Magnetic monopole

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A **magnetic monopole** is a hypothetical elementary particle in particle physics that is an isolated magnet with only one magnetic pole (a north pole without a south pole or vice-versa). In more technical terms, a magnetic monopole would have a net "magnetic charge". Modern interest in the concept stems from particle theories, notably the grand unified and superstring theories, which predict their existence. Magnetism in bar magnets and electromagnets does not arise from magnetic monopoles, and in fact there is no conclusive experimental evidence that magnetic monopoles exist at all in the universe. Some condensed matter systems contain effective (non-isolated) magnetic monopole quasi-particles, or contain phenomena that are mathematically analogous to magnetic monopoles.
http://en.wikipedia.org/wiki/Magnetic_monopole

REFERENCES
P.M. Dirac, Proc. Roy. Soc. (London) 133, 60-72
D.J. Thouless, Topological Quantum Numbers in Nonrelativistic Physics, World Scientific (1998)

In Maxwell's equation magnetic charges do not appear. We have

$$\nabla \cdot \boldsymbol{B} = 0$$

No magnetic charges have been confirmed to exist. Quantum mechanics does not require that magnetic charges exist, but it unambiguously requires the quantization of magnetic monopoles and predicts the unit of magnetic charge if they should ever be found.

We assume that the magnetic monoples exist and that a magnetic monopole is located at the origin.

$$\nabla \cdot \boldsymbol{B} = 4\pi\rho_M$$

$$\int \nabla \cdot \boldsymbol{B} dv = \int \boldsymbol{B} \cdot d\boldsymbol{a} = B_r 4\pi r^2 = 4\pi e_M$$

or

$$\boldsymbol{B} = \frac{e_M}{r^2}\boldsymbol{e}_r \,.$$

The vector potential $\boldsymbol{A}$ is defined as

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

$$\Phi = \int \boldsymbol{B} \cdot d\boldsymbol{a} = \int \nabla \times \boldsymbol{A} \cdot d\boldsymbol{a} = \int \boldsymbol{A} \cdot d\boldsymbol{r}$$

The vector potential has only a A_ϕ component, since $\boldsymbol{B} = \frac{e_M}{r^2} \boldsymbol{e}_r$

Note that

$$\nabla \times \boldsymbol{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & r\sin\theta \boldsymbol{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

$$= \boldsymbol{e}_r \frac{1}{r^2 \sin\theta}[\frac{\partial}{\partial \theta}(r\sin\theta A_\phi) - \frac{\partial}{\partial \phi}(rA_\theta)] + r\boldsymbol{e}_\theta \frac{1}{r^2 \sin\theta}[\frac{\partial}{\partial \phi}A_r - \frac{\partial}{\partial r}(r\sin\theta A_\phi)]$$

$$+ r\sin\theta \boldsymbol{e}_\phi \frac{1}{r^2 \sin\theta}[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial \theta}(A_r)]$$

or

$$\nabla \times \boldsymbol{A} = \boldsymbol{e}_r \frac{1}{r\sin\theta}[\frac{\partial}{\partial \theta}(\sin\theta A_\phi) - \frac{\partial}{\partial \phi}A_\theta] + \boldsymbol{e}_\theta[\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}A_r - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)] + \boldsymbol{e}_\phi \frac{1}{r}[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial \theta}(A_r)]$$

When $A_\theta = A_r = 0$, we have

$$\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\phi) = \frac{e_M}{r}$$

$$\frac{\partial}{\partial r}(rA_\phi) = 0$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\phi) = \frac{e_M}{r}$$

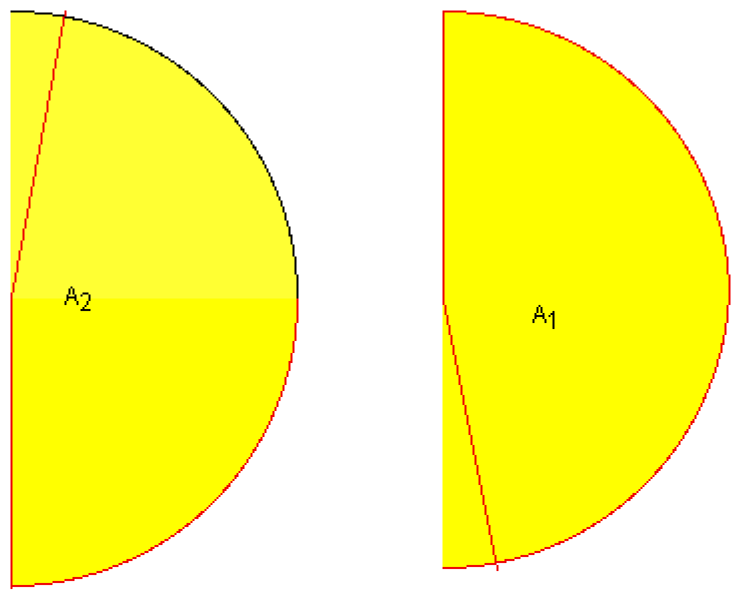
or

$$A_\phi = \frac{e_M}{r}\frac{(1-\cos\theta)}{\sin\theta} = \frac{e_M}{r}\tan(\frac{\theta}{2})$$

This value is singular on the negative z axis at $\theta = \pi$. If we consider $\boldsymbol{A}$ just a device for obtaining $\boldsymbol{B}$, then we can construct a pair of vector potentials

$$\boldsymbol{A}_1 = \frac{e_M}{r}\tan(\frac{\theta}{2})\boldsymbol{e}_\phi \qquad (\theta < \pi - \varepsilon),$$

$$\boldsymbol{A}_2 = -\frac{e_M}{r}\frac{1+\cos\theta}{\sin\theta}\boldsymbol{e}_\phi = -\frac{e_M}{r}\cot(\frac{\theta}{2})\boldsymbol{e}_\phi \qquad (\theta > \varepsilon)$$



which together yield the correct $\boldsymbol{B}$ everywhere. $\boldsymbol{A}_1$ can be used everywhere except inside a cone defined by $\theta = \pi - \varepsilon$ around the z axis, and $\boldsymbol{A}_2$ can be used everywhere except inside a cone defined by $\theta = \varepsilon$ around the positive z axis. In the overlap region ($\varepsilon \le \theta \le \pi - \varepsilon$) either $\boldsymbol{A}_1$ or $\boldsymbol{A}_2$ can be used. The two potentials lead to the same magnetic field and therefore related to each other by a gauge transformation.

$$\boldsymbol{A}_2 - \boldsymbol{A}_1 = -\frac{e_M}{r}\frac{1+\cos\theta}{\sin\theta}\boldsymbol{e}_\phi - \frac{e_M}{r}\frac{1-\cos\theta}{\sin\theta}\boldsymbol{e}_\phi = -\frac{2e_M}{r\sin\theta}\boldsymbol{e}_\phi = \nabla\chi$$

Note that

$$\nabla\chi = \boldsymbol{e}_r\frac{\partial\chi}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial\chi}{\partial\theta} + \boldsymbol{e}_\phi\frac{1}{r\sin\theta}\frac{\partial\chi}{\partial\phi}$$

Thus we have

$$\frac{1}{r\sin\theta}\frac{\partial\chi}{\partial\phi} = -\frac{2e_M}{r\sin\theta}$$

or

$$\frac{\partial \chi}{\partial \phi} = -2e_M$$

or

$$\chi = -2e_M \phi$$

The wave function of a charged particle depends on the particular gauge used.in the overlap region we have

$$\langle \mathbf{r} | \psi_2 \rangle = \langle \mathbf{r} | \hat{U} | \psi_1 \rangle = \exp(\frac{iq\chi}{c\hbar}) \langle \mathbf{r} | \psi_1 \rangle = \exp(-2e_M \phi \frac{iq}{c\hbar}) \langle \mathbf{r} | \psi_1 \rangle$$

Here we assume that $q = -e$ ($e>0$) is the charge of the particles.

$$\langle \mathbf{r} | \psi_2 \rangle = \exp(\frac{i2e_M \phi e}{c\hbar}) \langle \mathbf{r} | \psi_1 \rangle$$

or

$$\psi_2(r,\theta,\phi) = \exp(\frac{i2e_M \phi e}{c\hbar}) \psi_1(r,\theta,\phi)$$

The wave function must be single-valued. As we increase the azimuthal angle ϕ from 0 to 2π, the wave function must return to its original value.

$$\psi_2(r,\theta,\phi + 2\pi) = \exp[\frac{i2e_M e(\phi + 2\pi)}{c\hbar}] \psi_1(r,\theta,\phi + 2\pi)$$

and

$$\psi_2(r,\theta,\phi + 2\pi) = \psi_2(r,\theta,\phi) ,$$

and

$$\psi_1(r,\theta,\phi + 2\pi) = \psi_1(r,\theta,\phi)$$

Then we have

$$\exp[\frac{i2e_M 2\pi e}{c\hbar}] = 1$$

or

$$\frac{2e_M 2\pi e}{c\hbar} = 2\pi n \quad (n = 0, \pm 1, \pm 2, \pm 3, \ldots..)$$

or

$$\frac{2e_M e}{c\hbar} = n$$

or

The magnetic charge must be quantized in units of $\frac{c\hbar}{2e}$

$$e_M = n\frac{c\hbar}{2e}$$

Note that the quantum magnetic flux is

$$\Phi_0 = \frac{hc}{2e} = 2\pi\frac{\hbar c}{2e} \quad \text{(Gauss cm}^2\text{)}$$

((**Kittel**))

Suppose a magnetic monopole of strength e_M is situated just below the center of a superconducting ring. The magnetic flux through the ring is

$$(2\pi r^2)\frac{e_M}{r^2} = 2\pi e_M = n\Phi_0 = n(2\pi\frac{\hbar c}{2e})$$

or

$$e_M = n(\frac{\hbar c}{2e})$$