Schrödinger picture

Here we discuss the Rabi oscillation for the two level system (typically in the maser physics based on the Schrödinger picture

The Hamiltonian of the system is given by

$$\hat{H} = \begin{pmatrix} E_0 + \mu\varepsilon(t) & -A \\ -A & E_0 - \mu\varepsilon(t) \end{pmatrix}$$

The Schrodinger equation is given by

$$i\hbar \frac{\partial}{\partial t}|\psi_s\rangle = \hat{H}|\psi_s\rangle$$

We note that

$$|\psi_s\rangle = b_s(t)|s\rangle + b_a(t)|a\rangle$$

in the Schrodinger picture, and

$$|\psi_I\rangle = c_s(t)|s\rangle + c_a(t)|a\rangle$$

in the interaction picture. From the above Schrödinger equation, we get

$$\begin{aligned} i\hbar\dot{b}_s(t) &= \langle s|\hat{H}|\psi_s\rangle \\ &= \langle s|\hat{H}|s\rangle\langle s|\psi_s\rangle + \langle s|\hat{H}|a\rangle\langle a|\psi_s\rangle \\ &= \langle s|\hat{H}|s\rangle b_s(t) + \langle s|\hat{H}|a\rangle b_a(t) \end{aligned}$$

and

$$\begin{aligned} i\hbar\dot{b}_a(t) &= \langle a|\hat{H}|\psi_s\rangle \\ &= \langle a|\hat{H}|a\rangle\langle a|\psi_s\rangle + \langle a|\hat{H}|s\rangle\langle s|\psi_s\rangle \\ &= \langle a|\hat{H}|a\rangle b_a(t) + \langle a|\hat{H}|s\rangle b_s(t) \end{aligned}$$

Here we note that

$$|s\rangle = \hat{U}|1\rangle, \qquad |a\rangle = \hat{U}|2\rangle$$

where $\hat{U}$ is the unitary operator

$$\hat{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad \hat{U}^{+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\hat{U}^{+}\hat{H}\hat{U} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} E_0 + \mu\varepsilon(t) & -A \\ -A & E_0 - \mu\varepsilon(t) \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} E_0 - A & \mu\varepsilon(t) \\ \mu\varepsilon(t) & E_0 + A \end{pmatrix}$$

Thus we have

$$\langle a|\hat{H}|a\rangle = \langle 2|\hat{U}^{+}\hat{H}\hat{U}|2\rangle = E_0 + A$$

$$\langle a|\hat{H}|s\rangle = \langle 2|\hat{U}^{+}\hat{H}\hat{U}|1\rangle = \mu\varepsilon(t)$$

$$\langle s|\hat{H}|a\rangle = \langle 1|\hat{U}^{+}\hat{H}\hat{U}|2\rangle = \mu\varepsilon(t)$$

$$\langle s|\hat{H}|s\rangle = \langle 1|\hat{U}^{+}\hat{H}\hat{U}|1\rangle = E_0 - A$$

So we get the two equations for $b_s(t)$ and $b_a(t)$;

$$i\hbar\dot{b}_s(t) = (E_0 - A)b_s(t) + \mu\varepsilon(t)b_a(t)$$

$$i\hbar\dot{b}_a(t) = (E_0 + A)b_a(t) + \mu\varepsilon(t)b_s(t)$$

The coefficients $b_s(t)$ and and $b_a(t)$ for the Schrodinger picture are closely related to the coefficients $c_s(t)$ and and $c_a(t)$ as follows.

$$|\psi_s\rangle = \exp(-\frac{i}{\hbar}\hat{H}_0 t)|\psi_I\rangle$$

with

where

$$\hat{H}_0 = \begin{pmatrix} E_0 & -A \\ -A & E_0 \end{pmatrix} = E_0 \hat{1} - A\hat{\sigma}_x$$

under the basis of $|1\rangle$ and $|2\rangle$.

$$\hat{H}_0 |a\rangle = (E_0 + A)|a\rangle, \qquad \hat{H}_0 |s\rangle = (E_0 - A)|s\rangle$$

Thus we have

$$\begin{aligned} \langle a|\psi_s\rangle &= b_a(t) \\ &= \langle a|\exp(-\frac{i}{\hbar}\hat{H}_0 t)|\psi_I\rangle \\ &= \exp[-\frac{i}{\hbar}(E_0 + A)t]\langle a|\psi_I\rangle \\ &= \exp[-\frac{i}{\hbar}(E_0 + A)t]c_a(t) \end{aligned}$$

or

$$b_a(t) = \exp[-\frac{i}{\hbar}(E_0 + A)t]c_a(t). \qquad (1)$$

We also have

$$\begin{aligned} b_s(t) &= \langle s|\psi_s\rangle \\ &= \langle s|\exp(-\frac{i}{\hbar}\hat{H}_0 t)|\psi_I\rangle \\ &= \exp[-\frac{i}{\hbar}(E_0 - A)t]\langle s|\psi_I\rangle \\ &= \exp[-\frac{i}{\hbar}(E_0 - A)t]c_s(t) \end{aligned}$$

or

$$b_s(t) = \exp[-\frac{i}{\hbar}(E_0 - A)t]c_s(t). \qquad (2)$$

Using Eq.s(1) and (2),

$$i\hbar\dot{b}_a(t) = (E_0 + A)b_a(t) + i\hbar\exp[-\frac{i}{\hbar}(E_0 + A)t]\dot{c}_a(t)$$

or

$$i\hbar\dot{b}_a(t) - (E_0 + A)b_a(t) = \mu\varepsilon(t)b_s(t) = i\hbar\exp[-\frac{i}{\hbar}(E_0 + A)t]\dot{c}_a(t)$$

or

$$\mu\varepsilon(t)\exp[-\frac{i}{\hbar}(E_0 - A)t]c_s(t) = i\hbar\exp[-\frac{i}{\hbar}(E_0 + A)t]\dot{c}_a(t)$$

we get the differential equation for $c_a(t)$ and $c_s(t)$

$$i\hbar\dot{c}_a(t) = e^{i\omega_0 t}\mu\varepsilon(t)c_s(t)$$

Using Eq.s(1) and (2),

$$\dot{b}_s(t) = -\frac{i}{\hbar}(E_0 - A)b_s(t) + \exp[-\frac{i}{\hbar}(E_0 - A)t]\dot{c}_s(t)$$

or

$$i\hbar\dot{b}_s(t) - (E_0 - A)b_s(t) = \mu\varepsilon(t)b_a(t) = i\hbar\exp[-\frac{i}{\hbar}(E_0 - A)t]\dot{c}_s(t)$$

or

$$\mu\varepsilon(t)\exp[-\frac{i}{\hbar}(E_0 + A)t]c_a(t) = i\hbar\exp[-\frac{i}{\hbar}(E_0 - A)t]\dot{c}_s(t)$$

we get the differential equation for $c_a(t)$ and $c_s(t)$

$$i\hbar\dot{c}_s(t) = e^{-i\omega_0 t}\mu\varepsilon(t)c_a(t)$$

We suppose that $\varepsilon(t)$ is given by

$$\varepsilon(t) = 2\varepsilon_0 \cos\omega t = \varepsilon_0(e^{i\omega t} + e^{-i\omega t})$$

Then the two equations can be rewritten as

$$i\hbar\dot{c}_a(t) = e^{i\omega_0 t}\mu\varepsilon(t)c_s(t) = \mu\varepsilon_0[e^{i(\omega+\omega_0)t} + e^{i(-\omega+\omega_0)t}]c_s(t)$$

$$i\hbar\dot{c}_s(t) = e^{-i\omega_0 t}\mu\varepsilon(t)c_s(t) = \mu\varepsilon_0[e^{i(\omega-\omega_0)t} + e^{-i(\omega+\omega_0)t}]c_a(t)$$

We consider the case where $\omega \approx \omega_0$ (resonance).

$$i\hbar\dot{c}_a(t) = \mu\varepsilon_0 e^{-i(\omega-\omega_0)t}c_s(t)$$

$$i\hbar\dot{c}_s(t) = \mu\varepsilon_0 e^{i(\omega-\omega_0)t}c_a(t)$$

with the initial condition

$$c_s(t=0) = 0, \qquad c_a(t=0) = 1$$

REFERENCE

R.P. Feynman, R.,B. Leighton, and M. Sands, *The Feynman Lectures in Physics*, 6th edition (Addison Wesley, 1977).