The Hermitian operator and the rotation operator for photon polarization

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Here we discuss the Hermitian operator for photon polarization. These operators are derived from the projection operators. These operators are closely related to the Pauli matrices for spin 1/2 electron. The rotation operator for the photon polarization will be also discussed.

1. Basis $\{|x\rangle, |y\rangle\}$

(i) Horizontal state $|x\rangle$

$$|x\rangle = |\rightarrow\rangle = |H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1|x\rangle + 0|y\rangle ; \qquad \text{(horizontal state)}$$

$$\hat{P}_x = |x\rangle\langle x| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\hat{P}_x|x\rangle = |x\rangle\langle x|x\rangle = |x\rangle$$

(ii) Vertical state $|y\rangle$

$$|y\rangle = |\uparrow\rangle = |V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0|x\rangle + 1|y\rangle \qquad \text{(vertical state)}$$

$$\hat{P}_y = |y\rangle\langle y| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{P}_y|y\rangle = |y\rangle\langle y|y\rangle = |y\rangle$$

$$\hat{\Sigma}_z = \hat{P}_x - \hat{P}_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{(corresponding to the Pauli matrix } \hat{\sigma}_z\text{)}$$

$$\hat{\Sigma}_z|x\rangle = |x\rangle, \qquad \hat{\Sigma}_z|y\rangle = -|y\rangle$$

The commutation relation:

$$[\hat{P}_x, \hat{P}_y] = 0$$

since

$$\hat{P}_x \hat{P}_y = |x\rangle\langle x|y\rangle\langle y| = 0, \qquad \hat{P}_x \hat{P}_y = |y\rangle\langle y|x\rangle\langle x| = 0$$

The kets $|x\rangle$ and $|y\rangle$ are compatible. We note that $|x\rangle$ and $|y\rangle$ are orthogonal and form the complete set of basis.

$$\langle x|y\rangle = 0, \qquad |x\rangle\langle x| + |y\rangle\langle y| = \hat{1} \text{ (Closure relation, Completeness)}$$

Thus $|x\rangle$ and $|y\rangle$ are the eigenkets of the matrix $\hat{\Sigma}_z$ with the eigenvalues +1, and -1, respectively. $\hat{\Sigma}_z$ can be expressed by

$$\hat{\Sigma}_z = \hat{\Sigma}_z(|x\rangle\langle x| + |y\rangle\langle y|) = |x\rangle\langle x| - |y\rangle\langle y|$$

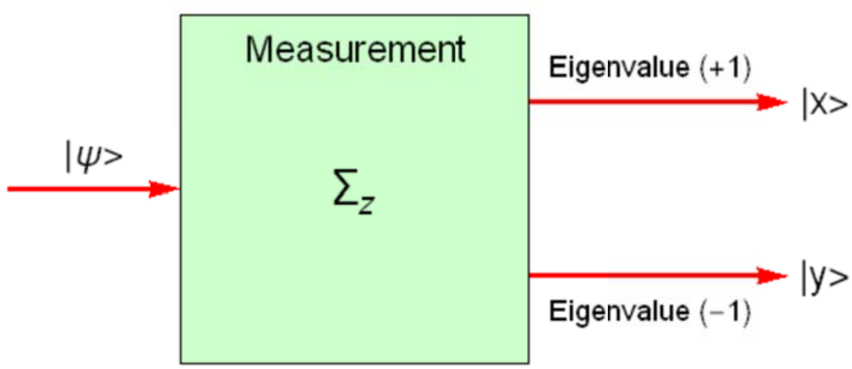


Fig. Measurement of $\hat{\Sigma}_z = \hat{P}_x - \hat{P}_y$. $\hat{\Sigma}_z|x\rangle = |x\rangle$. $\hat{\Sigma}_z|y\rangle = -|y\rangle$. The state $|\psi\rangle$ is the superposition of $|x\rangle$ and $|y\rangle$.

2. Basis { $|\theta\rangle$, $|\theta_\perp\rangle$ }

(i) Basis $|\theta\rangle$

We define the basis by

$$|\theta\rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \cos\theta|x\rangle + \sin\theta|y\rangle$$

The projection operator is defined by

$$\begin{aligned}\hat{P}_\theta &= |\theta\rangle\langle\theta| \\ &= \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\end{pmatrix} \\ &= \begin{pmatrix}\cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta\end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}\cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta)\end{pmatrix} \\ &= \frac{1}{2}\hat{I}_2 + \frac{1}{2}\Sigma_\theta\end{aligned}$$

$$\hat{P}_\theta|\theta\rangle = \langle\theta|\theta\rangle|\theta\rangle = |\theta\rangle$$

where

$$\hat{\Sigma}_\theta = \begin{pmatrix}\cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta)\end{pmatrix}$$

(ii) Basis $|\theta_\perp\rangle$

We define $|\theta_\perp\rangle$ **as**

$$|\theta_\perp\rangle = \left|\theta + \frac{\pi}{2}\right\rangle = \begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix} = -\sin\theta|x\rangle + \cos\theta|y\rangle$$

The projection operator is

$$\hat{P}_{\theta\perp} = |\theta_\perp\rangle\langle\theta_\perp|$$
$$= \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}\begin{pmatrix} -\sin\theta & \cos\theta \end{pmatrix}$$
$$= \begin{pmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$
$$= \frac{1}{2}\hat{I}_2 - \frac{1}{2}\hat{\Sigma}_\theta$$

$$\langle\theta|\theta_\perp\rangle = \begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix}\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = 0.$$

$$\hat{P}_\theta + \hat{P}_{\theta\perp} = \hat{1}.$$ (Closure relation, completeness)

$$\hat{P}_\theta|\theta_\perp\rangle = |\theta\rangle\langle\theta|\theta_\perp\rangle = 0, \qquad \hat{P}_{\theta\perp}|\theta\rangle = |\theta_\perp\rangle\langle\theta_\perp|\theta\rangle = 0.$$

The commutation relation;

$$[P_\theta, P_{\theta\perp}] = \begin{pmatrix} 0 & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & \sin 2\theta \\ -\sin 2\theta & 0 \end{pmatrix} \neq 0$$

We also get the Hermitian operator $\hat{\Sigma}_\theta$ as follows.

$$\hat{\Sigma}_\theta = \hat{P}_\theta - \hat{P}_{\theta\perp} = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}.$$

Then we have

$$\hat{\Sigma}_\theta|\theta\rangle = (\hat{P}_\theta - \hat{P}_{\theta\perp})|\theta\rangle = |\theta\rangle, \qquad \hat{\Sigma}_\theta|\theta_\perp\rangle = (\hat{P}_\theta - \hat{P}_{\theta\perp})|\theta\rangle = -|\theta_\perp\rangle$$

Note that $|\theta\rangle$ and $|\theta_\perp\rangle$ are orthogonal and form the complete set of basis; $|\theta\rangle$ and $|\theta_\perp\rangle$ are the eigenkets of $\hat{\Sigma}_\theta$ with the eigenvalues +1 and -1, respectively.

$$\hat{\Sigma}_\theta = \hat{\Sigma}_\theta(|\theta\rangle\langle\theta| + |\theta\perp\rangle\langle\theta\perp|) = |\theta\rangle\langle\theta| - |\theta_\perp\rangle\langle\theta_\perp|.$$

We note that

$$\hat{\Sigma}_\theta|\theta\rangle = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos(2\theta)\cos\theta + \sin(2\theta)\sin\theta \\ \sin(2\theta)\cos\theta - \cos(2\theta)\sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = |\theta\rangle,$$

$$\hat{\Sigma}_\theta|\theta_\perp\rangle = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \begin{pmatrix} -\cos(2\theta)\sin\theta + \sin(2\theta)\cos\theta \\ -\sin(2\theta)\sin\theta - \cos(2\theta)\cos\theta \end{pmatrix} = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix} = -|\theta_\perp\rangle$$

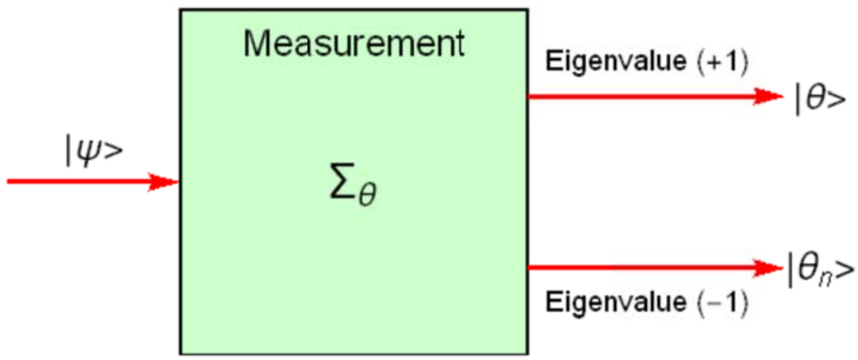


Fig. Measurement of $\hat{\Sigma}_\theta = |\theta\rangle\langle\theta| - |\theta_\perp\rangle\langle\theta_\perp|$, where $|\theta_n\rangle = |\theta_\perp\rangle$ for convenience. $\hat{\Sigma}_\theta|\theta\rangle = |\theta\rangle$. $\hat{\Sigma}_\theta|\theta_\perp\rangle = -|\theta_\perp\rangle$.

((**Note**))
Using the Pauli matrices, $\hat{\Sigma}_\theta$ can be expressed as

$$\hat{\Sigma}_\theta = \cos(2\theta)\hat{\sigma}_z + \sin(2\theta)\hat{\sigma}_x = \hat{\boldsymbol{\sigma}} \cdot \boldsymbol{n}$$

where $\boldsymbol{n} = (\sin(2\theta), 0, \cos(2\theta))$. We note that

$$|+\boldsymbol{n}\rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \cos\theta|x\rangle + \sin\theta|y\rangle, \quad |-\boldsymbol{n}\rangle = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = -\sin\theta|x\rangle + \cos\theta|y\rangle$$

3. Basis $\{\left|\frac{\pi}{4}\right\rangle, \left|-\frac{\pi}{4}\right\rangle\}$

We define the ket vectors as

$$\left|\frac{\pi}{4}\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}(|x\rangle + |y\rangle)$$

and

$$\left|-\frac{\pi}{4}\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{\sqrt{2}}(|x\rangle - |y\rangle)$$

We note that

$$\left\langle -\frac{\pi}{4}\middle|\frac{\pi}{4}\right\rangle = \frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = 0$$

The projection operators:

$$\hat{P}_{\pi/4} = \left|\frac{\pi}{4}\right\rangle\left\langle\frac{\pi}{4}\right| = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$$

$$\hat{P}_{-\pi/4} = \left|-\frac{\pi}{4}\right\rangle\left\langle-\frac{\pi}{4}\right| = \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}1 & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}$$

Then we have

$$\hat{P}_{\pi/4} + \hat{P}_{-\pi/4} = \left|\frac{\pi}{4}\right\rangle\left\langle\frac{\pi}{4}\right| + \left|-\frac{\pi}{4}\right\rangle\left\langle-\frac{\pi}{4}\right| = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$ (Closure relation, completeness)

The Hermitian operator is defined by

$$\hat{\Sigma}_x = \hat{P}_{\pi/4} - \hat{P}_{-\pi/4} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$$ (corresponding to the Pauli matrix $\hat{\sigma}_x$)

Then we have

$$\hat{\Sigma}_x\left|\frac{\pi}{4}\right\rangle = (\hat{P}_{\pi/4} - \hat{P}_{-\pi/4})\left|\frac{\pi}{4}\right\rangle = \left(\left|\frac{\pi}{4}\right\rangle\left\langle\frac{\pi}{4}\right| - \left|-\frac{\pi}{4}\right\rangle\left\langle-\frac{\pi}{4}\right|\right)\left|\frac{\pi}{4}\right\rangle = \left|\frac{\pi}{4}\right\rangle$$

$$\hat{\Sigma}_x\left|-\frac{\pi}{4}\right\rangle = (\hat{P}_{\pi/4} - \hat{P}_{-\pi/4})\left|-\frac{\pi}{4}\right\rangle = (\left|\frac{\pi}{4}\right\rangle\left\langle\frac{\pi}{4}\right| - \left|-\frac{\pi}{4}\right\rangle\left\langle-\frac{\pi}{4}\right|)\left|-\frac{\pi}{4}\right\rangle = -\left|-\frac{\pi}{4}\right\rangle$$

Note that $\left|\frac{\pi}{4}\right\rangle$ and $\left|-\frac{\pi}{4}\right\rangle$ are orthonogonal and form the complete set of basis; $\left|\frac{\pi}{4}\right\rangle$ and $\left|-\frac{\pi}{4}\right\rangle$ are the eigenkets of $\hat{\Sigma}_x$ with the eigenvalues +1 and -1, respectively.

Thus $\left|\frac{\pi}{4}\right\rangle$ and $\left|-\frac{\pi}{4}\right\rangle$ are the eigenkets of $\hat{\Sigma}_x$ with the eigenvalues +1, and -1, respectively.

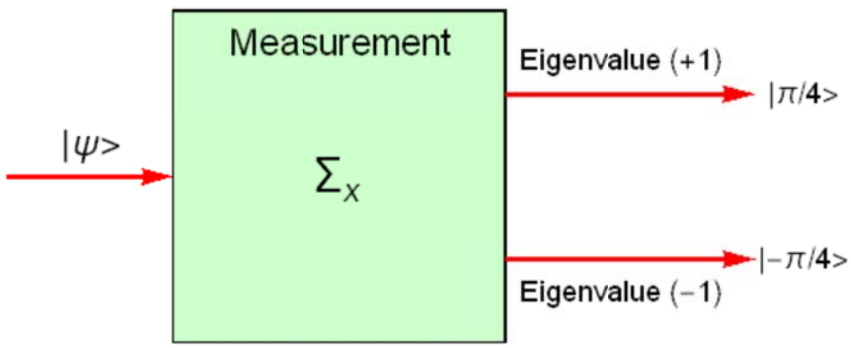


Fig. Measurement of $\hat{\Sigma}_x = \hat{P}_{\pi/4} - \hat{P}_{-\pi/4}$. $\hat{\Sigma}_x|\pi/4\rangle = |\pi/4\rangle$. $\hat{\Sigma}_x|-\pi/4\rangle = -|-\pi/4\rangle$

4. The basis $\{|R\rangle$ and $|L\rangle\}$

(i) Right- hand circularly polarized photon (clockwise)

$$|R\rangle = \alpha|x\rangle + \beta|y\rangle$$

where α and β are complex numbers,

$$\langle R|R\rangle = \begin{pmatrix}\alpha^* & \beta^*\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = |\alpha|^2 + |\beta|^2 = 1$$

We choose

$$\alpha = \frac{1}{\sqrt{2}}, \qquad \beta = \frac{i}{\sqrt{2}}$$

Then we have

$$|R\rangle = \frac{1}{\sqrt{2}}(|x\rangle + i|y\rangle)$$

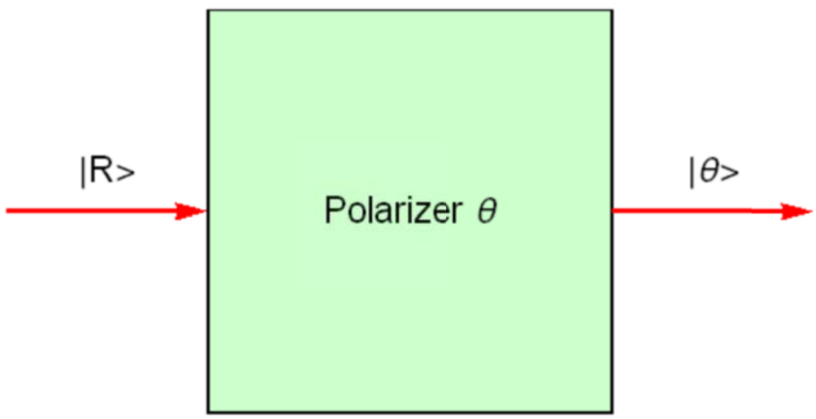
(ii) Left-hand circularly polarized photon (counter clockwise)
Similarly we get

$$|L\rangle = \frac{1}{\sqrt{2}}(|x\rangle - i|y\rangle)$$

We note that

$$\langle R|L\rangle = \frac{1}{2}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}1 \\ -i\end{pmatrix} = 0 \qquad \text{(orthogonal)}$$

((**Example**)) The RHC (right-hand circularly polarized light) passes the polarizer with angle θ.



Probability of finding the system in the state $|\theta\rangle$;

$$P_{\theta R} = |\langle\theta|R\rangle|^2 = \frac{1}{2}$$

since

$$\langle\theta|R\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}\cos\theta & \sin\theta\end{pmatrix}\begin{pmatrix}1 \\ i\end{pmatrix} = \frac{1}{\sqrt{2}}e^{i\theta}.$$

It should be noted that this probability $P_{\theta R}$ is independent of θ.

We define the projection operator:

$$\hat{P}_R = |R\rangle\langle R| = \frac{1}{2}\begin{pmatrix}1\\ i\end{pmatrix}\begin{pmatrix}1 & -i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}$$

$$\hat{P}_L = |L\rangle\langle L| = \frac{1}{2}\begin{pmatrix}1\\ -i\end{pmatrix}\begin{pmatrix}1 & i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}$$

Note that

$$\hat{P}_R|R\rangle = |R\rangle, \qquad \hat{P}_L|L\rangle = |L\rangle$$

and

$$\hat{P}_R + \hat{P}_L = \hat{1} \qquad \text{(Closure relation, completeness)}$$

We define the matrix

$$\hat{\Sigma}_y = \hat{P}_R - \hat{P}_L = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \qquad \text{(corresponding to the Pauli matrix } \hat{\sigma}_y\text{)}$$

with

$$\hat{\Sigma}_y{}^2 = \hat{1}.$$

Then we have

$$\hat{\Sigma}_y|R\rangle = (\hat{P}_R - \hat{P}_L)|R\rangle = |R\rangle, \qquad \hat{\Sigma}_y|L\rangle = (\hat{P}_R - \hat{P}_L)|L\rangle = -|L\rangle$$

Note that $|R\rangle$ and $|L\rangle$ are orthogonal and form the complete set of basis; $|R\rangle$ and $|L\rangle$ are the eigenkets of $\hat{\Sigma}_y$ with the eigenvalues +1 and -1, respectively. Thus $|R\rangle$ and $|L\rangle$ are the eigenkets of $\hat{\Sigma}_y$ with the eigenvalues +1 and -1, respectively. We use $\hat{\Sigma}_y$ instead of $\hat{\Sigma}$, because of the similarity with the Pauli matrix $\hat{\sigma}_y$

$$\hat{\Sigma}_y = \hat{\Sigma}_y(|R\rangle\langle R| + |L\rangle\langle L|) = |R\rangle\langle R| - |L\rangle\langle L|.$$

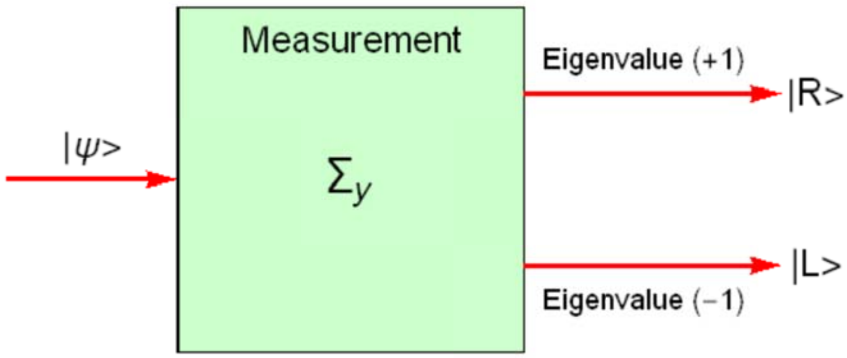


Fig. Measurement of $\hat{\Sigma}_y$. $\hat{\Sigma}_y|R\rangle = |R\rangle$. $\hat{\Sigma}_y|L\rangle = -|L\rangle$

5. Rotation operator

We now consider the rotation operator defined by $\exp(-i\hat{\Sigma}_y\theta)$

$$\exp(-i\hat{\Sigma}_y\theta)|R\rangle = e^{-i\theta}|R\rangle, \qquad \exp(-i\hat{\Sigma}_y\theta)|L\rangle = e^{i\theta}|L\rangle$$

since

$$\begin{aligned}
\exp(-i\hat{\Sigma}_y\theta) &= \hat{1} + \frac{1}{1!}(-i\theta)\hat{\Sigma}_y + \frac{1}{2!}(-i\theta)^2\hat{\Sigma}_y^{\ 2} + \frac{1}{3!}(-i\theta)^3\hat{\Sigma}_y^{\ 3} + \frac{1}{4!}(-i\theta)^4\hat{\Sigma}_y^{\ 4} + ... \\
&= \hat{1} + \frac{1}{1!}(-i\theta)\hat{\Sigma}_y + \frac{1}{2!}(-i\theta)^2\hat{1} + \frac{1}{3!}(-i\theta)^3\hat{\Sigma}_y + \frac{1}{4!}(-i\theta)^4\hat{1} + ... \\
&= \hat{1}\cos\theta - i\hat{\Sigma}_y\sin\theta \\
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}
\end{aligned}$$

This rotation operator can be also derived in a different way.

$$\exp(-i\hat{\Sigma}_y\theta) = \exp(-i\hat{\Sigma}_y\theta)[|R\rangle\langle R| + |L\rangle\langle L|]$$
$$= e^{-i\theta}|R\rangle\langle R| + e^{i\theta}|L\rangle\langle L|$$
$$= \frac{1}{2}e^{-i\theta}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} + \frac{1}{2}e^{i\theta}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

The rotation operator $\hat{S}(\theta)$ is defined by

$$\hat{S}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \hat{1}\cos\theta - i\sin\theta\hat{\Sigma}_y,$$

Note that

$$\hat{S}(\theta)|R\rangle = e^{-i\theta}|R\rangle, \qquad \hat{S}(\theta)|L\rangle = e^{i\theta}|L\rangle$$

$|R\rangle$ is the eigenket of $\hat{S}(\theta)$ with the eigenvalue $e^{-i\theta}$, and $|L\rangle$ is the eigenket of $\hat{S}(\theta)$ with the eigenvalue $e^{i\theta}$. Since the eigenket of $\hat{S}(\theta)$ is the same as that of $\hat{\Sigma}_y$, we have

$$\hat{S}(\theta)|R\rangle = (\hat{1}\cos\theta - i\sin\theta\hat{\Sigma}_y)|R\rangle = (\cos\theta - i\sin\theta)|R\rangle = e^{-i\theta}|R\rangle$$

$$\hat{S}(\theta)|L\rangle = (\hat{1}\cos\theta - i\sin\theta\hat{\Sigma}_y)|L\rangle = (\cos\theta + i\sin\theta)|L\rangle = e^{i\theta}|L\rangle$$

If we apply the rotation operator $\exp(-i\hat{\Sigma}_y\theta)$ to the ket vectors of the $\{|x\rangle$ and $|y\rangle\}$ basis, we get the rotated vectors $|\theta\rangle$ and $|\theta_\perp\rangle$.

$$\hat{S}(\theta)|x\rangle = (\hat{1}\cos\theta - i\sin\theta\hat{\Sigma}_y)|x\rangle = \cos\theta|x\rangle + \sin\theta|y\rangle = |\theta\rangle$$

$$\hat{S}(\theta)|y\rangle = (\hat{1}\cos\theta - i\sin\theta\hat{\Sigma}_y)|y\rangle = -\sin\theta|x\rangle + \cos\theta|y\rangle = |\theta_\perp\rangle$$

since

$$\hat{\Sigma}_y|x\rangle = i|y\rangle, \qquad \hat{\Sigma}_y|y\rangle = -i|x\rangle.$$

We also note that

$$
\begin{aligned}
\left|R_\theta\right\rangle &= \hat{S}(\theta)\left|R\right\rangle \\
&= \frac{1}{\sqrt{2}}\hat{S}(\theta)[\left|x\right\rangle + i\left|y\right\rangle] \\
&= \frac{1}{\sqrt{2}}[\left|\theta\right\rangle + i\left|\theta_\perp\right\rangle] \\
&= \frac{1}{\sqrt{2}}[\cos\theta\left|x\right\rangle + \sin\theta\left|y\right\rangle] + i\frac{1}{\sqrt{2}}[-\sin\theta\left|x\right\rangle + \cos\theta\left|y\right\rangle] = \\
&= \frac{1}{\sqrt{2}}(\cos\theta - i\sin\theta)\left|x\right\rangle + i\frac{1}{\sqrt{2}}(\cos\theta - i\sin\theta)\left|y\right\rangle \\
&= \frac{1}{\sqrt{2}}e^{-i\theta}[\left|x\right\rangle + i\left|y\right\rangle] \\
&= e^{-i\theta}\left|R\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
\left|L_\theta\right\rangle &= \hat{S}(\theta)\left|L\right\rangle \\
&= \frac{1}{\sqrt{2}}\hat{S}(\theta)[\left|x\right\rangle - i\left|y\right\rangle] \\
&= \frac{1}{\sqrt{2}}[\left|\theta\right\rangle - i\left|\theta_\perp\right\rangle] \\
&= \frac{1}{\sqrt{2}}[\cos\theta\left|x\right\rangle + \sin\theta\left|y\right\rangle] - i\frac{1}{\sqrt{2}}[-\sin\theta\left|x\right\rangle + \cos\theta\left|y\right\rangle] \\
&= \frac{1}{\sqrt{2}}(\cos\theta + i\sin\theta)\left|x\right\rangle - i\frac{1}{\sqrt{2}}(\cos\theta + i\sin\theta)\left|y\right\rangle \\
&= \frac{1}{\sqrt{2}}e^{i\theta}[\left|x\right\rangle - i\left|y\right\rangle] \\
&= e^{i\theta}\left|L\right\rangle
\end{aligned}
$$

Thus the ket vectors $\left|R_\theta\right\rangle$ and $\left|L_\theta\right\rangle$ differ from $\left|R\right\rangle$ and $\left|L\right\rangle$ by a phase factor only and they represent the same physical states.

6. Summary

In summary we show a list of basis which is based on the basis $\{\left|x\right\rangle$ and $\left|y\right\rangle\}$.

$$
\left|\theta = \frac{\pi}{4}\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}(\left|x\right\rangle + \left|y\right\rangle) \qquad (45°)
$$

$$\left|\theta = -\frac{\pi}{4}\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|x\rangle - |y\rangle), \qquad \text{(-45°)}$$

$$\left|\theta = \frac{3\pi}{4}\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(-|x\rangle + |y\rangle), \qquad \text{(135°)}$$

$$|R\rangle = \frac{1}{\sqrt{2}}(|x\rangle + i|y\rangle), \qquad \text{(RHC photon)}$$

$$|L\rangle = \frac{1}{\sqrt{2}}(|x\rangle - i|y\rangle). \qquad \text{(LHC photon)}$$

The rotation operator is given by

$$\hat{S}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \hat{1}\cos\theta - i\sin\theta\hat{\Sigma}_y .$$

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