Photon polarization

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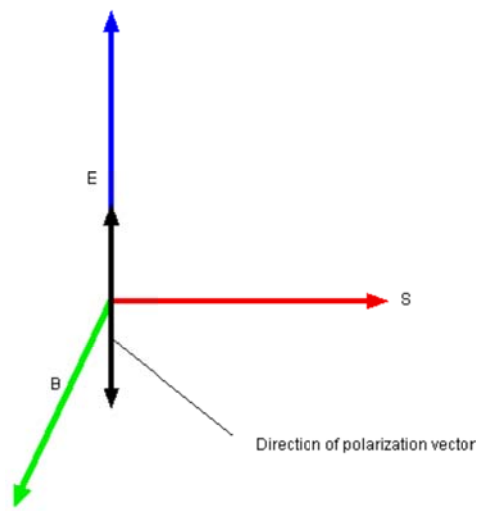
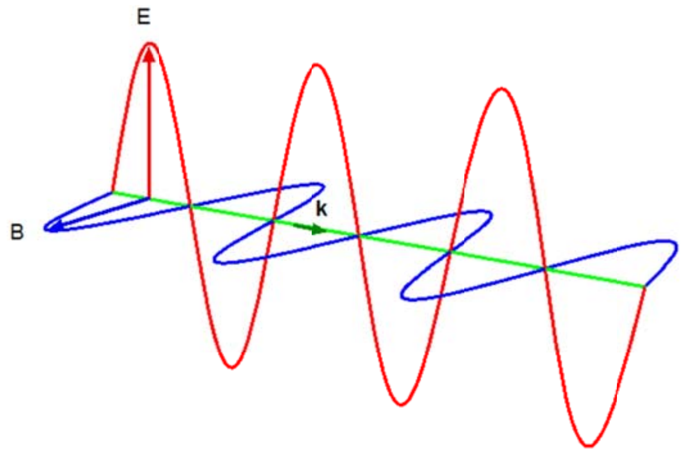
1. Polarization

The electric and magnetic vectors associated with an electromagnetic wave are perpendicular to each other and to the direction of wave propagation. Polarization is a property that specifies the directions of the electric and magnetic fields associated with an EM wave. The direction of polarization is defined to be the direction in which the *electric field is vibrating.*



The plane containing the ***E***-vector is called the plane of oscillation of the wave. Hence the wave is said to be plane polarized in the y direction. We can represent the wave's polarization by showing the direction of electric field oscillations in a head-on view of the plane of oscillation.



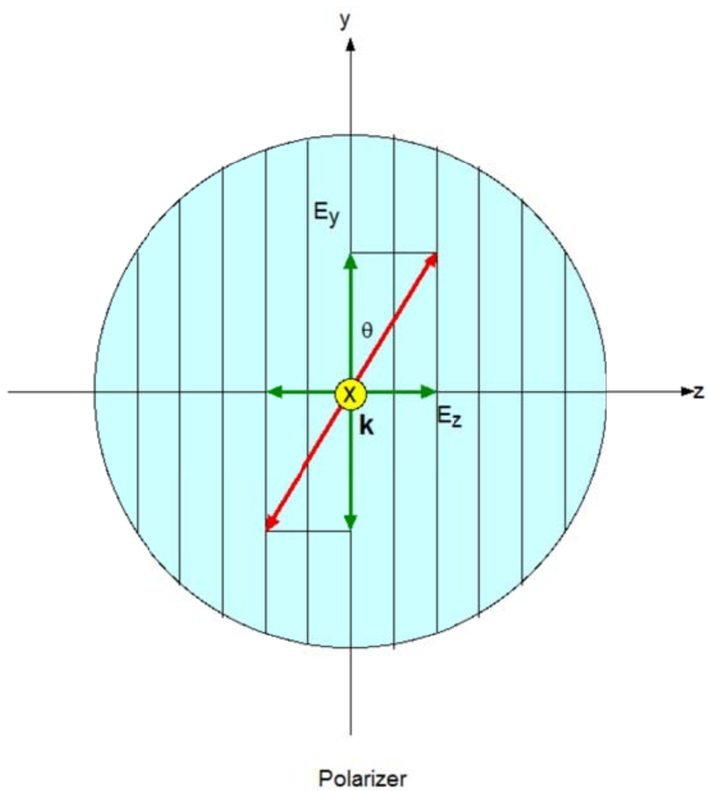
2. Unpolarized light

All directions of vibration from a wave source are possible. The resultant EM wave is a superposition of waves vibrating in many different directions. This is an unpolarized wave. The arrows show a few possible directions of the waves in the beam. The representing unpolarized light is the superposition of two polarized waves (E_x and E_y) whose planes of oscillation are perpendicular to each other.

3 Intensity of transmitted polarized light

(1) Malus' law

An electric field component parallel to the polarization direction is passed (transmitted) by a polarizing sheet. A component perpendicular to it is absorbed.



The electric field along the direction of the polarizing sheet is given by

$$E_y = E\cos\theta .$$

Then the intensity I of the polarized light with the polarization vector parallel to the y axis is given by

$$I = I_0 \cos^2\theta \quad \text{(Malus' law)},$$

where

$$I = S_{avg} = \frac{E_{rms}{}^2}{c\mu_0} = I_0 \cos^2\theta .$$

((Note)) Etienne Louis Malus (1775 – 1812).

Étienne-Louis Malus (23 July 1775 – 24 February 1812) was a French officer, engineer, physicist, and mathematician. Malus was born in Paris, France. He participated in Napoleon's expedition into Egypt (1798 to 1801) and was a member of the mathematics section of the Institut d'Égypte. Malus became a member of the Académie des Sciences in 1810. In 1810 the Royal Society of London awarded him the Rumford Medal. His mathematical work was almost entirely concerned with the study of light. He studied geometric systems called *ray systems*, closely connected to Julius Plücker's *line geometry*. He conducted experiments to verify Christiaan Huygens' theories of light and rewrote the theory in analytical form. His discovery of the polarization of light by reflection was published in 1809 and his theory of double refraction of light in crystals, in 1810. Malus attempted to identify the relationship between the polarizing angle of reflection that he had discovered, and the refractive index of the reflecting material. While he deduced the correct relation for water, he was unable to do so for glasses due to the low quality of materials available to him (most glasses at that time showing a variation in refractive index between the surface and the interior of the glass). It was not until 1815 that Sir David Brewster was able to experiment with higher quality glasses and correctly formulate what is known as Brewster's law. Malus is probably best remembered for Malus' law, giving the resultant intensity, when a polarizer is placed in the path of an incident beam. His name is one of the 72 names inscribed on the Eiffel tower.

http://en.wikipedia.org/wiki/%C3%89tienne-Louis_Malus

((**Note**))

I made a simulation of the experiment of polarization of light using the Mathematica

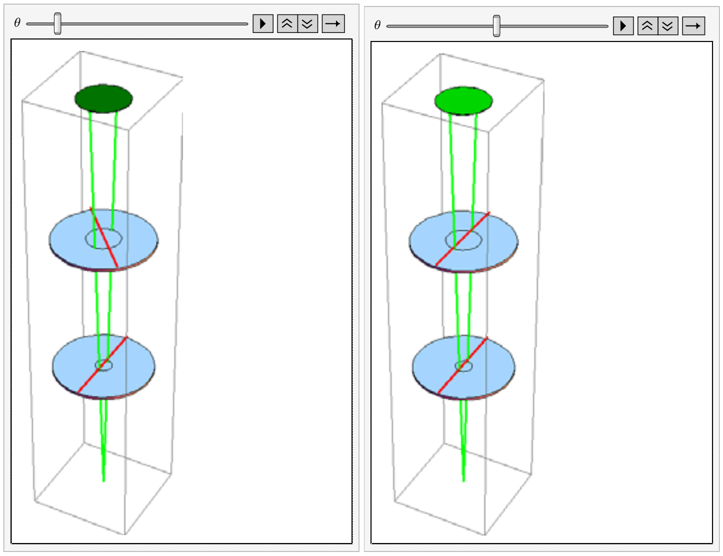


Fig. Demonstration for the role of two polarizers. The light passes when the directions of the two polarizers are the same. The light does not pass when the directions of two polarizers are perpendicular to each other.

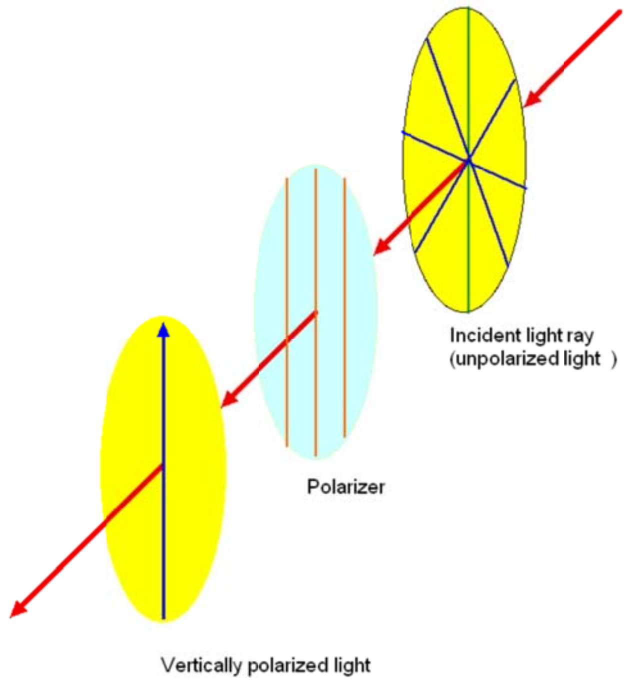
(2) One-half rule for unpolarized light

When the light reaching a polarization sheet is unpolarized, we get a polarized light with the intensity

$$\boxed{I = \frac{I_0}{2}}, \qquad \text{(one-half rule)}$$

since

$$I = \frac{1}{2\pi}\int_0^{2\pi} I_0 \cos^2\theta d\theta = \frac{I_0}{2\pi}\frac{1}{2}\int_0^{2\pi}[1-\cos(2\theta)]d\theta = \frac{I_0}{2}.$$



4. Definition of the Polarization

We define α as

$$\alpha = kz - \omega t\ .$$

We use the conventional notation for the plane wave propagating along the positive z directyion, such that

$$\psi(z,t) = \exp[i(kz - \omega t)] = \exp(i\alpha)\ .$$

((**Note**))
In physics, usually we use this notation. This corresponds to the plane wave travelling along the positive *z* direction. This definition is significant to the understanding of the right hand and left hand circularly polarization waves.

(1) $\left|\psi\right\rangle = \left|x\right\rangle$ (0° polarization)

The electric field for the 0° polarization is given by

$$\mathrm{Re}[E_0 e^{i\alpha}\boldsymbol{e}_x] = (E_0\cos\alpha)\boldsymbol{e}_x$$

(2) $\quad |\psi\rangle = |y\rangle$ (90° polarization)

The electric field for the 90° polarization is given by

$$\mathrm{Re}[E_0 e^{i\alpha} \boldsymbol{e}_y] = (E_0 \cos\alpha)\boldsymbol{e}_y .$$

(3) $\quad |\psi\rangle = \frac{1}{\sqrt{2}}(|x\rangle + |y\rangle)$ (45° polarization)

The electric field for the 45° polarization is given by

$$\mathrm{Re}[E_0 e^{i\alpha}\left(\frac{\boldsymbol{e}_x + \boldsymbol{e}_y}{\sqrt{2}}\right)] = E_0 \cos\alpha \left(\frac{\boldsymbol{e}_x + \boldsymbol{e}_y}{\sqrt{2}}\right).$$

(4) $\quad |\psi\rangle = \frac{1}{\sqrt{2}}(|x\rangle - |y\rangle)$ (-45° polarization)

The electric field for the (-45°) polarization is given by

$$\mathrm{Re}[E_0 e^{i\alpha}\left(\frac{\boldsymbol{e}_x - \boldsymbol{e}_y}{\sqrt{2}}\right)] = E_0 \cos\alpha \left(\frac{\boldsymbol{e}_x - \boldsymbol{e}_y}{\sqrt{2}}\right).$$

(5) $\quad |R\rangle = \frac{1}{\sqrt{2}}(|x\rangle + i|y\rangle)$ (right-circularly polarized wave)

$$\begin{aligned}\mathrm{Re}[E_0 e^{i\alpha} \frac{(\boldsymbol{e}_x + i\boldsymbol{e}_y)}{\sqrt{2}}] &= \frac{E_0}{\sqrt{2}}[(\cos\alpha)\boldsymbol{e}_x - (\sin\alpha)\boldsymbol{e}_y] \\ &= \frac{E_0}{\sqrt{2}}[\cos(kz - \omega t)\boldsymbol{e}_x - \sin(kz - \omega t)\boldsymbol{e}_y] \\ &= \frac{E_0}{\sqrt{2}}[\cos(\omega t - kz)\boldsymbol{e}_x + \sin(\omega t - kz)\boldsymbol{e}_y]\end{aligned}$$

with

$$\boldsymbol{e}_+ = \frac{\boldsymbol{e}_x + i\boldsymbol{e}_y}{\sqrt{2}} .$$

The electric field rotates in clock-wise direction with z (wavenumber k) (as the wave propagates forward). The electric field rotates in counter clock-wise direction with time (angular frequency ω).

(6) $\quad |L\rangle = \frac{1}{\sqrt{2}}(|x\rangle - i|y\rangle)$ (left-circularly polarized wave)

$$\mathrm{Re}[E_0 e^{i\alpha} \frac{(\boldsymbol{e}_x - i\boldsymbol{e}_y)}{\sqrt{2}}] = \frac{E_0}{\sqrt{2}}[(\cos\alpha)\boldsymbol{e}_x + (\sin\alpha)\boldsymbol{e}_y]$$
$$== \frac{E_0}{\sqrt{2}}[\cos(kz-\omega t)\boldsymbol{e}_x + \sin(kz-\omega t)\boldsymbol{e}_y]$$
$$= \frac{E_0}{\sqrt{2}}[\cos(\omega t - kz)\boldsymbol{e}_x - \sin(\omega t - kz)\boldsymbol{e}_y]$$

with

$$\boldsymbol{e}_- = \frac{\boldsymbol{e}_x - i\boldsymbol{e}_y}{\sqrt{2}}.$$

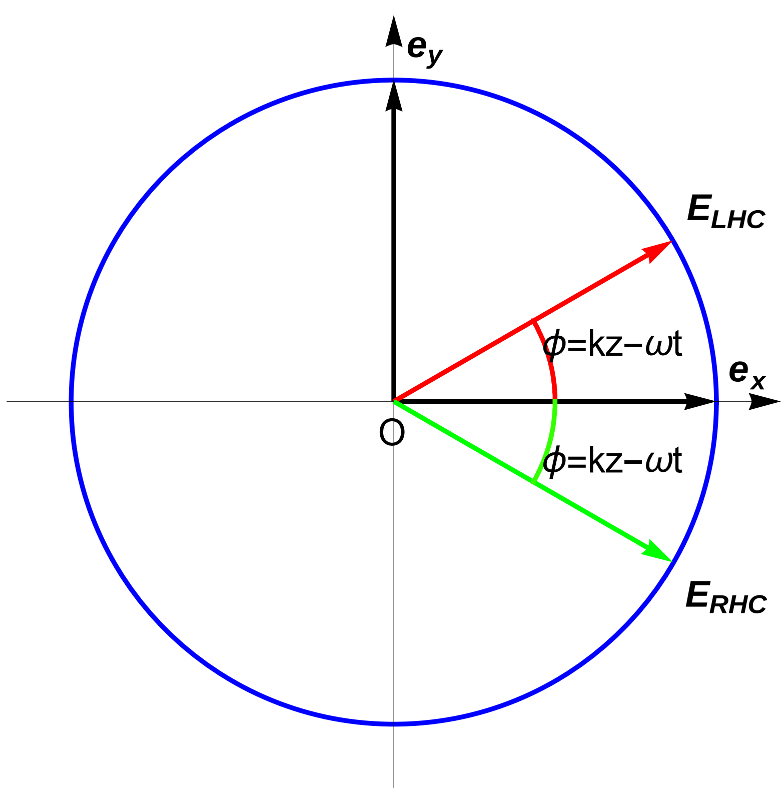


Fig. Direction of the electric field for the RHC (right-hand circularly) photon and LHC (left-hand circularly) photon (x-y plane which is perpendicular to the propagation direction z axis). The phase angle is given by $\phi = kz - \omega t$.

The electric field rotates in counter clock-wise direction with z (wavenumber k) (as the wave propagates forward). The electric field rotates in clock-wise direction with time (angular frequency ω).

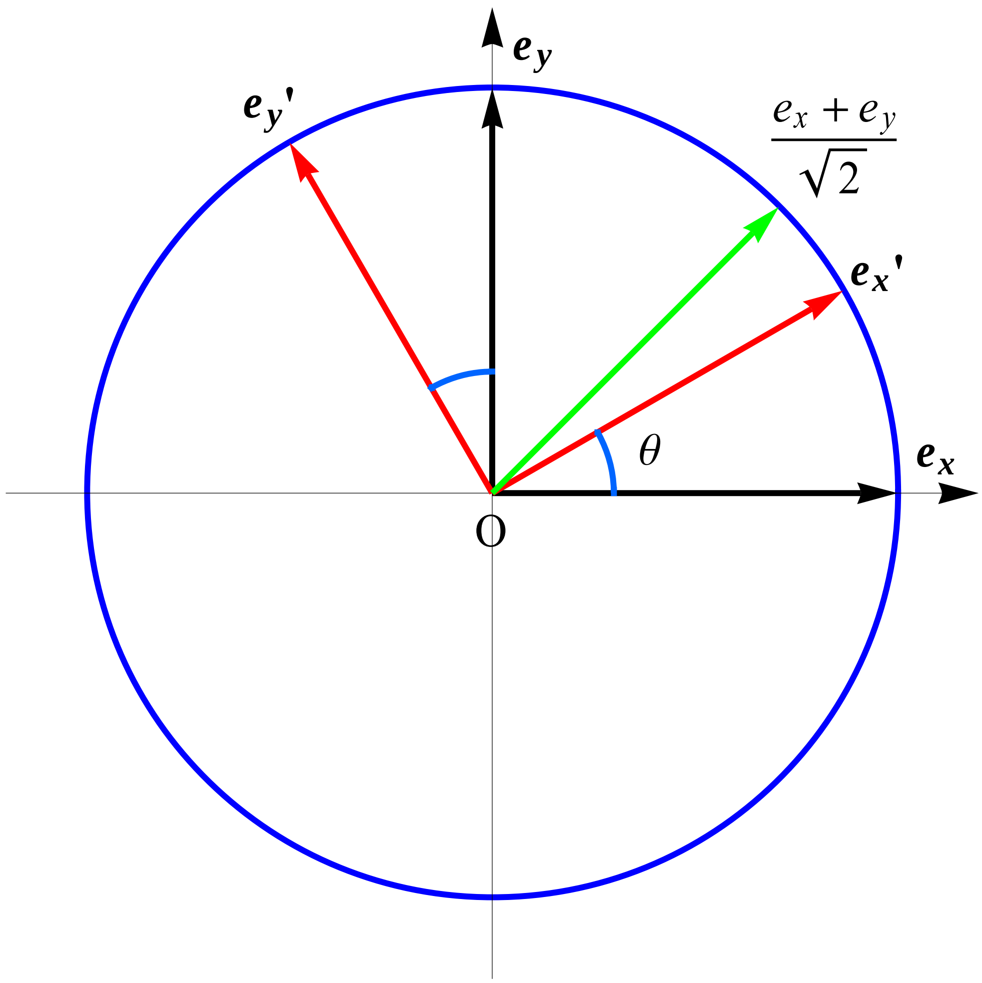


Fig. The direction of the electric field for various kinds of polarization. The phase angle θ is fixed (θ does not change with time t).

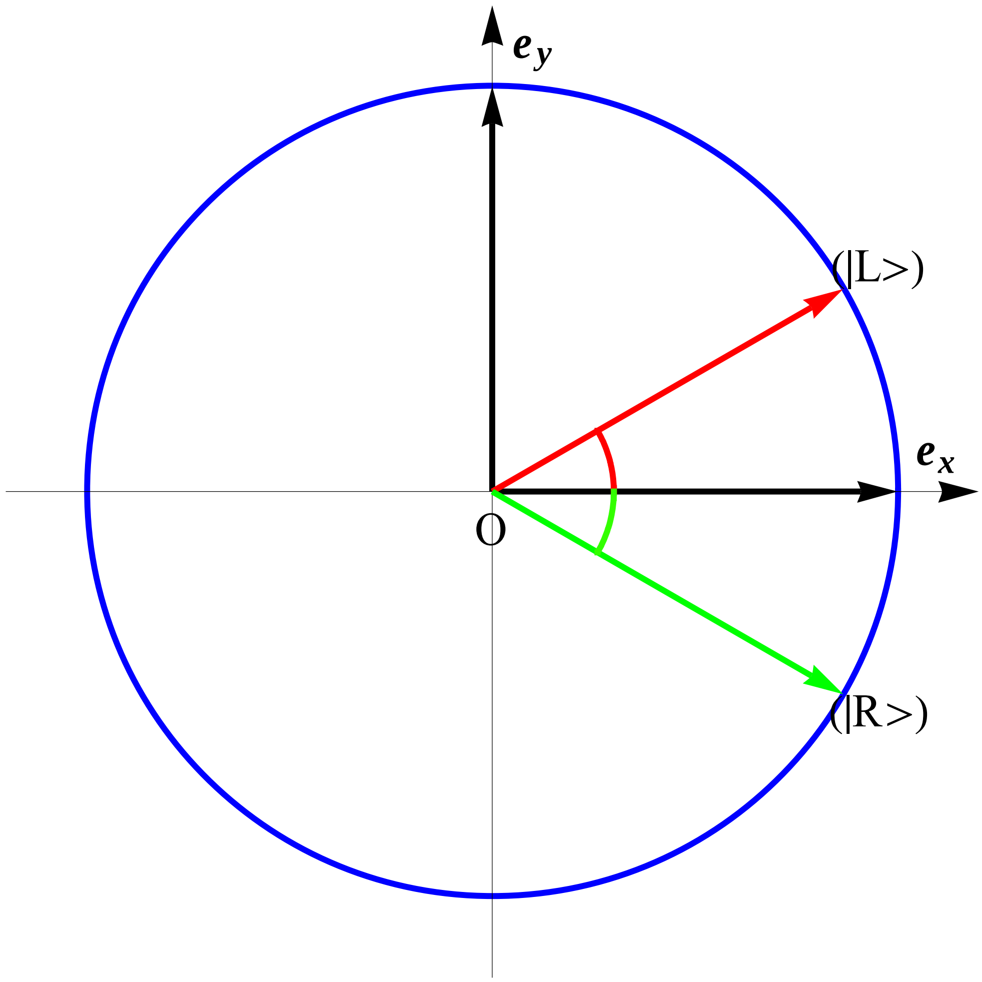


Fig. The direction of the electric field for the right-hand and left-hand circularly polarizations. The phase angle is given by $\alpha = kz - \omega t$. At the fixed z, when t is changed, α is actually negative. $\frac{1}{\sqrt{2}}[\cos(\alpha)\boldsymbol{e}_x + \sin(\alpha)\boldsymbol{e}_y]$ for the right-hand circular polarization (which rotates with time in clock-wise) and $\frac{1}{\sqrt{2}}[\cos(\alpha)\boldsymbol{e}_x - \sin(\alpha)\boldsymbol{e}_y]$ for the left-hand circularly polarization (which rotates with time in counter-clock wise). Note that the electric field rotates in a counter-clock wise as the wave propagates forward, for the right-hand circularly polarized light and that it rotates in a clock-wise as the wave propagates forward, for the left-hand circularly light.

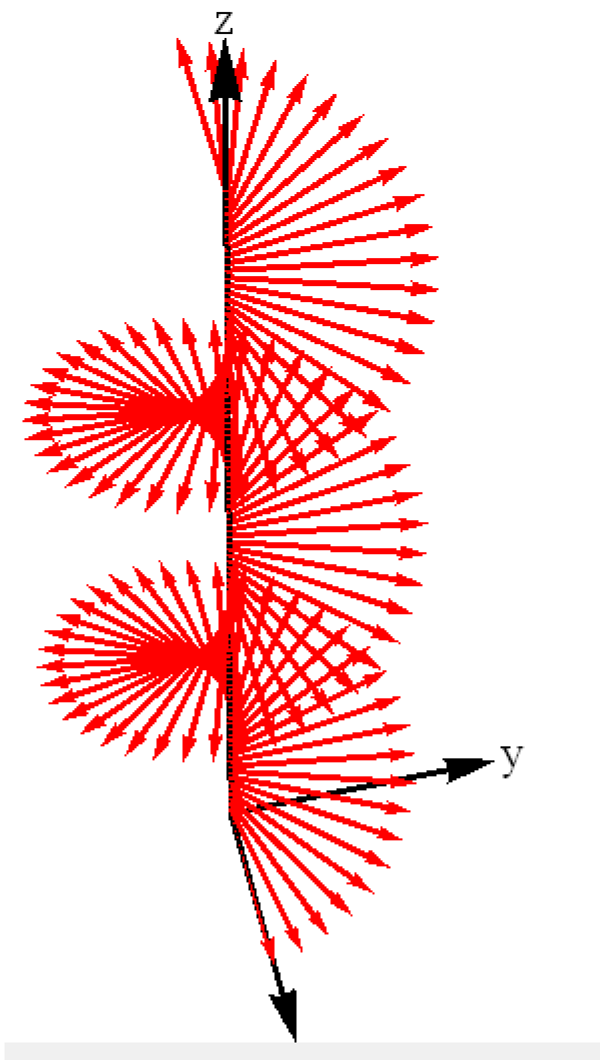


Fig. Left circularly polarization with z. ParametricPlot3D of $\{\frac{1}{\sqrt{2}}\cos(kz), \frac{1}{\sqrt{2}}\sin(kz), z\}$, where z is varied as a parameter between $z = 0$ and $z = 5.k = \pi$. Note that t is fixed. In this case $t = 0$. The electric field rotates in a counterclock-wise direction as z is increased.

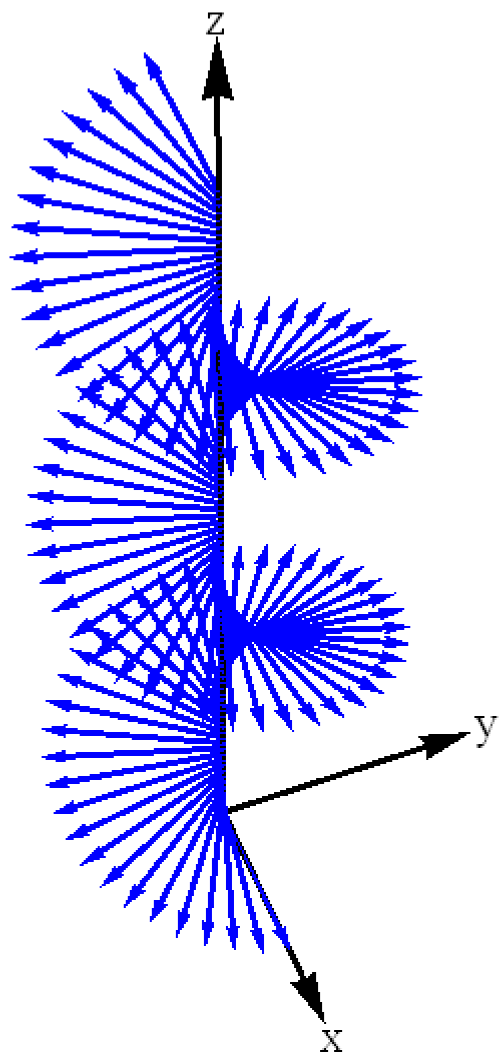


Fig. Right circularly polarization with z. ParametricPlot3D of $\{\frac{1}{\sqrt{2}}\cos(kz), -\frac{1}{\sqrt{2}}\sin(kz), z\}$, where z is varied as a parameter between $z = 0$ and $z = 5.k = \pi$. Note that t is fixed. In this case, $t = 0$. The electric field rotates in a clock-wise direction as z is increased.

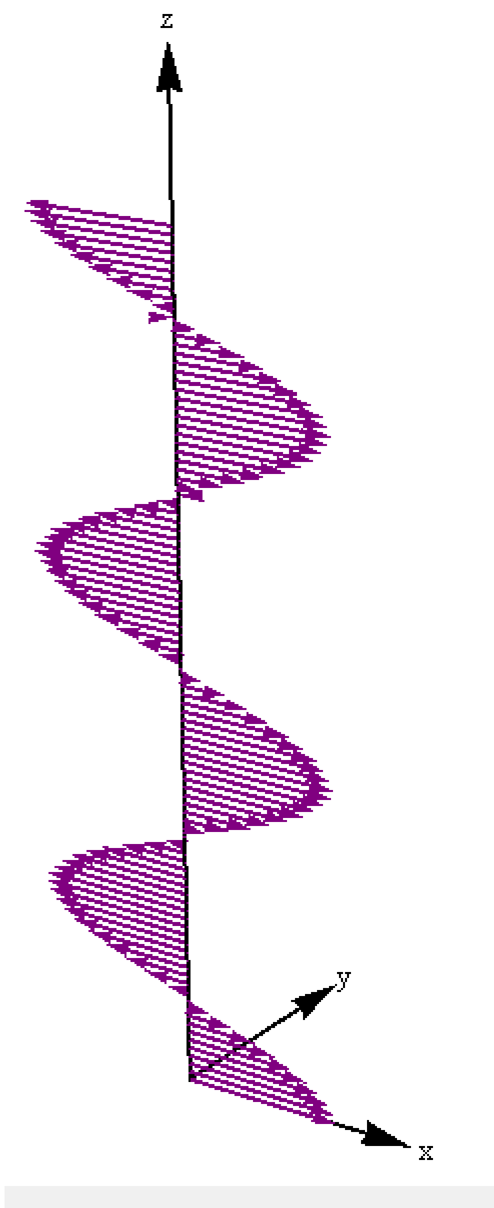


Fig. Linear polarization with z. ParametricPlot3D of $\{\cos(kz), 0, z\}$, where z is varied as a parameter between $z = 0$ and $z = 5.k = \pi$. Note that t is fixed. In this case, $t = 0$.

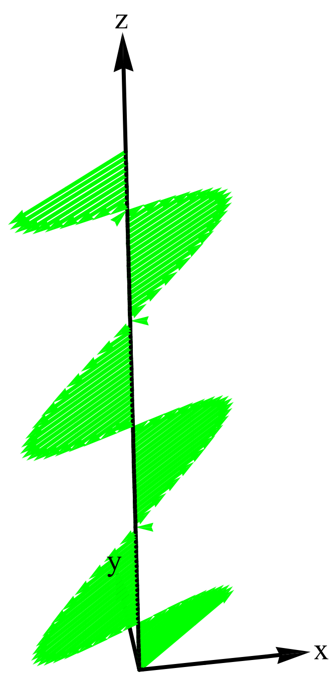


Fig. Linear polarization (45° polarization). ParametricPlot3D of {cos(kz), cos(kz), z}, where z is varied as a parameter between z = 0 and z = 5.k = π. In this case, t = 0.

5. Definition of photon polarization in the quantum mechanics

We define the state vector of the photon polarization.

$$|\psi\rangle = C_x|x\rangle + C_y|y\rangle,$$

where the basis is given by

$$|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}: \qquad x\text{-polarization}$$

$$|y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}: \qquad y\text{-polarization}$$

Note that

$$\langle x|\psi\rangle = C_x\langle x|x\rangle + C_y\langle x|y\rangle = C_x ,$$

$$\langle y|\psi\rangle = C_x\langle y|x\rangle + C_y\langle y|y\rangle = C_y .$$

where

$$\langle x|x\rangle = 1 , \qquad \langle x|y\rangle = 0 , \qquad \langle y|x\rangle = 0 , \qquad \langle y|y\rangle = 1 .$$

From the relation

$$\begin{aligned} |\psi\rangle &= |x\rangle\langle x|\psi\rangle + |y\rangle\langle y|\psi\rangle \\ &= (|x\rangle\langle x| + |y\rangle\langle y|)|\psi\rangle \end{aligned}$$

we have the closure relation (completeness),

$$|x\rangle\langle x| + |y\rangle\langle y| = \hat{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

((Note))

$$|x\rangle\langle x| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$|y\rangle\langle y| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$|x\rangle\langle y| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$|y\rangle\langle x| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

6. Polarization vectors $|x'\rangle$ and $|y'\rangle$

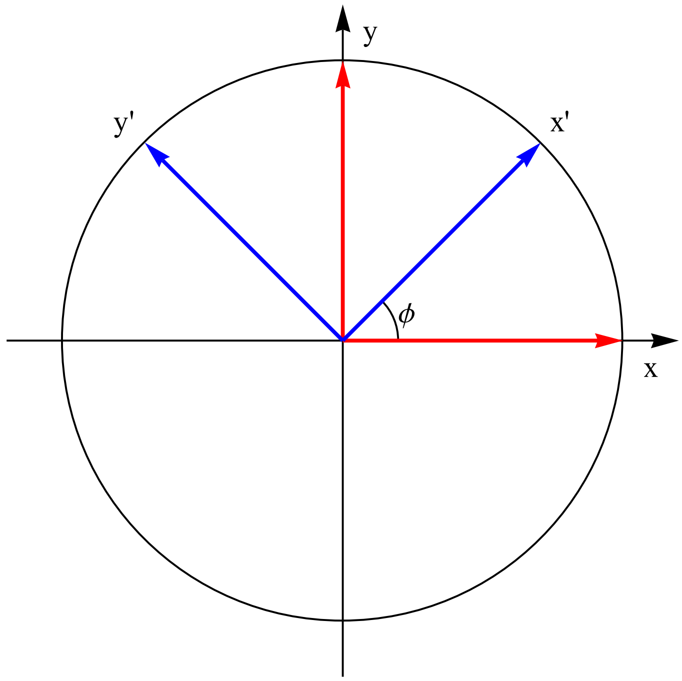


Fig. The orientation of x'- axis and y'-axis.

We now consider more general case of the polarization with the angle ϕ. Since (x', y') basis is the plane polarized basis rotated through an angle ϕ from the (x, y) basis.

(i) $|x'\rangle$ polarization:

$$\mathrm{Re}[E_0 e^{i\alpha}(\boldsymbol{e}_x \cos\phi + \boldsymbol{e}_y \sin\phi)] = E_0 \cos\alpha(\boldsymbol{e}_x \cos\phi + \boldsymbol{e}_y \sin\phi)$$

$$|x'\rangle = \cos\phi|x\rangle + \sin\phi|y\rangle .$$

(ii) $|y'\rangle$ polarization:

$$\mathrm{Re}\{E_0 e^{i\alpha}\left(\boldsymbol{e}_x \cos(\phi + \frac{\pi}{2}) + \boldsymbol{e}_y \sin(\phi + \frac{\pi}{2})\right)] = E_0 \cos\alpha(-\boldsymbol{e}_x \sin\phi + \boldsymbol{e}_y \cos\phi)$$

$$|y'\rangle = -\sin\phi|x\rangle + \cos\phi|y\rangle .$$

Using $|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we have

$$|x'\rangle = \cos\phi|x\rangle + \sin\phi|y\rangle = \begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix},$$

$$|y'\rangle = -\sin\phi|x\rangle + \cos\phi|y\rangle = \begin{pmatrix}-\sin\phi\\ \cos\phi\end{pmatrix},$$

or

$$|x'\rangle = \begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix},$$

$$|y'\rangle = \begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}-\sin\phi\\ \cos\phi\end{pmatrix},$$

Inversely,

$$|x\rangle = \cos\phi|x'\rangle - \sin\phi|y'\rangle,$$

$$|y\rangle = \sin\phi|x'\rangle + \cos\phi|y'\rangle,$$

or

$$|x\rangle = \begin{pmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}\cos\phi\\ -\sin\phi\end{pmatrix},$$

$$|y\rangle = \begin{pmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}\sin\phi\\ \cos\phi\end{pmatrix}.$$

((**Note**)) The above notations can be obtained from the analogy of the unit vectors in the *x*-*y* plane.

$$\boldsymbol{e}_x' = a_{11}\boldsymbol{e}_x + a_{12}\boldsymbol{e}_y,$$

$$\boldsymbol{e}_y' = a_{21}\boldsymbol{e}_x + a_{22}\boldsymbol{e}_y,$$

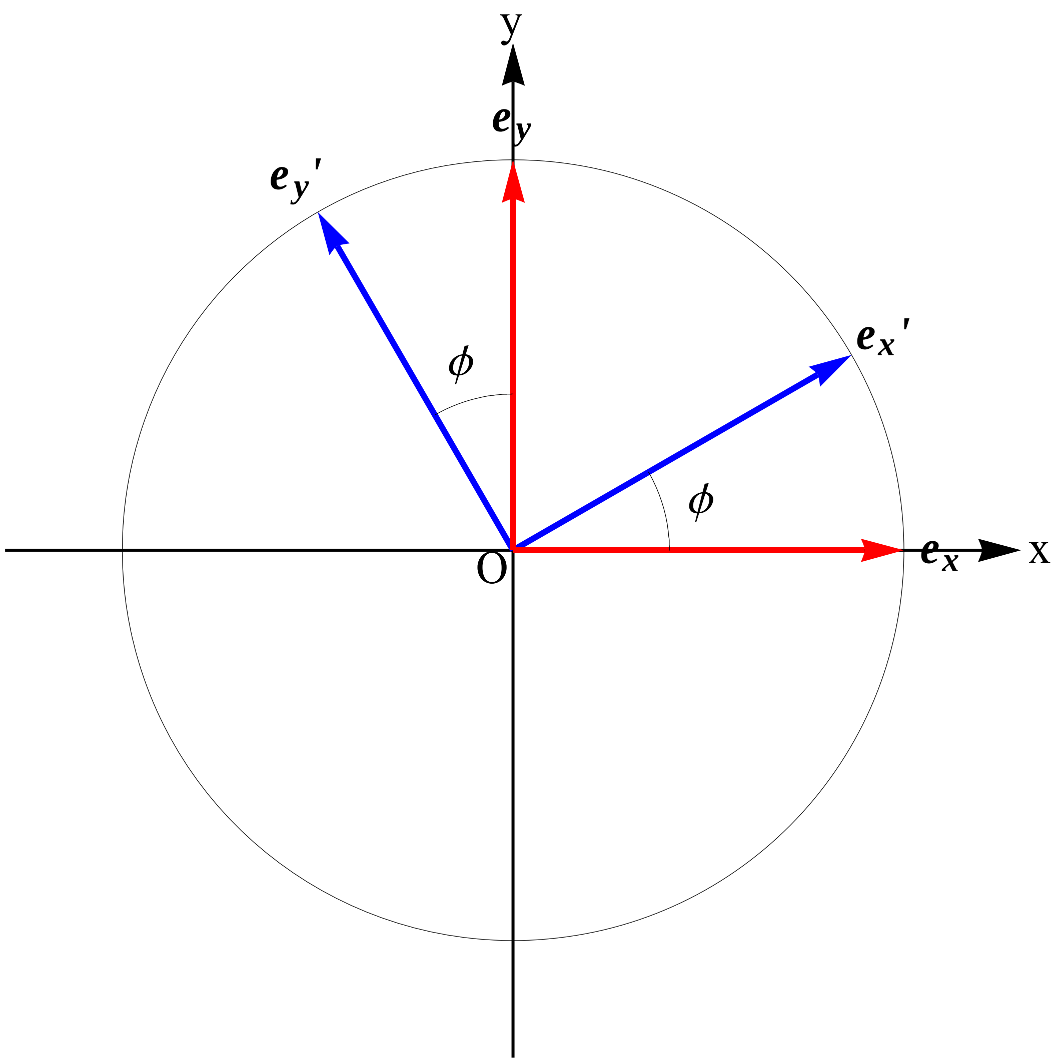
with

$$a_{11} = \boldsymbol{e}_x{}' \cdot \boldsymbol{e}_x = \cos\phi ,$$

$$a_{12} = \boldsymbol{e}_x{}' \cdot \boldsymbol{e}_y = \cos(\frac{\pi}{2} - \phi) = \sin\phi ,$$

$$a_{21} = \boldsymbol{e}_y{}' \cdot \boldsymbol{e}_x = \cos(\frac{\pi}{2} + \phi) = -\sin\phi ,$$

$$a_{22} = \boldsymbol{e}_y{}' \cdot \boldsymbol{e}_y = \cos\phi .$$



Then we have

$$\boldsymbol{e}_x' = (\cos\phi, \sin\phi) = \cos\phi\boldsymbol{e}_x + \sin\phi\boldsymbol{e}_y ,$$

$$\boldsymbol{e}_y' = (-\sin\phi, \cos\phi) = -\sin\phi\boldsymbol{e}_x + \cos\phi\boldsymbol{e}_y .$$

7. Rotation operator $\hat{S}(\phi)$

((G. **Baym, Lecture on quantum mechanics**))

We now consider any ket vector

$$\left|\psi_2\right\rangle = \left|x\right\rangle\left\langle x\middle|\psi_2\right\rangle + \left|y\right\rangle\left\langle y\middle|\psi_2\right\rangle .$$

Then we have

$$\begin{aligned}\left\langle x'\middle|\psi_2\right\rangle &= \left\langle x'\middle|x\right\rangle\left\langle x\middle|\psi_2\right\rangle + \left\langle x'\middle|y\right\rangle\left\langle y\middle|\psi_2\right\rangle \\ &= \cos\phi\left\langle x\middle|\psi_2\right\rangle + \sin\phi\left\langle y\middle|\psi_2\right\rangle\end{aligned}$$

$$\begin{aligned}\left\langle y'\middle|\psi_2\right\rangle &= \left\langle y'\middle|x\right\rangle\left\langle x\middle|\psi_2\right\rangle + \left\langle y'\middle|y\right\rangle\left\langle y\middle|\psi_2\right\rangle \\ &= -\sin\phi\left\langle x\middle|\psi_2\right\rangle + \cos\phi\left\langle y\middle|\psi_2\right\rangle\end{aligned}$$

or

$$\begin{pmatrix}\left\langle x'\middle|\psi_2\right\rangle \\ \left\langle y'\middle|\psi_2\right\rangle\end{pmatrix} = \begin{pmatrix}\cos\phi & \sin\phi \\ -\sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}\left\langle x\middle|\psi_2\right\rangle \\ \left\langle y\middle|\psi_2\right\rangle\end{pmatrix}.$$

The left hand side can be rewritten as

$$\begin{pmatrix}\left\langle x'\middle|\psi_2\right\rangle \\ \left\langle y'\middle|\psi_2\right\rangle\end{pmatrix} = \begin{pmatrix}\left\langle x\middle|\psi_1\right\rangle \\ \left\langle y\middle|\psi_1\right\rangle\end{pmatrix},$$

from the analogy of the algebraic relation between the new co-ordinates and old-coordinates under the rotation around the z axis,

or

$$\begin{pmatrix} \langle x|\psi_1\rangle \\ \langle y|\psi_1\rangle \end{pmatrix} = \begin{pmatrix} \langle x'|x\rangle & \langle x'|y\rangle \\ \langle y'|x\rangle & \langle y'|y\rangle \end{pmatrix} \begin{pmatrix} \langle x|\psi_2\rangle \\ \langle y|\psi_2\rangle \end{pmatrix}$$
$$= \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \langle x|\psi_2\rangle \\ \langle y|\psi_2\rangle \end{pmatrix}$$

or

$$\langle x|\psi_1\rangle = \cos\phi\langle x|\psi_2\rangle + \sin\phi\langle y|\psi_2\rangle,$$

$$\langle y|\psi_1\rangle = -\sin\phi)\langle x|\psi_2\rangle + \cos\phi\langle y|\psi_2\rangle].$$

using the new ket vector $|\psi_1\rangle$. Note that the ket vector $|\psi_1\rangle$ is the ket vector $|\psi_2\rangle$ rotated clockwise by ϕ,

$$\begin{aligned}\langle x|\psi_1\rangle + i\langle y|\psi_1\rangle &== [\cos\phi\langle x|\psi_2\rangle + \sin\phi\langle y|\psi_2\rangle] + i[-\sin\phi)\langle x|\psi_2\rangle + \cos\phi\langle y|\psi_2\rangle] \\ &= (\cos\phi - i\sin\phi)[\langle x|\psi_2\rangle + i\langle y|\psi_2\rangle] \\ &= e^{-i\phi}[\langle x|\psi_2\rangle + i\langle y|\psi_2\rangle]\end{aligned}$$

Then we have

$$|\psi_1\rangle = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} |\psi_2\rangle,$$

or

$$|\psi_2\rangle = \hat{S}(\phi)|\psi_1\rangle = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} |\psi_1\rangle,$$

$\hat{S}(\phi)$ is the rotation operator,

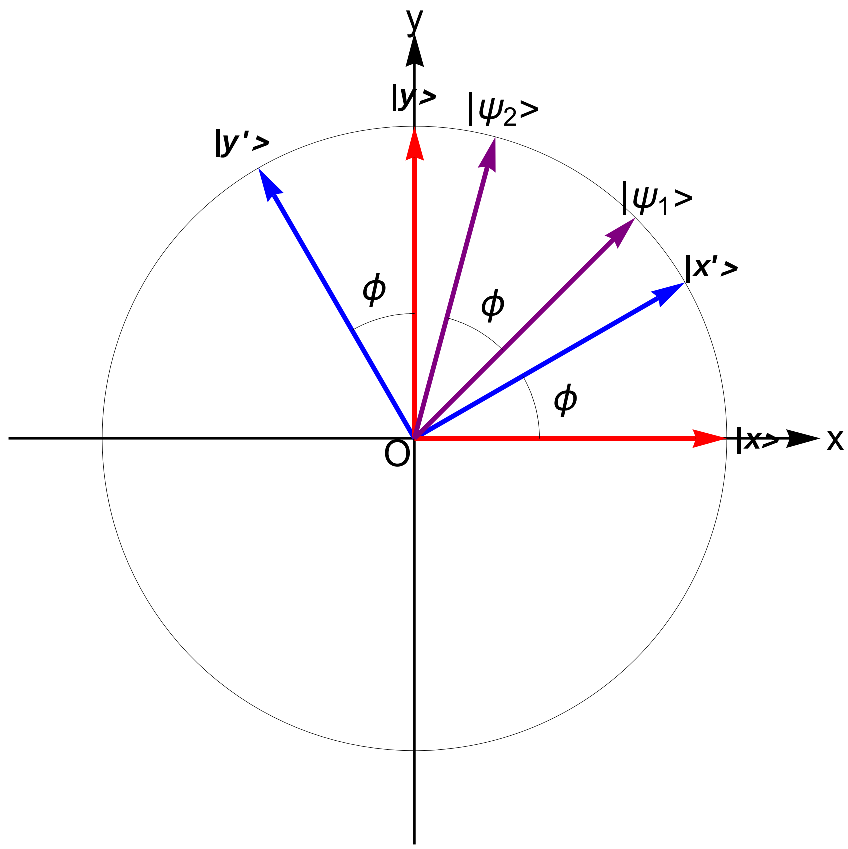
$$\hat{S}(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

and it is a unitary operator

$$\hat{S}(\phi)\hat{S}^+(\phi) = \hat{S}^+(\phi)\hat{S}(\phi) = \hat{1},$$

Here we note that

$$\hat{S}(\phi) \Rightarrow \begin{pmatrix} \langle x|\hat{S}|x\rangle & \langle x|\hat{S}|y\rangle \\ \langle y|\hat{S}|x\rangle & \langle y|\hat{S}|y\rangle \end{pmatrix}. \qquad \text{(under the basis of } \{|x\rangle, \text{ and } |y\rangle\}.$$

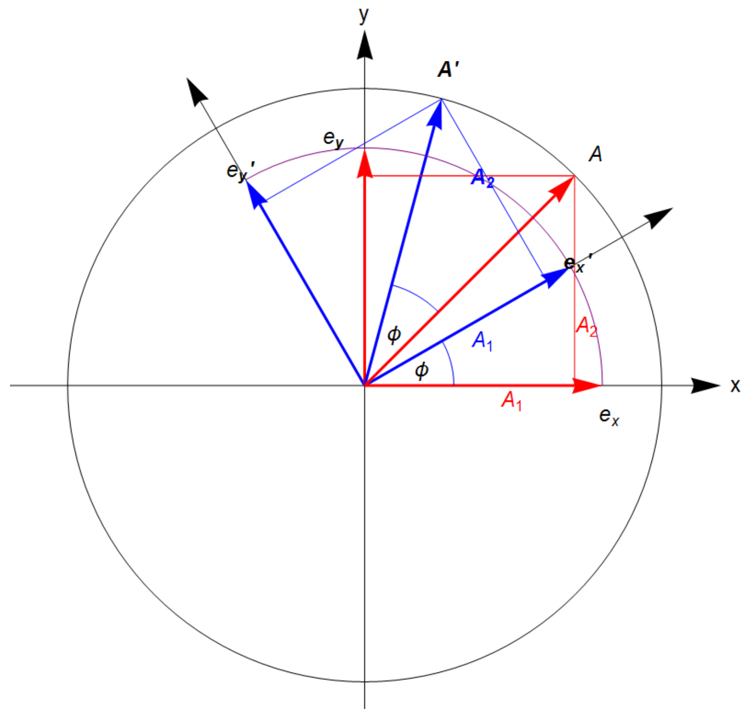


((**Note**))

$$\hat{S}(\phi)|x\rangle = |x'\rangle = \cos\phi|x\rangle + \sin\phi|y\rangle,$$

$$\hat{S}(\phi)|y\rangle = |y'\rangle = -\sin\phi|x\rangle + \cos\phi|y\rangle.$$

8. Note on the derivation of $\hat{S}(\phi)$ from the analogy of the 2D rotation in the *x*-*y* plane



(i)
We consider the position vector $\boldsymbol{A}$ in the x-y plane,

$$\boldsymbol{A} = A_1\boldsymbol{e}_x + A_2\boldsymbol{e}_y .$$

$\Rightarrow$ Correspondingly, in quantum mechanics we can write

$$|\psi_1\rangle = A_1|x\rangle + A_2|y\rangle ,$$

from the analogy, where

$$A_1 = \langle x|\psi_1\rangle, \text{ and } A_2 = \langle y|\psi_1\rangle .$$

(ii) We also consider the position vector $\boldsymbol{A}$' in the x-y plane. The rotation of the vector $\boldsymbol{A}$ by the rotation angle ϕ around the origin leads to the new vector $\boldsymbol{A}$',

$$\boldsymbol{A}' = A_1'\boldsymbol{e}_x + A_2'\boldsymbol{e}_y = A_1\boldsymbol{e}_x' + A_2\boldsymbol{e}_y' ,$$

(see the above figure). $\Rightarrow$ Correspondingly, in quantum mechanics we can write

$$\left|\psi_2\right\rangle = A_1'\left|x\right\rangle + A_2'\left|y\right\rangle = A_1\left|x'\right\rangle + A_2\left|y'\right\rangle,$$

with

$$A_1' = \left\langle x\middle|\psi_2\right\rangle, \text{ and } A_2' = \left\langle y\middle|\psi_2\right\rangle,$$

$$A_1 = \left\langle x'\middle|\psi_2\right\rangle, \text{ and } A_2' = \left\langle y'\middle|\psi_2\right\rangle.$$

We note that

$$\boldsymbol{e}_x' = \cos\phi\boldsymbol{e}_x + \sin\phi\boldsymbol{e}_y, \qquad \boldsymbol{e}_y' = -\sin\phi\boldsymbol{e}_x + \cos\phi\boldsymbol{e}_y,$$

$\Rightarrow$

$$\left|x'\right\rangle = \cos\phi\left|x\right\rangle + \sin\phi\left|y\right\rangle, \qquad \left|y'\right\rangle = -\sin\phi\left|x\right\rangle + \cos\phi\left|y\right\rangle.$$

Then we have

$$\begin{aligned} A_1'\boldsymbol{e}_x + A_2'\boldsymbol{e}_y &= A_1(\cos\phi\boldsymbol{e}_x + \sin\phi\boldsymbol{e}_y) + A_2(-\sin\phi\boldsymbol{e}_x + \cos\phi\boldsymbol{e}_y) \\ &= (A_1\cos\phi - A_2\sin\phi)\boldsymbol{e}_x + (A_1\sin\phi + A_2\cos\phi)\boldsymbol{e}_y \end{aligned}$$

or

$$\begin{pmatrix} A_1' \\ A_2' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix},$$

or

$$\begin{pmatrix} A_1' \\ A_2' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix},$$

or

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} A_1' \\ A_2' \end{pmatrix},$$

Correspondingly, in quantum mechanics, we have

$$\begin{pmatrix} \langle x|\psi_1\rangle \\ \langle y|\psi_1\rangle \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \langle x|\psi_2\rangle \\ \langle y|\psi_2\rangle \end{pmatrix},$$

or

$$\begin{pmatrix} \langle x|\psi_2\rangle \\ \langle y|\psi_2\rangle \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \langle x|\psi_1\rangle \\ \langle y|\psi_1\rangle \end{pmatrix},$$

leading to the expression

$$|\psi_2\rangle = \hat{S}(\phi)|\psi_1\rangle,$$

with

$$\hat{S}(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

9. Eigenvalue problem for the rotation operator $\hat{S}(\phi)$

((Townsend))

We now consider the eigenvalue problem for the rotation operator given by $\hat{S}(\phi)$

$$\hat{S}(\phi)|\psi\rangle = \lambda|\psi\rangle.$$

Eigenvalue problem:

$$\hat{S}(\phi)|R\rangle = \lambda_R|R\rangle, \qquad \hat{S}(\phi)|L\rangle = \lambda_L|L\rangle,$$

with

$$\hat{S}(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

We show that $|R\rangle$ and $|L\rangle$ are the eigenket of $\hat{S}(\phi)$.

$$|R\rangle = \hat{U}|x\rangle = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} U_{11} \\ U_{21} \end{pmatrix},$$

$$|L\rangle = \hat{U}|y\rangle = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} U_{12} \\ U_{22} \end{pmatrix}.$$

where $\hat{U}$ is a unitary operator to be determined.

$$\hat{U} = \hat{U}(|x\rangle\langle x| + |y\rangle\langle y|) = |R\rangle\langle x| + |L\rangle\langle y|,$$

and

$$\langle x|\hat{U}|x\rangle = \langle x|R\rangle, \qquad \langle x|\hat{U}|y\rangle = \langle x|L\rangle,$$

$$\langle y|\hat{U}|x\rangle = \langle y|R\rangle, \qquad \langle y|\hat{U}|y\rangle = \langle y|L\rangle.$$

The equation for the eigenvalues;

$$\det\begin{pmatrix} \cos\phi - \lambda & -\sin\phi \\ \sin\phi & \cos\phi - \lambda \end{pmatrix} = 0,$$

or

$$(\cos\phi - \lambda)^2 + \sin^2\phi = 0,$$

or

$$\lambda = \cos\phi \pm i\sin\phi = e^{\pm i\phi}.$$

(a) Eigenvalue: $\lambda_R = e^{-i\phi}$

$$|R\rangle = \hat{U}|x\rangle = \begin{pmatrix} U_{11} \\ U_{21} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}}[|x\rangle + i|y\rangle],$$

where

$$\hat{S}(\phi)|R\rangle = e^{-i\phi}|R\rangle, \qquad \text{or} \qquad \hat{S}^{+}(\phi)|R\rangle = e^{i\phi}|R\rangle.$$

(b) Eigenvalue: $\lambda_L = e^{i\phi}$,

$$|L\rangle = \hat{U}|y\rangle = \begin{pmatrix} U_{21} \\ U_{22} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}}[|x\rangle - i|y\rangle],$$

where

$$\hat{S}(\phi)|L\rangle = e^{i\phi}|L\rangle, \qquad \text{or} \qquad \hat{S}^{+}(\phi)|L\rangle = e^{-i\phi}|L\rangle.$$

The unitary operator;

$$\hat{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \qquad \hat{U}^{+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix},$$

$$\hat{U}^{+}\hat{S}(\phi)\hat{U} = \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}, \qquad \text{(diagonal matrix)}$$

under the basis of $\{|x\rangle$ and $|y\rangle\}$

((**Note**))

$$\begin{aligned} \hat{U} &= \hat{U}(|x\rangle\langle x| + |y\rangle\langle y|) \\ &= |R\rangle\langle x| + |L\rangle\langle y| \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ i & 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & -i \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \end{aligned}$$

10. Mathematica

We solve the eigenvalue problem using the Mathematica.

```
Clear["Global`*"];

expr_* := expr /. {Complex[a_, b_] :> Complex[a, -b]};

S = ( Cos[ϕ]  -Sin[ϕ]
      Sin[ϕ]   Cos[ϕ] );
eq1 = Eigensystem[S] // TrigToExp
```

$\{\{e^{-i\phi}, e^{i\phi}\}, \{\{-i, 1\}, \{i, 1\}\}\}$

```
R1 = i eq1[[2, 1]] // Normalize
```

$\left\{\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right\}$

```
L1 = -i eq1[[2, 2]] // Normalize
```

$\left\{\frac{1}{\sqrt{2}}, -\frac{i}{\sqrt{2}}\right\}$

```
UT = {R1, L1}
```

$\left\{\left\{\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2}}, -\frac{i}{\sqrt{2}}\right\}\right\}$

```
U = Transpose[UT]
```

$\left\{\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{i}{\sqrt{2}}, -\frac{i}{\sqrt{2}}\right\}\right\}$

```
UH = UT*
```

$\left\{\left\{\frac{1}{\sqrt{2}}, -\frac{i}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right\}\right\}$

```
UH.S.U // FullSimplify
```

$\{\{e^{-i\phi}, 0\}, \{0, e^{i\phi}\}\}$

11. Derivation for the eigenkets of $\hat{S}(\phi)$ (alternative way I).

We start with

$$\hat{S}(\phi)\left|x\right\rangle = \left|x'\right\rangle = \cos\phi\left|x\right\rangle + \sin\phi\left|y\right\rangle = \begin{pmatrix}\cos\phi \\ \sin\phi\end{pmatrix}, \qquad (1)$$

$$\hat{S}(\phi)\left|y\right\rangle = \left|y'\right\rangle = -\sin\phi\left|x\right\rangle + \cos\phi\left|y\right\rangle = \begin{pmatrix}-\sin\phi \\ \cos\phi\end{pmatrix}. \qquad (2)$$

where

$$\hat{S}(\phi) = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}.$$

Clearly, $\left|x\right\rangle$ and $\left|y\right\rangle$ are not the eigenkets of the operator $\hat{S}(\phi)$. In order to get the eigenkets of $\hat{S}(\phi)$, we consider the superposition of $\left|x\right\rangle$ and $\left|y\right\rangle$. From the sum of Eqs.(1) and Eq.(2) x *i*, we have

$$\begin{aligned}\hat{S}(\phi)[\frac{\left|x\right\rangle + i\left|y\right\rangle}{\sqrt{2}}] &= \frac{1}{\sqrt{2}}[\cos\phi\left|x\right\rangle + \sin\phi\left|y\right\rangle] + \frac{i}{\sqrt{2}}[-\sin\phi\left|x\right\rangle + \cos\phi\left|y\right\rangle] \\ &= e^{-i\phi}\frac{(\left|x\right\rangle + i\left|y\right\rangle}{\sqrt{2}}\end{aligned}$$

From the sum of Eq.(1) and Eq.(2) x (-*i*), we have

$$\begin{aligned}\hat{S}(\phi)\frac{(\left|x\right\rangle - i\left|y\right\rangle}{\sqrt{2}} &= \frac{1}{\sqrt{2}}[\cos\phi\left|x\right\rangle + \sin\phi\left|y\right\rangle] - \frac{i}{\sqrt{2}}[-\sin\phi\left|x\right\rangle + \cos\phi\left|y\right\rangle] \\ &= e^{i\phi}\frac{(\left|x\right\rangle - i\left|y\right\rangle}{\sqrt{2}}\end{aligned}$$

Then it is found that $\left|R\right\rangle$ and $\left|L\right\rangle$ are the eigenket of $\hat{S}(\phi)$ with the eigenvalues +1 and (-1),

$$\hat{S}(\phi)\left|R\right\rangle = e^{-i\phi}\left|R\right\rangle, \qquad \hat{S}(\phi)\left|L\right\rangle = e^{i\phi}\left|L\right\rangle,$$

where

$$|R\rangle = \frac{1}{\sqrt{2}}[|x\rangle + i|y\rangle], \qquad |L\rangle = \frac{1}{\sqrt{2}}[|x\rangle - i|y\rangle].$$

We also note that the Hermitian conjugate operator $\hat{S}^{+}(\phi)$ satisfies the eigenvalue problem

$$\hat{S}^{+}(\phi)|R\rangle = e^{i\phi}|R\rangle, \qquad \hat{S}^{+}(\phi)|L\rangle = e^{-i\phi}|L\rangle.$$

12. Circular polarization ((alternative method II))

$\hat{S}(\phi)$ can be rewritten as

$$\begin{aligned}\hat{S}(\phi) &= \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \\ &= \cos\phi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\sin\phi \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= \cos\phi\hat{1} - i\sin\phi\hat{\Sigma}\end{aligned}$$

under the basis of $\{|x\rangle, |y\rangle\}$,where $\hat{\Sigma}$ is called the spin operator of photon. The form of $\hat{\Sigma}$ is the same as that of Pauli matrix $\hat{\sigma}_y$,

$$\hat{S}^{+}(\phi)\hat{S}(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{1},$$

and

$$\hat{\Sigma} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \text{and} \qquad \hat{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We note that

$$\hat{\Sigma}^2 = \hat{1}.$$

Then we have

$$|x'\rangle = \hat{S}(\phi)|x\rangle = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} = \cos\phi|x\rangle + \sin\phi|x\rangle,$$

or

$$|x'\rangle = \hat{S}(\phi)|x\rangle = (\cos\phi\hat{1} - i\sin\phi\hat{\Sigma})|x\rangle = \cos\phi|x\rangle - i\sin\phi\hat{\Sigma}|x\rangle.$$

We also have

$$|y'\rangle = \hat{S}(\phi)|y\rangle = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix} = -\sin\phi|x\rangle + \cos\phi|y\rangle,$$

or

$$|y'\rangle = \hat{S}\ (\phi)|y\rangle = (\cos\phi\hat{1} - i\sin\phi\hat{\Sigma})|y\rangle = \cos\phi|y\rangle - i\sin\phi\hat{\Sigma}|y\rangle.$$

We consider the eigenvalue problem given by

$$\hat{\Sigma}|\psi\rangle = \lambda|\psi\rangle,$$

with

$$\hat{\Sigma} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

$|\psi\rangle$ is the eigenket of $\hat{\Sigma}$ under the basis of $\{|x\rangle, |y\rangle\}$. Since $\hat{\Sigma}^2 = \hat{1}$, we get

$$\hat{\Sigma}^2|\psi\rangle = \lambda\hat{\Sigma}|\psi\rangle = \lambda^2|\psi\rangle = |\psi\rangle,$$

leading to $\lambda = \pm 1$. The eigenket is obtained as

$$|\psi_+\rangle = |R\rangle = \frac{1}{\sqrt{2}}[|x\rangle + i|y\rangle], \qquad |\psi_-\rangle = |L\rangle = \frac{1}{\sqrt{2}}[|x\rangle - i|y\rangle],$$

since the form of $\hat{\Sigma}$ is the same as that of the Pauli matrix $\hat{\sigma}_y$.

(a) For $\lambda = 1$,

$$\hat{\Sigma}|R\rangle = |R\rangle,$$

with

$$|R\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}.$$ right-hand circular polarization

We also have

$$\hat{S}(\phi)|R\rangle = (\cos\phi\hat{1} - i\sin\phi\hat{\Sigma})|R\rangle = \cos\phi|R\rangle - i\sin\phi|R\rangle = e^{-i\phi}|R\rangle,$$

or

$$\hat{S}(\phi)|R\rangle = e^{-i\phi}|R\rangle.$$

(b) For λ = -1,

$$\hat{\Sigma}|L\rangle = -|L\rangle,$$

with

$$|L\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}.$$ left circular polarization

We also have

$$\hat{S}(\phi)|L\rangle = (\cos\phi\hat{1} - i\sin\phi\hat{\Sigma})|L\rangle = \cos\phi|L\rangle + i\sin\phi|L\rangle = e^{i\phi}|L\rangle,$$

or

$$\hat{S}(\phi)|L\rangle = e^{i\phi}|L\rangle.$$

13. Outer products; closure relation

$$|x\rangle\langle x| = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \qquad |y\rangle\langle y| = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},$$

$$|x\rangle\langle y| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad |y\rangle\langle x| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$|R\rangle\langle R| = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \quad |L\rangle\langle L| = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix},$$

$$|R\rangle\langle L| = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad |L\rangle\langle R| = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix},$$

$$|x\rangle\langle x| + |y\rangle\langle y| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{1}.$$

(completeness relation, closure relation)

$$|x\rangle\langle x| - |y\rangle\langle y| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$|R\rangle\langle R| + |L\rangle\langle L| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{1},$$

$$|R\rangle\langle R| - |L\rangle\langle L| = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Using the closure relation, we get

$$\begin{aligned} |\psi\rangle &= (|R\rangle\langle R| + |R\rangle\langle R|)|\psi\rangle \\ &= |R\rangle\langle R|\psi\rangle + |L\rangle\langle L|\psi\rangle \end{aligned},$$

for any $|\psi\rangle$.

14. $\hat{S}(\phi)$ and the unitary operator $\hat{U}$

We start with the relations,

$$\hat{S}(\phi)|R\rangle = e^{-i\phi}|R\rangle, \qquad \hat{S}(\phi)|L\rangle = e^{i\phi}|L\rangle.$$

Using

$$|R\rangle = \hat{U}|x\rangle = \frac{1}{\sqrt{2}}[|x\rangle + i|y\rangle],$$

$$|L\rangle = \hat{U}|y\rangle = \frac{1}{\sqrt{2}}[|x\rangle - i|y\rangle],$$

we get

$$|x'\rangle = \cos\phi|x\rangle + \sin\phi|y\rangle = \hat{S}(\phi)|x\rangle,$$

$$|y'\rangle = -\sin\phi|x\rangle + \cos\phi|y\rangle = \hat{S}(\phi)|y\rangle.$$

The rotation operators $\hat{S}(\phi)$ and $\hat{S}^{+}(\phi)$ can be written as

$$\hat{S}(\phi) = \hat{S}(\phi)(|x\rangle\langle x| + |y\rangle\langle y|) = |x'\rangle\langle x| + |y'\rangle\langle y|,$$

$$\hat{S}^{+}(\phi) = |x\rangle\langle x'| + |y\rangle\langle y'|,$$

The unitary operators $\hat{U}$ and $\hat{U}^{+}$ can be written as

$$\hat{U} = \hat{U}(|x\rangle\langle x| + |y\rangle\langle y|) = |R\rangle\langle x| + |L\rangle\langle y|,$$

$$\hat{U}^{+} = |x\rangle\langle R| + |y\rangle\langle L|.$$

15. Matrix elements of $\hat{S}(\phi)$

$$\hat{S}(\phi) = \begin{pmatrix} \langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix},$$

$$\hat{S}^{+}(\phi) = \begin{pmatrix} \langle x'|x\rangle & \langle x'|y\rangle \\ \langle y'|x\rangle & \langle y'|y\rangle \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}.$$

16. $|R'\rangle$ and $|L'\rangle$

$$|R'\rangle = |x'\rangle + i|y'\rangle = \hat{S}(\phi)[|x\rangle + i|y\rangle] = \hat{S}(\phi)|R\rangle = e^{-i\phi}|R\rangle,$$

$$|L'\rangle = |x'\rangle - i|y'\rangle = \hat{S}(\phi)[|x\rangle - i|y\rangle] = \hat{S}(\phi)|L\rangle = e^{i\phi}|L\rangle,$$

with

$$|x'\rangle = \hat{S}(\phi)|x\rangle, \qquad |y'\rangle = \hat{S}(\phi)|x\rangle.$$

17. Angular momentum operator $\hat{J}_z$

We note that $\hat{S}(\phi)$ is related to the corresponding angular momentum $\hat{J}_z$ through

$$\hat{S}(\phi) = \exp(-\frac{i}{\hbar}\hat{J}_z\phi).$$

(see the Appendix I)

(i) Right-hand circularly polarization

$$\hat{S}(\phi)|R\rangle = e^{-i\phi}|R\rangle,$$

In the limit of $\phi \to 0$, we use the Taylor expansion,

$$\hat{S}(\phi)|R\rangle = e^{-i\phi}|R\rangle,$$

or

$$(1-\frac{i}{\hbar}\hat{J}_z\phi)|R\rangle = (1+i\phi)|R\rangle,$$

or

$$\hat{J}_z|R\rangle = \hbar|R\rangle.$$

In other words, $|R\rangle$ is the eigenket of $\hat{J}_z$ with the eigenvalue ($\hbar$)

(ii) Left-hand circularly polarization

$$\hat{S}(\phi)|L\rangle = e^{i\phi}|L\rangle,$$

or

$$\hat{S}(\phi)|L\rangle = e^{i\phi}|L\rangle.$$

In the limit of $\phi \to 0$, we use the Taylor expansion,

$$(1-\frac{i}{\hbar}\hat{J}_z\phi)|L\rangle = (1+i\phi)|L\rangle,$$

or

$$\hat{J}_z|L\rangle = -\hbar|L\rangle.$$

In other words, $|L\rangle$ is the eigenket of J_z with the eigenvalue ($-\hbar$)

Using the closure relation, we have

$$\begin{aligned}\hat{J}_z &= \hat{J}_z(|R\rangle\langle R| + |L\rangle\langle L|) \\ &= \hat{J}_z|R\rangle\langle R| + \hat{J}_z|L\rangle\langle L| \\ &= \hbar(|R\rangle\langle R| - |L\rangle\langle L|)\end{aligned}$$

If the photon is right circularly polarized, then $J_z = \hbar$, and we can say that the photon definitely has angular momentum $\hbar$. Similarly if the photon is left circularly polarized, then $J_z = -\hbar$, and we can say that the photon definitely has angular momentum $-\hbar$. Photons have intrinsic spin of 1 instead of 1/2. The absence of the 0-eigenvalue for $\hat{J}_z$ for a photon turns out to be a special characteristic of a massless particle, which moves at speed *c*.

((**Note**))
((Bellac L.M. Quantum Physics))

Why does the state $|j=1, m=0\rangle$ not exist for the photon state with $j = 1$?

The states $|R\rangle$ and $|L\rangle$ are identified as the states $|j=1, m=1\rangle$ and $|j=1, m=-1\rangle$, respectively. The angular momentum quantization axis Oz is taken to lie along the photon propagation direction. The value of m is called the photon *helicity*: m = +1 corresponds to positive helicity and $m = -1$ to negative helicity. Since the angular momentum $j = 1$ corresponds to three possible values of the magnetic quantum number, $m = +1$, 0, and −1, we might wonder what has happened to the value $m = 0$ for the photon. A general analysis due to Wigner shows that for a particle of zero mass and spin j, the only allowed eigenvalues of J_z are $m = j$ and $m=-j$, where the axis Oz is taken to lie along the particle propagation direction. When parity is not a symmetry of the Hamiltonian, the two possible values are independent. If the spin-1/2 neutrino had zero mass, it would always have $m = -1/2$, while the antineutrino, which is a *different* particle, would always have $m = +1/2$. The photon interactions conserve parity as they are electromagnetic interactions, and so the same particle can have both $m = 1$ or $m=-1$.

18. Consideration

We have

$$\hat{J}_z|R\rangle = \hbar|R\rangle, \qquad \hat{J}_z|L\rangle = -\hbar|L\rangle.$$

In the basis of $\{|R\rangle, |L\rangle\}$, the matrix of $\hat{J}_z$ can be described as

$$\hat{J}_z = \hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We also note that

$$\exp(-\frac{i}{\hbar}\hat{J}_z\phi) = \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}.$$

What is the expression of this matrix under the basis of $\{|x\rangle, |y\rangle\}$?

$$|R\rangle = \hat{U}|x\rangle, \qquad |x\rangle = \hat{U}^+|R\rangle,$$

$$|L\rangle = \hat{U}|y\rangle, \qquad |y\rangle = \hat{U}^+|L\rangle.$$

Therefore we have the matrix elements as

$$\langle x|\exp(-\frac{i}{\hbar}\hat{J}_z\phi)|x\rangle = \langle R|\hat{U}\exp(-\frac{i}{\hbar}\hat{J}_z\phi)\hat{U}^+|R\rangle,$$

$$\langle x|\exp(-\frac{i}{\hbar}\hat{J}_z\phi)|y\rangle = \langle R|\hat{U}\exp(-\frac{i}{\hbar}\hat{J}_z\phi)\hat{U}^+|L\rangle,$$

$$\langle y|\exp(-\frac{i}{\hbar}\hat{J}_z\phi)|x\rangle = \langle L|\hat{U}\exp(-\frac{i}{\hbar}\hat{J}_z\phi)\hat{U}^+|R\rangle,$$

$$\langle y|\exp(-\frac{i}{\hbar}\hat{J}_z\phi)|y\rangle = \langle L|\hat{U}\exp(-\frac{i}{\hbar}\hat{J}_z\phi)\hat{U}^+|R\rangle L.$$

Using Mathematica, we can calculate

$$\hat{U}\exp(-\frac{i}{\hbar}\hat{J}_z\phi)\hat{U}^+ = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = \hat{S}(\phi),$$

under the basis of of $\{|x\rangle, |y\rangle\}$, using the Unitary matrix

$$\hat{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \qquad \hat{U}^+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

19. Summary

The above results are summarized as follows.

(a) The state vectors,

$$|x'\rangle = \cos\phi|x\rangle + \sin\phi|y\rangle, \qquad |y'\rangle = -\sin\phi|x\rangle + \cos\phi|y\rangle.$$

(b) Rotation operator: $\hat{S}(\phi)$

$$\hat{S}(\phi)|x\rangle = |x'\rangle, \qquad \hat{S}(\phi)|y\rangle = |y'\rangle.$$

(c) $|R\rangle$ and $|L\rangle$:

$$\hat{S}(\phi)|R\rangle = e^{-i\varphi}|R\rangle, \qquad \hat{S}(\phi)|L\rangle = e^{i\varphi}|L\rangle.$$

(d) Unitary operator $\hat{U}$:

$$|R\rangle = \hat{U}|x\rangle, \qquad |L\rangle = \hat{U}|y\rangle.$$

where

$$\hat{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

(d) Angular momentum

$$\hat{S}(\phi) = \exp(-\frac{i}{\hbar}\hat{J}_z\phi),$$

where $\hat{J}_z$ is the spin operator of photon,

$$\hat{J}_z|R\rangle = \hbar|R\rangle, \qquad \hat{J}_z|L\rangle = -\hbar|R\rangle,$$

$$\begin{aligned}\hat{J}_z &= \hat{J}_z(|R\rangle\langle R| + |L\rangle\langle L|) \\ &= \hbar(|R\rangle\langle R| - \hbar|L\rangle\langle L|) \\ &= \hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\end{aligned}$$

under the basis of $\{|x\rangle, |y\rangle\}$. In this case

$$\hat{S}(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}. \qquad \{|x\rangle, |y\rangle\} \text{ basis}$$

((**Mathematica**))

```
Clear["Global`*"];
exp_* :=
  exp /. {Complex[re_, im_] :> Complex[re, -im]};
σy = (0  -ⅈ
      ⅈ   0);
S1 = MatrixExp[-ⅈ/ħ ϕ ħ σy] // Simplify;
S1 // MatrixForm
```

$$\begin{pmatrix} \text{Cos}[\phi] & -\text{Sin}[\phi] \\ \text{Sin}[\phi] & \text{Cos}[\phi] \end{pmatrix}$$

Note that the matrix of $\hat{J}_z$ is given by

$$\hat{J}_z = \hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

under the basis of $\{|R\rangle, |L\rangle\}$. In this case

$$\hat{S}(\phi) = \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}, \qquad\qquad \{|R\rangle, |L\rangle\} \text{ basis}$$

(e) Projection operator

$$|x'\rangle\langle x'| = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}\begin{pmatrix} \cos\phi & \sin\phi \end{pmatrix} = \begin{pmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{pmatrix},$$

$$|y'\rangle\langle y'| = \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix}\begin{pmatrix} -\sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} \sin^2\phi & -\sin\phi\cos\phi \\ -\sin\phi\cos\phi & \cos^2\phi \end{pmatrix},$$

$$|R\rangle\langle R| = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix},$$

$$|L\rangle\langle L| = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

20. Double refraction (physics)

The optical properties are isotropic in the x, y plane. So we can choose the axes so that the beam is propagating in the y-z plane without the loss of generality. This allows us to split the polarization of the light into two orthogonal components, one of which is polarized along the x axis, and the other polarized at an angle of $90° + \theta$ to the optic axis. The former is the ordinary ray, and the latter is the extraordinary axis.

The refractive index will be different for light which is polarized along the z axis or in the x, y plane. Thus the o-ray experiences a different refractive index to the e-ray, and will be refracted differently: hence double refraction. The two refractive indices are usually labeled as n_o and n_e. On the other hand, if the beam propagates along the optic axis, so that $\theta = 0$, the electric-field vector of the light will always fall in the x, y plane. In this case, no double refraction will be observed because x and y directions are equivalent and there is no e-ray.

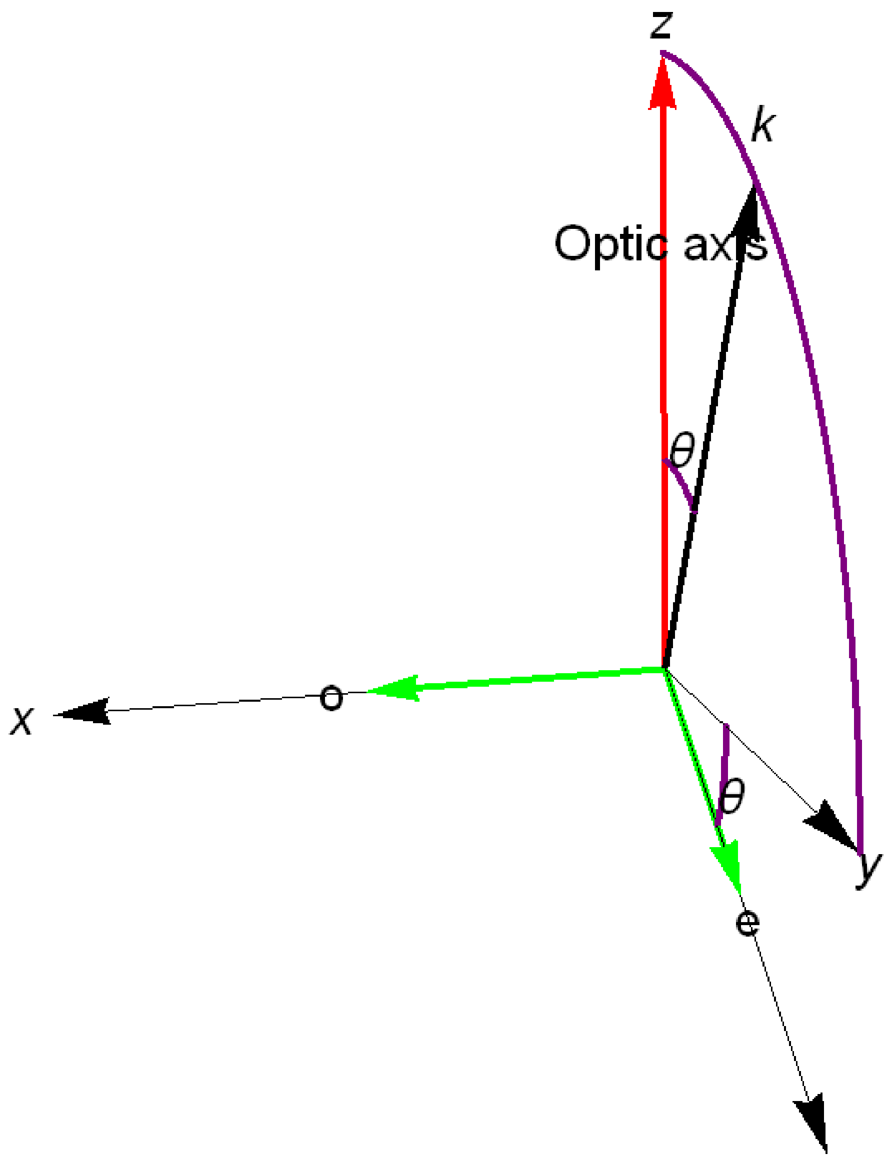


Fig. Electric field vector of ray propagating in a uniaxial crystal with its optic axis along the z axis. The ray makes an angle of θ with respect to the optic axis (the z-axis). The x and y axes are chosen so that the beam is propagating in the y, z plane. The polarization can be resolved into (a) a component along the x axis (the o-ray), and (b) a component at an angle of $(90° + \theta)$ to the optic axis (the e-ray). $|o\rangle$ for o-ray and $|e\rangle$ for e-ray. The polarization vector for the e-ray lies in the y-z plane, while the polarization for the o-ray is along the x axis.

21. **Birefringence**: polarized light in Calcite

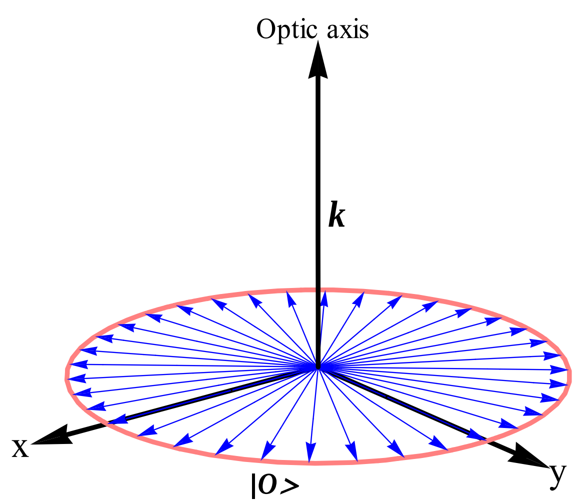
We discuss a phenomenon of double refraction occurring in a material such as calcite (Iceland spar). In this effect an un-polarized light ray is separated into two rays which emerge displaced from each other. The two rays are called ordinary and extraordinary. These two rays have mutually perpendicular polarizations and travel at different speeds through the materials and are orthogonally polarized to each other. These two speeds corresponds to two indices of refraction, no for the ordinary ray and ne for the extraordinary rays.

There is one direction, called the optic axis, along which the ordinary and extraordinary rays have the same speed. If light enters a birefringent material at an angle to the optic axis, however, the different indices of refraction will cause the two polarized rays to split and travel in a different directions.

The index of refraction no for the ordinary ray is the same in all directions, while the index of refraction n_e varies with the direction of propagation. In calcite, n_o = 1.658 and ne varies from 1.658 along the optic axis to 1.486 perpendicular to the optic axis.

(1) The wave propagates along the optic axis

Suppose that that the optic axis (the z axis) is parallel to the direction of wave vector $\boldsymbol{k}$. The polarization vector lie in the x-y plane which is perpendicular to the optic axis. All the directions in the x-y plane are equivalent. There is no difference between the o-ray and e-ray The optical properties are isotropic in the x, y plane. Note that for convenience we use two states $|o\rangle$ and $|e\rangle$ in spite of the isotropy of states in the x-y plane perpendicular to the optic axis.



(2) The case when the direction of the wave vector deviates from the optic axis

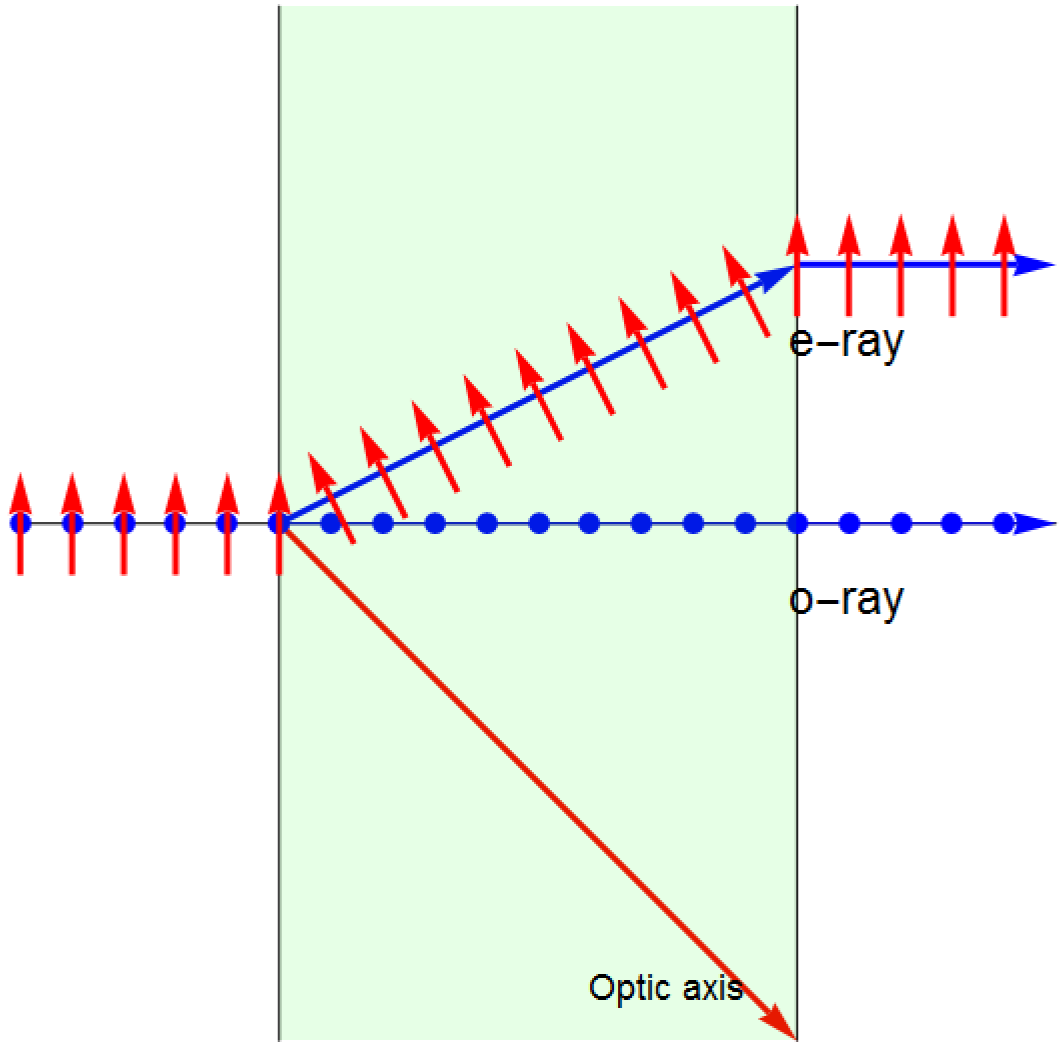


Fig. A light with two orthogonal field components traversing a calcite section.

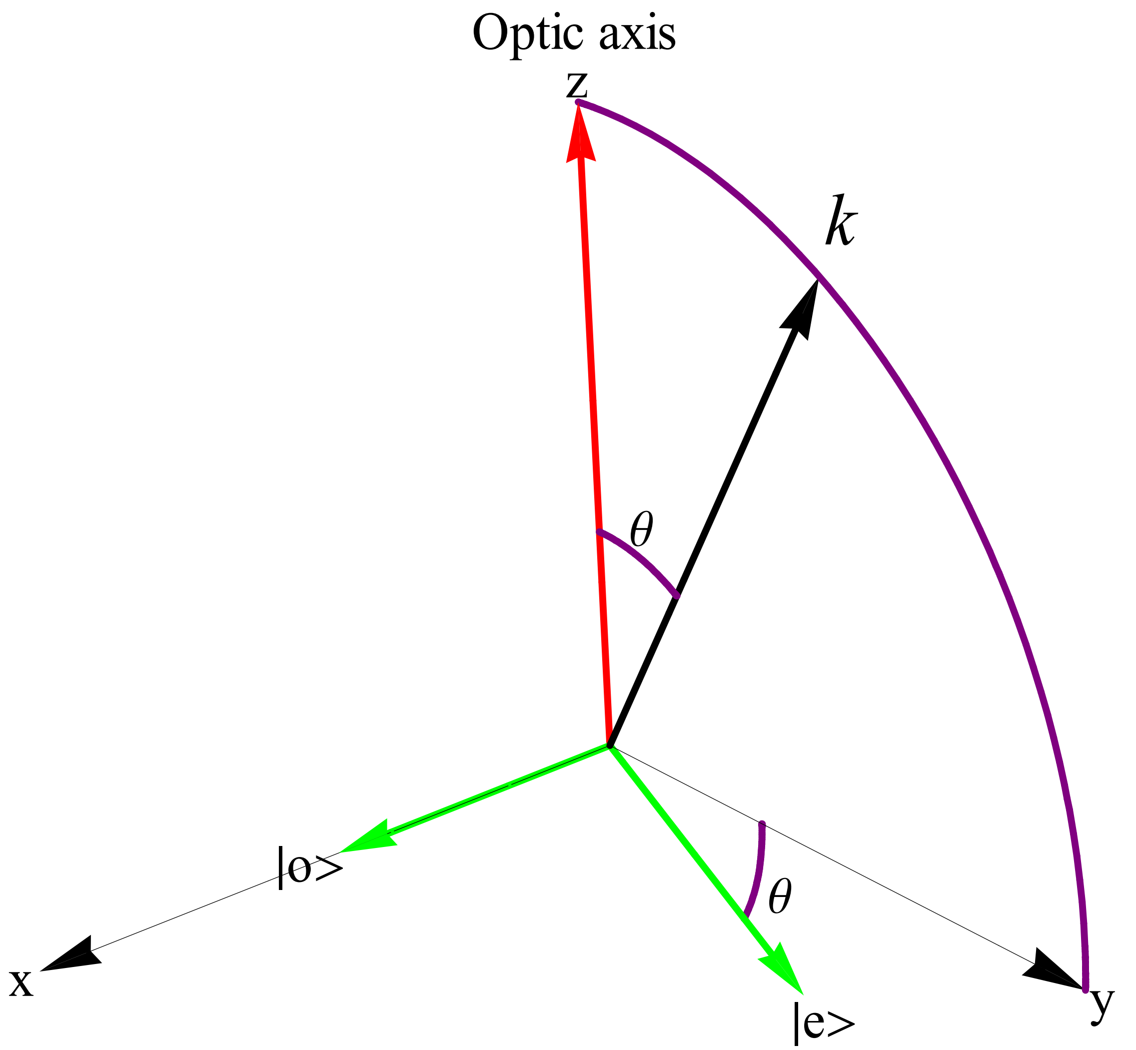


Fig. Electric field vector of ray propagating in a uniaxial crystal with its optic axis along the z axis. The ray with the wavevector $\boldsymbol{k}$ makes an angle of θ with respect to the optic axis (the z-axis). The x and y axes are chosen so that the beam is propagating in the y-z plane.

The polarization can be resolved into (a) a component along the x axis (the o-ray), and (b) a component at an angle of 90° + θ to the optic axis (the e-ray). $|o\rangle$ for o-ray and $|e\rangle$ for e-ray. The polarization vector for the e-ray lies in the y-z plane, while the polarization for the o-ray is along the x axis. The essence of the state $|e\rangle$ appears as the angle θ increases.

As shown in the above figure, we can choose the axes so that the beam is propagating in the y-z plane without the loss of generality. This allows us to split the polarization of the light into two orthogonal components, one of which is polarized along the x axis, and the other polarized at an angle of 90° + θ to the optic axis. The former is the ordinary ray, and the latter is the extraordinary ray.

The refractive index will be different for light which is polarized along the z axis or in the x, y plane. Thus the o-ray experiences a different refractive index to the e-ray, and will be refracted differently: hence double refraction. The two refractive indices are usually labeled as n_o and n_e. On the other hand, if the beam propagates along the optic axis, so that $\theta = 0$, the electric-field vector of the light will always fall in the x, y plane. In this case, no double refraction will be observed because x and y directions are equivalent and there is no e-ray.

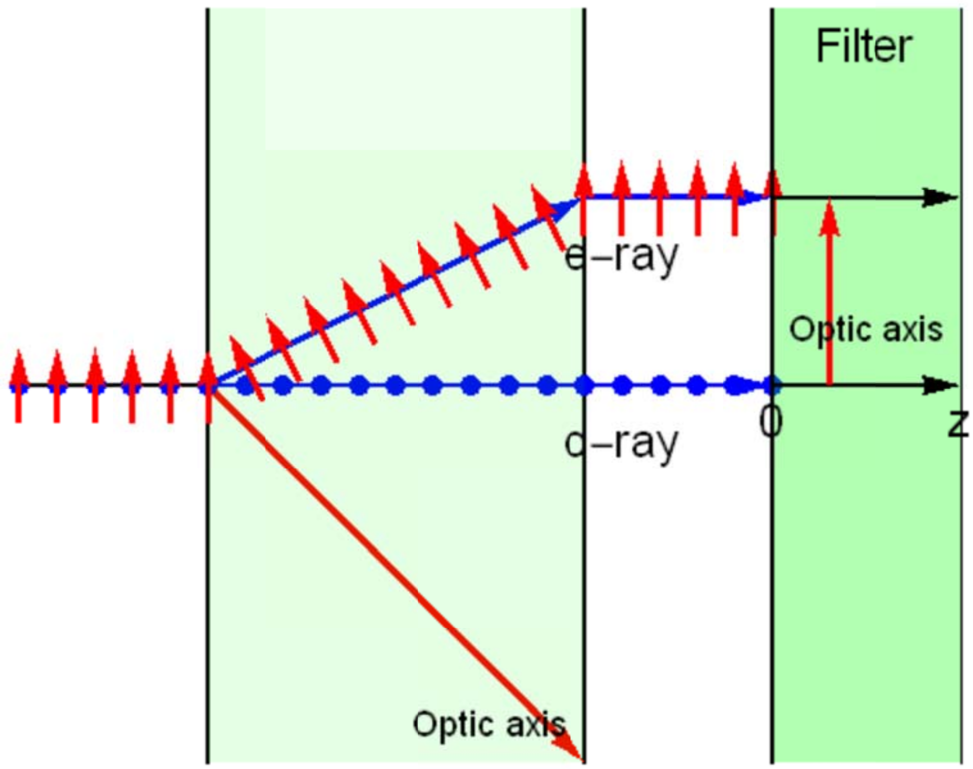
22. Wavenumber matrix

In vacuum, a photon wave function is given by

$$\psi(\boldsymbol{r}, t) = \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)].$$

In matter, the wavenumber k also depends on the index of refraction $n(k)$;

$$k = n\frac{\omega}{c},$$



We consider the propagation of the light through a filter along the z axis (the direction of wavevector $\boldsymbol{k}$). We assume that the optic axis is the z axis. Photons polarized to the x axis are ordinary rays. Those polarized perpendicular to the optic axis (the z axis) are extraordinary rays (the y-axis). The o-ray is along the x-axis, and the e-ray is along the y axis. The optic axis of the filter is parallel to the y-axis.

$$k_0 = n_o \frac{\omega}{c}, \qquad k_e = n_e \frac{\omega}{c}.$$

Suppose that the photon comes into the calcite as $|\psi_{in}\rangle$,

$$|\psi_{in}\rangle = (|e\rangle\langle e| + |o\rangle\langle o|)|\psi_{in}\rangle = |e\rangle\langle e|\psi_{in}\rangle + |o\rangle\langle o|\psi_{in}\rangle,$$

since

$$|e\rangle\langle e| + |o\rangle\langle o| = \hat{1}.$$

If the plate thickness is z, then we have the final state as

$$\left|\psi_z\right\rangle = e^{ik_e z}\left|e\right\rangle\left\langle e\middle|\psi_{in}\right\rangle + e^{ik_o z}\left|o\right\rangle\left\langle o\middle|\psi_{in}\right\rangle = \hat{U}_z\left|\psi_{in}\right\rangle,$$

where

$$\hat{U}_z = e^{ik_e z}\left|e\right\rangle\left\langle e\right| + e^{ik_o z}\left|o\right\rangle\left\langle o\right| = e^{ik_e z}\hat{P}_e + e^{ik_o z}\hat{P}_O .$$

We notice that $\hat{U}_z$ obeys the simple property

$$\hat{U}_{z+a} = \hat{U}_a \hat{U}_z .$$

((**Proof**))

$$\begin{aligned}\hat{U}_z \hat{U}_a &= (e^{ik_e z}\hat{P}_e + e^{ik_o z}\hat{P}_o)(e^{ik_e a}\hat{P}_e + e^{ik_o a}\hat{P}_o) \\ &= e^{ik_e(z+a)}\hat{P}_e + e^{ik_o(z+a)}\hat{P}_o\end{aligned}$$

since

$$\hat{P}_e\hat{P}_o = \hat{P}_o\hat{P}_e = 0, \qquad \hat{P}_e^{\,2} = \hat{P}_e, \qquad \hat{P}_o^{\,2} = \hat{P}_o.$$

Using this property, we get

$$\left|\psi_{z+a}\right\rangle = \hat{U}_{z+a}\left|\psi_{in}\right\rangle = \hat{U}_a\hat{U}_z\left|\psi_{in}\right\rangle = \hat{U}_a\left|\psi_z\right\rangle .$$

Suppose that $k_e a \ll 1$ and $k_o a \ll 1$,

$$\hat{U}_a = e^{ik_e a}\left|e\right\rangle\left\langle e\right| + e^{ik_o a}\left|o\right\rangle\left\langle o\right| \approx (1 + ik_e a)\left|e\right\rangle\left\langle e\right| + (1 + ik_o a)\left|o\right\rangle\left\langle o\right| .$$

Noting that

$$\left|e\right\rangle\left\langle e\right| + \left|o\right\rangle\left\langle o\right| = \hat{1},$$

we get

$$\hat{U}_a = \hat{1} + ia(k_e\left|e\right\rangle\left\langle e\right| + k_o)\left|o\right\rangle\left\langle o\right|) = \hat{1} + ia\hat{K} ,$$

where $\hat{K}$ is the wave number matrix,

$$\hat{K} = k_e|e\rangle\langle e| + k_o|o\rangle\langle o| = \begin{pmatrix} k_e & 0 \\ 0 & k_o \end{pmatrix}.$$

Then we have

$$|\psi_{z+a}\rangle = \hat{U}_a|\psi_z\rangle = (\hat{1} + ia\hat{K})|\psi_z\rangle .$$

In the limit $(a \to 0)$, we have

$$\frac{d}{dz}|\psi_z\rangle = i\hat{K}|\psi_z\rangle .$$

The solution of this differential equation is

$$|\psi_z\rangle = \exp(i\hat{K}z)|\psi_{z=0}\rangle .$$

((**Mathematica**))

```
Clear["Global`*"]; K1 = ( ke  0
                          0   ko );
MatrixExp[ⅈ K1 z]
{{ⅇ^(ⅈ ke z), 0}, {0, ⅇ^(ⅈ ko z)}}
```

23. Example

Suppose that

$$|\psi_{in}\rangle = |x'\rangle = \cos\phi|x\rangle + \sin\phi|y\rangle] .$$

where

$$|e\rangle = |y\rangle, \qquad |o\rangle = |x\rangle,$$

$$k_e = k_y, \qquad k_o = k_x,$$

$$\hat{K} = k_e|e\rangle\langle e| + k_o|o\rangle\langle o| = k_x|x\rangle\langle x| + k_y|y\rangle\langle y| = \begin{pmatrix} k_x & 0 \\ 0 & k_y \end{pmatrix}.$$

Then we have

$$\left|\psi_z\right\rangle = \exp(i\hat{K}z)\left|x'\right\rangle = \exp(i\hat{K}z)\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix} = \begin{pmatrix}e^{ik_x z} & 0\\ 0 & e^{ik_y z}\end{pmatrix}\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix} = \begin{pmatrix}e^{ik_x z}\cos\phi\\ e^{ik_y z}\sin\phi\end{pmatrix},$$

or

$$\left|\psi_z\right\rangle = e^{ik_x z}\cos\phi\left|x\right\rangle + e^{ik_y z}\sin\phi\left|y\right\rangle .$$

We can calculate the probability amplitudes as

$$\left\langle R\middle|\psi_z\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}e^{ik_x z}\cos\phi\\ e^{ik_y z}\sin\phi\end{pmatrix} = \frac{1}{\sqrt{2}}(e^{ik_x z}\cos\phi - ie^{ik_y z}\sin\phi)\,,$$

$$\left\langle L\middle|\psi_z\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\end{pmatrix}\begin{pmatrix}e^{ik_x z}\cos\phi\\ e^{ik_y z}\sin\phi\end{pmatrix} = \frac{1}{\sqrt{2}}(e^{ik_x z}\cos\phi + ie^{ik_y z}\sin\phi)\,.$$

24. Quarter-wave plate

A quarter-wave plate consists of a carefully adjusted thickness of a birefringent material such that the light associated with the larger index of refraction is retarded by 90° in phase (a quarter wavelength) with respect to that associated with the smaller index. The material is cut so that the optic axis is parallel to the front and back plates of the plate. Any linearly polarized light which strikes the plate will be divided into two components with different indices of refraction. One of the useful applications of this device is to convert linearly polarized light to circularly polarized light and vice versa. This is done by adjusting the plane of the incident light so that it makes 45° angle with the optic axis. This gives equal amplitude o- and e-waves. When the o-wave is slower (n_o is large), as in calcite, the o-wave will fall behind by 90° in phase, producing circularly polarized light. For calcite, n_e = 1.4864 and n_o = 1.6583.

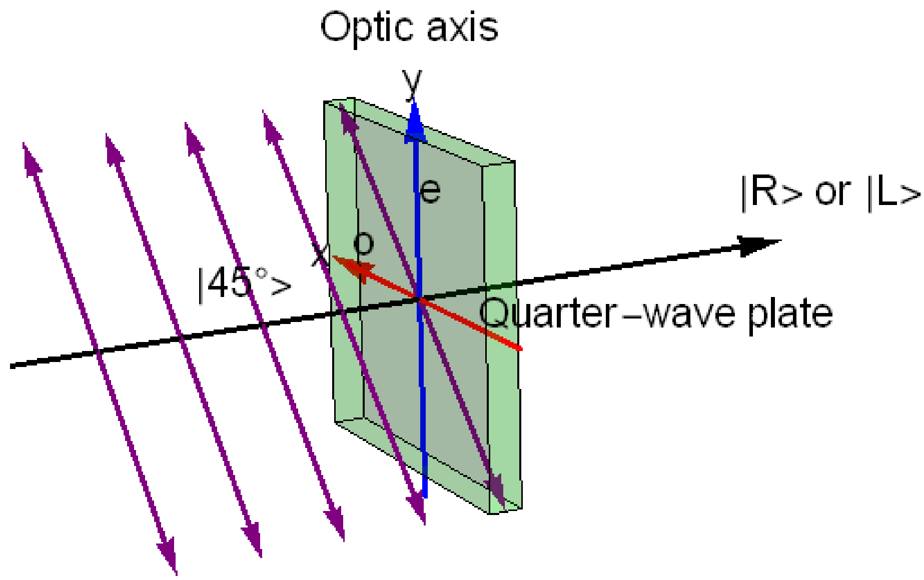
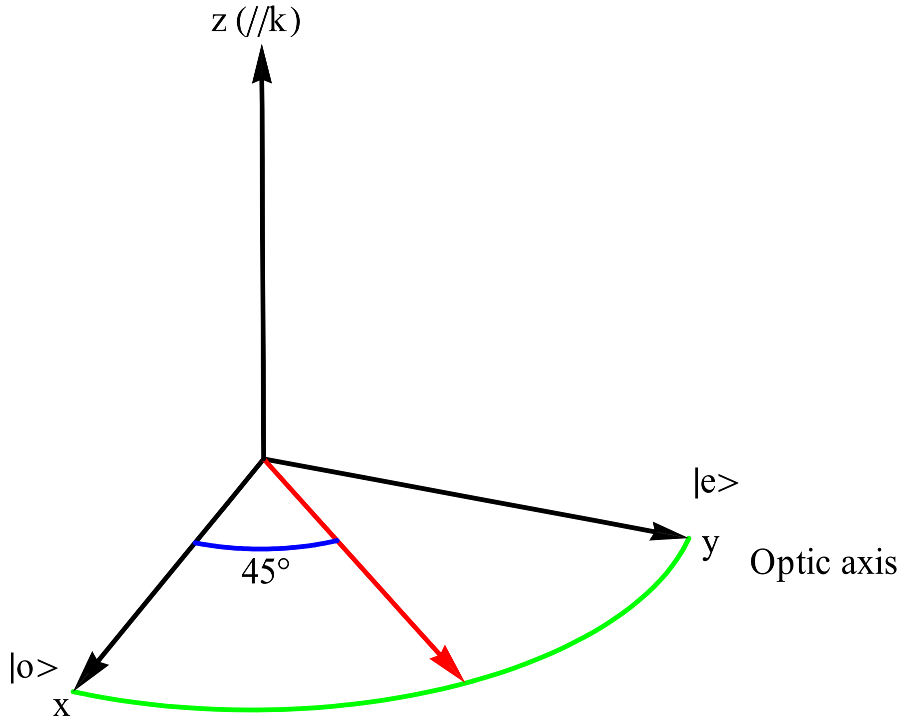


Fig. If linearly polarized light is incident on a quarter-wave plate at 45° to the optic axis, then the light is divided into two equal electric field components. One of these is retarded by a quarter wavelength by the plate. This produces circularly polarized light. Incident circularly polarized light will be changed to linearly polarized light.

Light polarized parallel to the optic axis in a birefringent crystal has a different index of refraction than does light polarized perpendicular to the optic axis. Suppose that the optic axis is along the y axis and the direction of the propagation of the ray is along the z axis. The x axis is perpendicular to the optic axis. Denoting the different indices of refraction by n_x and n_y, the light polarized parallel to the x axis (o-ray) will pick up a phase ($\frac{\omega n_x z}{c}$) in traversing a distance z through the crystal. The light polarized parallel to the y axis (e-ray) will gain a phase ($\frac{\omega n_y z}{c}$). Suppose a beam of linearly polarized light incident on a crystal with its polarization axis declined at 45° to the x axis will have equal magnitudes for the x and y components of the electric field.

$$|\psi_{in}\rangle = \frac{1}{\sqrt{2}}(|o\rangle + |e\rangle).$$



The index refraction is n_y for the *e*-ray and n_x for the *o*-ray; $n_y>n_x$. The crystal can be cut to a particular thickness z , called a quarter-wave plate.

$$\begin{aligned}
\left|\psi\right\rangle &= \frac{1}{\sqrt{2}}[\exp(\frac{i\omega n_x z}{c})\left|o\right\rangle + \exp(\frac{i\omega n_y z}{c})\left|e\right\rangle] \\
&= \exp(\frac{i\omega n_x z}{c})\frac{1}{\sqrt{2}}[\left|0\right\rangle + \exp[\frac{i\omega(n_y - n_x)z}{c}]\left|e\right\rangle\} \\
&= \exp(\frac{i\omega n_x z}{c})\frac{1}{\sqrt{2}}[\left|0\right\rangle + e^{i\phi_0}\left|e\right\rangle\}
\end{aligned}$$

When

$$\phi_0 = \frac{\omega(n_y - n_x)z}{c} = \frac{\pi}{2},$$

we have the right-hand circularly polarized light,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|e\rangle),$$

which corresponds to the right-hand circularly wave when $|0\rangle = |x\rangle$ and $|e\rangle = |y\rangle$.

25. **Quater-wave plate** (λ/4 - plate)

We use the same discussion as is used in the above. At $z = 0$,

$$|\psi_{z=0}\rangle = \cos\phi|o\rangle + \sin\phi|e\rangle.$$

For simplicity we use

$$|o\rangle = |x\rangle, \qquad |o\rangle = |y\rangle.$$

At z,

$$|\psi_z\rangle = e^{ik_x z}\cos\phi|x\rangle + e^{ik_y z}\sin\phi|y\rangle$$
$$= e^{ik_x z}[\cos\phi|x\rangle + e^{i(k_y - k_x)z}\sin\phi|y\rangle]$$

Here we can neglect the phase factor in front of the parenthesis. Then

$$|\psi_z\rangle = \cos\phi|x\rangle + e^{i(k_y - k_x)z}\sin\phi|y\rangle.$$

After passing through a quarter-wave plate (λ/4 - plate, 90° phase change),

$$\phi_0 = (k_y - k_x)z = \frac{\omega(n_y - n_x)z}{c} = \frac{\pi}{2}.$$

We get

$$|\psi\rangle = \cos\phi|x\rangle + \exp(i\frac{\pi}{2})\sin\phi|y\rangle$$
$$= \cos\phi|x\rangle + i\sin\phi|y\rangle$$
$$= \frac{1}{\sqrt{2}}(\cos\phi + \sin\phi)|R\rangle + \frac{1}{\sqrt{2}}(\cos\phi - \sin\phi)|L\rangle$$

leading to the generation of the superposition of two waves with $|R\rangle$ and $|L\rangle$, where

$$\langle R|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}\cos\phi \\ i\sin\phi\end{pmatrix} = \frac{1}{\sqrt{2}}(\cos\phi + \sin\phi)\,,$$

$$\langle L|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\end{pmatrix}\begin{pmatrix}\cos\phi \\ i\sin\phi\end{pmatrix} = \frac{1}{\sqrt{2}}(\cos\phi - \sin\phi)\,.$$

26. Example (Townsend problem 2-21)

Linearly polarized light of wavelength 589 nm is incident normally on a birefringent crystal that has its optic axis parallel to the face of the crystal, along the *x* axis. If the incident light is polarized at an angle of 45° to the *x* and *y* axes, what is the probability that the photons exiting a crystal of thickness 100 μm will be right-circularly polarized? The index of refraction for light of this wavelength polarized along *y* axis (perpendicular to the optic axis) is 1.66 and the index of refraction for light polarized along the *x* axis (parallel to the optic axis) is 1.49.

((**Solution**)) See the text in Chapter 2 for the detail of the experiment.

λ = 589.0 nm.

$$\begin{aligned}|\psi\rangle &= \frac{1}{\sqrt{2}}[\exp(\frac{i\omega n_x z}{c})|x\rangle + \exp(\frac{i\omega n_y z}{c})|y\rangle] \\ &= \exp(\frac{i\omega n_x z}{c})\frac{1}{\sqrt{2}}[|x\rangle + \exp[\frac{i\omega(n_y - n_x)z}{c}]|y\rangle\}\end{aligned}$$

where the dispersion relation of the light is given by

$$\omega = \frac{c}{n}k\,, \qquad \text{or} \qquad k = \frac{n\omega}{c}\,.$$

Since

$$|R\rangle = \frac{1}{\sqrt{2}}(|x\rangle + i|y\rangle)\,, \qquad |L\rangle = \frac{1}{\sqrt{2}}(|x\rangle - i|y\rangle)\,,$$

we have

$$\langle R|\psi\rangle = \frac{1}{2}\exp(\frac{i\omega n_x z}{c})\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}1 \\ \exp[\frac{i\omega(n_y-n_x)z}{c}]\end{pmatrix}$$

$$= \frac{1}{2}\exp(\frac{i\omega n_x z}{c})\{1 - i\exp[\frac{i\omega(n_y-n_x)z}{c}]\}$$

$$= \frac{1}{2}\exp(\frac{i\omega n_x z}{c})\{1 + \exp[\frac{i\omega(n_y-n_x)z}{c} - i\frac{\pi}{2}]\}$$

The probability is

$$|\langle R|\psi\rangle|^2 = \frac{1}{4}\{1+\exp[\frac{i\omega(n_y-n_x)z}{c} - i\frac{\pi}{2}]\}\{1+\exp[\frac{-i\omega(n_y-n_x)z}{c} + i\frac{\pi}{2}]\}$$

$$= \frac{1}{4}\{2+\exp[\frac{i\omega(n_y-n_x)z}{c} - i\frac{\pi}{2}] + \exp[\frac{-i\omega(n_y-n_x)z}{c} + i\frac{\pi}{2}]\}$$

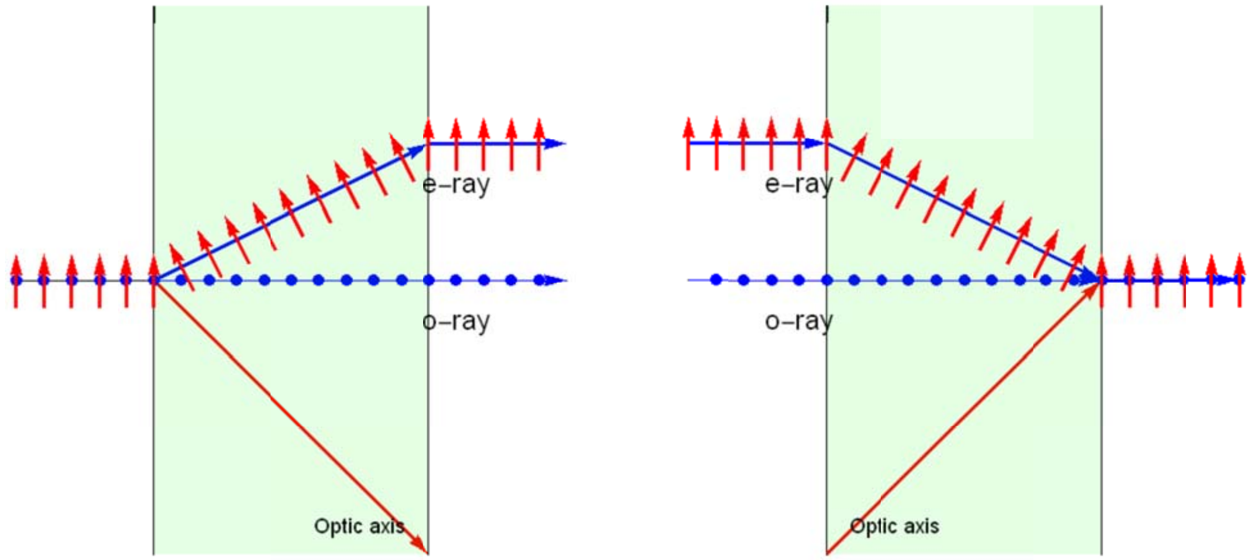
$$= \frac{1}{4}\{2+2\cos[\frac{\omega(n_y-n_x)z}{c} - \frac{\pi}{2}]\}$$

$$= \frac{1}{2}\{1+\sin[\frac{\omega(n_y-n_x)z}{c}]\}$$

For z = 100 μm, λ = 589 nm, n_y = 1.66 and n_x = 1.49, and $\omega = ck = \frac{2\pi c}{\lambda}$

$$|\langle R|\psi\rangle|^2 = 0.119754$$

28. Polarizer and analyzer



(a) Polarizer. (b) Analyzer

Fig. Decomposition and recombination of polarizations using birefringent plates.

This recombination of amplitudes is possible because two beams from the same source are coherent. Of course, it would be impossible to add the amplitudes of two polarized beams from different sources; the situation is identical to that in the case of interference. The addition of two polarization states can be illustrated using the apparatus of Fig. The two beams are recombined by a second birefringent plate, symmetrically located relative to the first with respect to a vertical plane, before the beam passes through the analyzer. In order to simplify the arguments, we shall neglect the phase difference originating from the difference between the ordinary and extraordinary indices in the birefringent. Under these conditions the light wave at the exit of the second birefringent plate is polarized in the ˆn direction. The recombination of the two x and y beams gives the initial light beam polarized in the x direction.

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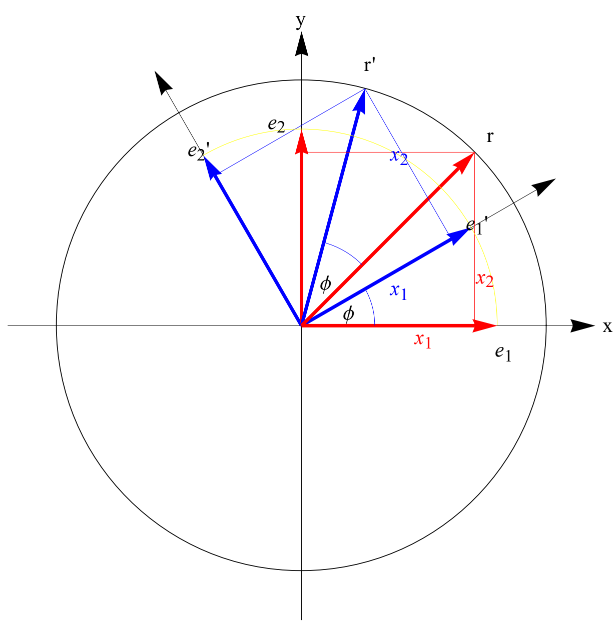
APPENDIX-I 2D rotation matrix

Suppose that the vector $\boldsymbol{r}$ is rotated through θ (counter-clock wise) around the z axis. The position vector $\boldsymbol{r}$ is changed into $\boldsymbol{r}'$ in the same orthogonal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$.



In this Fig, we have

$$\mathbf{e}_1 \cdot \mathbf{e}_1' = \cos\phi \qquad \mathbf{e}_2 \cdot \mathbf{e}_1' = \sin\phi$$
$$\mathbf{e}_1 \cdot \mathbf{e}_2' = -\sin\phi, \qquad \mathbf{e}_2 \cdot \mathbf{e}_2' = \cos\phi,$$

We define $\boldsymbol{r}$ and $\boldsymbol{r}'$ as

$$\mathbf{r}' = x_1'\mathbf{e}_1 + x_2'\mathbf{e}_2 = x_1\mathbf{e}_1' + x_2\mathbf{e}_2',$$

and

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 .$$

Using the relation

$$\begin{aligned}\boldsymbol{e}_1 \cdot \boldsymbol{r}' &= \boldsymbol{e}_1 \cdot (x_1'\boldsymbol{e}_1 + x_2'\boldsymbol{e}_2) = \boldsymbol{e}_1 \cdot (x_1\boldsymbol{e}_1' + x_2\boldsymbol{e}_2') \\ \boldsymbol{e}_2 \cdot \boldsymbol{r}' &= \boldsymbol{e}_2 \cdot (x_1'\boldsymbol{e}_1 + x_2'\boldsymbol{e}_2) = \boldsymbol{e}_2 \cdot (x_1\boldsymbol{e}_1' + x_2\boldsymbol{e}_2')\end{aligned},$$

we have

$$\begin{aligned}x_1' &= \boldsymbol{e}_1 \cdot (x_1\boldsymbol{e}_1' + x_2\boldsymbol{e}_2') = x_1\cos\phi - x_2\sin\phi \\ x_2' &= \boldsymbol{e}_2 \cdot (x_1\boldsymbol{e}_1' + x_2\boldsymbol{e}_2') = x_1\sin\phi + x_2\cos\phi\end{aligned},$$

or

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \Re(\phi)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

(i) This rotation matrix is related to the definition of the angular momentum

$$\Re(\phi) \rightarrow \exp(-\frac{i\hat{J}_z\phi}{\hbar}) = \hat{R}_z(\phi) .$$

(ii)

$$\hat{R}_z(\phi) = \hat{S}(\phi) ,$$

with

$$\hat{S}(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

APPENDIX-II Malus's law

When completely plane polarized light is incident on the analyzer, the intensity I of the light transmitted by the analyzer is directly proportional to the square of the cosine of angle between the transmission axes of the analyzer and the polarizer.

After the first polarizer with the x axis as a preferred axis, the state of the light is expressed by $|x\rangle$. The next polarizer has the preferred axis with x' axis. The final state after passing the polarizer becomes $|x'\rangle$ state. The corresponding probability P is

$$P = |\langle x|x'\rangle|^2 = \cos^2\theta, \qquad \text{(Malus's law)}$$

where

$$|x'\rangle = |\theta\rangle = \cos\theta|x\rangle + \sin|y\rangle.$$

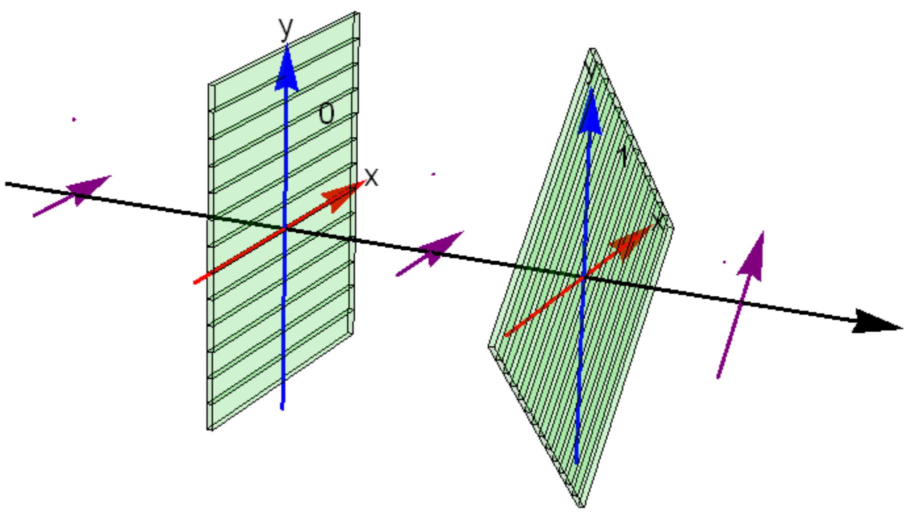


Fig. The light with the polarization $|x\rangle$ passes through a filter with $|\theta\rangle$.

The projection operator for $|\theta\rangle$ is defined by

$$\hat{P}(\theta) = |\theta\rangle\langle\theta| == \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}.$$

For the incident light with the polarization $|x\rangle$, we have

$$\hat{P}(\theta)|x\rangle == \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta \\ \sin\theta\cos\theta \end{pmatrix}$$

$$= \cos\theta \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \cos\theta|\theta\rangle$$

The probability of finding the light with the polarization $|\theta\rangle$ is

$$P = \left|\langle\theta|\hat{P}(\theta)|x\rangle\right|^2 = \cos^2\theta.$$

which is in agreement with the Malus' law.

For the light with $|y\rangle$, we have

$$\hat{P}(\theta)|y\rangle == \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \sin\theta\cos\theta \\ \sin^2\theta \end{pmatrix}$$

$$= \sin\theta \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \sin\theta|\theta\rangle$$

The probability of finding the light with the polarization $|\theta\rangle$ is

$$P = \left|\langle\theta|\hat{P}(\theta)|y\rangle\right|^2 = \sin^2\theta.$$

APPENDIX-III Notation of the polarization vectors

A. Jones vectors (R.C. Jones, 1941)

In 1941 R. Clark Jones developed a method to represent polarization of light using vectors. Named after its inventor, these vector are called Jones vectors

Jones suggested the orthogonal unit vectors

$$\boldsymbol{e}_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \boldsymbol{e}_y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$e_+ = \frac{e_x + ie_y}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \qquad \text{(right circularly polarized)}$$

$$e_- = \frac{e_x - ie_y}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \qquad \text{(left circularly polarized)}$$

Note that

$$e_+ \cdot e_+ = 0, \qquad e_- \cdot e_- = 0.$$

$$e_+ \cdot e_- = 1, \qquad e_- \cdot e_+ = 1.$$

B. Schwinger (2001, p.470)

In the book of Schwinger, Schwinger uses the following notations,

$$e_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad e_y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$e_+ = \frac{e_x + ie_y}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \qquad \text{(right circularly polarized)}$$

$$e_- = \frac{e_y + ie_x}{\sqrt{2}} = i\left(\frac{e_x - ie_y}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} = i\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \qquad \text{(left circularly polarized)}$$

C. Grynberg et al (2010, p.125)

In the book of Grynberg et al (2010), they use the following notations,

$$\boldsymbol{e}_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \boldsymbol{e}_y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\boldsymbol{e}_+ = -\left(\frac{\boldsymbol{e}_x + i\boldsymbol{e}_y}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \qquad \text{(right circularly polarized)}$$

$$\boldsymbol{e}_- = \frac{\boldsymbol{e}_x - i\boldsymbol{e}_y}{\sqrt{2}} = \left(\frac{\boldsymbol{e}_x - i\boldsymbol{e}_y}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \qquad \text{(left circularly polarized)}$$

In this notation, we have the following expressions for the electric field for the right-circularly polarized wave and the left-circularly polarized wave. The direction of the rotation is the same as that for the Jones vector.

(i) $$\boldsymbol{e}_+ = -\left(\frac{\boldsymbol{e}_x + i\boldsymbol{e}_y}{\sqrt{2}}\right) \qquad \text{(right-circularly polarized wave)}$$

$$\begin{aligned}
\mathrm{Re}[E_0 e^{i\alpha}(-1)\frac{(\boldsymbol{e}_x + i\boldsymbol{e}_y)}{\sqrt{2}}] &= \mathrm{Re}[E_0 e^{i(\alpha+\pi)}\frac{(\boldsymbol{e}_x + i\boldsymbol{e}_y)}{\sqrt{2}}] \\
&= \frac{E_0}{\sqrt{2}}[\cos(\alpha+\pi)\boldsymbol{e}_x - \sin(\alpha+\pi)\boldsymbol{e}_y] \\
&= \frac{E_0}{\sqrt{2}}[\cos(kz-\omega t+\alpha)\boldsymbol{e}_x - \sin(kz-\omega t+\alpha)\boldsymbol{e}_y]
\end{aligned}$$

with

$$\boldsymbol{e}_+ = -\left(\frac{\boldsymbol{e}_x + i\boldsymbol{e}_y}{\sqrt{2}}\right)$$

The electric field rotates in clock-wise direction with z (wavenumber k) (as the wave propagates forward). The electric field rotates in counter clock-wise direction with time (angular frequency ω).

(ii) $\quad \boldsymbol{e}_{-} = \dfrac{\boldsymbol{e}_x - i\boldsymbol{e}_y}{\sqrt{2}}$ (left-circularly polarized wave)

$$\mathrm{Re}[E_0 e^{i\alpha} \frac{(\boldsymbol{e}_x - i\boldsymbol{e}_y)}{\sqrt{2}}] = \frac{E_0}{\sqrt{2}}[(\cos\alpha)\boldsymbol{e}_x + (\sin\alpha)\boldsymbol{e}_y]$$
$$== \frac{E_0}{\sqrt{2}}[\cos(kz-\omega t)\boldsymbol{e}_x + \sin(kz-\omega t)\boldsymbol{e}_y]$$
$$= \frac{E_0}{\sqrt{2}}[\cos(\omega t-kz)\boldsymbol{e}_x - \sin(\omega t-kz)\boldsymbol{e}_y]$$

with

$$\boldsymbol{e}_{-} = \frac{\boldsymbol{e}_x - i\boldsymbol{e}_y}{\sqrt{2}}$$

The electric field rotates in counter clock-wise direction with z (wavenumber k) (as the wave propagates forward). The electric field rotates in clock-wise direction with time (angular frequency ω).

APPENDIX-IV Projection operators

Construction of the projection operators out of bras and kets for x-polarized and y-polarized photons.

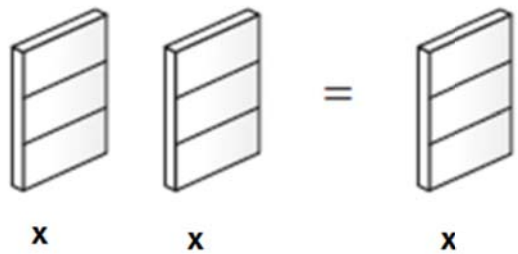
$$\hat{P}_x = |x\rangle\langle x|, \qquad \hat{P}_y = |y\rangle\langle y|,$$

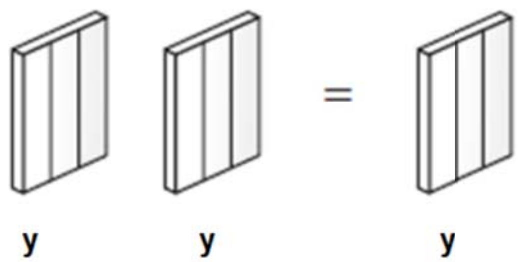
$$\hat{P}_x^{\,2} = (|x\rangle\langle x|)(|x\rangle\langle x|) = |x\rangle\langle x|x\rangle\langle x| = |x\rangle\langle x| = \hat{P}_x,$$

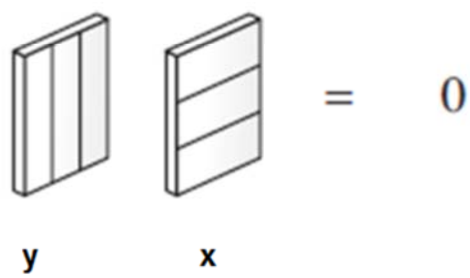
$$\hat{P}_y^{\,2} = (|y\rangle\langle y|)(|y\rangle\langle y|) = |y\rangle\langle y|y\rangle\langle y| = |y\rangle\langle y| = \hat{P}_y,$$

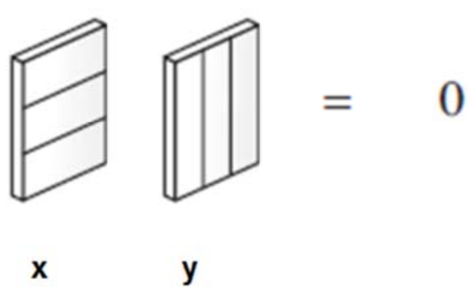
$$\hat{P}_x\hat{P}_y = (|x\rangle\langle x|)(|y\rangle\langle y|) = |x\rangle\langle x|y\rangle\langle y| = 0,$$

$$\hat{P}_y\hat{P}_x = (|y\rangle\langle y|)(|x\rangle\langle x|) = |y\rangle\langle y|x\rangle\langle x| = 0.$$









The polarization is denoted by the projection operators in quantum mechanics,such as

$$\hat{P}(\theta)=|\theta\rangle\langle\theta|=\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\end{pmatrix}=\begin{pmatrix}\cos^2\theta & \sin\theta\cos\theta\\ \sin\theta\cos\theta & \sin^2\theta\end{pmatrix},$$

$$\hat{P}(\theta_\perp) = |\theta_\perp\rangle\langle\theta_\perp| = \begin{pmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}.$$

For example, we have

$$\hat{P}(0°) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \hat{P}(90°) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \hat{P}(45°) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

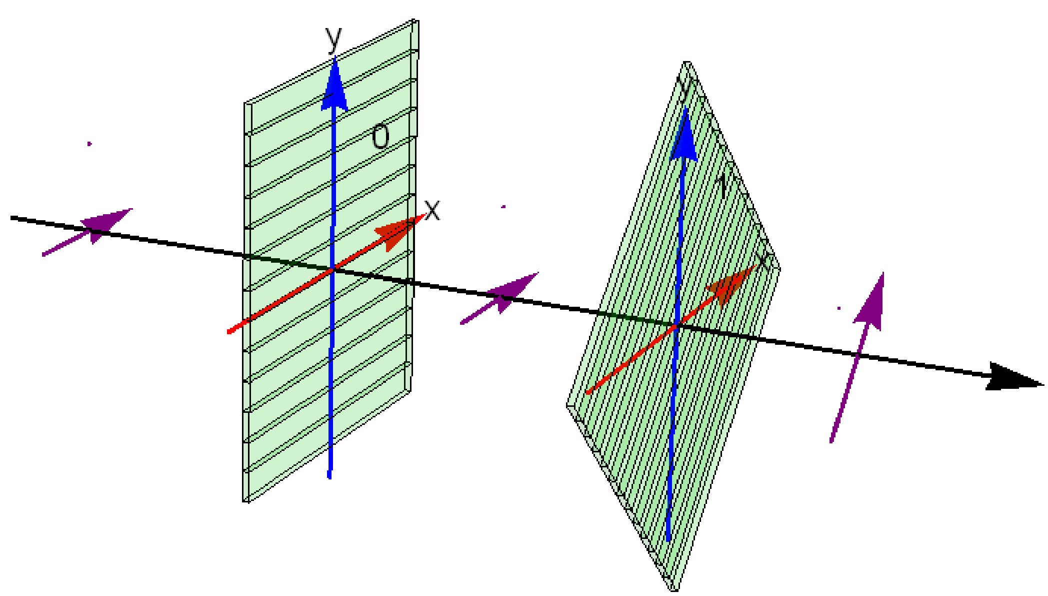
$$\hat{P}(135°) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad \hat{P}(-45°) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Note that

$$|45°\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad |x\rangle = |0^\circ\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |y\rangle = |90^\circ\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

APPENDIX-V
Quantum interference with single photon with two kind of filters

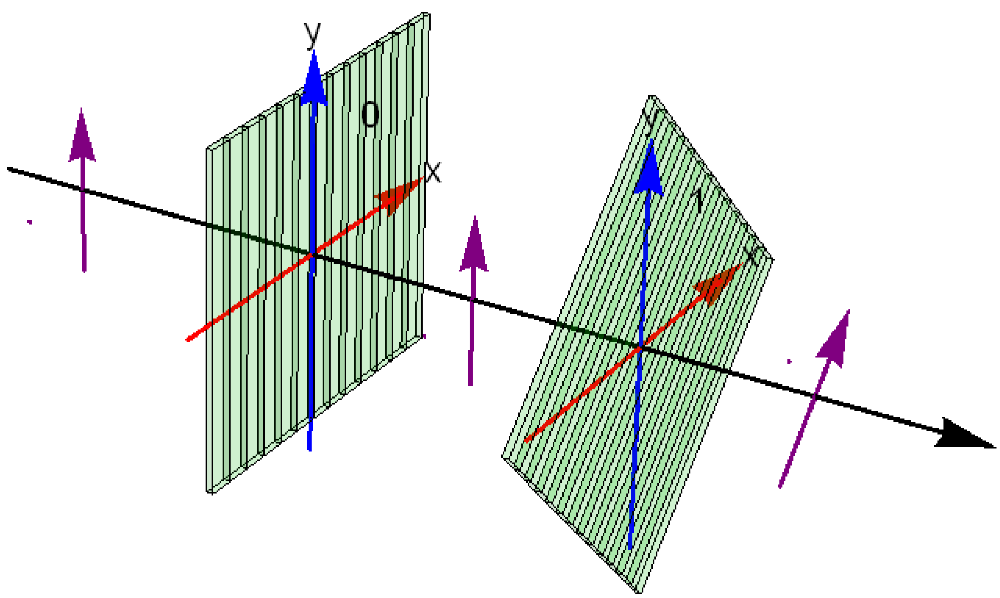
(a)



$$\hat{P}(45°)|x\rangle == \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}|45^{\circ}\rangle$$

Fig. x-45° filters. The polarized light with $|x\rangle$ enters the x-axis filter (0). The probability of finding the system in the state $|x\rangle$ is 100% between the filters 0 and 1. After passing through the 45° filter (1), the polarized light becomes in the state $|45°\rangle$ with the resultant probability of finding the system as 50%.

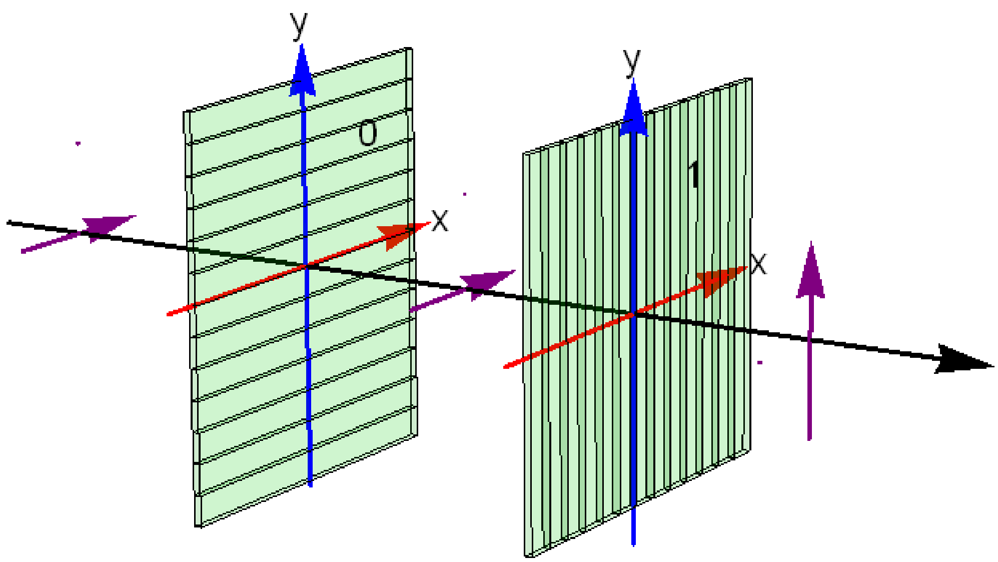
(b)



$$\hat{P}(45°)|y\rangle == \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}|45^{\circ}\rangle.$$

Fig. y-45° filters. The polarized light with $|y\rangle$ enters the y-axis filter (0). The probability of finding the system in the state $|y\rangle$ is 100% between the filters 0 and 1. After passing through the 45° filter (1), the polarized light becomes in the state $|45°\rangle$ with the resultant probability of finding the system as 50%.

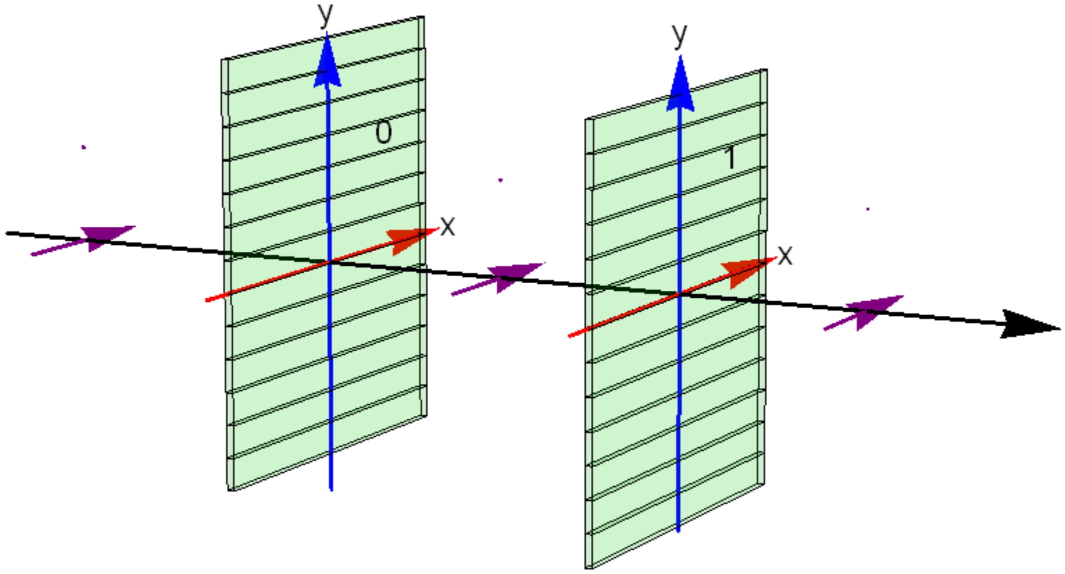
(c)



$$\hat{P}(90°)\left|x\right\rangle == \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\,.$$

Fig. *x-y* filters. The polarized light with $\left|x\right\rangle$ enters the *x*-axis filter (0). The probability of finding the system in the state $\left|x\right\rangle$ is 100% between the filters 0 and 1. After passing through the *y*-axis fiter (1), the light disappears, which means that the resultant probability of finding the system in the $\left|y\right\rangle$ is 0 %.

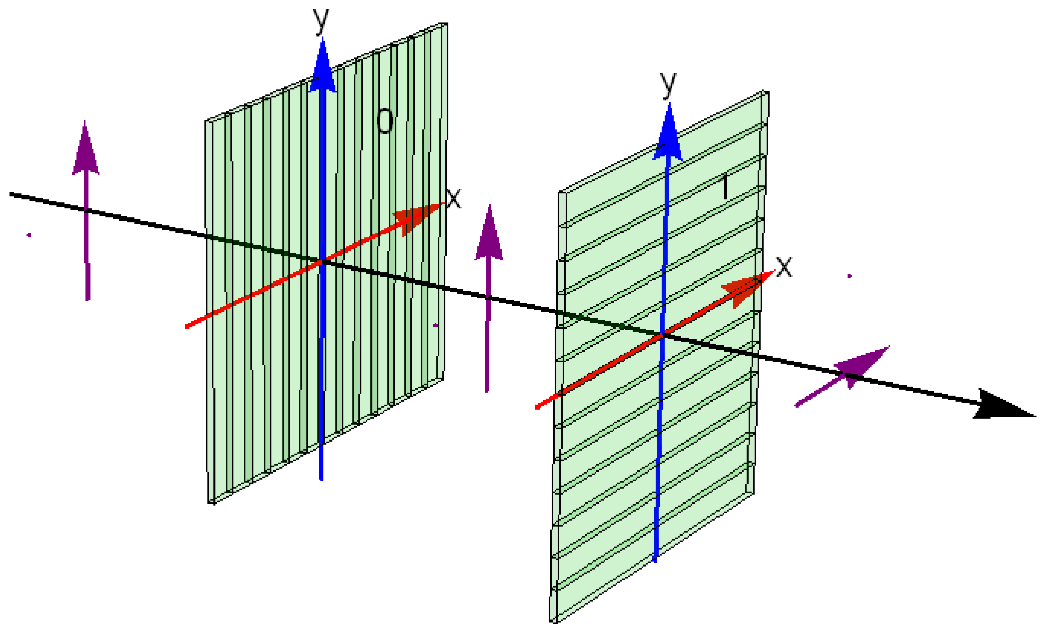
(d)



$$\hat{P}(0°)|x\rangle == \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |x\rangle .$$

Fig. x-x filers. The polarized light with $|x\rangle$ enters the x-axis filter (0). The probability of finding the system in the state $|x\rangle$ is 100% between the filters 0 and 1. After passing through the x-axis filter (1), the light remains in the same state ($|x\rangle$). Th resultant probability of finding the system in the $|x\rangle$ is 100 %.

(e)

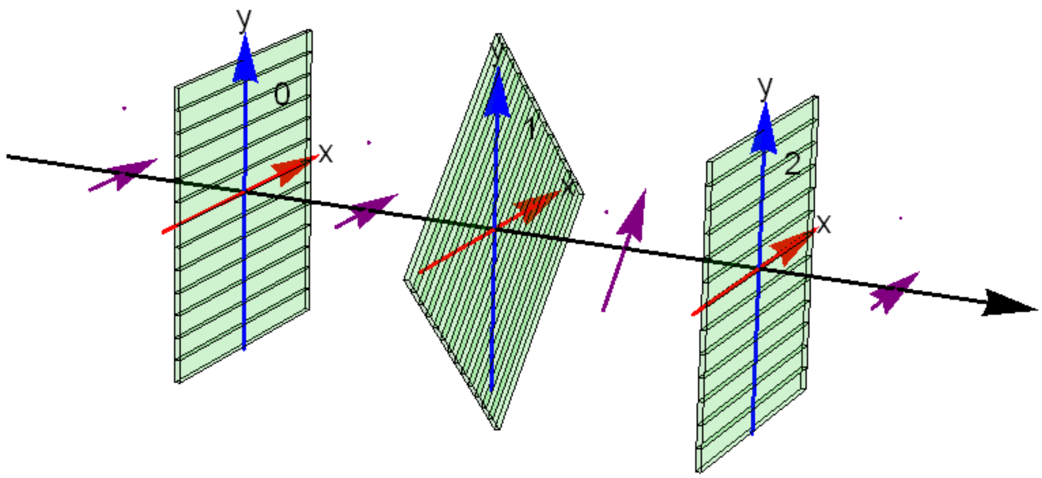


$$\hat{P}(0°)|y\rangle == \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 .$$

Fig. *y-x* filters. The polarized light with $|y\rangle$ enters the *x*-axis filter (0). The probability of finding the system in the state $|y\rangle$ is 100% between the filters 0 and 1. After passing through the *x*-axis fiter (1), the light disappears, which means that the resultant probability of finding the system in the $|x\rangle$ is 0 %.

APPENDIX-VII Quantum interference with single photon with three kind of filters

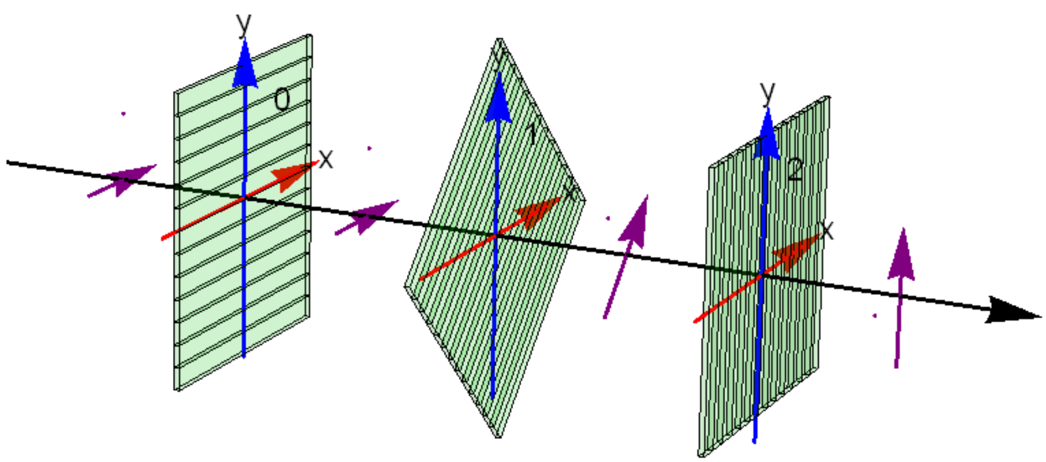
(a)



$$\hat{P}(0°)\hat{P}(45°)|x\rangle == \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} |x\rangle .$$

Fig. *x*-45°-*x* filters. The polarized light with $|x\rangle$ enters the *x*-axis filter (0). The probability of finding the photon in the state $|x\rangle$ is 100% between the filters 0 and 1. After passing through the 45° fiter (1), only the light with $|45°\rangle$ appears between the 0 and 1 filters. The probability of finding the photon in the $|45°\rangle$ is 50 %. After passing through the *x*-filter (2), only the light with $|x\rangle$ appears. The resultant probability of finding the photon in the $|x\rangle$ is 25 % after the *x*-filter (2).

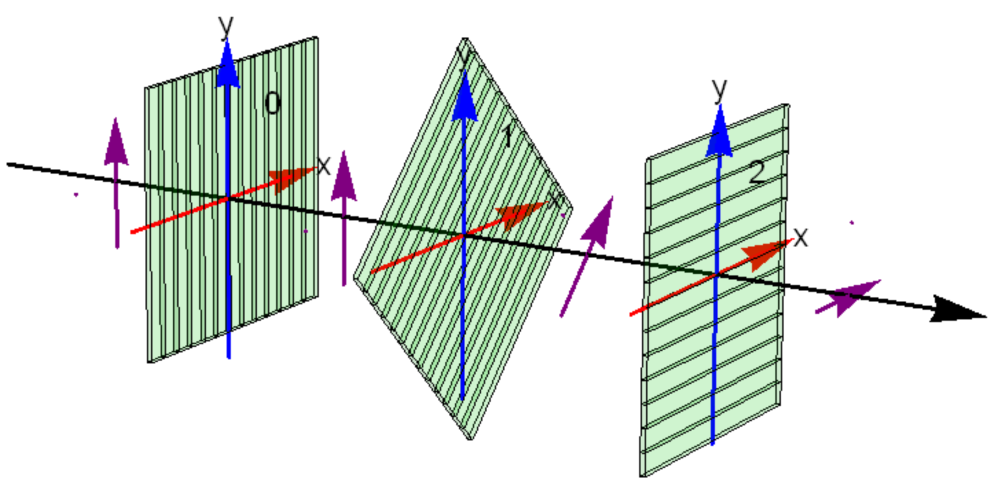
(b)



$$\hat{P}(90°)\hat{P}(45°)|x\rangle == \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2} |y\rangle .$$

Fig. *x*-45°-*y* filters. The polarized light with $|x\rangle$ enters the *x*-axis filter (0). The probability of finding the photon in the state $|x\rangle$ is 100% between the filters 0 and 1. After passing through the 45° fiter (1), only the light with $|45°\rangle$ appears between the 0 and 1 filters. The probability of finding the photon in the $|45°\rangle$ is 50 %. After passing through the *y*-filter (2), only the light with $|y\rangle$ appears. The resultant probability of finding the photon in the $|y\rangle$ is 25 % after the *x*-filter (2).

(c)

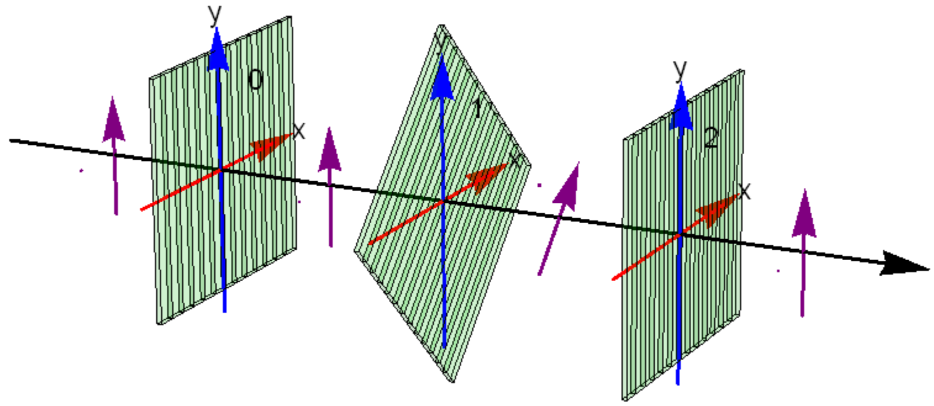


$$\hat{P}(0°)\hat{P}(45°)|y\rangle == \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} |x\rangle .$$

Fig. *y*-45°-*x* filters. The polarized light with $|y\rangle$ enters the *y*-axis filter (0). The probability of finding the photon in the state $|y\rangle$ is 100% between the filters 0 and 1. After passing through the 45° fiter (1), only the light with $|45°\rangle$ appears between the 0 and 1 filters. The probability of finding the photon in the $|45°\rangle$ is 50 %. After passing through the *x*-

filter (2), only the light with $|x\rangle$ appears. The resultant probability of finding the photon in the $|x\rangle$ is 25 % after the x-filter (2).

(d)



$$\hat{P}(90°)\hat{P}(45°)|y\rangle == \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}|y\rangle .$$

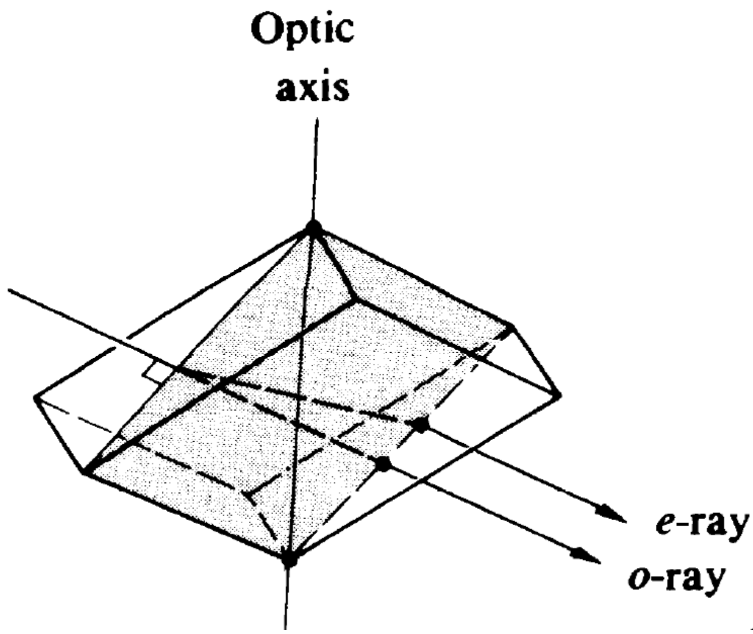
Fig. y-45°-y filters. The polarized light with $|y\rangle$ enters the y-axis filter (0). The probability of finding the photon in the state $|y\rangle$ is 100% between the filters 0 and 1. After passing through the 45° fiter (1), only the light with $|45°\rangle$ appears between the 0 and 1 filters. The probability of finding the photon in the $|45°\rangle$ is 50 %. After passing through the y-filter (2), only the light with $|y\rangle$ appears. The resultant probability of finding the photon in the $|y\rangle$ is 25 % after the x-filter (2).

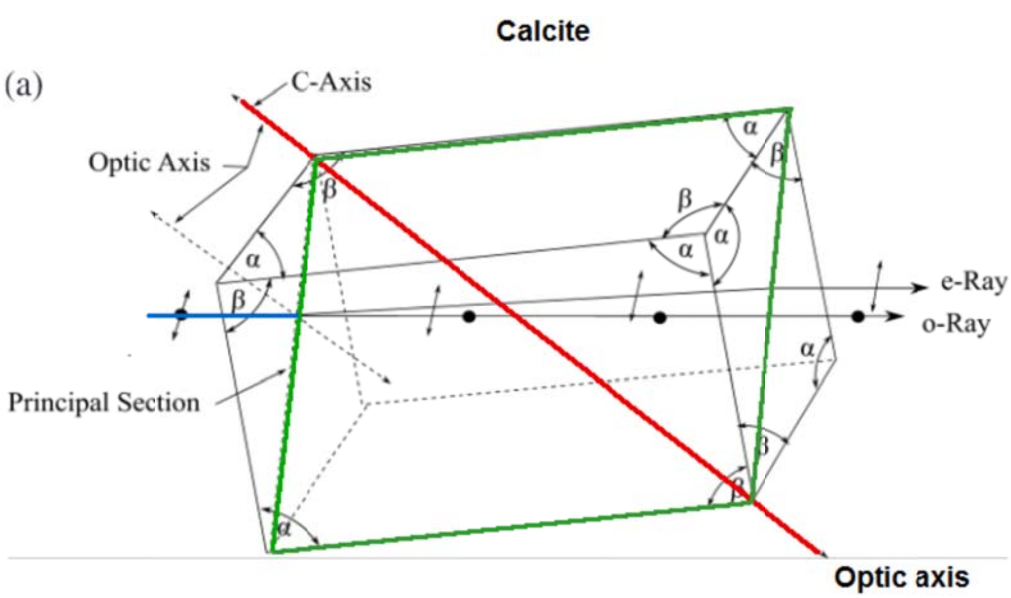
APPENDIX-VI Calcite

Calcite is transparent to opaque and may occasionally show phosphorescence or fluorescence. A transparent variety called *Iceland spar* is used for optical purposes. Acute scalenohedral crystals are sometimes referred to as "dogtooth spar" while the rhombohedral form is sometimes referred to as "nailhead spar". Single calcite crystals display an optical property called birefringence (double refraction). This strong birefringence causes objects viewed through a clear piece of calcite to appear doubled. The birefringent effect (using calcite) was first described by the Danish scientist Rasmus Bartholin in 1669. At a wavelength of ~590 nm calcite has ordinary and extraordinary refractive indices of 1.658 and 1.486, respectively. Between 190 and

1700 nm, the ordinary refractive index varies roughly between 1.9 and 1.5, while the extraordinary refractive index varies between 1.6 and 1.4.
http://en.wikipedia.org/wiki/Calcite

A birefringence in calcite can be observed in calcite.





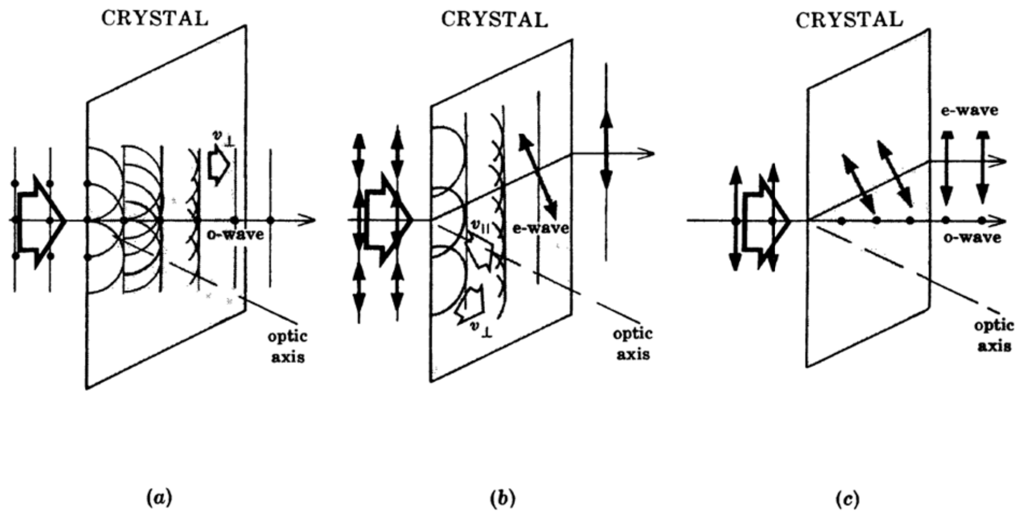


Fig. Schematic diagram of the propagating *o*-ray and *e*-ray in calcite crystal. E. Hecht, *Schaum's outline of theory and problems of optics* (McGraw-Hill, 1975).

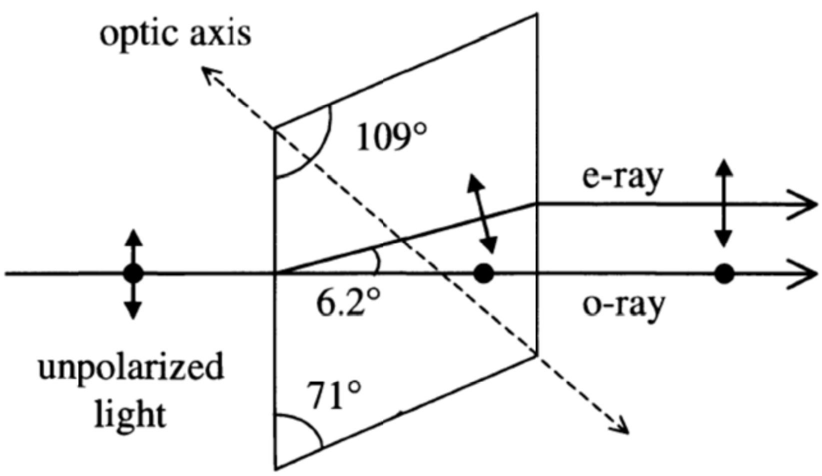


Fig. A light with two orthogonal field components traversing a calcite section. (M. Fox, *Optical Properties of Solids* (Oxford, 2001).

Double refraction in a natural calcite crystal. The shape of the crystal and the orientation of the optic axis is determined by the cleavage planes of calcite. An un-polarized incident light ray is split into two spatially separated orthogonally polarized rays. The ● symbol for the o-ray indicates that it is polarized with its ***E***-field pointing out of the page.

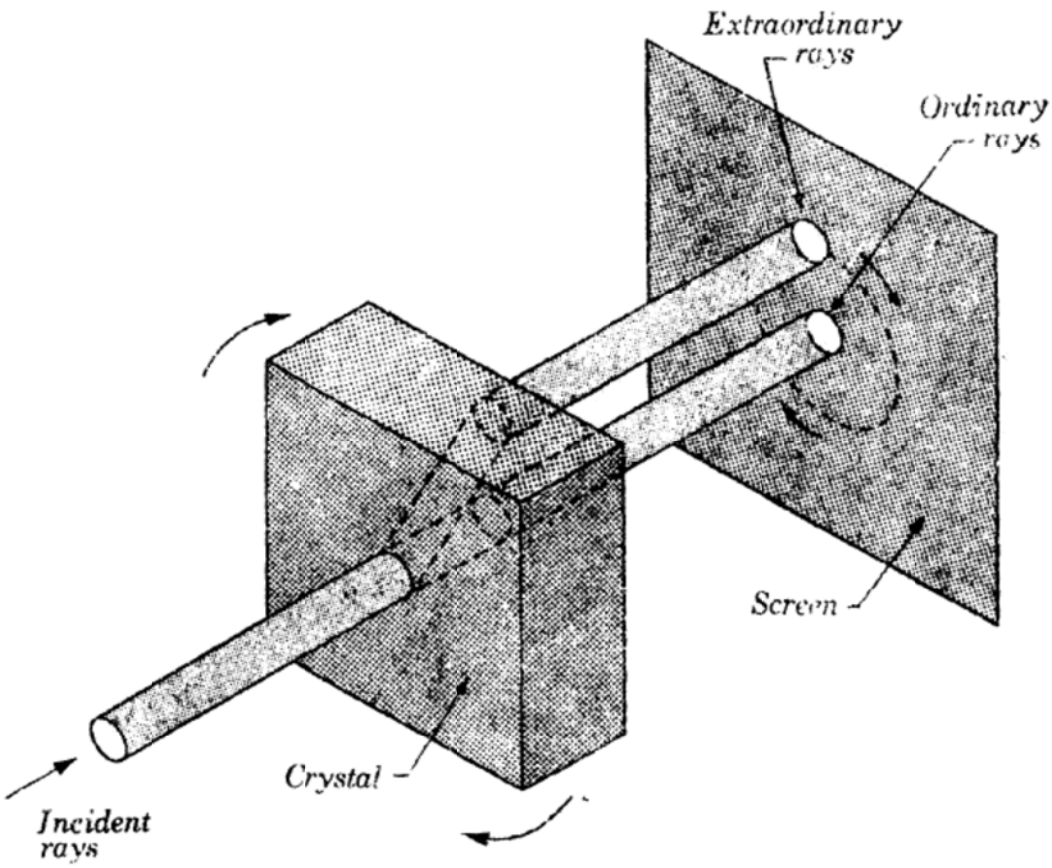


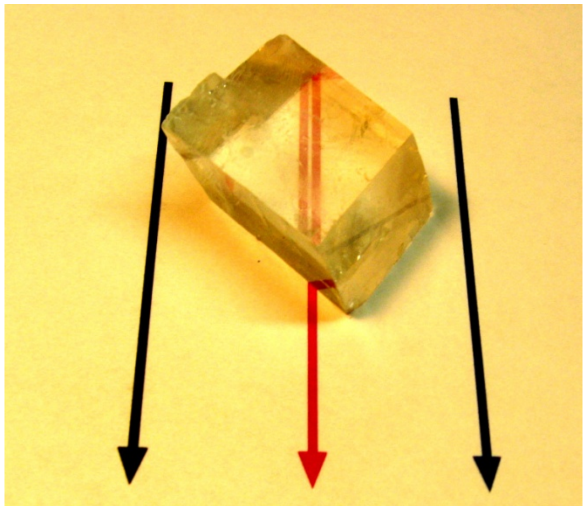
Fig. A narrow beam of natural light can be split into two beams by a doubly refracting crystal. (F.W. Sears, Optics) p.180. Fig.7.11

((**Calcite**))
Index of refraction for calcite:

$n_o = 1.6583$,
$n_e = 1.4864$,
$\alpha = 102°$, $\beta = 78°$.

((Youtube))
Double refraction in calcite
http://www.youtube.com/watch?v=MoZar-gCj3E

Circular polarization
http://www.youtube.com/watch?v=ycY2mUZHS84



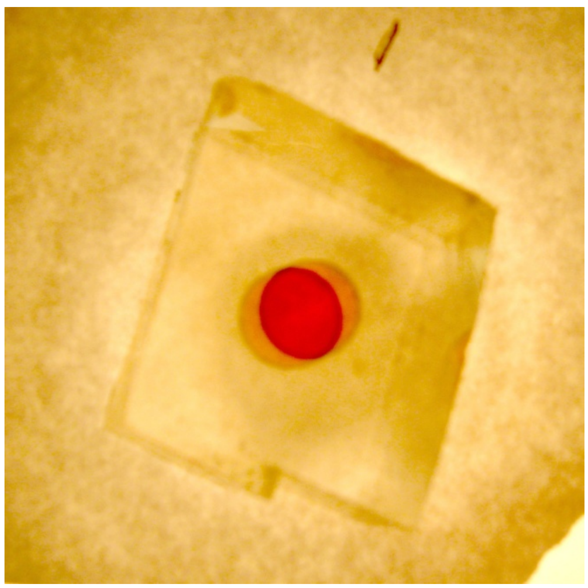
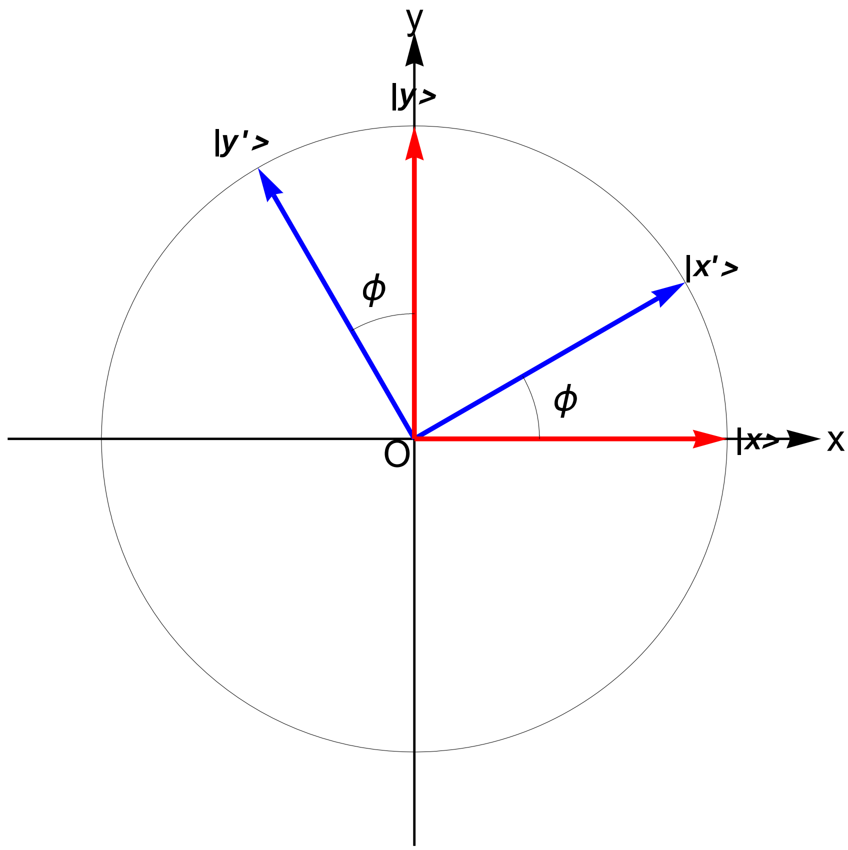


Fig. Calcite [(pictures taken by the author (M.S.)]

APPENDIX-VII Rotation operator $\hat{S}(\phi)$



$$\left|x'\right\rangle = \hat{S}(\phi)\left|x\right\rangle = \cos\phi\left|x\right\rangle + \sin\phi\left|y\right\rangle = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix},$$

$$\left|y'\right\rangle = \hat{S}(\phi)\left|y\right\rangle = -\sin\phi\left|x\right\rangle + \cos\phi\left|y\right\rangle = \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix}$$

where

$$\begin{aligned}
\hat{S}(\phi) &= \begin{pmatrix} \left\langle x\right|\hat{S}(\phi)\left|x\right\rangle & \left\langle x\right|\hat{S}(\phi)\left|y\right\rangle \\ \left\langle y\right|\hat{S}(\phi)\left|x\right\rangle & \left\langle y\right|\hat{S}(\phi)\left|y\right\rangle \end{pmatrix} \\
&= \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \\
&= \cos\phi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\sin\phi \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
&= \cos\phi\hat{1} - i\sin\phi\hat{\sigma}_y
\end{aligned}$$

We now consider the eigenvalue problem,

$$\begin{aligned}\hat{S}(\phi)|+y\rangle &= (\cos\phi\hat{1} - i\sin\phi\hat{\sigma}_y)|+y\rangle \\ &= (\cos\phi - i\sin\phi)|+y\rangle \\ &= e^{-i\phi}|+y\rangle\end{aligned}$$

$$\begin{aligned}\hat{S}(\phi)|-y\rangle &= (\cos\phi\hat{1} - i\sin\phi\hat{\sigma}_y)|-y\rangle \\ &= (\cos\phi + i\sin\phi)|-y\rangle \\ &= e^{i\phi}|+y\rangle\end{aligned}$$

where

$|R\rangle$ is the eigenket of $\hat{S}(\phi)$ with the eigenvalue $e^{-i\phi}$

$$|R\rangle = |+y\rangle = \frac{1}{\sqrt{2}}[|x\rangle + i|y\rangle] = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$$

$|L\rangle$ is the eigenket of $\hat{S}(\phi)$ with the eigenvalue $e^{i\phi}$

$$|L\rangle = |-y\rangle = \frac{1}{\sqrt{2}}[|x\rangle - i|y\rangle] = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$$

$$\hat{S}(\phi) = \exp(-\frac{i}{\hbar}\hat{J}_z\phi)$$

$\hat{J}_z$ is the angular momentum for photon.

$$\hat{S}(\phi)|R\rangle = e^{-i\phi}|R\rangle, \qquad \hat{S}(\phi)|L\rangle = e^{-i\phi}|L\rangle$$

In the limit of $\phi \to \delta\phi$,

$$(\hat{1} - \frac{i}{\hbar}\hat{J}_z\delta\phi)|R\rangle = (1 - i\delta\phi)|R\rangle, \qquad (\hat{1} - \frac{i}{\hbar}\hat{J}_z\delta\phi)|L\rangle = (1 + i\delta\phi)|L\rangle$$

or

$$\hat{J}_z|R\rangle = \hbar|R\rangle, \qquad \hat{J}_z|L\rangle = -\hbar|L\rangle$$

$|R\rangle$ is the eigenket of $\hat{J}_z$ with the eigenvalue $(+\hbar)$. $|L\rangle$ is the eigenket of $\hat{J}_z$ with the eigenvalue $(-\hbar)$.