Neutrino Oscillation

Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
(Date: October 10, 2015)

The mass of neutrino is about $1/10^7$ of the electron mass. Such a small mass of neutrino can be measured experimentally by means of the neutrino oscillation. There are three kinds of neutrino; electron neutrino, muon neutrino, and tau neutrino. For simplicity we have only two neutrinos, electron neutrino ν_e and muon neutrino ν_μ. Nuetrino generated by weak interaction can be expressed in terms of the linear combination of ν_e and ν_μ. If the masses of ν_e and ν_μ are different, the probability of finding ν_e in the system decreases, while the probability of finding ν_μ increases with increasing time. Such a phenomena is called the neutrino oscillation. Here the neutrino oscillation is discussed in terms of quantum mechanics.

Observation of neutrino oscillation in a superKamiokande

Super-Kamiokande finds neutrinos apparently "disappearing". Since it is unlikely that momentum and energy are actually vanishing from the universe, a more plausible explanation is that the types of neutrinos we can detect are changing into types we cannot detect. This phenomenon is known as neutrino oscillation.

The neutrinos observed by Super-Kamiokande are without exception produced at great distances from the detector. Neutrinos produced in the atmosphere arrive at the detector from distances of about 40 km (if produced above it) to 12,000 km (if produced on the other side of the Earth, the radius of Earth = 6,371 km). These distances are significantly greater than any measurements made to date with neutrinos from accelerators or nuclear reactors on Earth. Such great distances not only allow one to detect effects which would be invisible with a closer neutrino source. They also allow one to measure the behavior of neutrinos produced over a great *range* of distances. These advantages lead to some of the most dramatic evidence that oscillations are occurring.

The probability of a neutrino changing type is related to the distance travelled by the neutrino from its point of production to its point of detection. As a general rule, neutrinos travelling greater distances will exhibit greater depletion from oscillation. Moreover, the oscillation probability varies smoothly over increasing distance.

The reason neutrino oscillation is relevant to the question of neutrino mass is that massless neutrinos cannot oscillate. The observation of oscillation implies that the masses of the neutrinos involved cannot be equal to one another. Since they cannot be equal to one another, they cannot both be zero. In fact it is quite likely that if any neutrinos have non-zero mass, all of them do.

1. Introduction: significance of neutrino oscillation

In the standard model, neutrinos are assumed to be massless. Only the left-handed fields appear while the right-hand counterparts have vanishing coupling constants are invisible in standard model interactions. The experimental evidence indicates that neutrinos have tiny masses, raising many questions about how the standard model must be extended to accommodate neutrino mass. The evidence all concerns oscillation of neutrino flavor as neutrino propagates through matter and vacuum over macroscopic distance. There are three neutrino flavors described by a complex matrix; (i) the electron neutrino, (ii) the muon neutrino, and (iii) the tau neutrino. The electron neutrinos sourced in solar fusion reactions in the core of the Sun disappear in route to the Earth. They become muon and tau neutrinos. The oscillation is affected by interaction of the neutrinos with the material of the Sun.

((**Standard model**))

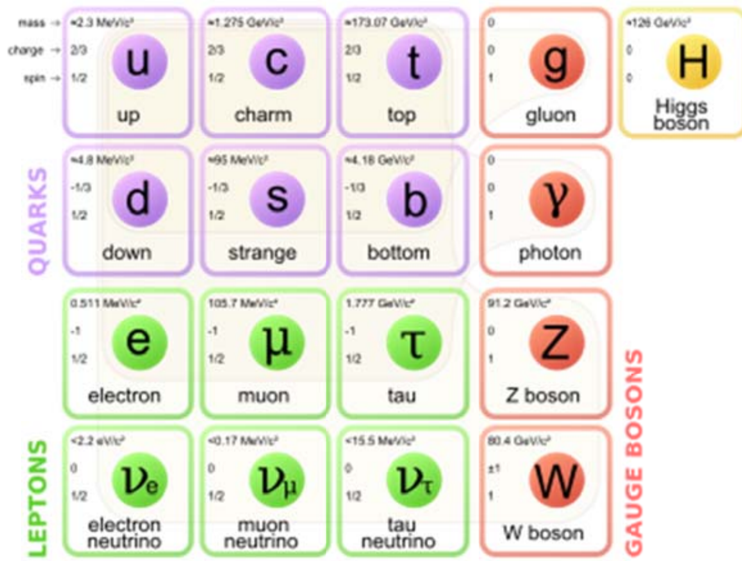


Fig. The Standard Model of elementary particles (more schematic depiction), with the three generations of matter, gauge bosons in the fourth column, and the Higgs boson in the fifth.
https://en.wikipedia.org/wiki/Standard_Model

2. Discovery of ν_e, ν_μ, and ν_τ

The concept of the neutrino was introduced in 1930 by W. Pauli in order to explain the energy spectrum of the β rays. In 1956, the existence of the neutrino was experimentally confirmed. A team led by F. Reines and C.L. Cowan detected neutrinos from the Savanna River Nuclear Reactor (South Carolina). In 1961, a Columbia Univ. and BNL group, led by L.M. Lederman, M. Schwartz and J. Steinberger, carried out an experiment striking aluminum targets with energetic neutrinos produced in $\pi \rightarrow \mu + \nu$ decay. They found that only muons were produced in the neutrino induced reactions. This indicates that the neutrino produced in the pion decay is different from the one produced in the β decays in reactor. This was the discovery of the second neutrino, now called the muon neutrino (ν_μ). The neutrino produced from nuclear β decay is now called the electron antineutrino ($\bar{\nu}_e$). Around 1969, R. Davis successfully detected solar neutrinos for the first time and found that the solar neutrino flux was significantly less than the prediction obtained from the standard solar model (SSM). After this observation, a number of experiments were carried out to measure the solar neutrinos with different energy thresholds and all of them confirmed that the solar neutrino flux is significantly less than the predicted value. The discrepancy between the measurements and the prediction was called the "solar neutrino problem" and remained unsolved until the solar neutrino oscillation was established. In 1976, M. Perl discovered the τ lepton in the $e+$ $e-$ collider experiment at SLAC and deduced the existence of the third neutrino (ν_τ) from the large missing energy and momentum of its decay. The ν_τ was directly identified by DONUT (Direct Observation of Nu Tau; Fermi Lab) group using a nuclear emulsion detector.

3. Experimental evidence

Neutrino oscillation is one of the most exciting subjects in elementary particle physics today. It was first confirmed in 1998 by the Super-Kamiokande group from their studies of atmospheric neutrinos. Experimental studies of neutrino oscillation have been rapidly progressing since then, and a number of positive oscillation results have been observed in atmospheric, solar, accelerator, and reactor neutrinos. The implication of the existence of neutrino oscillation is that neutrinos have finite masses and mixings, which are not accounted for in the framework of the standard model of elementary particles. Therefore, the standard model now must be extended to include the new information. Because the neutrino masses are extremely small, it is considered to be unnatural to be included in the standard model similar to the way quark and charged lepton masses are. Therefore, the neutrino oscillation is believed to provide an important new concept that will be a big step toward the unified understanding of elementary particle physics.

The neutrino oscillation is unique because neutrinos travel with ultra-relativistic velocity and the oscillation length is very long. Using neutrino oscillations it is possible to study a very low mass scale regime which other experiments struggle to reach. The neutrino oscillation is not incorporated into the standard model. Like other oscillations and particle masses, the neutrino oscillations and masses can be understood to be generated by the transitions between the neutrino flavors. However, we do not know the origin of the very small observed transition amplitudes yet. It is expected that new physics will evolve from studies of neutrino oscillations.

The evidence for neutrino mass first came in the 1990s from measurements in an enormous underground detector in Japan called Super Kamiokande, and was reinforced in 2001 and 2002 by measurements at SNO (Sudbery Neutrino Observatory, Ontario Canada). The neutrinos are created high in the atmosphere by cosmic rays. Because the earth is so nearly transparent to neutrinos,

There is a deficit of muons from underfoot relative to those from overhead. Some of neutrinos that travel farther – some eight thousand miles farther-are getting lost. How does this imply the neutrinos have mass?

The quantum state of the neutrino is the linear combination of of two states $|\nu_1\rangle$ and $|\nu_2\rangle$ where the mass of the particle in the state $|\nu_1\rangle$ (denoted by mass m_1) is different from that in the state $|\nu_1\rangle$ (denoted by mass m_2). The amplitudes of the linear combinations are different depending on the type of neutrinos such as the electron-neutrino, muon-neutrino, and the tau-neutrino. The quantum waves associated with the two mixed particles oscillate at different frequencies (the frequency is related to the mass). This makes the muon neutrino turn gradually into an electron neutrino. This phenomenon is called the neutrino oscillation.

The Sun is emitting as many neutrinos. Through neutrino oscillation, electron neutrinos are getting transformed into muon and tau neutrinos. The neutrinos have masses that are not zero and not equal.

4. Flavor eigenstates and mass eigenstates

Part of the problem was that the standard model of particle physics assumed that neutrinos were massless. If neutrinos are massless they must propagate at the speed of light, and all flavors of neutrinos must then have the same speed. We now believe that neutrinos have very small but finite masses. Furthermore, the flavor eigenstates ($|\nu_e\rangle, |\nu_\mu\rangle$, and $|\nu_\tau\rangle$ are not the same as the mass eigenstates ($|\nu_1\rangle, |\nu_2\rangle$, and $|\nu_3\rangle$); the flavor eigenstates are linear combinations of mass eigenstates. Since the different mass eigenstates have different masses, they can propagate at different speeds. Neutrino oscillations arise from the fact that as a linear combination of mass eigenstates propagates, the phase difference between the mass states changes, resulting in a different linear combination.

Imagine the flavor eigenstate $|\nu_e\rangle$, for example, being made up of a linear combination of $|\nu_1\rangle$ and $|\nu_2\rangle$. It is the mass eigenstates that are the energy eigenstates (more on this below), and which propagate at particular speeds. A neutrino that starts as $|\nu_e\rangle$ at the Sun may end up as $|\nu_\mu\rangle$ on Earth because the original linear combination of $|\nu_1\rangle$ and $|\nu_2\rangle$ has changed on propagation. If your detector is sensitive to ν_e's, but not ν_μ's, you would perceive this as a deficit of ν_e's unless you account for the oscillations. If you think neutrinos are massless, you won't account for the oscillation because there shouldn't be any. SNO was able to confirm the observation of neutrino oscillations, and solve the solar neutrino problem, because it was sensitive to all three neutrino
flavors.

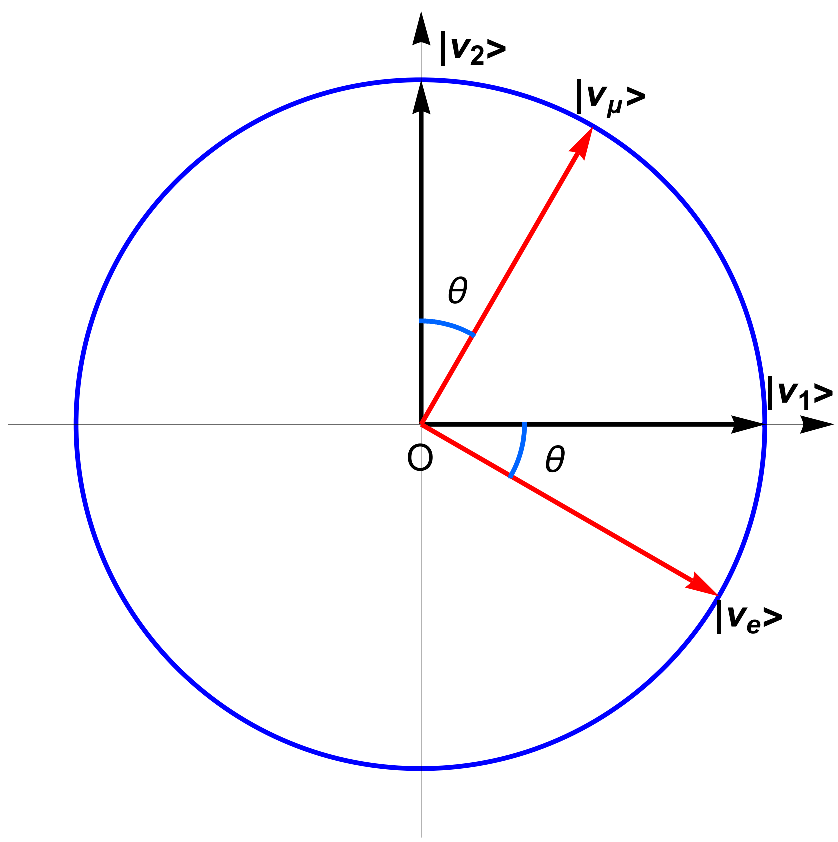
(**Mark Beck, Quantum Mechanics, Theory and Experiment (Oxford, 2012))**

4. Two flavor approximation

A lovely example of quantum-mechanical dynamics leading to interference in a two-state system, based on current physics research, is provided by the phenomenon known as neutrino oscillations. Neutrinos are elementary particles with no charge and very small mass, much smaller than that of an electron. They are known to occur in nature in three distinct "flavors," although for this discussion it suffices to consider only two of them. These two flavors are identified by their interactions, which may be either with electrons, in which case we write ν_e, or with muons, that is ν_μ. These are in fact eigenstates of a Hamiltonian that controls those interactions. On the other hand, it is possible (and, in fact, is now known to be true) that neutrinos have some other interactions, in which case their energy eigenvalues correspond to states that have a well-defined mass. These "mass eigenstates" would have eigenvalues E_1 and E_2, say, corresponding to masses m_1 and m_2, and might be denoted as the mass eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$. The flavor eigenstates $|\nu_e\rangle$ and $|\nu_\mu\rangle$ .are related to these through a simple unitary transformation, specified by some mixing angle θ, as follows:

$$|\nu_e\rangle = \hat{S}(\theta)|\nu_1\rangle = \cos\theta|\nu_1\rangle - \sin\theta|\nu_2\rangle$$

$$|\nu_\mu\rangle = \hat{S}(\theta)|\nu_2\rangle = \sin\theta|\nu_1\rangle + \cos\theta|\nu_2\rangle$$



where

$$\hat{S}(\theta) = \begin{pmatrix} \langle \nu_1 | \hat{S}(\theta) | \nu_1 \rangle & \langle \nu_1 | \hat{S}(\theta) | \nu_2 \rangle \\ \langle \nu_2 | \hat{S}(\theta) | \nu_1 \rangle & \langle \nu_2 | \hat{S}(\theta) | \nu_2 \rangle \end{pmatrix} = \begin{pmatrix} \langle \nu_1 | \nu_e \rangle & \langle \nu_1 | \nu_\mu \rangle \\ \langle \nu_2 | \nu_e \rangle & \langle \nu_2 | \nu_\mu \rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Eigenvalue problem:

$$\begin{aligned} \hat{S}(\theta) &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= \cos\theta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\sin\theta \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= \cos\theta \hat{1} + i\sin\theta \hat{\sigma}_y \end{aligned}$$

$$\hat{S}(\theta)|+y\rangle = (\cos\theta\hat{1} + i\sin\theta\hat{\sigma}_y)|+y\rangle = e^{i\theta}|+y\rangle$$

$$\hat{S}(\theta)|-y\rangle = (\cos\theta\hat{1} + i\sin\theta\hat{\sigma}_y)|-y\rangle = e^{-i\theta}|-y\rangle$$

with

$$|+y\rangle = \frac{1}{\sqrt{2}}[|\nu_1\rangle + i|\nu_2\rangle], \qquad \text{(eigenvalue, } e^{i\theta}\text{)}$$

$$|-y\rangle = \frac{1}{\sqrt{2}}[|\nu_1\rangle - i|\nu_2\rangle], \qquad \text{(eigenvalue, } e^{-i\theta}\text{)}$$

The energies E_1 and E_2 are given by

$$\frac{E^2}{c^2} = p^2 + m^2c^2 = p^2(1 + \frac{m^2c^2}{p^2}),$$

or

$$E = cp(1 + \frac{m^2c^2}{p^2})^{1/2} \approx cp(1 + \frac{m^2c^2}{2p^2}) = cp + \frac{m^2c^3}{2p}.$$

We consider the time dependence of the state $|\alpha, t\rangle$ with the initial state $|\nu_e\rangle$

$$\begin{aligned}\left|\alpha,t\right\rangle &= \exp(-\frac{i\hat{H}t}{\hbar})\left|\nu_e\right\rangle \\ &= \exp(-\frac{iE_1t}{\hbar})\cos\theta\left|\nu_1\right\rangle - \exp(-\frac{iE_2t}{\hbar})\sin\theta\left|\nu_2\right\rangle \\ &= \exp(-\frac{ipct}{\hbar})[\exp(-\frac{im_1{}^2c^3t}{2p\hbar})\cos\theta\left|\nu_1\right\rangle - \sin\theta\exp(-\frac{im_2{}^2c^3t}{2p\hbar})\left|\nu_2\right\rangle]\end{aligned}$$

where

$$\exp(-\frac{i\hat{H}t}{\hbar})\left|\nu_1\right\rangle = \exp(-\frac{iE_1t}{\hbar})\left|\nu_1\right\rangle, \qquad \exp(-\frac{i\hat{H}t}{\hbar})\left|\nu_2\right\rangle = \exp(-\frac{iE_2t}{\hbar})\left|\nu_2\right\rangle$$

and

$$E_1 = cp + \frac{m_1{}^2c^3}{2p}, \qquad E_2 = cp + \frac{m_2{}^2c^3}{2p}.$$

The probability for finding the system in a state $\left|\nu_e\right\rangle$,

$$P(\nu_e \to \nu_e) = \left|\left\langle \nu_e \middle| \alpha,t\right\rangle\right|^2$$

where

$$\begin{aligned}\left\langle \nu_e \middle| \alpha,t\right\rangle &= \begin{pmatrix}\cos\theta & -\sin\theta\end{pmatrix}\begin{pmatrix}\exp(-\frac{ipct}{\hbar})[\exp(-\frac{im_1{}^2c^3t}{2p\hbar})\cos\theta \\ -\exp(-\frac{ipct}{\hbar})[\exp(-\frac{im_2{}^2c^3t}{2p\hbar})\sin\theta\end{pmatrix} \\ &= \exp(-\frac{ipct}{\hbar})[\exp(-\frac{im_1{}^2c^3t}{2p\hbar})\cos^2\theta + \exp(-\frac{im_2{}^2c^3t}{2p\hbar})\sin^2\theta] \\ &= \exp(-\frac{ipct}{\hbar})\exp(-\frac{im_1{}^2c^3t}{2p\hbar})[\cos^2\theta + \exp(\frac{i(m_1{}^2 - m_2{}^2)c^3t}{2p\hbar})\sin^2\theta]\end{aligned}$$

then we have

$$P(\nu_e \to \nu_e) = \left| \exp(-\frac{i m_1{}^2 c^3 t}{2p\hbar}) \cos^2\theta + \exp(-\frac{i m_2{}^2 c^3 t}{2p\hbar}) \sin^2\theta \right|^2$$

$$= \cos^4\theta + \sin^4\theta + 2\cos^2\theta \sin^2\theta \cos[\frac{\Delta m_{12}{}^2 c^3 t}{2p\hbar}]$$

$$= 1 - 2\cos^2\theta \sin^2\theta (1 - \cos[\frac{\Delta m_{12}{}^2 c^3 t}{2p\hbar}])$$

$$= 1 - \sin^2(2\theta) \sin^2[\frac{\Delta m_{12}{}^2 c^3 t}{4p\hbar}]$$

When $L = ct$ and $E_0 = cp$

$$P(\nu_e \to \nu_e) = 1 - \sin^2(2\theta) \sin^2(\frac{\Delta m_{12}{}^2 c^4 L}{4E_0 \hbar c}) .$$

where

$$\Delta m_{12}{}^2 = m_1{}^2 - m_2{}^2 ,$$

E is the neutrino energy, L is the distance between the neutrino source and detector, m_1 and m_2 are the neutrino masses of the mass eigenstates and θ is the mixing angle between flavor eigenstates and mass eigenstates.

The probability for finding the system in a state $|\nu_\mu\rangle$,

$$P = \left| \langle \nu_\mu | \alpha, t \rangle \right|^2$$

$$= \sin^2\theta \cos^2\theta \left| \exp(-\frac{i m_1{}^2 c^3 t}{2p\hbar}) - \exp(-\frac{i m_2{}^2 c^3 t}{2p\hbar}) \right|^2$$

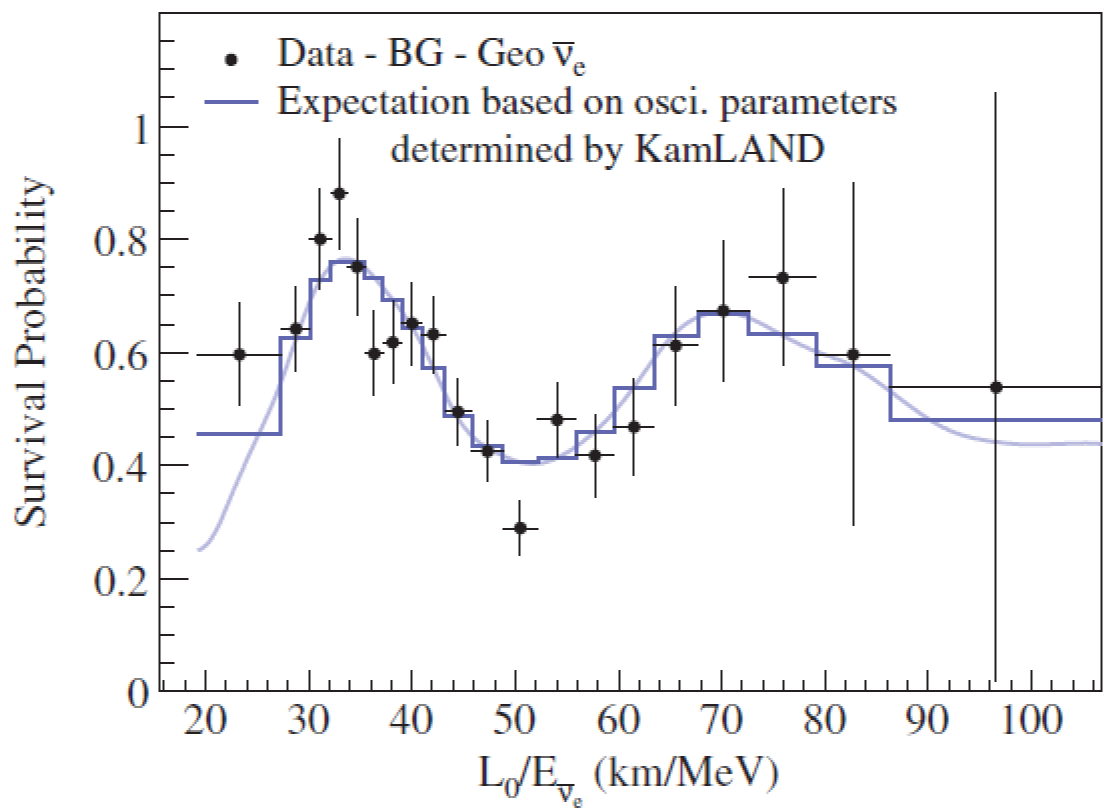
$$= \sin^2\theta \cos^2\theta \left| \exp(-\frac{i \Delta m_{12}{}^2 c^3 t}{2p\hbar}) - 1 \right|^2$$

$$= \sin^2(2\theta) \sin^2(\frac{\Delta m_{12}{}^2 c^3 t}{4p\hbar})$$

where

$$\langle \nu_e | \alpha, t\rangle = \begin{pmatrix}\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix} \exp(-\frac{ipct}{\hbar})[\exp(-\frac{im_1^2c^3t}{2p\hbar})\cos\theta \\ -\exp(-\frac{ipct}{\hbar})[\exp(-\frac{im_2^2c^3t}{2p\hbar})\sin\theta \end{pmatrix}$$

$$= \exp(-\frac{ipct}{\hbar})\sin\theta\cos\theta[\exp(-\frac{im_1^2c^3t}{2p\hbar}) - \exp(-\frac{im_2^2c^3t}{2p\hbar})]$$



((**Note**))

$$A = \frac{(eV)^2 km}{4\hbar c(MeV)} = 1.26693 \text{ x } 10^3.$$

$$\Delta m_{12}{}^2 c^4 = a_0 \text{ x } 10^{-4} \text{ (eV)}^2.$$

$$(\Delta m_{12}{}^2 c^4) A = 0.126693 a_0$$

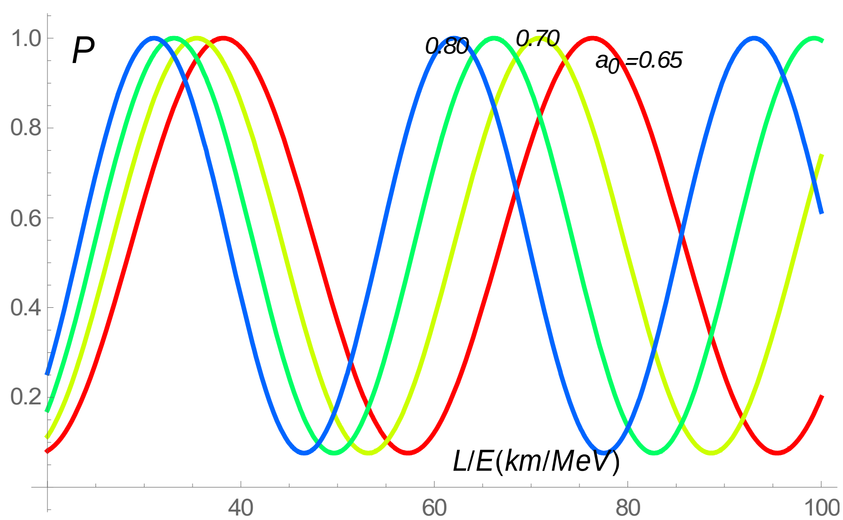
$$P(\nu_e \to \nu_e) = 1 - \sin^2(2\theta)\sin^2(0.126693 a_0 \frac{L(km)}{E(MeV)})$$

$\theta = 37^0$, $\Delta m_{12}{}^2 c^4 \approx 0.70 \text{ x } 10^{-4}\ (\text{eV})^2$.

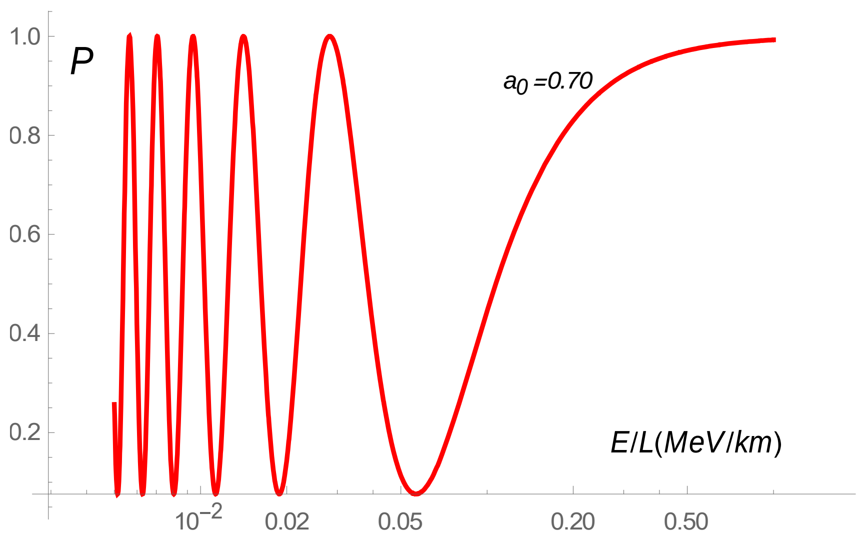
((**Mathematica**)) Numerical calculation in the cgs units.

```
Clear["Global`*"];
rule1 = {c → 2.99792 × 10^10, ℏ → 1.054571628 10^-27, eV → 1.602176487 × 10^-12,
  MeV → 1.602176487 × 10^-6, GeV → 1.602176487 × 10^-3 , km → 10^5};
A1 = (eV)^2 km / (4 ℏ c MeV) //. rule1
1266.93

θ = 37 °; f1[a0_] := 1 - Sin[2 θ]^2 Sin[a0 0.126693 x]^2;
h1 = Plot[Evaluate[Table[f1[a0], {a0, 0.65, 0.80, 0.05}]],
  {x, 20, 100}, PlotStyle → Table[{Hue[0.2 i], Thick}, {i, 0, 3}]];
h2 = Graphics[{Text[Style["P", Black, 15, Italic], {22, 0.95}],
   Text[Style["L/E(km/MeV)", Black, 12, Italic], {90, 0.05}],
   Text[Style["a0=0.65", Black, 10, Italic], {80, 1}],
   Text[Style["0.70", Black, 10, Italic], {70, 1}],
   Text[Style["0.80", Black, 10, Italic], {60, 1}]}];
Show[h1, h2]
```



```
θ = 37 °;
g1[a0_] := 1 - Sin[2 θ]^2 Sin[a0 0.126693 1/y]^2;
h11 = LogLinearPlot[Evaluate[Table[g1[a0], {a0, 0.70, 0.70, 0.05}]],
  {y, 0.005, 1}, PlotStyle → Table[{Hue[0.2 i], Thick}, {i, 0, 3}]];
h12 = Graphics[{Text[Style["P", Black, 15, Italic], {0.005, 0.95}],
   Text[Style["E/L(MeV/km)", Black, 12, Italic], {0.08, 0.15}],
   Text[Style["a0=0.70", Black, 10, Italic], {0.039, 0.8}]}];
Show[h11, h12]
```



4. General case

$$|\nu_e\rangle = \hat{U}|\nu_1\rangle, \qquad |\nu_\mu\rangle = \hat{U}|\nu_2\rangle, \qquad |\nu_\tau\rangle = \hat{U}|\nu_3\rangle$$

$$|\nu_1\rangle = \hat{U}^+|\nu_e\rangle, \qquad |\nu_2\rangle = \hat{U}^+|\nu_\mu\rangle, \qquad |\nu_2\rangle = \hat{U}^+|\nu_\mu\rangle$$

where $\hat{U}$ is the unitary operator, $\hat{U}^+\hat{U} = 1$, and it is called as Pontecorvo–Maki–Nakagawa–Sakata (PMNS) mixing matrix, in honor of the pioneering contributions of these physicists to neutrino mixing and oscillations,

$$|\nu_e\rangle = U_{11}|\nu_1\rangle + U_{21}|\nu_2\rangle + U_{31}|\nu_3\rangle,$$

$$\left|\nu_\mu\right\rangle = U_{12}\left|\nu_1\right\rangle + U_{22}\left|\nu_2\right\rangle + U_{32}\left|\nu_3\right\rangle$$

$$\left|\nu_\tau\right\rangle = U_{13}\left|\nu_1\right\rangle + U_{23}\left|\nu_2\right\rangle + U_{33}\left|\nu_3\right\rangle$$

and

$$\left|\nu_1\right\rangle = U_{11}{}^*\left|\nu_e\right\rangle + U_{12}{}^*\left|\nu_\mu\right\rangle + U_{13}{}^*\left|\nu_\tau\right\rangle,$$

$$\left|\nu_2\right\rangle = U_{21}{}^*\left|\nu_e\right\rangle + U_{22}{}^*\left|\nu_\mu\right\rangle + U_{23}{}^*\left|\nu_\tau\right\rangle$$

$$\left|\nu_3\right\rangle = U_{31}{}^*\left|\nu_e\right\rangle + U_{32}{}^*\left|\nu_\mu\right\rangle + U_{33}{}^*\left|\nu_\tau\right\rangle$$

Note that

$$\hat{H}\left|\nu_i\right\rangle = E_i\left|\nu_i\right\rangle, \qquad \exp(-\frac{i}{\hbar}\hat{H}t)\left|\nu_i\right\rangle = \exp(-\frac{i}{\hbar}E_i t)\left|\nu_i\right\rangle$$

The mass eigenstates $\left|\nu_i\right\rangle$ (I = 1, 2, 3), are the neutrino states that have definite mass m_i.

$$E_i = cp(1+\frac{m_i{}^2c^2}{p^2})^{1/2} \approx cp(1+\frac{m_i{}^2c^2}{2p^2}) = cp + \frac{m_i{}^2c^3}{2p}$$

At $t = 0$,

$$\left|\alpha, t=0\right\rangle = \left|\nu_e\right\rangle = U_{11}\left|\nu_1\right\rangle + U_{21}\left|\nu_2\right\rangle + U_{31}\left|\nu_3\right\rangle$$

$$\begin{aligned}\left|\alpha, t\right\rangle &= \exp(-\frac{i\hat{H}t}{\hbar})\left|\nu_e\right\rangle \\ &= \exp(-\frac{iE_1t}{\hbar})U_{11}\left|\nu_1\right\rangle + \exp(-\frac{iE_2t}{\hbar})U_{21}\left|\nu_2\right\rangle + \exp(-\frac{iE_3t}{\hbar})U_{31}\left|\nu_3\right\rangle \\ &= \exp(-\frac{ipct}{\hbar})[\exp(-\frac{im_1{}^2c^3t}{2p\hbar})U_{11}\left|\nu_1\right\rangle + \exp(-\frac{im_2{}^2c^3t}{2p\hbar})U_{21}\left|\nu_2\right\rangle \\ &\quad + \exp(-\frac{im_3{}^2c^3t}{2p\hbar})U_{31}\left|\nu_3\right\rangle]\end{aligned}$$

$$\langle \nu_e | \alpha, t\rangle = \begin{pmatrix} U_{11}^{\ *} & U_{21}^{\ *} & U_{31}^{\ *} \end{pmatrix} \begin{pmatrix} \exp(-\frac{ipct}{\hbar})[\exp(-\frac{im_1^{\ 2}c^3t}{2p\hbar})U_{11} \\ \exp(-\frac{ipct}{\hbar})[\exp(-\frac{im_2^{\ 2}c^3t}{2p\hbar})U_{21} \\ \exp(-\frac{ipct}{\hbar})[\exp(-\frac{im_3^{\ 2}c^3t}{2p\hbar})U_{31} \end{pmatrix}$$

$$= \exp(-\frac{ipct}{\hbar})[\exp(-\frac{im_1^{\ 2}c^3t}{2p\hbar})|U_{11}|^2 + \exp(-\frac{im_2^{\ 2}c^3t}{2p\hbar})|U_{21}|^2 + \exp(-\frac{im_3^{\ 2}c^3t}{2p\hbar}))|U_{31}|^2]$$

The probability is calculated as

$$P(\nu_e \to \nu_e) = |\langle \nu_e | \alpha, t\rangle|^2$$

$$= |\exp(-\frac{im_1^{\ 2}c^3t}{2p\hbar})|U_{11}|^2 + \exp(-\frac{im_2^{\ 2}c^3t}{2p\hbar})|U_{21}|^2 + \exp(-\frac{im_3^{\ 2}c^3t}{2p\hbar}))|U_{31}|^2\ |^2$$

$$= |U_{11}|^4 + |U_{21}|^4 + |U_{31}|^4 + 2\,\mathrm{Re}[\exp(-\frac{im_1^{\ 2}c^3t}{2p\hbar})\exp(\frac{im_2^{\ 2}c^3t}{2p\hbar})]|U_{11}|^2|U_{21}|^2$$

$$+ 2\,\mathrm{Re}[\exp(-\frac{im_2^{\ 2}c^3t}{2p\hbar})\exp(\frac{im_3^{\ 2}c^3t}{2p\hbar})]|U_{21}|^2|U_{31}|^2$$

$$+ 2\,\mathrm{Re}[\exp(-\frac{im_3^{\ 2}c^3t}{2p\hbar})\exp(\frac{im_1^{\ 2}c^3t}{2p\hbar})]|U_{31}|^2|U_{11}|^2$$

$$= |U_{11}|^4 + |U_{21}|^4 + |U_{31}|^4 + 2|U_{11}|^2|U_{21}|^2\cos[\frac{(m_1^{\ 2} - m_2^{\ 2})c^3t}{2p\hbar}]$$

$$+ 2|U_{21}|^2|U_{31}|^2\cos[\frac{(m_2^{\ 2} - m_3^{\ 2})c^3t}{2p\hbar}] + 2|U_{31}|^2|U_{31}|^2\cos[\frac{(m_3^{\ 2} - m_1^{\ 2})c^3t}{2p\hbar}]$$

or

$$P(\nu_e \to \nu_e) = |U_{11}|^4 + |U_{21}|^4 + |U_{31}|^4 + 2|U_{11}|^2|U_{21}|^2[1 - 2\sin^2[\frac{(m_1^{\ 2} - m_2^{\ 2})c^3t}{4p\hbar}] + 2|U_{21}|^2|U_{31}|^2[1 - 2\sin^2[\frac{(m_2^{\ 2} - m_3^{\ 2})c^3t}{4p\hbar}]$$

$$+ 2|U_{31}|^2|U_{31}|^2[1 - 2\sin^2[\frac{(m_3^{\ 2} - m_1^{\ 2})c^3t}{4p\hbar}]$$

$$= 1 - 4\{|U_{11}|^2|U_{21}|^2\sin^2[\frac{\Delta m_{12}^{\ 2}c^3t}{4p\hbar}]$$

$$+ |U_{21}|^2|U_{31}|^2\sin^2[\frac{\Delta m_{23}^{\ 2}c^3t}{4p\hbar}] + |U_{31}|^2|U_{31}|^2\sin^2[\frac{\Delta m_{31}^{\ 2}c^3t}{4p\hbar}]\}$$

where

$$|U_{11}|^2 + |U_{21}|^2 + |U_{31}|^2 = 1$$

$$|U_{11}|^4 + |U_{21}|^4 + |U_{31}|^4 = 1 - 2(|U_{11}|^2|U_{21}|^2 + |U_{21}|^2|U_{31}|^2 + |U_{31}|^2|U_{31}|^2)$$

$$\Delta m_{12}{}^2 = m_1{}^2 - m_2{}^2, \qquad \Delta m_{23}{}^2 = m_2{}^2 - m_3{}^2, \qquad \Delta m_{31}{}^2 = m_3{}^2 - m_1{}^2$$

REFERENCES

K. Ford The Quantum World, Quantum Physics for Everyone (Harvard University Press, 2004)

Y. Suzuki, M. Nakahara, Y. Itow, M. Shiozawa, and Y. Obayashi, The Fourth Workshop on Neutron Oscillations and Their Origin (World Scientific, 2004)

F. Suekane, Neutrino Oscillation, A Practical Guide to Basics and Applications (Springer, 2015).

J.P. Ochoa-Ricoux, A Search for Muon Neutrino to Electron Neutrino Oscillations in the MINOS Experiment (Springer, 2011).

T. Ahrens, From Dirac to Neutrino Oscillations (Springer, 2000).

M. Fukugita and Y. Yanagida, Physics of Nuetrinos and Applications to Astrophysics (Springer, 2003).

Z. Maki, M. Nakagawa, and S. Sakata, Prog. Theor. Phys. 28, 870 (1962).

J.A. Thomas and P.L. Vahle, Neutrino Oscillation Present Status and Future Plans (World Scientific, 2008).

F. Boehm and P. Vogel, Physics of Massive Neutrinos (Cambridge, 1992).

S. Bilenky, Introduction to the Physics of Massive and Mixed Neutrinos (Springer, 2010).

D. Carlsmith, Particle Physics (Pearson, 2010).

Mark Beck, Quantum Mechanics, Theory and Experiment (Oxford, 2012)

APPENDIX

$$H = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}, \qquad |\nu_e\rangle = \hat{S}(\theta)|\nu_1\rangle, \qquad |\nu_\mu\rangle = \hat{S}(\theta)|\nu_2\rangle$$

where

$$\hat{S}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad \hat{S}^+(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\exp(-\frac{i}{\hbar}\hat{H}t) = \begin{pmatrix} e^{-\frac{i}{\hbar}E_1 t} & 0 \\ 0 & e^{-\frac{i}{\hbar}E_2 t} \end{pmatrix}$$

with

$$E_1 = \sqrt{p^2c^2 + m_1{}^2c^4} = cp + \frac{m_1{}^2c^3}{2p},$$

$$E_2 = \sqrt{p^2c^2 + m_2{}^2c^4} = cp + \frac{m_2{}^2c^3}{2p}.$$

The time dependence of the state vector

$$\left|\alpha(t)\right\rangle = \exp(-\frac{i}{\hbar}\hat{H}t)\hat{S}(\theta)\left|\nu_1\right\rangle$$

Then we have the probabilities,

$$P_e = \left|\left\langle \nu_e \middle| \alpha(t)\right\rangle\right|^2, \qquad P_\mu = \left|\left\langle \nu_\mu \middle| \alpha(t)\right\rangle\right|^2$$

with

$$\left\langle \nu_e \middle| \alpha(t)\right\rangle = \left\langle \nu_1\right|\hat{S}^+(\theta)\exp(-\frac{i}{\hbar}\hat{H}t)\hat{S}(\theta)\left|\nu_1\right\rangle,$$

$$\left\langle \nu_\mu \middle| \alpha(t)\right\rangle = \left\langle \nu_2\right|\hat{S}^+(\theta)\exp(-\frac{i}{\hbar}\hat{H}t)\hat{S}(\theta)\left|\nu_1\right\rangle.$$

((**Mathematica**))

```
Clear["Global`*"];
exp_* :=
 exp /. {Complex[re_, im_] :> Complex[re, -im]};
```

$$S1 = \begin{pmatrix} \text{Cos}[\theta] & \text{Sin}[\theta] \\ -\text{Sin}[\theta] & \text{Cos}[\theta] \end{pmatrix};$$

$$S1H = \begin{pmatrix} \text{Cos}[\theta] & -\text{Sin}[\theta] \\ \text{Sin}[\theta] & \text{Cos}[\theta] \end{pmatrix};$$

$$EPH1 = \begin{pmatrix} \text{Exp}\left[\frac{-i\, E1\, t}{\hbar}\right] & 0 \\ 0 & \text{Exp}\left[\frac{-i\, E2\, t}{\hbar}\right] \end{pmatrix};$$

$$rule1 = \left\{E1 \to p\, c + \frac{m1^2\, c^3}{2\, p},\ E2 \to p\, c + \frac{m2^2\, c^3}{2\, p}\right\};$$

```
h1 = S1H.EPH1.S1 // FullSimplify;

p11 = h1[[1, 1]] h1[[1, 1]]* /. rule1 // FullSimplify
```

$$\text{Cos}[\theta]^4 + 2\,\text{Cos}[\theta]^2\,\text{Cos}\left[\frac{c^3\,(m1 - m2)\,(m1 + m2)\,t}{2\,p\,\hbar}\right]\text{Sin}[\theta]^2 + \text{Sin}[\theta]^4$$

```
p21 = h1[[2, 1]] h1[[2, 1]]* /. rule1 // FullSimplify
```

$$\text{Sin}[2\,\theta]^2\,\text{Sin}\left[\frac{c^3\,(m1 - m2)\,(m1 + m2)\,t}{4\,p\,\hbar}\right]^2$$