Linear potential: Neutron Bouncing problem
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(Date: September 16, 2015)

The basic idea of the experiment is to let neutrons''flow''—with a certain horizontal velocity—between a mirror below and an absorber/scatterer above. The absorber acts as a selector for the vertical velocity component. Then one measures the number N of transmitted neutrons as a function of the absorber height h. This measurement should allow the identification of the neutron quantum states because a clear ''signature'' should appear: the classical dependence $N_{\text{Classical}}(h)$ is modified into a stepwise quantum-mechanical dependence $N_{\text{QM}}(h)$ at small absorber heights h.

1. Theory

We consider a neutron particle in the presence of a potential energy given by

$$V(x) = mg\left|x\right|,$$

where g (>0) is the gravitational acceleration. The Schrödinger equation is given by

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi(x)$$

or

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + [V(x) - E]\psi = 0$$

where E is the energy of a particle with a mass m. Then the Schrödinger equation is expressed by

$$\frac{d^2\psi(x)}{dx^2} - \frac{2m^2}{\hbar^2}g(x-a)\psi(x) = 0$$

where

$$E = mga$$

and $x = a$ is a classical turning point. Since the potential $V(x) = mg\left|x\right|$ is even function of x, the wavefunction should be either an even function of x (even parity), or an odd function of x. The boundary condition for the wavefunction is as follows.

$\psi(x = 0) = 0$ for the wavefunction with the odd parity

$$\psi'(x=0)=0 \qquad \text{for the wavefunction with the even parity}$$

Here we put

$$z=\left(\frac{2m^2 g}{\hbar^2}\right)^{1/3}(x-a).$$

where z is the dimensionless variable. We note that

$$\frac{d}{dx}=\frac{dz}{dx}\frac{d}{dz}=\left(\frac{2m^2 g}{\hbar^2}\right)^{1/3}\frac{d}{dz}$$

$$\frac{d^2}{dx^2}=\left(\frac{2m^2 g}{\hbar^2}\right)^{2/3}\frac{d^2}{dz^2}$$

Then we get

$$\frac{d^2\psi(z)}{dz^2}-z\psi(z)=0.$$

The solution of this equation is given by

$$\psi(z)=C_1 A_i(z)+C_2 B_i(z)$$

where C_1 and C_2.

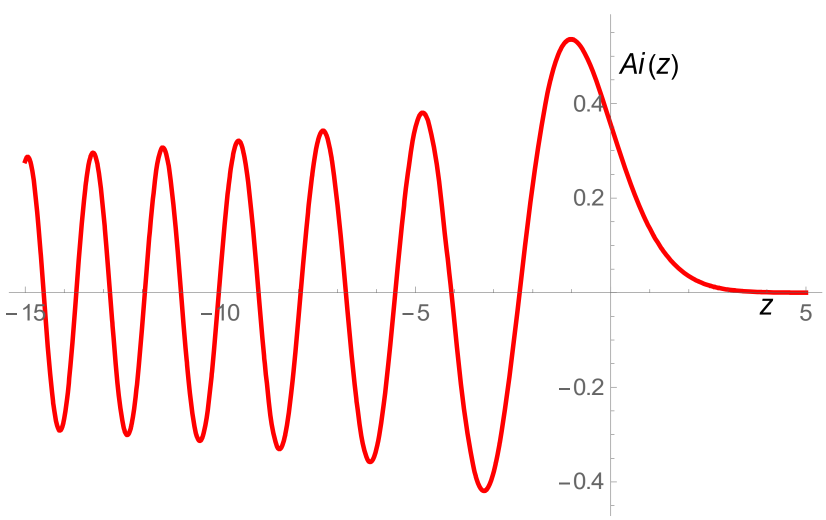


Fig. Plot of the $A_i(z)$ (red) as a function of z.

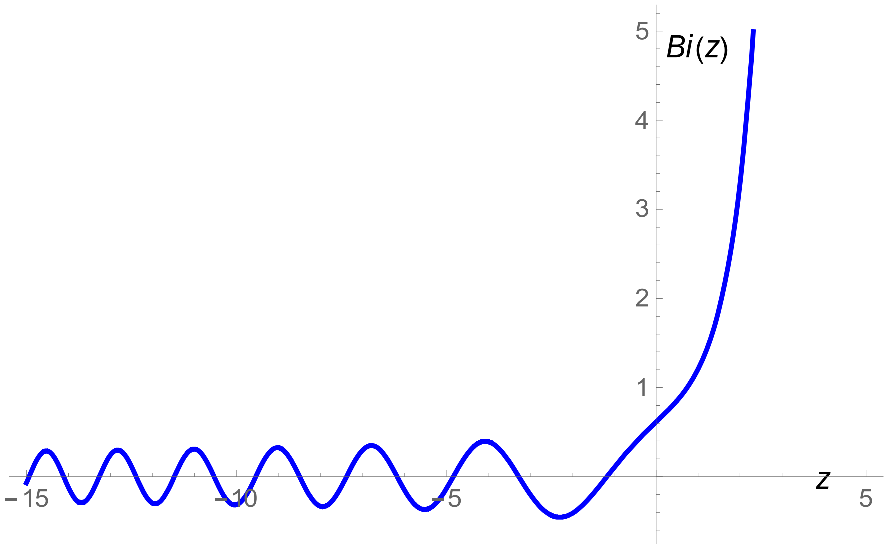


Fig. Plot of the $B_i(z)$ (blue) as a function of z.

Since $z \to \infty$, $B_i(z)$ becomes infinity. So we have $C_2 = 0$.

$$\psi(z) = C_1 A_i(z)$$

where

$$z = \left(\frac{2m^2 g}{\hbar^2}\right)^{1/3}(x - a)$$

We determine the energy eigenvalue from the boundary condition.

(i) Odd parity case

$$\psi(x = 0) = 0 \qquad \text{for the wavefunction with the odd parity}$$

When the n-th zero points of $A_i(z)$ is z_n,

$$z_n = \left(\frac{2m^2 g}{\hbar^2}\right)^{1/3}(-a_n)$$

or

$$a_n = -\frac{z_n}{\left(\frac{2m^2 g}{\hbar^2}\right)^{1/3}}$$ (the discrete height)

$$E_n = mga_n = -mg\frac{z_n}{\left(\frac{2m^2 g}{\hbar^2}\right)^{1/3}}$$ (the energy eigenvalue).

or

$$\begin{aligned}\frac{E_n}{mg} &= a_n \\ &= -\frac{z_n}{2^{1/3}}\left(\frac{\hbar^2}{m^2 g}\right)^{1/3} \\ &= -0.793701 z_n h_c \\ &= (0.79370 \times 7.39401 \mu m) z_n \\ &= 5.868625(-z_n)\mu m.\end{aligned}$$

The energy eigenvalue can be expressed by

$$E_n = mga_n = \frac{1}{(2m)^{1/3}}(\hbar mg)^{2/3}(-z_n)$$

where $z = z_n$, $x = 0$, and $a = a_n$ in the expression

$$z = \left(\frac{2m^2 g}{\hbar^2}\right)^{1/3}(x-a)$$

The characteristic length h_c is given by

$$h_c = \left(\frac{\hbar^2}{m^2 g}\right)^{1/3} = 7.39401\ \mu\text{m}.$$

The value of z_n for the odd parity:

n	z_n	a_n (μm)
1	-2.33811	13.7215
2	-4.08795	23.9906
3	-5.52056	32.3981
4	-6.78671	39.8286
5	-7.94413	46.6211
6	-9.02265	52.9506
7	-10.0402	58.922
8	-11.0085	64.6049
9	-11.936	70.048
10	-12.8288	75.2873
11	-13.6915	80.3502
12	-14.5278	85.2584
13	-15.3408	90.0291
14	-16.1327	94.6767
15	-16.9056	99.2128

The normalized wave function is obtained as follows.

$$a_n = \int_{Z_n}^{\infty} [A_i(z)]^2 dz$$

$$\psi_n{}^2(x) = \frac{A_i{}^2(z)}{a_n} |_{x=x+z_n}$$

(ii) Even parity case

$\psi'(x=0) = 0$ for the wavefunction with the even parity

The *n*-th zero points of $A_i'(z)$ is y_n. Then the value of the height a_n and the energy eigenvalue are given by

$$a_n = -\frac{y_n}{\left(\frac{2m^2 g}{\hbar^2}\right)^{1/3}}$$ (the discrete height)

$$E_n = mga_n = -mg\frac{y_n}{\left(\frac{2m^2g}{\hbar^2}\right)^{1/3}}$$ (the energy eigenvalue).

or

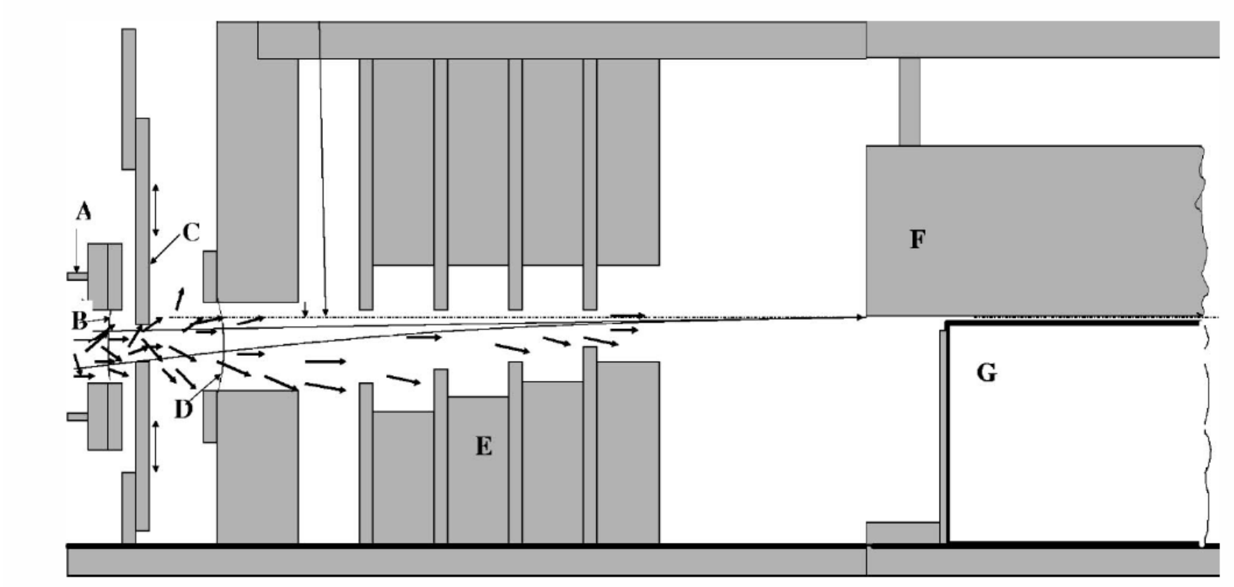
$$\frac{E_n}{mg} = a_n = -\frac{y_n}{2^{1/3}}\left(\frac{\hbar^2}{m^2g}\right)^{1/3}$$

The value of y_n vs n for the even parity:

n	y_n (even parity)	a_n (μm)
1	-1.01879	5.978896
2	-3.2482	19.06247
3	-4.8201	28.28736
4	-6.16331	36.17016
5	-7.37218	43.26456
6	-8.48849	49.81576
7	-9.53545	55.96000
8	-10.5277	61.78312
9	-11.4751	67.34306
10	-12.3841	72.67764
11	-13.2622	77.83088
12	-14.1115	82.81510
13	-14.9359	87.65320
14	-15.7382	92.36160
15	-16.5205	96.95262

2. Experiment

We consider a neutron with mass m at a height x above the floor. The potential energy is given by mgx. We note that the potential energy becomes infinity at $x = 0$, since the neutron bounces at $x = 0$. So the wavefunction becomes zero at $x = 0$. This means that the odd parity solution of the wavefunction is allowed.


A
B
C
D
E
F
G

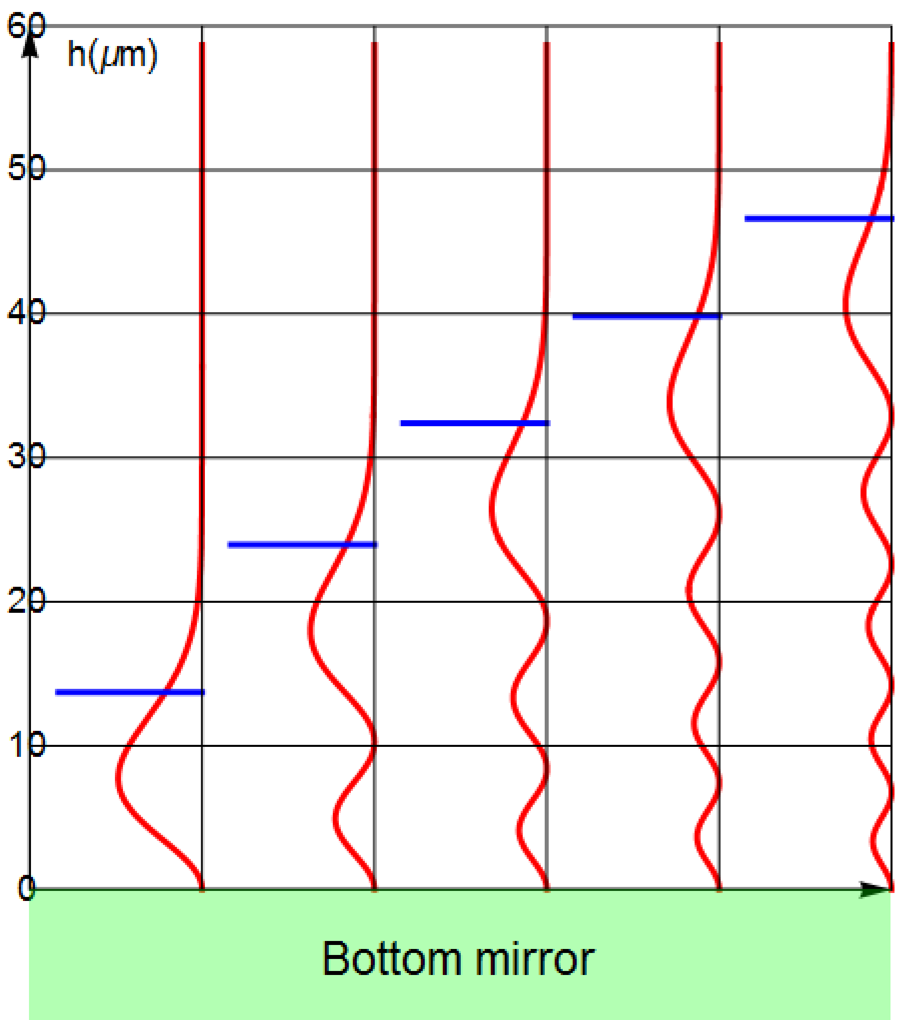
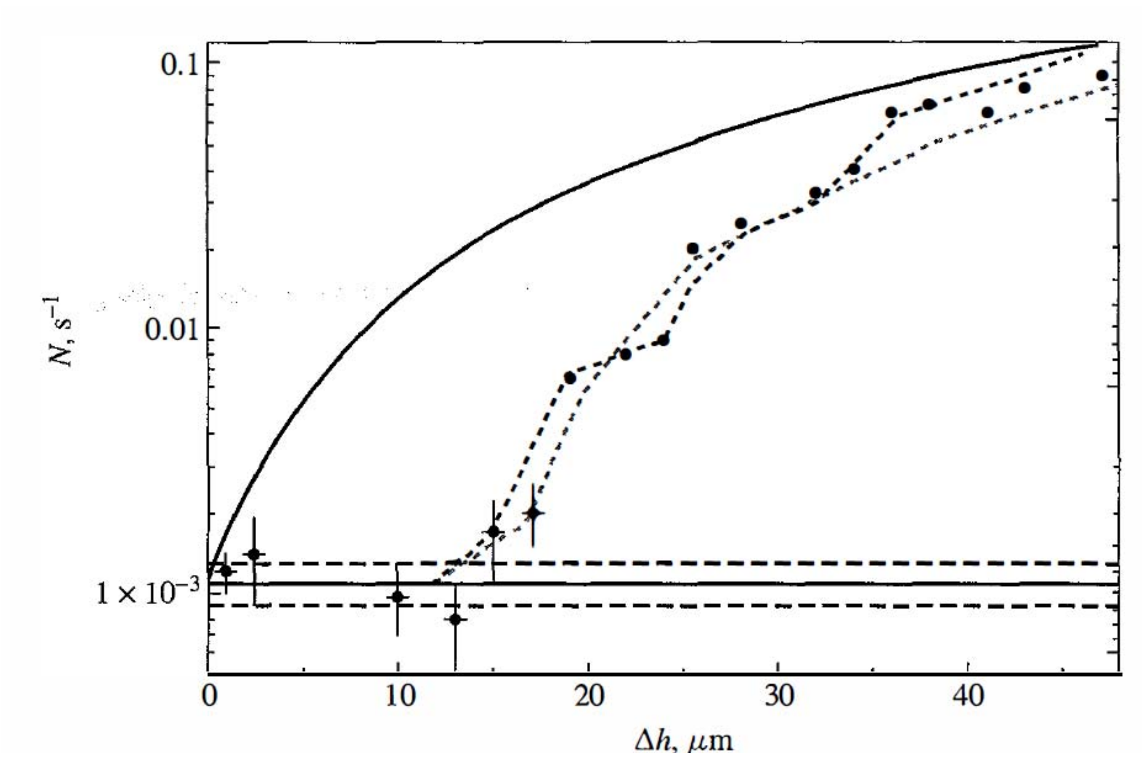

h(μm)
60
50
40
30
20
10
0
Bottom mirror

Fig. Probability amplitude $\psi_n^2(x)$ for the n-th state (n = 1, 2, 3,…). The blue lines denote the turning point in the classical limit. The energy eigenvalue of the n-th state is given by mga_n where a_n is the height of the turning point. The wave function has an odd parity since it should be zero at $x = 0$ because of the infinite potential at $x = 0$.



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V.V. Nesvizhevsky and A. Voronin, Surprising Quantum Bounces (Imperial College Press, 2015).

APPENDIX

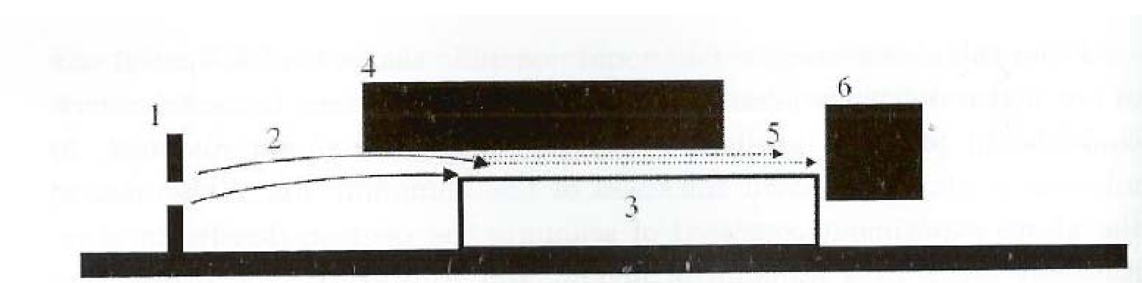

Fig: A simplified scheme of measuring gravitational quantum states of neutrons.

1: neutron collimator. 2: illustration of classical trajectories of neutrons upstream entrance of installation. 3: mirror. 4: absorber/scatterer. 5: illustration of horizontal component of neutron velocity in quantum regime of motion. 6: neutron detector. (From **Nesvizhevsky and Voronin**).

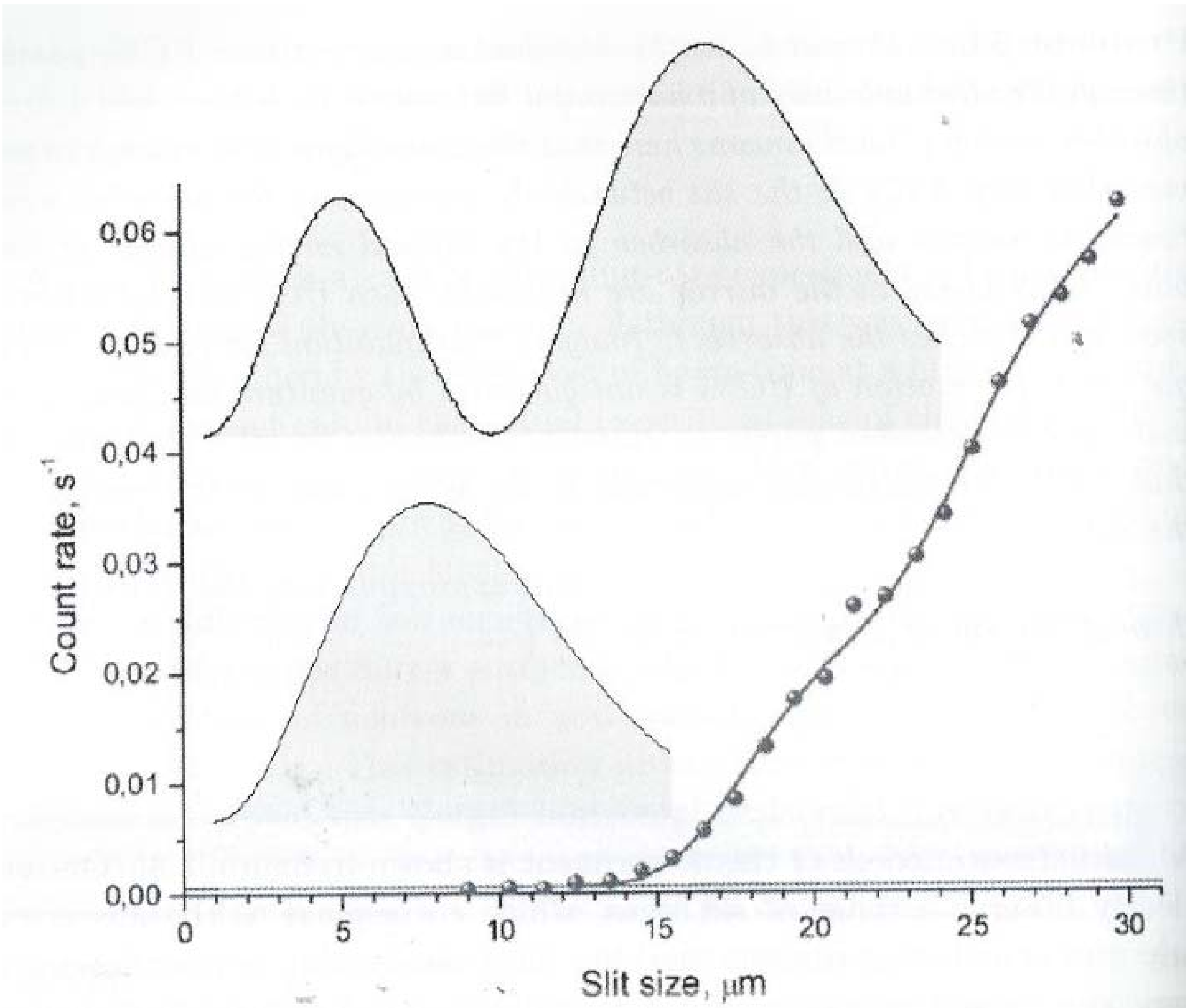

Fig. A typical result of first experiments for gravitational quantum states of neutrons. The measured neutron count in the detector is shown, with circles as a function of the size of slit between a flat polished horizontal mirror on bottom and an absorber with microscopically rough and macroscopically flat surface on top. A theoretical curve fits the data. One clearly observes the main feature of the measured data, which consists of the fact that neutrons do not pass through the slit as long as the slit size is smaller than the characteristic size of the gravitational quantum state. One also observes irregularities corresponding to second and third quantum states. Predicted probability density is shown in the insert for first and second quantum states.
(From **Nesvizhevsky and Voronin**).