Spin angular momentum of photon

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1. Spin angular momentum of photon

The spin angular momentum for photon is given by

$$\begin{aligned}\left(\hat{\boldsymbol{S}}_{EM}\right)_i &= \frac{i}{4\pi c\hbar}\int d\boldsymbol{r}(-i\hbar\varepsilon_{ijk}\hat{E}_j\hat{A}_k) \\ &= \frac{i}{4\pi c\hbar}\int d\boldsymbol{r}\hat{E}_j(-i\hbar\varepsilon_{ijk})\hat{A}_k \\ &= \frac{i}{4\pi c\hbar}\int d\boldsymbol{r}\hat{E}_j(\hat{S}_i)_{jk}\hat{A}_k\end{aligned}$$

$(S_i)_{jk}$ comes from the cross product operator, however, it can be seen to be a quantum spin operator, that couples different components of ***E*** and ***A***. This operator is defined here by its matrices, one for each component *i*, where *j* and *k* are the column and row

$$(\hat{S}_i)_{jk} = -i\hbar\varepsilon_{ijk},$$

with

$$\hat{S}_x = -i\hbar\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad \hat{S}_y = -i\hbar\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \hat{S}_z = -i\hbar\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We note that

$$[\hat{S}_x,\hat{S}_y] = i\hbar\hat{S}_z, \qquad [\hat{S}_y,\hat{S}_z] = i\hbar\hat{S}_x, \qquad [\hat{S}_z,\hat{S}_x] = i\hbar\hat{S}_y$$

$$\hat{S}_x^{\ 2} + \hat{S}_y^{\ 2} + \hat{S}_z^{\ 2} = 2\hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2\hbar^2\hat{1}$$

None of the matrices $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$ are diagonal when expressed in Cartesian. This just means that the Cartesian axes, to which these correspond, are not the good quantization axes.

((Eigenvalue problems))

(a) $\hat{S}_x$

Eigenvalue: $(+\hbar)$ Eigenket: $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$

Eigenvalue: (0) Eigenket: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Eigenvalue: $(-\hbar)$ Eigenket: $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}$

(b) $\hat{S}_y$

Eigenvalue: $(-\hbar)$ Eigenket: $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$

Eigenvalue: (0) Eigenket: $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Eigenvalue: $(+\hbar)$ Eigenket: $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix}$

(c) $\hat{S}_z$

Eigenvalue: $(+\hbar)$ Eigenket: $|u_+\rangle = |+\rangle = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$

Eigenvalue: (0) Eigenket: $|u_0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Eigenvalue: $(-\hbar)$ Eigenket: $|u_{-1}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$

The unitary operator $\hat{U}_z$ and its Hermite conjugate $\hat{U}_z^+$ are defined by

$$\hat{U}_z = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}, \qquad \hat{U}_z^+ = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}$$

We have

$$\hat{U}_z^+ \hat{S}_z \hat{U}_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \hat{J}_z$$

$$\hat{U}_z^+ \hat{S}_x \hat{U}_z = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \hat{J}_x$$

$$\hat{U}_z^+ \hat{S}_y \hat{U}_z = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} = \hat{J}_y$$

where $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_x$ are the conventional angular momentum with the magnitude $\hbar$.

2. Rotation operator

We now calculate the rotation operators

$$\hat{R}_z(\theta) = \exp[-\frac{i}{\hbar}\hat{S}_z\theta] = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{R}_z(\theta)|u_+\rangle = e^{-i\theta}|u_+\rangle, \quad \hat{R}_z(\theta)|u_0\rangle = |u_0\rangle, \qquad \hat{R}_z(\theta)|u_-\rangle = e^{i\theta}|u_-\rangle$$

Note that

$$\hat{R}_z^{+}(\theta)\hat{S}_x\hat{R}_z(\theta) = \hat{S}_x\cos\theta - \hat{S}_y\sin\theta$$

$$\hat{R}_z^{+}(\theta)\hat{S}_y\hat{R}_z(\theta) = \hat{S}_x\sin\theta + \hat{S}_y\cos\theta$$

We also have

$$\hat{R}_x(\theta) = \exp[-\frac{i}{\hbar}\hat{S}_x\theta] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$\hat{R}_y(\theta) = \exp[-\frac{i}{\hbar}\hat{S}_y\theta] = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

If we choose the eigenvectors of the spin operator $\hat{S}_z$,

$$\begin{aligned}\hat{E}_j(\hat{S}_z)_{jk}\hat{A}_k &= E_+\langle u_+|\hat{S}_z|u_+\rangle A_+ + E_-\langle u_-|\hat{S}_z|u_-\rangle A_- + E_0\langle u_0|\hat{S}_z|u_0\rangle A_0 \\ &= \hbar E_+A_+ - \hbar E_-A_-\end{aligned}$$

The two states $|+\rangle$ and $|-\rangle$ correspond to states where the A and E fields are rotating around the z axis. It is typical to consider waves propagating along the z axis.

((Mathematica))

```
Clear["Global`*"]; Sx = -ⅈ ℏ {{0, 0, 0}, {0, 0, 1}, {0, -1, 0}};
Sy = -ⅈ ℏ {{0, 0, -1}, {0, 0, 0}, {1, 0, 0}};
Sz = -ⅈ ℏ {{0, 1, 0}, {-1, 0, 0}, {0, 0, 0}};
```

$$Sx = -i\hbar\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix};\quad Sy = -i\hbar\begin{pmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix};\quad Sz = -i\hbar\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix};$$

```
Sx.Sx + Sy.Sy + Sz.Sz // MatrixForm
```

$$\begin{pmatrix}2\hbar^2 & 0 & 0\\ 0 & 2\hbar^2 & 0\\ 0 & 0 & 2\hbar^2\end{pmatrix}$$

```
Sx.Sy - Sy.Sx - ⅈ ℏ Sz
```

{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

```
Sy.Sz - Sz.Sy - ⅈ ℏ Sx
```

{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

```
Sz.Sx - Sx.Sz - ⅈ ℏ Sy
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

Eigensystem[Sx]
{{-ℏ, ℏ, 0}, {{0, ⅈ, 1}, {0, -ⅈ, 1}, {1, 0, 0}}}

Eigensystem[Sy]
{{-ℏ, ℏ, 0}, {{-ⅈ, 0, 1}, {ⅈ, 0, 1}, {0, 1, 0}}}

Eigensystem[Sz]
{{-ℏ, ℏ, 0}, {{ⅈ, 1, 0}, {-ⅈ, 1, 0}, {0, 0, 1}}}
```

When $\hat{\boldsymbol{a}}$ is a vector operator, the commutation relation

$$[\hat{L}_i, \hat{a}_k] = i\hbar\, \varepsilon_{ikl}\, \hat{a}_l \,.$$

where $\hat{\boldsymbol{L}}$ is an orbital angular momentum.

((Example))

(i) $\hat{\boldsymbol{a}} = \hat{\boldsymbol{r}}$

(ii) $\hat{\boldsymbol{a}} = \hat{\boldsymbol{p}}$

(iii) $\hat{\boldsymbol{a}} = \hat{\boldsymbol{n}} = \dfrac{\hat{\boldsymbol{r}}}{|\hat{\boldsymbol{r}}|} \to \boldsymbol{n} = \dfrac{\boldsymbol{r}}{r}$

(iv) $\hat{\boldsymbol{a}} = |\hat{\boldsymbol{r}}|\hat{\boldsymbol{p}} \to \dfrac{\hbar}{i} r\nabla$

(v) $\hat{\boldsymbol{a}} = \hat{\boldsymbol{n}} \times (|\hat{\boldsymbol{r}}|\hat{\boldsymbol{p}}) \to \dfrac{\hbar}{i} \boldsymbol{n} \times (r\nabla)$

((**Proof**)) Using the Mathematica, we show that $\hat{\boldsymbol{a}}$ for each case above shown is the vector operator.

```
Clear["Global`*"]; px := ℏ/ⅈ D[#, x] &; py := ℏ/ⅈ D[#, y] &;
pz := ℏ/ⅈ D[#, z] &; Lx := (y pz[#] - z py[#]) &;
Ly := (z px[#] - x pz[#]) &; Lz := (x py[#] - y px[#]) &;
nx = x/Sqrt[x^2 + y^2 + z^2]; ny = y/Sqrt[x^2 + y^2 + z^2]; nz = z/Sqrt[x^2 + y^2 + z^2];
Qx := Sqrt[x^2 + y^2 + z^2] px[#] &; Qy := Sqrt[x^2 + y^2 + z^2] py[#] &;
Qz := Sqrt[x^2 + y^2 + z^2] pz[#] &; hx := ( ny Qz[#] - nz Qy[#]) &;
hy := ( nz Qx[#] - nx Qz[#]) &;
hz := ( nx Qy[#] - ny Qx[#]) &;
```

Commutation [Li, ak] with a={nx,ny,nz}; **n**=**r**/|**r**|

```
Lx[ny ψ[x, y, z]] - ny Lx[ψ[x, y, z]] - ⅈ ℏ nz ψ[x, y, z] //
 Simplify

0

Ly[nz ψ[x, y, z]] - nz Ly[ψ[x, y, z]] - ⅈ ℏ nx ψ[x, y, z] //
 Simplify

0

Lz[nx ψ[x, y, z]] - nx Lz[ψ[x, y, z]] - ⅈ ℏ ny ψ[x, y, z] //
 Simplify

0
```

Commutation [Li, ak] with a={Qx,Qy,Qz}, **Q**=(|**r**|**p**)

```
Lx[Qy [ψ[x, y, z]]] - Qy [Lx[ψ[x, y, z]]] -
  ⅈ ℏ Qz [ψ[x, y, z]] // Simplify
```

0

```
Ly[Qz [ψ[x, y, z]]] - Qz [Ly[ψ[x, y, z]]] -
  ⅈ ℏ Qx [ψ[x, y, z]] // Simplify
```

0

```
Lz[Qx [ψ[x, y, z]]] - Qx [Lz[ψ[x, y, z]]] -
  ⅈ ℏ Qy [ψ[x, y, z]] // Simplify
```

0

Commutation [Li, ak] with a={hx,hy,hz}, **h** = **n** X (|**r**|**p**)

```
Lz[hx [ψ[x, y, z]]] - hx [Lz[ψ[x, y, z]]] -
  ⅈ ℏ hy[ ψ[x, y, z]] // Simplify
```

0

```
Lx[hy [ψ[x, y, z]]] - hy [Lx[ψ[x, y, z]]] -
  ⅈ ℏ hz[ ψ[x, y, z]] // Simplify
```

0

```
Ly[hz[ψ[x, y, z]]] - hz[Ly[ψ[x, y, z]]] -
  iℏ hx[ψ[x, y, z]] // Simplify
```

0

Commutation [Li, ak] with a={px,py,pz},

```
Lx[py[ψ[x, y, z]]] - py[Lx[ψ[x, y, z]]] -
  iℏ pz[ψ[x, y, z]] // Simplify
```

0

```
Ly[pz[ψ[x, y, z]]] - pz[Ly[ψ[x, y, z]]] -
  iℏ px[ψ[x, y, z]] // Simplify
```

0

```
Lz[px[ψ[x, y, z]]] - px[Lz[ψ[x, y, z]]] -
   iℏ py[ψ[x, y, z]] // Simplify // FullSimplify
```

0

Commutation [Li, ak] with a={x,y,z},

```
Lx[y ψ[x, y, z]] - y Lx[ψ[x, y, z]] - ⅈ ℏ z ψ[x, y, z] //
 Simplify
```

0

```
Ly[z ψ[x, y, z]] - z Ly[ψ[x, y, z]] - ⅈ ℏ x ψ[x, y, z] //
 Simplify
```

0

```
Lz[x ψ[x, y, z]] - x Lz[ψ[x, y, z]] - ⅈ ℏ y ψ[x, y, z] //
 Simplify
```

0

$$(\hat{S}_i)_{jk} = -i\hbar\varepsilon_{ijk}$$

$$[\hat{L}_i, \hat{a}_k] = i\hbar\varepsilon_{ikl}\hat{a}_l = -\hat{S}_i\hat{a}_k$$

$$\hat{L}_i\hat{a}_k - \hat{a}_k\hat{L}_i = i\hbar\varepsilon_{ikl}\hat{a}_l = -\hat{S}_i\hat{a}_k$$

$$\hat{J}_i\hat{a}_k = (\hat{L}_i + \hat{S}_i)\hat{a}_k = \hat{a}_k\hat{L}_i$$

Cartesian spin one operator
(Kelly)

$$\hat{\boldsymbol{S}} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$$

$$\boldsymbol{a} = (a_x, a_y, a_z)$$

$$\hat{\boldsymbol{S}}\cdot\boldsymbol{a} = a_x\hat{S}_x + a_y\hat{S}_y + a_z\hat{S}_z = i\hbar\begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix}$$

$$(\hat{\boldsymbol{S}}\cdot\boldsymbol{a})\boldsymbol{F} = i\hbar\begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix}\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = i\hbar\begin{pmatrix} -a_zF_y + a_yF_z \\ -a_xF_z + a_zF_x \\ -a_yF_x + a_xF_y \end{pmatrix} = i\hbar\boldsymbol{a}\times\boldsymbol{F}$$

or

$$(\hat{\boldsymbol{S}}\cdot\boldsymbol{a})\boldsymbol{F} = i\hbar\boldsymbol{a}\times\boldsymbol{F}$$

REFERENCES

V.B. Berestetskii, E.M. Lifshitz, and L.P. Pitaevskii, Quantum Electrodynamics, 2nd edition (Pergamon Press, 1982).

Das

$$\langle\psi'|\hat{A}_i|\psi'\rangle = \langle\psi|\hat{R}^+\hat{A}_i\hat{R}|\psi\rangle = \sum_j \Re_{ij}\langle\psi|\hat{A}_j|\psi\rangle,$$

$$\Re_{ij} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Re^{-1}{}_{ij} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{R}|\boldsymbol{r}\rangle = |\Re\boldsymbol{r}\rangle \qquad \langle\Re\boldsymbol{r}| = \langle\boldsymbol{r}|\hat{R}^+$$

$$\hat{R}^+|\boldsymbol{r}\rangle = |\Re^{-1}\boldsymbol{r}\rangle \qquad \langle\boldsymbol{r}|\hat{R} = \langle\Re^{-1}\boldsymbol{r}|$$

Infinitesimal transformation:

$$\hat{R}|\boldsymbol{r}\rangle = |\Re\boldsymbol{r}\rangle = |x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta\rangle = |x-\varepsilon y, \varepsilon x + y\rangle$$

$$\hat{R}^+|\boldsymbol{r}\rangle = |\Re^{-1}\boldsymbol{r}\rangle = |x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta\rangle = |x+\varepsilon y, -\varepsilon x + y\rangle$$

$$\begin{aligned}
\hat{R}|\psi\rangle &= \hat{R}\int d\boldsymbol{r}_0 |\boldsymbol{r}_0\rangle\langle\boldsymbol{r}_0|\psi\rangle \\
&= \int d\boldsymbol{r}_0 \hat{R}|\boldsymbol{r}_0\rangle\langle\boldsymbol{r}_0|\psi\rangle \\
&= \int d\boldsymbol{r}_0 |\Re\boldsymbol{r}_0\rangle\langle\boldsymbol{r}_0|\psi\rangle \\
&= \int d\boldsymbol{r} |\boldsymbol{r}\rangle\langle\Re^{-1}\boldsymbol{r}|\psi\rangle
\end{aligned}$$

Here we use a new variable; $\Re\boldsymbol{r}_0 = \boldsymbol{r}$.

$$\int d\boldsymbol{r}_0 |\Re\boldsymbol{r}_0\rangle\langle\boldsymbol{r}_0|\psi\rangle = \int d\boldsymbol{r} |\boldsymbol{r}\rangle\langle\Re^{-1}\boldsymbol{r}|\psi\rangle$$

since

$$d\boldsymbol{r}_0 = \frac{\partial(x_0, y_0, z_0)}{\partial(x, y, z)} d\boldsymbol{r} = d\boldsymbol{r}$$

where the Jacobian

$$\frac{\partial(x_0, y_0, z_0)}{\partial(x, y, z)} = 1$$

For the infinitesimal rotation, we get

$$\hat{R}|\psi\rangle = \int d\boldsymbol{r} |\boldsymbol{r}\rangle\langle\Re^{-1}\boldsymbol{r}|\psi\rangle$$

or

$$\langle\psi|\hat{R}^{+} = \int d\boldsymbol{r} \langle\Re^{-1}\boldsymbol{r}|\psi\rangle^{*}\langle\boldsymbol{r}|$$

Evaluation of $\langle\psi|\hat{R}^{+}\hat{A}_i\hat{R}|\psi\rangle$

$$\langle\psi|\hat{R}^{+}\hat{A}_i\hat{R}|\psi\rangle = \int d\boldsymbol{r}'\langle\Re^{-1}\boldsymbol{r}'|\psi\rangle^{*}\langle\boldsymbol{r}'|\hat{A}_i\int d\boldsymbol{r}|\boldsymbol{r}\rangle\langle\Re^{-1}\boldsymbol{r}'|\psi\rangle$$
$$= \iint d\boldsymbol{r}'d\boldsymbol{r}\langle\Re^{-1}\boldsymbol{r}'|\psi\rangle^{*}\langle\Re^{-1}\boldsymbol{r}|\psi\rangle\langle\boldsymbol{r}'|\hat{A}_i|\boldsymbol{r}\rangle$$
$$= \iint d\boldsymbol{r}'d\boldsymbol{r}\langle\Re^{-1}\boldsymbol{r}'|\psi\rangle^{*}\langle\Re^{-1}\boldsymbol{r}|\psi\rangle A_i(\boldsymbol{r})\delta(\boldsymbol{r}-\boldsymbol{r}')$$
$$= \int d\boldsymbol{r}\langle\Re^{-1}\boldsymbol{r}|\psi\rangle^{*}\langle\Re^{-1}\boldsymbol{r}|\psi\rangle A_i(\boldsymbol{r})$$
$$= \int d\boldsymbol{r}\langle\boldsymbol{r}|\psi\rangle^{*}\langle\boldsymbol{r}|\psi\rangle A_i(\Re\boldsymbol{r})$$
$$= \int d\boldsymbol{r}|\langle\boldsymbol{r}|\psi\rangle|^{2} A_i(\Re\boldsymbol{r})$$

and

$$\langle\psi|\hat{A}_j|\psi\rangle = \iint d\boldsymbol{r}d\boldsymbol{r}'\langle\psi|\boldsymbol{r}'\rangle\langle\boldsymbol{r}'|\hat{A}_j|\boldsymbol{r}\rangle\langle\boldsymbol{r}|\psi\rangle$$
$$= \iint d\boldsymbol{r}d\boldsymbol{r}'\langle\psi|\boldsymbol{r}'\rangle\hat{A}_j(\boldsymbol{r})\langle\boldsymbol{r}'|\boldsymbol{r}\rangle\langle\boldsymbol{r}|\psi\rangle$$
$$= \iint d\boldsymbol{r}d\boldsymbol{r}'\langle\psi|\boldsymbol{r}'\rangle\hat{A}_j(\boldsymbol{r})\delta(\boldsymbol{r}-\boldsymbol{r}')\langle\boldsymbol{r}|\psi\rangle$$
$$= \int d\boldsymbol{r}\langle\psi|\boldsymbol{r}\rangle\hat{A}_j(\boldsymbol{r})\langle\boldsymbol{r}|\psi\rangle$$
$$= \int d\boldsymbol{r}\hat{A}_j(\boldsymbol{r})|\langle\boldsymbol{r}|\psi\rangle|^{2}$$

Then we have

$$A_i(\Re\boldsymbol{r}) = \Re_{ij}A_j(\boldsymbol{r})$$

or

$$A_i(\boldsymbol{r}) = \Re_{ij}A_j(\Re^{-1}\boldsymbol{r})$$

Infinitesimal rotation

$$
\begin{aligned}
A_x(\boldsymbol{r}) &= A_x(\Re^{-1}\boldsymbol{r}) - \varepsilon A_y(\Re^{-1}\boldsymbol{r}) \\
&= A_x(x+\varepsilon y, -\varepsilon x + y, z) - \varepsilon A_y(x+\varepsilon y, -\varepsilon x + y, z) \\
&= A_x(x,y,z) + \varepsilon\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)A_x(x,y,z) \\
&\quad - \varepsilon_x\left[A_y(x,y,z) + \varepsilon\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)A_y(x,y,z)\right] \\
&= A_x(x,y,z) + \varepsilon\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)A_x(x,y,z) \\
&\quad - \varepsilon\left[A_y(x,y,z) + \varepsilon\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)A_y(x,y,z)\right] \\
&\approx A_x(x,y,z) + \varepsilon\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)A_x(x,y,z) - \varepsilon A_y(x,y,z)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
A_y(\boldsymbol{r}) &= \varepsilon A_x(\Re^{-1}\boldsymbol{r}) + A_y(\Re^{-1}\boldsymbol{r}) \\
&= \varepsilon A_x(x+\varepsilon y, -\varepsilon x + y, z) + A_y(x+\varepsilon y, -\varepsilon x + y, z) \\
&= \varepsilon A_x(x,y,z) + \\
&\quad + A_y(x,y,z) + \varepsilon\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)A_y(x,y,z)] \\
&\approx A_y(x,y,z) + \varepsilon\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)A_y(x,y,z) + \varepsilon A_x(x,y,z)
\end{aligned}
$$

$$
\begin{aligned}
A_z(\boldsymbol{r}) &= A_z(x+\varepsilon y, -\varepsilon x + y, z) \\
&= A_z(x,y,z) + \varepsilon\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)A_z(x,y,z)
\end{aligned}
$$

$$
\begin{pmatrix} A_x(\boldsymbol{r}) \\ A_y(\boldsymbol{r}) \\ A_z(\boldsymbol{r}) \end{pmatrix} = [1 + \varepsilon\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) + \varepsilon\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}]\begin{pmatrix} A_x(x,y,z) \\ A_y(x,y,z) \\ A_z(x,y,z) \end{pmatrix}
$$

REFERENCE

M.E. Rose p.102 Elementary Theory of Angular Momentum
Das p.168