases}$$

or

$$\overline{W}_T(\omega) = \frac{A_{21}}{B_{12}} \bar{n},$$

where the Bose-Einstein distribution function is given by

$$\bar{n} = \frac{1}{e^{\beta\hbar\omega} - 1}$$

or

$$\frac{A_{21}}{B_{21} \overline{W}_T(\omega)} = e^{\beta\hbar\omega} - 1$$

((Example)) Suppose that $\hbar\omega = k_B T$

For T = 300 K, $\nu_T = 6 \times 10^{12}$ Hz = 6 THz.

For $\hbar\omega \ll k_B T$, $A_{21} \ll B_{21} \overline{W}_T(\omega)$ ($\nu \ll \nu_T$)

For $\hbar\omega \gg k_B T$, $A_{21} \gg B_{21} \overline{W}_T(\omega)$ ($\nu \gg \nu_T$)

For optical experiments that use electromagnetic radiation in the near-infrared, we have visible, ultraviolet region of the spectrum ($\nu \gg 5$ THz).

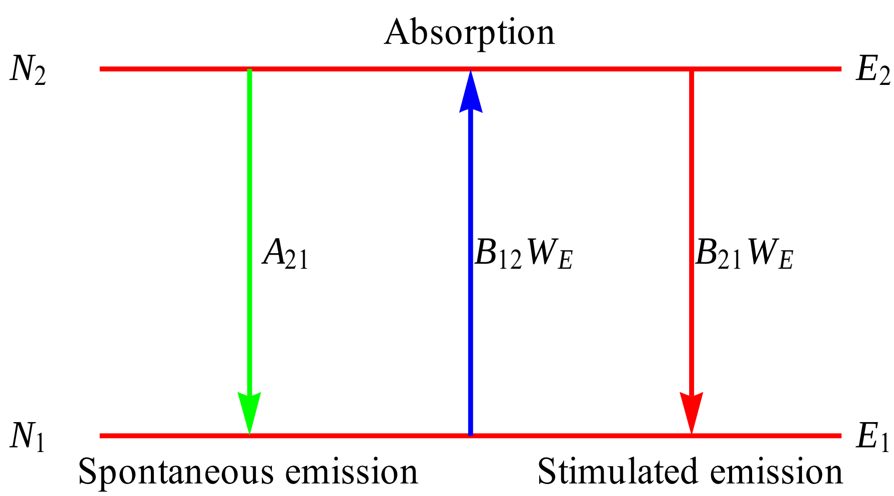
Then we have

(i) $A_{21} \gg B_{21}\overline{W}_T(\omega)$

A_{21}: spontaneous emission rate
B_{21}: rate of thermally stimulated emission

(ii) $\overline{W}(\omega) = \overline{W}_T(\omega) + \overline{W}_E(\omega) \cong \overline{W}_E(\omega)$

Therefore the radioactive process of interest involve the absorption and stimulated emission associated with the external source.



((**Note**))
The unit of the Einstein A and B coefficients:

A: [1/s]
B [m^3/Js^2]

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4. Determination of the Einstein *A* coefficient

$$\frac{dN_1}{dt} = -\frac{dN_2}{dt} = A_{21}N_2 + (N_2 - N_1)B_{12}\overline{W}(\omega)$$

If the incident beam is now turned off, the excited atoms return to their ground state. When $\overline{W}(\omega)$=0 (the beam energy density is zero),

$$\frac{dN_2}{dt} = -A_{21}N_2 = -\frac{N_2}{\tau_{12}}$$

with

$$A_{21} = \frac{1}{\tau_{21}}$$

or

$$N_2 = N_2{}^0 \exp(-\frac{t}{\tau_{12}})$$

where τ_{21} is called the fluorescent or radioactive life time of the transition. The observation of the fluorescent emission is an experimental means of measuring the Einstein *A* coefficient.

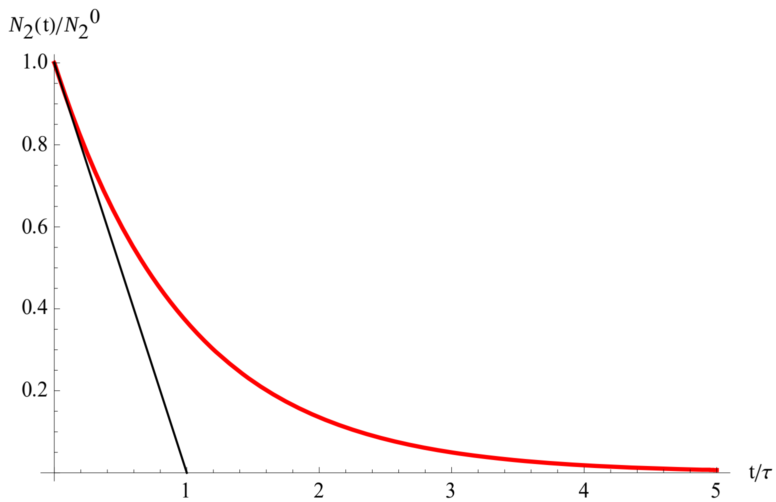


Fig. Plot of $N_2(t)/N_2{}^0$ vs *t*. Exponential decay of the form exp($-t/\tau$).

5. The time dependence of N_1 and N_2 for the two levels

The rate of change of N_1 and N_2 is given by

$$\frac{dN_1}{dt} = A_{21}N_2 - N_1B_{12}\overline{W}(\omega) + N_2B_{21}\overline{W}(\omega)$$
$$= -N_1B_{12}\overline{W}(\omega) + N_2[A_{21} + B_{21}\overline{W}(\omega)]$$

$$\frac{dN_2}{dt} = -A_{21}N_2 + N_1B_{12}\overline{W}(\omega) - N_2B_{21}\overline{W}(\omega)$$
$$= N_1B_{12}\overline{W}(\omega) - [A_{21} + N_2B_{21}\overline{W}(\omega)]N_2$$

where

$$N_1 = N\,, \qquad N_2 = 0 \qquad \text{at } t = 0 \qquad \text{(initial condition)},$$

and

$$N_1 + N_2 = N \ (= \text{const}).$$

The solution of these equations are given as follows.

$$N_1 = \frac{NB\overline{W}}{A+2B\overline{W}}\exp[-(A+2B\overline{W})t] + \frac{N(A+B\overline{W})}{A+2B\overline{W}},$$

$$N_2 = N - N_1$$
$$= \frac{NB\overline{W}}{A+2B\overline{W}}\{1-\exp[-(A+2B\overline{W})t]\}$$

and

$$\frac{N_2}{N_1} = \frac{\{1-\exp[-(A+2B\overline{W})t]\}}{(1+\frac{A}{B\overline{W}})+\exp[-(A+2B\overline{W})t]}$$

(i) For $(A+2B\overline{W})t \ll 1$,

$$N_2 = NB\overline{W}$$

The atoms contain a constant amount of stored energy

$$N_2\hbar\omega = \frac{NB\overline{W}\hbar\omega}{A+2B\overline{W}} = \frac{N\overline{W}\hbar\omega}{\frac{A}{B}+2\overline{W}} = \frac{N\overline{W}\hbar\omega}{\frac{\hbar\omega^3}{\pi^2c^3}+2\overline{W}}$$

(ii) For $(A+2B\overline{W})t \gg 1$

$$N_2 = \frac{NB\overline{W}}{A + 2B\overline{W}} = N \frac{\frac{B\overline{W}}{A}}{1 + 2\frac{B\overline{W}}{A}}$$

When $A >> B\overline{W}$, the most of atoms remain in their ground state.

$$N_2 = \frac{NB\overline{W}}{A + 2B\overline{W}} = N \frac{B\overline{W}}{A} \propto \overline{W}$$

When $\frac{B\overline{W}}{A} >> 1$, the dependence of N_2 on the beam intensity becomes nonlinear. This nonlinear behavior is called *saturation* of the atomic transition.

In the steady state ($t = \infty$), we have

$$N_1 = \frac{N(A + B\overline{W})}{A + 2B\overline{W}}, \qquad N_2 = \frac{NB\overline{W}}{A + 2B\overline{W}},$$

and

$$\frac{N_2}{N_1} = \frac{\frac{B\overline{W}}{A}}{1 + \frac{B\overline{W}}{A}}.$$

We make a plot of N_2/N as a function of $\frac{B\overline{W}}{A}$. In the limit of $\frac{B\overline{W}}{A} \rightarrow \infty$, N_2/N becomes 1/2. Which means that the population inversion ($N_2 > N_1$) does not occur in the two-level system.

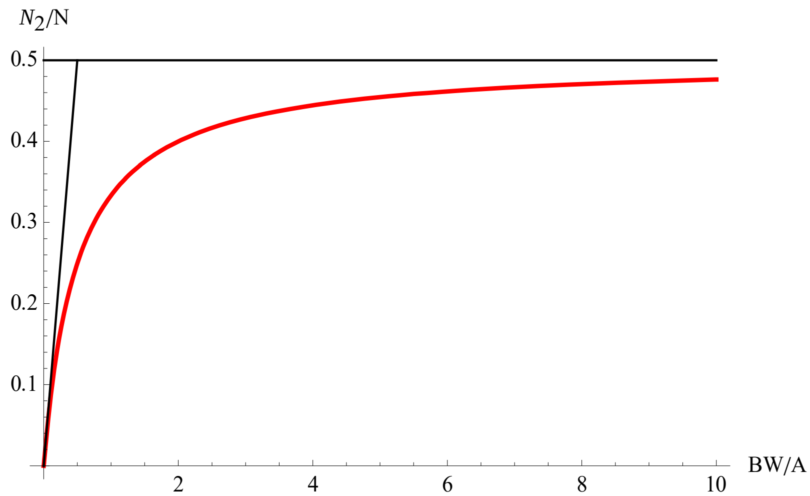


Fig. Plot of N_2/N vs $\dfrac{B\overline{W}}{A}$.

6. Population inversion and negative temperature

In thermal equilibrium, $N_2<N_1$. If we have a means of inverting the normal population of states so that $N_2>N_1$ (negative temperature, population inversion). Then the emission would exceed the absorption rate. This means that the applied radiation of the energy $\hbar\omega$ will be amplified in intensity by the interaction process.

7. Three-level laser (I)

To achieve non-equilibrium conditions, an indirect method of populating the excited state must be used. To understand how this is done, we may use a slightly more realistic model, that of a *three-level laser*. Again consider a group of N atoms, this time with each atom able to exist in any of three energy states, levels 1, 2 and 3, with energies E_1, E_2, and E_3, and populations N_1, N_2, and N_3, respectively. Note that $E_1 < E_2 < E_3$; that is, the energy of level 2 lies between that of the ground state and level 3. The population inversion can be achieved for levels 1 and 2 by experiments that make use of the other energy levels (level 3).

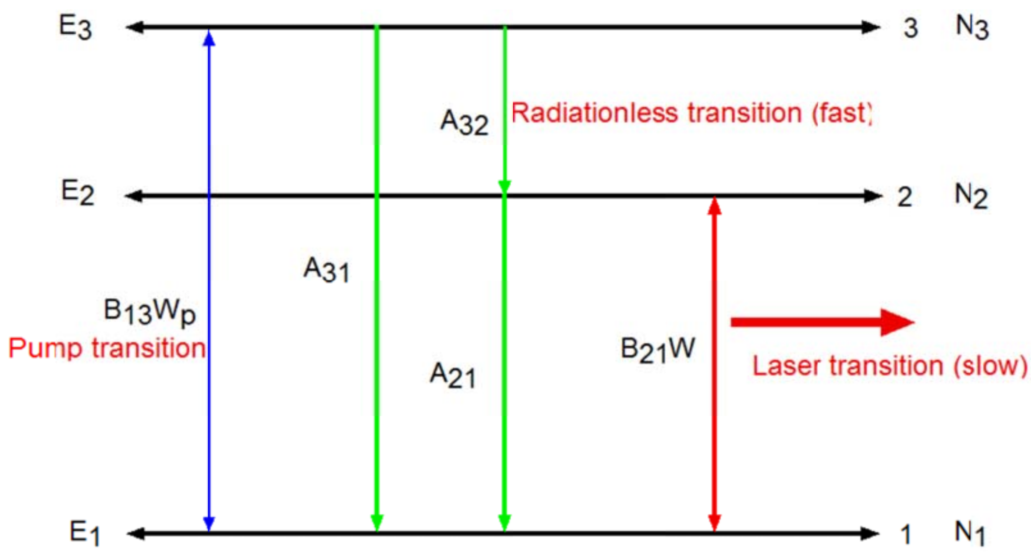


Fig. A transition in the three-level laser. The lasing transition takes the system from the metastable state (E_2) to the ground state (E_1). The popolation inversion occurs when A_{32}>>A_{21} (or τ_{21}>>τ_{32})

We consider the rate of change in N_1, N_2, and N_3,

$$\frac{dN_1}{dt} = B_{13}\overline{W}_p(N_3 - N_1) + A_{21}N_2 + A_{31}N_3 + B_{21}\overline{W}(N_2 - N_1)$$

$$\frac{dN_2}{dt} = -A_{21}N_2 + B_{21}\overline{W}(N_1 - N_2) + A_{32}N_3$$

$$\frac{dN_3}{dt} = -(A_{32} + A_{31})N_3 + B_{13}\overline{W}_p(N_1 - N_3)$$

where

$$N_1 + N_2 + N_3 = N$$

We consider the case of the thermal equilibrium ($dN_1/dt = 0$, $dN_2/dt = 0$, and $dN_3/dt = 0$). The net rate at which atoms are promoted from state 1 (ground state) into state 3 by the pump,

$$Nr = \overline{W}_p B_{23}(N_1 - N_3) = \frac{N(A_{31} + A_{32})(A_{21} + B_{21}\overline{W})B_{13}\overline{W}_p}{(2A_{31} + 2A_{32})B_{21}\overline{W} + (A_{32} + 2A_{21} + 3B_{21}\overline{W})B_{13}\overline{W}_p + A_{21}(A_{31} + A_{32})}$$

$$N_1 = \frac{A_{31} + A_{32} + B_{13}\overline{W_p}}{(A_{31} + A_{32})B_{13}\overline{W_p}} Nr$$

$$N_2 = \frac{(A_{31} + A_{32})B_{21}\overline{W} + (A_{32} + B_{21}\overline{W})B_{13}\overline{W_p}}{(A_{31} + A_{32})(A_{21} + B_{21}\overline{W})B_{13}\overline{W_p}} Nr$$

$$\Delta N = N_2 - N_1 = \frac{(A_{32} - A_{21})B_{13}\overline{W_p} - A_{21}(A_{31} + A_{32})}{(A_{31} + A_{32})(A_{21} + B_{21}\overline{W})B_{13}\overline{W_p}} Nr$$

The inversion population occurs when

$$\frac{N_2}{N_1} > 1$$

or

$$B_{13}\overline{W_p} > \frac{A_{21}(A_{31} + A_{32})}{A_{32} - A_{21}} = \frac{\frac{1}{\tau_{31}} + \frac{1}{\tau_{32}}}{\tau_{21}(\frac{1}{\tau_{32}} - \frac{1}{\tau_{21}})}$$

where the relaxation times are related by the constant *A* for each transition,

$$A_{32} = \frac{1}{\tau_{32}}, \qquad A_{21} = \frac{1}{\tau_{21}}, \qquad A_{31} = \frac{1}{\tau_{31}}$$

In the limit of $\overline{W}_p \to \infty$, the difference ΔN (= N_2 - N_1) is given

$$\Delta N = \frac{(A_{32} - A_{21})N}{2A_{21} + A_{32} + 3B_{21}\overline{W}} > 0$$

The appropriate condition for $\Delta N > 0$ is that

$$\tau_{21} > \tau_{32}$$

In summary, atoms in the ground state (E_1) are pumped to a higher state (E_3) by some external source of energy. The atom then decay quickly to the state (E_2). A large number of atoms can exist for a relatively long time at E_2, where they just waiting for a photon to come along and stimulate the transition to E_1. In the normal operation many more atoms will be in the excited (metastable state) than in the ground state: a population inversion, which is the essential feature for the lasing.

((**Potential difficulty**))

There is a potential difficulty with the three-level system. What happens to an atom after it has been returned to the ground state E_1 by the stimulated emission. The level 1 (E_1) is the ground state. In thermal equilibrium, a large fraction of atoms is in the ground state. Since $N_1>N_2$ in thermal equilibrium, sufficiently strong power supply is needed for the pumping, leading to the population inversion ($N_2>N_1$). This is a disadvantage for the three-level laser.

8. Ruby laser

The first laser was built by Theiredore Maiman. It consists of small rod of ruby (a few cm) surrounded by a helical gaseous flashtube. The ends of ruby rod are flat and perpendicular to the axis of the rod. Ruby is a transparent crystal of Al_2O_3 containing a small amount of Cr. It appears red because of Cr^{3+}.

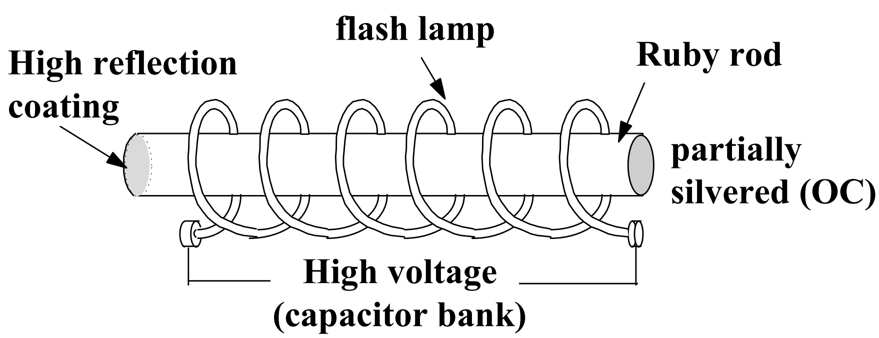


Fig. The first laser invented (in 1960). Pump source: flash lamp. Classic 3-level laser. Very low repetition rate ~1 pulse/min. Since its repetition rate is so slow, nobody wants to use this type of laser these days. Laser wavelength 694.3 nm, pulse width ~ 10-8 sec.

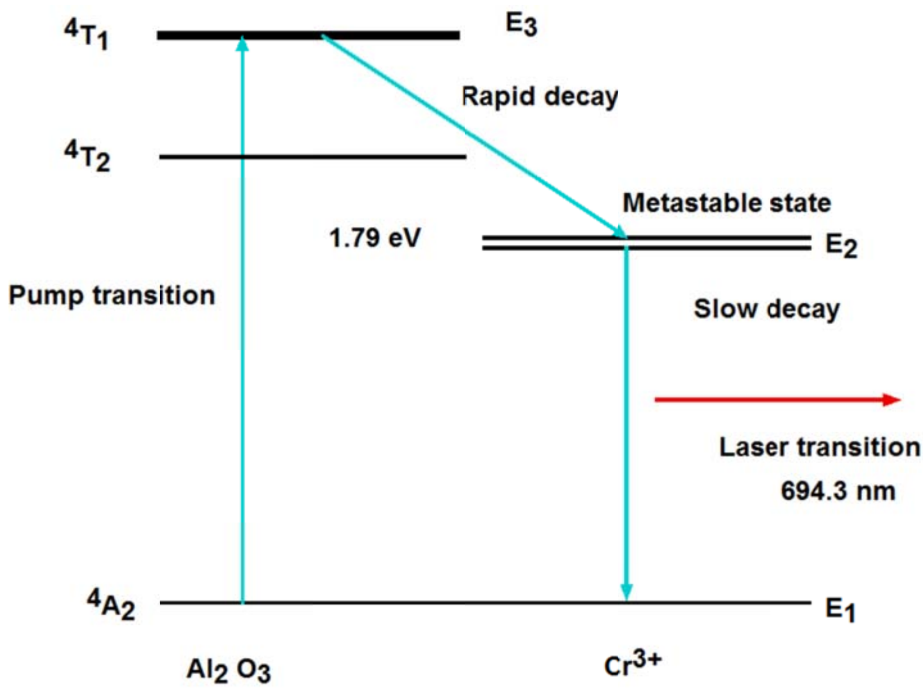
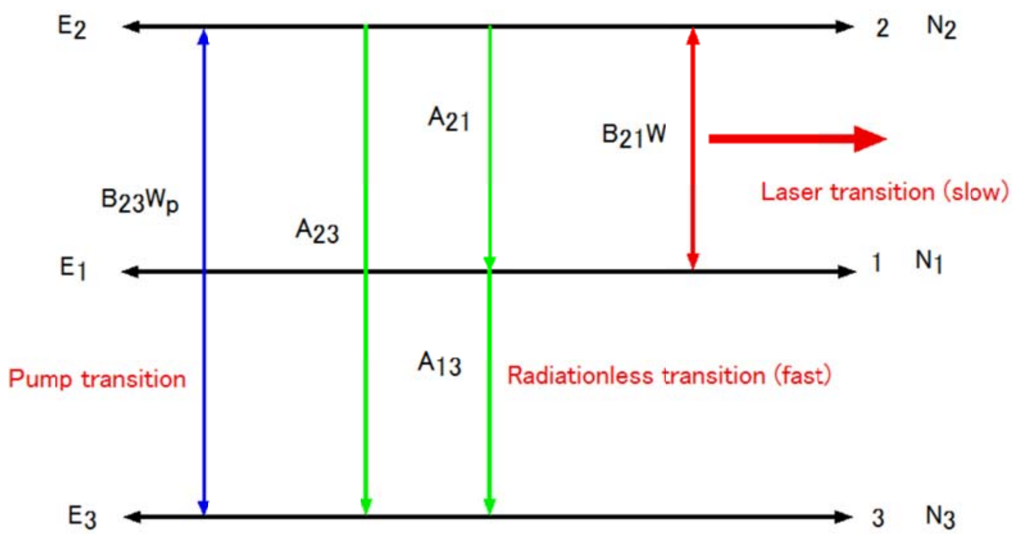


Fig. Laser transition for Ruby laser. λ = 694.3 nm.

When the mercury- or xenon-filled flashtube is fired, there is an intense burst of light lasting a few ms. Absorption excites many of the Cr^{3+} ions to the bands of energy levels called *pump levels*. The excited Cr^{3+} ions give up their energy to the crystal in nonradiative transitions and drop down to a pair of metastable states labeled E_2. These metastable states are about 1.79 eV above the ground state. If the flash is intense enough, more atoms will make the transition to the states E_2 than remain in the ground state. As a result, the populations of the ground state and the metastable states become inverted. When some of the atoms in the states E_2 decay to the ground state by spontaneous emission, they emit photons of energy 1.79 eV and wavelength 694.3 nm. Some of these photons then stimulate other excited atoms to emit photons of the same energy by the stimulated emisison, and moving in the same direction with the same phase.

9. Three-level laser (II)

We consider another type of the three-level laser.



The rate of change in N_1, N_2, and N_3;

$$\frac{dN_2}{dt} = -N_2A_{21} - N_2A_{23} - \overline{W}B_{21}(N_2 - N_1) + \overline{W_p}B_{23}(N_3 - N_2)$$

$$\frac{dN_1}{dt} = N_2A_{21} - N_1A_{13} + \overline{W}B_{21}(N_2 - N_1)$$

$$\frac{dN_3}{dt} = N_2A_{23} + N_1A_{13} - \overline{W_p}B_{23}(N_3 - N_2)$$

The net rate at which atoms are promoted into state 2 by the pump,

$$Nr = \overline{W}_pB_{23}(N_3 - N_2) = \frac{B_{23}N[A_{23}B_{21}\overline{W} + A_{13}(A_{21} + A_{23} + B_{21}\overline{W})]\overline{W}_p}{A_{23}B_{21}\overline{W} + B_{23}(A_{21} + 3B_{21}\overline{W})\overline{W}_p + A_{13}(A_{21} + A_{23} + B_{21}\overline{W} + 2B_{23}\overline{W}_p)}$$

Then we have

$$N_1 = \frac{(A_{21} + B_{21}\overline{W})Nr}{(A_{13} + A_{23})B_{21}\overline{W} + A_{13}(A_{21} + A_{23})}$$

$$N_2 = \frac{(A_{13} + B_{21}\overline{W})Nr}{(A_{13} + A_{23})B_{21}\overline{W} + A_{13}(A_{21} + A_{23})}$$

$$\Delta N = N_2 - N_1 = \frac{(A_{13} - A_{21})Nr}{(A_{23} + A_{13})B_{21}\overline{W} + A_{13}(A_{21} + A_{23})}$$

$$\frac{N_2}{N_1} = \frac{(A_{13} + B_{21}\overline{W})}{(A_{21} + B_{21}\overline{W})}$$

For $N_2 > N_1$ (the population inversion), the condition

$$A_{13} > A_{21} \qquad \text{or} \qquad \tau_{21} > \tau_{13}$$

must be satisfied.

10. He-Ne laser

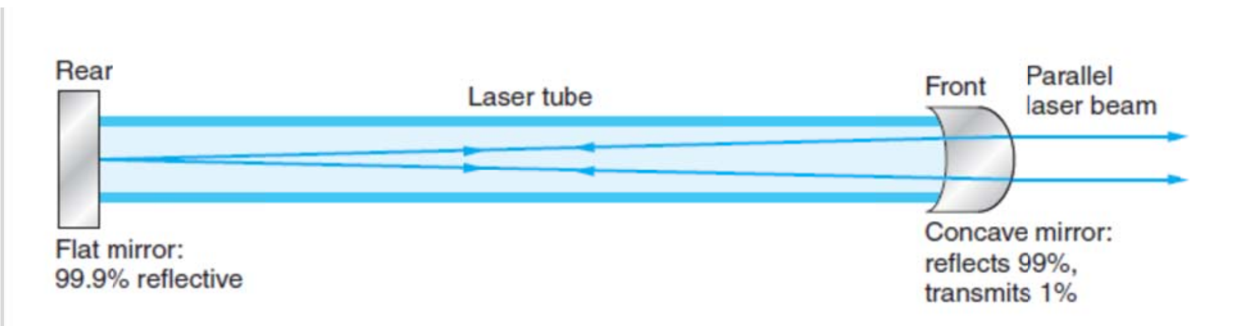


Fig. Schematic diagram of He-Ne laser. $\lambda = 632.8$ nm

The red He-Ne laser ($\lambda = 632.8$ nm) use transitions between energy levels in both He and Ne. The applied voltage excites a He atom from its ground state to an excited levels at 19.72 eV and 20.61 eV above the ground state. An excited He atom will occasionally collide with a Ne atom and transfer its excess energy to the energy states of Ne atom (19.83 eV and 20.66 eV) above the ground state. The population inversion is thus achieved between the $2p^5 5s^1$ level (metastable) (20.66 eV) and the $2p^5 3s^1$ state (18.70 eV). The lasing process then proceeds, resulting in a coherent beam of red light. Emission from the 19.83 eV to 18.70 eV level also produce laser output about 1100 nm.

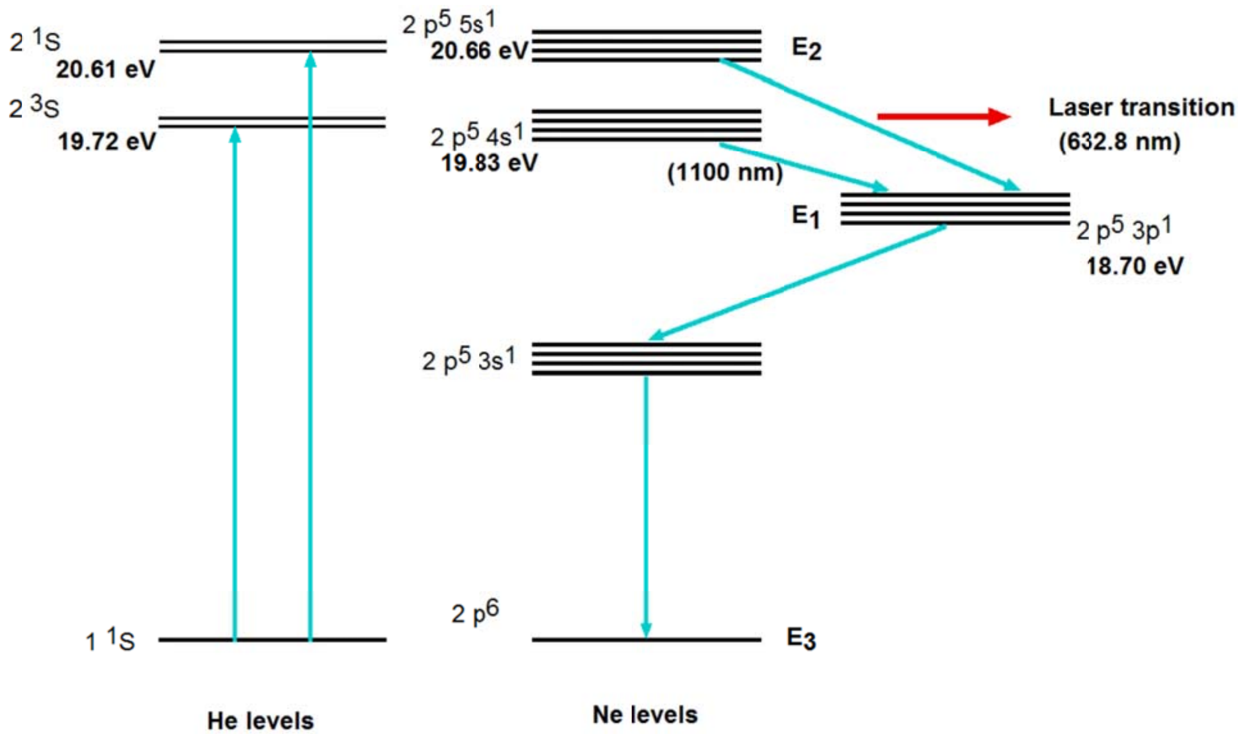
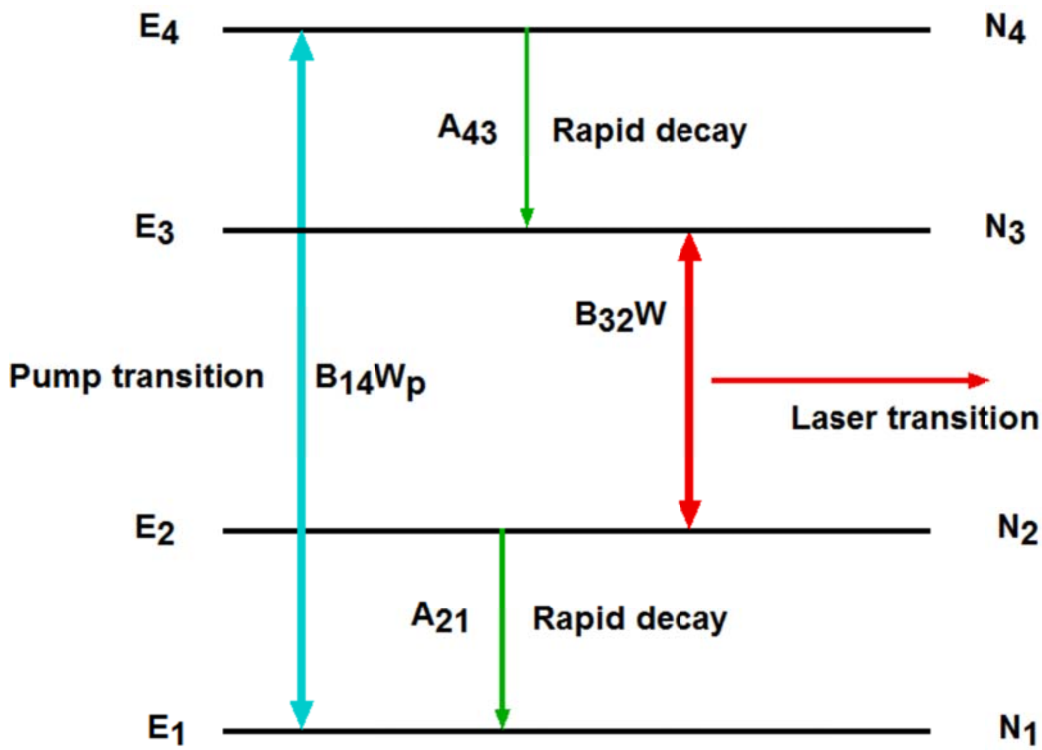


Fig. Laser transition of He-Ne laser. $\lambda = 632.8$ nm.

11. Four-level laser

We consider the case of four-level laser. Nd^{3+}:YAG laser is one of typical example of the four level laser system. We have E_1, N_1 at the ground state. The pumping process raise the atoms from E_1 to E_4, pumping rate is $B_{14}\overline{W}_p$. Atoms at E_4 have rapid decay to E_3, decay time is $1/A_{43}$. Lasing transition occurs between E_3 and E_2. Atoms at E_2 then decay very fast to E_1, the decay time is $1/A_{21}$.

The energy diagram of the four level laser is simplified as follows.



The rate of change in N_1, N_2, N_3, and N_4 is given by

$$\frac{dN_1}{dt} = A_{21}N_2 - \overline{W_p}B_{14}(N_1 - N_4),$$

$$\frac{dN_2}{dt} = -A_{21}N_2 + \overline{W}B_{32}(N_3 - N_2),$$

$$\frac{dN_3}{dt} = A_{43}N_4 - \overline{W}B_{32}(N_3 - N_2),$$

$$\frac{dN_4}{dt} = -A_{43}N_4 + \overline{W_p}B_{14}(N_1 - N_4).$$

We consider the case of the thermal equilibrium ($dN_1/dt = 0$, $dN_2/dt = 0$, $dN_3/dt = 0$, $dN_4/dt = 0$), where

$$N_1 + N_2 + N_3 + N_4 = N$$

$$N_1 = \frac{NA_{21}(A_{43} + B_{14}\overline{W_p})B_{32}\overline{W}}{2(A_{21} + A_{43})B_{14}B_{32}\overline{W}\,\overline{W_p} + A_{21}A_{43}(B_{32}\overline{W} + B_{14}\overline{W_p})}$$

$$N_2 = \frac{NA_{43}B_{14}B_{32}\overline{W_p}\overline{W}}{2(A_{21}+A_{43})B_{14}B_{32}\overline{W}\overline{W_p} + A_{21}A_{43}(B_{32}\overline{W} + B_{14}\overline{W_p})}$$

$$N_3 = \frac{NA_{43}(A_{21}+B_{32}\overline{W})B_{14}\overline{W_p}}{2(A_{21}+A_{43})B_{14}B_{32}\overline{W}\overline{W_p} + A_{21}A_{43}(B_{32}\overline{W} + B_{14}\overline{W_p})}$$

$$N_4 = \frac{NA_{21}B_{14}\overline{W_p}B_{32}\overline{W}}{2(A_{21}+A_{43})B_{14}B_{32}\overline{W}\overline{W_p} + A_{21}A_{43}(B_{32}\overline{W} + B_{14}\overline{W_p})}$$

It is found that the population inversion ($N_3>N_2$) occurs in this system, since

$$\Delta N = N_3 - N_2 = \frac{NA_{21}A_{43}B_{14}\overline{W_p}}{2(A_{21}+A_{43})B_{14}B_{32}\overline{W}\overline{W_p} + A_{21}A_{43}(B_{32}\overline{W} + B_{14}\overline{W_p})} > 0,$$

or

$$\frac{N_3}{N_1} = 1 + \frac{A_{21}}{B_{32}\overline{W}} > 1.$$

In the limit of $\overline{W_p} \to \infty$, we have

$$\lim_{\overline{W_p}\to\infty} \Delta N = \frac{NA_{21}A_{43}}{A_{21}A_{43} + 2(A_{21}+A_{43})B_{32}\overline{W}}$$

((**Discussion**))

For simplification, we assume the relaxation times

$$\tau_{43} = \frac{1}{A_{43}}, \qquad \text{and} \qquad \tau_{21} = \frac{1}{A_{21}}$$

short enough that the pumped atoms to E_4 immediately decay to E_3 and that atoms at E_2 decay so fast.

$$\frac{\Delta N}{N} = \frac{B_{14}\overline{W_p}}{B_{32}\overline{W} + [2(\frac{A_{21}+A_{43}}{A_{21}A_{43}})B_{32}\overline{W} + 1]B_{14}\overline{W_p}} \approx \frac{B_{14}\overline{W_p}}{B_{32}\overline{W} + B_{14}\overline{W_p}} = \frac{\frac{B_{14}\overline{W_p}}{B_{32}\overline{W}}}{1 + \frac{B_{14}\overline{W_p}}{B_{32}\overline{W}}}$$

The levels 2 and 3 are not the ground state. The population inversion ($N_3 > N_2$) can occur readily in comparison with the case of three-level laser, since the population of N_3 and N_4 are much smaller than that of the ground state.

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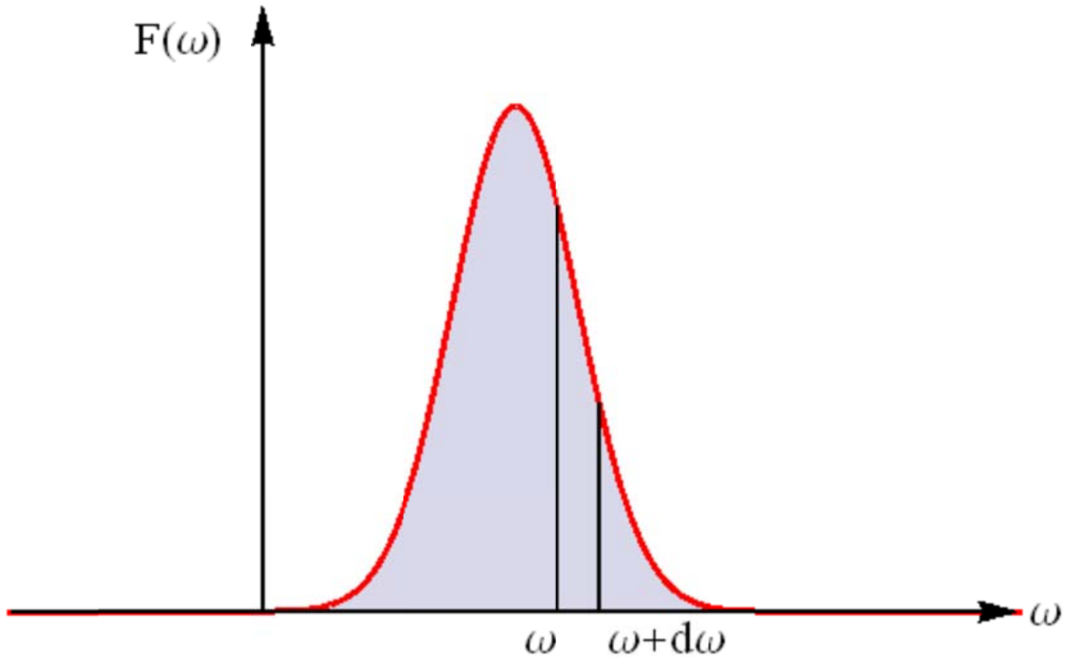
12. Amplification of laser

We consider the two levels where the population of atoms in the upper state is larger than that in the lower state. The inversion population occurs in this system.

$N_2 A\hbar\omega$: the rate at which energy is scattered out of the beam by spontaneous emission.

$B\frac{\overline{W}}{\eta^2}N_1\hbar\omega$: the rate at which energy is taken out of beam by absorption

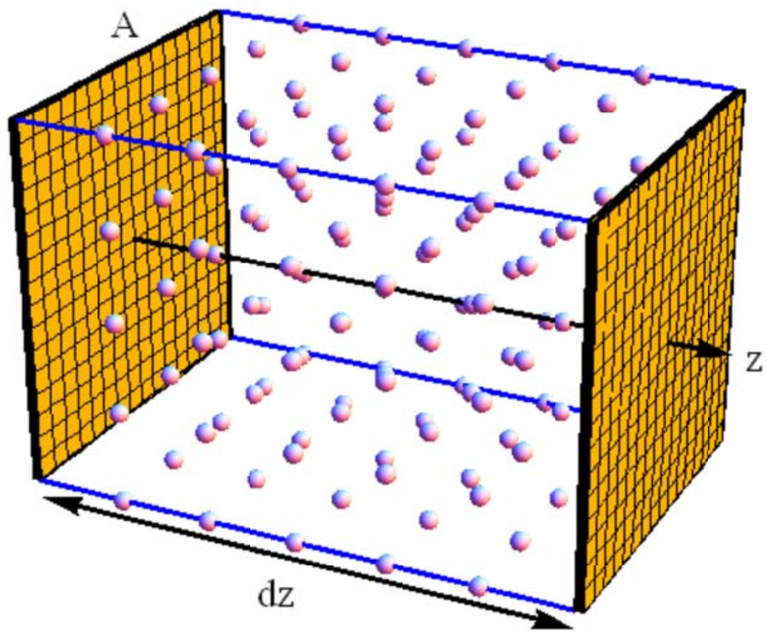
$B\frac{\overline{W}}{\eta^2}N_2\hbar\omega$: the rate at which energy is put back in the beam by stimulated emission.



Here η is the index of refraction of the medium in a cavity. The rate of change of the beam energy in the frequency (ω - ω + dω is equal to

$$B\frac{\overline{W}}{\eta^2}(N_2 - N_1)\hbar\omega F(\omega)\hbar\omega$$

where $F(\omega)$ is the distribution of atomic transition frequencies.



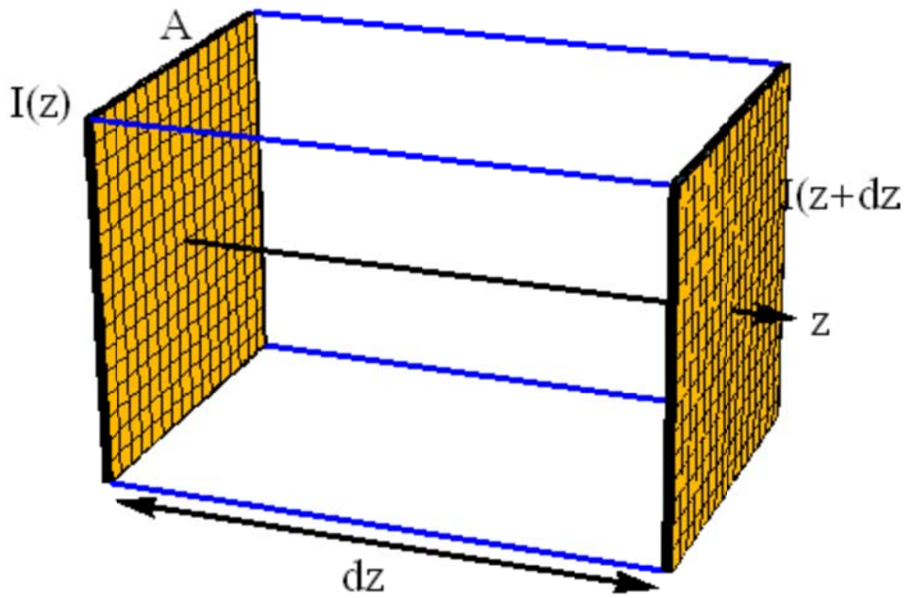
$\overline{W}Adzd\omega$;   beam energy in the slice in the frequency range (ω - ω +dω)

If V is the volume of cavity, $\frac{Adz}{V}$ is the fractional number of atoms that are located in the slice considered.

$$\frac{\partial}{\partial t}(\overline{W}d\omega Adz) = -(N_1 - N_2)F(\omega)d\omega\frac{B\overline{W}}{\eta^2}\hbar\omega\frac{Adz}{V}$$

or

$$\frac{\partial}{\partial t}(\overline{W}) = -(N_1 - N_2)F(\omega)B\overline{W}\frac{\hbar\omega}{v\eta^2}$$



$\bar{I}(z)$: energy crossing unit area in unit time.

$$[\bar{I}(z) - \bar{I}(z+dz)]Ad\omega = -\frac{\partial \bar{I}(z)}{\partial z} dzAd\omega$$

Then we get

$$-\frac{\partial}{\partial t}(\overline{W}Adzd\omega) = -\frac{\partial \bar{I}}{\partial z} dzd\omega A,$$

or

$$\frac{\partial}{\partial t}\overline{W} = \frac{\partial \bar{I}}{\partial z}$$

Using the relation

$$c\overline{W} = \eta \bar{I},$$

Substitution of the above relations into the origional equation yields

or

$$\frac{\partial \bar{I}}{\partial z} = -(N_1 - N_2)F(\omega)B\frac{\eta}{c}\bar{I}\frac{\hbar\omega}{v\eta^2}$$

or

$$\frac{\partial \bar{I}}{\partial z} = (N_2 - N_1)F(\omega)\frac{B\hbar\omega}{Vc\eta}\bar{I}$$

((Note))

$$\overline{W} = \frac{1}{2}\varepsilon_0\eta^2|E|^2, \qquad \bar{I} = \frac{1}{2}\varepsilon_0 c\eta|E|^2$$

13. Amplification for the two-level laser

We consider the case of the two-level laser, where no population inversion occurs.

$$N_1 = \frac{N(A + \frac{B\overline{W}}{\eta^2})}{A + 2\frac{B\overline{W}}{\eta^2}}, \qquad N_2 = \frac{NB\frac{\overline{W}}{\eta^2}}{A + 2B\frac{\overline{W}}{\eta^2}}$$

and

$$N_2 - N_1 = -\frac{NA}{A + 2B\frac{\overline{W}}{\eta^2}}$$

$$\frac{1}{\bar{I}}\frac{\partial \bar{I}}{\partial z} = -\frac{NA}{A + 2B\frac{\overline{W}}{\eta^2}}F(\omega)\frac{B\hbar\omega}{Vc\eta}$$

or

$$\frac{1}{\bar{I}}(1+\frac{2B\bar{I}}{A\eta c})\frac{\partial \bar{I}}{\partial z} = -F(\omega)\frac{NB\hbar\omega}{Vc\eta}$$

For simplicity we put

$$\frac{1}{\bar{I}}(1+\frac{\bar{I}}{\bar{I}_c})\frac{\partial \bar{I}}{\partial z} = -K$$

where

$$K = F(\omega)\frac{NB\hbar\omega}{Vc\eta}, \qquad \bar{I}_c = \frac{A\eta c}{2B}$$

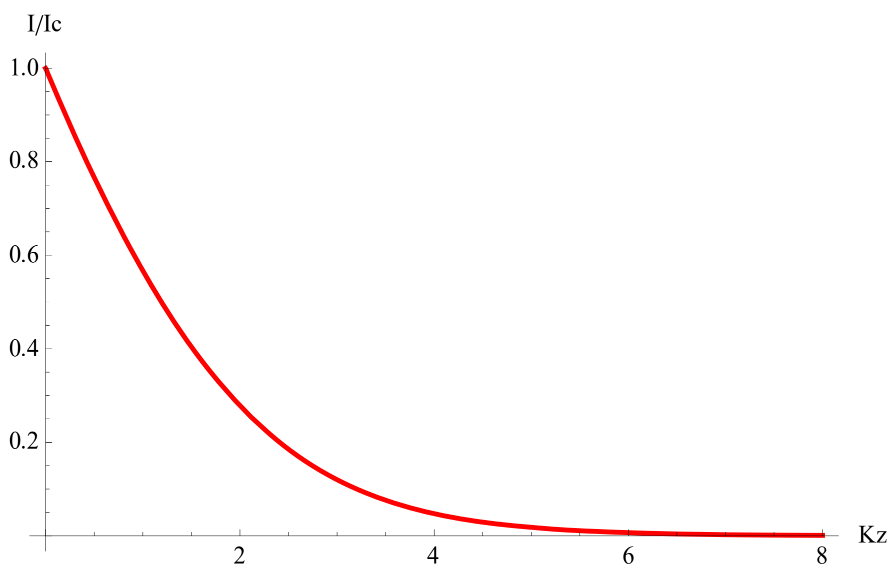


Fig. Plot of $I(z)/I_c$ as a function of Kz, where $I(z=0)/I_c = 1$. The intensity falls with increasing the distance z.

14. Amplification for the three-level laser

Next we discuss the z dependence of the intensity $I(z)$ for the three-level laser where a population inversion occurs. From the above discussion, it is found that that

$$\begin{aligned}\frac{1}{N_2 - N_1} &= \frac{A_{13}(A_{21}+A_{23})}{(A_{13}-A_{21})Nr}[1+\frac{(A_{13}+A_{23})B_{21}c\overline{W}}{cA_{13}(A_{21}+A_{23})}] \\ &= \frac{A_{13}(A_{21}+A_{23})}{(A_{13}-A_{21})Nr}(1+\frac{\bar{I}}{I_c})\end{aligned}$$

where

$$I_c = \frac{A_{13}(A_{21}+A_{23})}{(A_{13}+A_{23})}\frac{c}{B_{21}}$$

and

$$G = \frac{r(A_{13}-A_{21})}{A_{13}(A_{23}+A_{21})}\frac{NB_{21}\hbar\omega F(\omega)}{Vc}$$

Then we have a nonlinear differential equation

$$\frac{1}{\bar{I}}(1+\frac{\bar{I}}{I_C})\frac{\partial I}{\partial z} = G$$

A light amplifier of the type described above is called three level laser. The laser can also act as a self-sustaining oscillator, since even if no $\overline{W}$ is present initially, some radiation at w can appear owing to A_{21}.

We put

$$y = \frac{\bar{I}}{I_c} \qquad Gz = x,$$

and

$$y = y_0 \qquad \text{at } x = 0.$$

Then we get

$$\frac{1}{y}(1+y)\frac{\partial y}{\partial(x)} = 1.$$

The solution of this equation is obtained as

$$y = e^x \qquad \text{for x<<1}$$

and

$$y = 1 + y_0 x \qquad \text{for } x >> 1$$

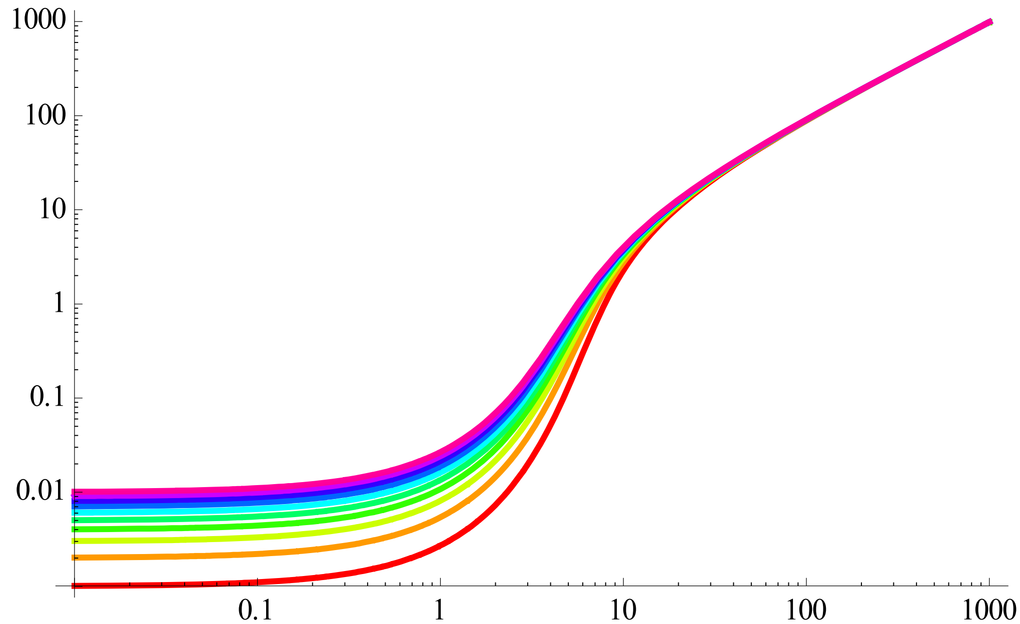


Fig. Plot of $y = \frac{\bar{I}}{I_c}$ vs $x = Gz$. $y_0 = \frac{\bar{I}}{I_c}$ at $t = 0$, is changed as a parameter between 0.001 and 0.01. The intensity increases with increasing the distance z.

REFERENCES

R. Loudon, The Quantum Theory of Light, second edition (Oxford Science Publications).

APPENDIX

A.1 Interaction with the classical radiation field (cgs units)

classical radiation field

⇒ electric or magnetic field derivable from a classical radiation field as opposed to quantized field

$$\hat{H} = \frac{1}{2m}\hat{\mathbf{p}}^2 - e\phi(\hat{\mathbf{r}}) + \frac{|e|}{mc}\mathbf{A}\cdot\hat{\mathbf{p}}$$

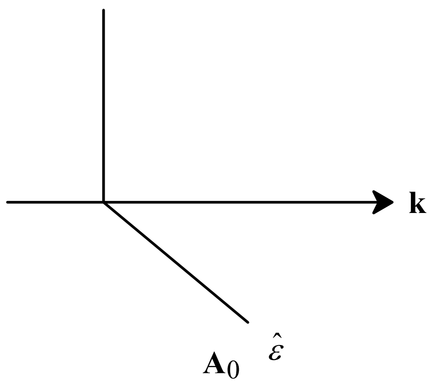
($e > 0$) ⇐ We use q=-|e| (|e| > 0),
which is justified if

$\nabla \cdot \mathbf{A} = 0$.

We work with a monochromatic field of the plane wave

$$\mathbf{A} = 2|\mathbf{A}_0|\boldsymbol{\varepsilon}\cos(\mathbf{k}\cdot\mathbf{r} - \omega t)$$

$$\mathbf{k} = \frac{\omega}{c}\mathbf{n}, \qquad \boldsymbol{\varepsilon}\cdot\mathbf{k} = 0$$



($\boldsymbol{\varepsilon}$ and $\mathbf{n}$ are the (linear) polarization and propagation directions.)

or

$$\mathbf{A} = |\mathbf{A}_0|\boldsymbol{\varepsilon}\left[e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right]$$

$\Rightarrow \qquad \hat{H} = \hat{H}_0 + \hat{H}_1$

$\hat{H}_1$: time dependent perturbation

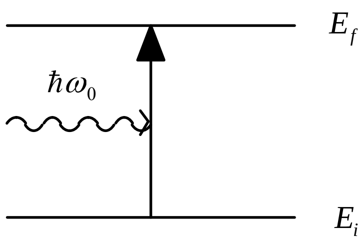
$$\begin{aligned}\hat{H}_1 &= \frac{e}{mc}|\mathbf{A}_0|\boldsymbol{\varepsilon}\cdot\hat{\mathbf{p}}\left[e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] \\ &= \hat{H}_1^+ e^{-i\omega t} + \hat{H}_1 e^{i\omega t}\end{aligned}$$

$$\left(\hat{H}_1^+\right)_{fi} = \frac{e|\mathbf{A}_0|}{mc}\left\langle \varphi_f \left| e^{i\mathbf{k}\cdot\mathbf{r}}\boldsymbol{\varepsilon}\cdot\hat{\mathbf{p}}\right|\varphi_i\right\rangle$$

and

$$W_{i\to f} = \frac{2\pi}{\hbar}\frac{e^2}{m^2c^2}|\mathbf{A}_0|^2\left|\langle\varphi_f|e^{i\mathbf{k}\cdot\mathbf{r}}\boldsymbol{\varepsilon}\cdot\hat{\mathbf{p}}|\varphi_i\rangle\right|^2\delta\left(E_f - E_i - \hbar\omega\right)$$

(Fermi's golden rule)



(absorption)

$$E_f - E_i = \hbar\omega$$

2. Stimulated emission and absorption (cgs units)

$$W_{i\to f} = \frac{2\pi}{\hbar}\frac{e^2}{m^2c^2}|\mathbf{A}_0|^2\left|\langle\varphi_f|e^{i\mathbf{k}\cdot\mathbf{r}}\boldsymbol{\varepsilon}\cdot\hat{\mathbf{p}}|\varphi_i\rangle\right|^2\delta\left(E_f - E_i \pm \hbar\omega\right)$$

$$\begin{cases} u = \dfrac{\omega^2}{2\pi c^2}|\mathbf{A}_0|^2 \ (\text{erg/cm}^3) \to \overline{W}(\omega)d\omega\left(\text{erg}\dfrac{\text{s}}{\text{cm}^3}\dfrac{1}{\text{s}}\right) \\ s = cu\left(\text{erg}\dfrac{\text{s}}{\text{cm}^3}\right) \to I(\omega)d\omega \quad \left(I(\omega) = c\overline{W}(\omega)\right) \quad I = \left[\dfrac{\text{erg}\cdot\text{s}}{\text{cm}^3}\cdot\dfrac{\text{cm}}{\text{s}}\right] = \left[\text{erg}\dfrac{1}{\text{cm}^2}\right] \end{cases}$$

$$\frac{\omega^2}{2\pi c^2}|\mathbf{A}_0|^2 \to \overline{W}(\omega)d\omega = \frac{1}{c}I(\omega)d\omega$$

$$W_{i\to f} = \frac{2\pi}{\hbar}\int\frac{e^2}{m^2c^2}\left[\frac{2\pi c^2}{\omega^2}\overline{W}(\omega)\right]d\omega\left|\langle\varphi_f|e^{i\mathbf{k}\cdot\mathbf{r}}\boldsymbol{\varepsilon}\cdot\hat{\mathbf{p}}|\varphi_i\rangle\right|^2\delta\left(E_f - E_i - \hbar\omega\right)$$

Since $\delta\left(E_f - E_i - \hbar\omega\right) = \dfrac{1}{\hbar}\delta\left(\omega_0 \pm \omega\right)$

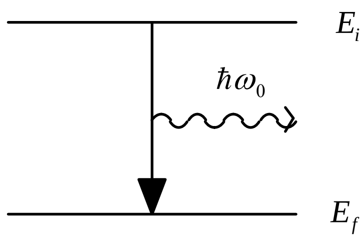
$$W_{i\to f}^{(a)} = \frac{4\pi^2e^2}{\hbar^2m^2\omega_0^2}\overline{W}(\omega_0)\left|\langle\varphi_f|e^{i\mathbf{k}\cdot\mathbf{r}}\boldsymbol{\varepsilon}\cdot\hat{\mathbf{p}}|\varphi_i\rangle\right|^{22}$$

absorption

Similarly

$$W_{i\to f}^{(e)} = \frac{4\pi^2 e^2}{\hbar^2 m^2 \omega_0^2}\overline{W}(\omega_0)\left|\left\langle \varphi_f \left| e^{i\mathbf{k}\cdot\mathbf{r}}\boldsymbol{\varepsilon}\cdot\hat{\mathbf{p}}\right|\varphi_i\right\rangle\right|^2$$

stimulated emission



(stimulated emission)

A.3. Electric dipole approximation

$$e^{i\mathbf{k}\cdot\mathbf{r}} = 1 + i\mathbf{k}\cdot\mathbf{r} + \cdots \cong 1$$

$$\begin{aligned}
W_{i\to f} &= \frac{4\pi^2 e^2}{\hbar^2 m^2 \omega_0^2}\overline{W}(\omega_0)\left|\left\langle \varphi_f \left| e^{i\mathbf{k}\cdot\mathbf{r}}\boldsymbol{\varepsilon}\cdot\hat{\mathbf{p}}\right|\varphi_i\right\rangle\right|^2 \\
&= \frac{4\pi^2 e^2}{\hbar^2 m^2 \omega_0^2}\overline{W}(\omega_0) m^2\omega_0^2\left|\left\langle \varphi_f \left| \hat{\mathbf{r}}_\varepsilon\right|\varphi_i\right\rangle\right|^2 \\
&= \frac{4\pi^2 e^2}{\hbar^2}\overline{W}(\omega_0)\left|\left\langle \varphi_f \left| \hat{\mathbf{r}}_\varepsilon\right|\varphi_i\right\rangle\right|^2 = B_{12}\overline{W}(\omega_0)
\end{aligned}$$

$$\Rightarrow \begin{aligned}
B_{12} = B_{21} &= \frac{4\pi^2 e^2}{\hbar^2}\left|\left\langle \varphi_f \left| \hat{\mathbf{r}}_\varepsilon\right|\varphi_i\right\rangle\right|^2 \\
&\cong \frac{4\pi^2 e^2}{3\hbar^2}\left|\left\langle \varphi_f \left| \hat{\mathbf{r}}\right|\varphi_i\right\rangle\right|^2
\end{aligned}$$

(average)